Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xxxxxxx

pp. **X–XX**

ANOMALOUS TIME-SCALING OF EXTREME EVENTS IN INFINITE SYSTEMS AND BIRKHOFF SUMS OF INFINITE OBSERVABLES

STEFANO GALATOLO, MARK HOLLAND, TOMAS PERSSON AND YIWEI ZHANG

Stefano Galatolo

Dipartimento di Matematica
Via Buonattoti 1
56127 Pisa
Italy

Mark Holland

Mathematics (CEMPS)
Harrison Building (327)
North Park Road
EXETER, EX4 4QF
United Kingdom

Tomas Persson

Centre for Mathematical Sciences
Lund University
Box 118, 221 00 Lund
Sweden

Yiwei Zhang

School of Mathematics and Statistics
Center for Mathematical Sciences
Hubei Key Laboratory of Engineering Modeling and Scientific Computing
Huazhong University of Sciences and Technology
Wuhan 430074
China

(Communicated by the associate editor name)

2020 *Mathematics Subject Classification.* Primary: 37A40, 37A50, 60G70; Secondary: 37A25, 37D25.

Key words and phrases. Conservative dynamics; infinite invariant measure; Birkhoff sum; Logatithm law; hitting time; extreme values; Borel–Cantelli; intermittent system; Run length.

S. Galatolo thanks GNAMPA-INdAM for partial support during this research. M. Holland acknowledges the support by the EPSRC: EP/P034489/1, and the support from the mathematics departments at the University of Pisa, and the University of Houston. T. Persson thanks institut Mittag-Leffler and the program "Fractal Geometry and Dynamics", during which parts of this paper was written. Y. Zhang acknowledges the support by the NSFC: 11701200 and 11871262, and the support from the mathematics departments at the University of Warwick, the University of Exeter, the University of Pisa and the Centre Physique Théorique, Marseille.

Abstract. We establish quantitative results for the statistical behaviour of *infinite systems*. We consider two kinds of infinite system:

i) a conservative dynamical system (f, X, μ) preserving a σ-finite measure μ such that $\mu(X) = \infty$;
ii) the case where μ is a probability measure but we consider the statistical behaviour of an observable $\phi\colon X \to [0, \infty)$ which is non-integrable: $\int \phi \, d\mu = \infty$.

In the first part of this work we study the behaviour of Birkhoff sums of systems of the kind ii). For certain weakly chaotic systems, we show that these sums can be strongly oscillating. However, if the system has superpolynomial decay of correlations or has a Markov structure, then we show this oscillation cannot happen. In this case we prove a general relation between the behavior of ϕ, the local dimension of μ, and the scaling rate of the growth of Birkhoff sums of ϕ as time tends to infinity. We then establish several important consequences which apply to infinite systems of the kind i). This includes showing anomalous scalings in extreme event limit laws, or entrance time statistics. We apply our findings to non-uniformly hyperbolic systems preserving an infinite measure, establishing anomalous scalings for the power law behavior of entrance times (also known as logarithm laws), dynamical Borel–Cantelli lemmas, almost sure growth rates of extremes, and dynamical run length functions.

Contents

1. Introduction.

1.1. Non Integrable Observables.

We consider a dynamical system preserving a probability measure (f, X, μ), together with an observable function $\phi\colon X \to [0, \infty)$. Let us consider the case where the observable ϕ is non-integrable, i.e. $\int \phi \, d\mu = \infty$, and the Birkhoff sum

$$S_n(x) := \sum_{k=0}^{n-1} \phi(f^k(x)).$$

The pointwise ergodic theorem implies that $S_n(x)$ grows to infinity faster than any linear increasing speed, for almost each $x \in X$. For these systems, Aaronson [2, Theorem 2.3.2] has shown that for any sequence $b(n) > 0$, if $\lim_{n\to\infty} \frac{b(n)}{n} = \infty$ then either

$$\limsup_{n\to\infty} \frac{S_n(x)}{b(n)} = \infty \quad \text{a.e.} \qquad \text{or} \qquad \liminf_{n\to\infty} \frac{S_n(x)}{b(n)} = 0 \quad \text{a.e.} \tag{1}$$

Thus, for these kind of systems a kind of pointwise ergodic theorem cannot hold for the asymptotic behaviour of the ratio $\frac{S_n(x)}{b(n)}$ for every possible rescaling sequence $b(n)$.[1] It is then natural to investigate the speed of growth of such Birkhoff sums quantitatively from a coarser point of view. We approach this problem in the first part of the paper finding general estimates on the scaling behaviour of such Birkhoff sums growth. In the second part of the paper we consider applications of these studies to understand several quantitative ergodic features of systems preserving an infinite measure. We set up a general framework and give examples of application to a family of intermittent, non uniformly hyperbolic maps, finding a kind of anomalous time-scaling for several quantitative statistical properties of the dynamics related to extreme events and hitting times. The understanding of the asymptotic behaviour of Birkhoff sums of infinite observables also has other important applications. We mention as an example the works [50, 51] where this is used to estimate the speed of mixing on area preserving flows on surfaces.

To obtain estimates from above on the behaviour of S_n, the following general result is useful.

Proposition 1 (Aaronson [1, Proposition 2.3.1])**.** *If $a(x)$ is increasing,*

$$\lim_{x\to\infty} \frac{a(x)}{x} = 0$$

and

$$\int a(\phi(x)) \, d\mu(x) < \infty,$$

then for μ-a.e. $x \in X$

$$\lim_{n\to\infty} \frac{a(S_n)}{n} = 0.$$

[1] A similar phenomenon occurs for L^1 observables in systems preserving an infinite measure, see [2], or [40] for a discussion and recent developments.

Remark 1. Consider the case where $\phi(x) = d(x, \tilde{x})^{-k}$ for some $\tilde{x} \in X$ and $k \geq 0$, and denote by $d_\mu(\tilde{x})$ the local dimension of μ at $\tilde{x}$. From Proposition 1, if we let $a(x) = x^{\frac{d_\mu(\tilde{x})}{k} - \varepsilon_1}$ for some $\varepsilon_1 > 0$ we get $\int a(\phi(x))\, d\mu(x) < \infty$. This implies that for each $\varepsilon > 0$ and μ-almost all x, we have eventually (as $n \to \infty$)

$$S_n(x) \leq n^{\frac{k}{d_\mu(\tilde{x})} + \varepsilon}. \tag{2}$$

(See the proof of Theorem 2.4 for more details on this estimate).

As it will be shown in Theorem 2.5 and in Section 8, there are systems for which the asymptotic behaviour of S_n is strongly oscillating, or far from the estimate given in (2).

Thus, establishing convergence (or finding the typical growth rate) of S_n is in general non-trivial, and suitable assumptions are needed on the system to get a definite asymptotic behaviour for S_n. Lower bound estimates on the growth rates of S_n have been given in [12, 22] under assumptions related to hitting time statistics and recurrence. These assumptions include having a logarithm law for the hitting time or a dynamical Borel–Cantelli property for certain shrinking target sets. We now review these connections in greater detail.

1.1.1. *Known relations between Birkhoff sums of infinite observables, hitting times and Borel–Cantelli properties.* Our first main result, Theorem 2.4 establishes almost sure bounds on the growth rate of S_n under mild assumptions on the system (f, X, μ) and the non integrable observable function ϕ. We include the case where the system (f, X, μ) has super-polynomial decay of correlations for Lipschitz continuous functions, and allow the observable $\phi(x)$ to be quite general in the sense that merely a regularity assumption is imposed on the level sets $\{\phi(x) = u\}$. In particular our results allow for the fact these sets might not be homeomorphic to balls (in a given Riemannian metric), e.g. $\{\phi(x) \geq u\}$ might be a *tube* or another regular set.[2]

Let us now briefly discuss the hitting time scaling behaviour indicators considered in [22] and their relation with S_n. Let $B(\tilde{x}, r)$ be a closed ball with centre $\tilde{x}$ and radius r. We define the first hitting (or entrance) time of the orbit of x to $B(\tilde{x}, r)$ by

$$\tau_r(x, \tilde{x}) := \min\{\, n \in \mathbb{N} : n > 0,\ f^n(x) \in B(\tilde{x}, r)\,\}.$$

Then define the hitting time indicators as

$$\overline{H}(x, \tilde{x}) := \limsup_{r \to 0} \frac{\log(\tau_r(x, \tilde{x}))}{-\log(r)}, \qquad \underline{H}(x, \tilde{x}) := \liminf_{r \to 0} \frac{\log(\tau_r(x, \tilde{x}))}{-\log(r)}.$$

To help in understanding the sense of these definitions, we remark that according to the definitions, $\tau_r(x, \tilde{x})$ scales like $r^{H(x, \tilde{x})}$. If observables of the form $\phi(x) = d(x, \tilde{x})^{-k}$ are considered then relations between $\overline{H}$, $\underline{H}$ and the behaviour of Birkhoff sums of infinite observables are proved in [22]. Among these, it is shown that for each $\varepsilon > 0$, eventually $(n \to \infty)$,

$$S_n(x) \geq n^{\frac{k}{\overline{H}(x, \tilde{x})} - \varepsilon}$$

holds μ-a.e. We recall that $\overline{H}$ and $\underline{H}$ have been estimated in many systems (see e.g. [23, 25, 28, 29, 30, 31, 39] and references therein) and are related to the local dimension of the invariant measure in strongly chaotic systems, while in weakly

[2] Growth of sums, and related hitting statistics for these types of observables has been the focus of recent interest, see [20, 24, 37, 42].

chaotic or non chaotic ones they also have relations with the arithmetical properties of the system. In particular it is proved that in fastly mixing systems

$$\overline{H}(x,\tilde{x}) = d_\mu(\tilde{x})$$

holds for μ-a.e. x (see Proposition 7 for a precise statement) hence implying for μ-almost every x, the lower estimate

$$S_n(x) \geq n^{\frac{k}{d_\mu(\tilde{x})}-\varepsilon}$$

holds for the observable $\phi(x) = d(x,\tilde{x})^{-k}$, and for large n (compare with (2)).

In the recent paper [12], it is supposed that the system has absolutely continuous invariant measure, on a space of dimension $D \in \mathbb{N}$ and to satisfy a strong Borel–Cantelli assumption.[3] Under this assumption, it is shown that for each $\varepsilon > 0$ and μ-almost every x, for the observables of the kind $\phi(x) = d(x,\tilde{x})^{-k}$ with $k \geq 0$, we have eventually $(k \to \infty)$

$$n^{\frac{k}{D}-\varepsilon} \leq S_n(x) \leq n^{\frac{k}{D}} (\log n)^{\frac{k}{D}+\varepsilon}.$$

Other similar results are given in the case the system is exponentially mixing and the invariant measure has density in L^p, or in particular cases of intermittent maps.

In this paper we generalize this kind of results to systems having invariant measures which are not absolutely continuous and a much larger class of observables which are not necessarily related to the distance from a point. As we will see in the next section this is motivated by several applications to systems preserving an infinite measure.

1.1.2. *Growth of Birkhoff sums and extremes.* Given a measure preserving system (f, X, μ) consider the maximum process

$$M_n(x) := \max_{0\leq k\leq n-1} \phi(f^k(x)), \tag{3}$$

where $\phi\colon X \to \mathbb{R}$ is an observable function. In the case where $\phi \geq 0$ on all of X, it is clear that $S_n(x) \geq M_n(x)$. Hence $M_n(x)$ can provide a lower bound for $S_n(x)$. In [16] it is proved that if a process (X_n) is generated by i.i.d. random variables, and $\|X_1\|_1 < \infty$ then $M_n/S_n \to 0$ (μ-almost surely). Conversely, in the case of infinite observables the behaviour of M_n gives good lower bounds in many interesting systems, approaching the general upper bound given in Proposition 1. This is indeed the strategy used to get lower bounds to $S_n(x)$ in [12, 22] and in the present paper to get Theorem 2.4.

In the classical probabilistic literature the statistical properties of such M_n are of interest to those working in *extreme value theory*, [16, 21]. For dynamical systems preserving a probability measure, the distributional properties of M_n are known in some cases (e.g. [42], [19]). For certain dynamical systems, almost sure growth rates of M_n have also been investigated [22, 34, 36]. In this article, we give precise quantification on the almost sure behaviour of M_n for a general class of infinite observables. The process M_n is indeed strongly related to the hitting time $\tau_r(x,\tilde{x})$. In the case $\phi(x) = \psi(d(x,\tilde{x}))$, for some monotone decreasing function $\psi\colon [0,\infty) \to \mathbb{R}$, then the event

$$\{M_n(x) \leq u\} \tag{4}$$

[3]The reader can find in Section 2.4 precise definitions about the strong or weak Borel–Cantelli assumption. We remark that these assumptions are strictly related to the hitting time behaviour, as it is shown in [25].

corresponds to the event $\{\tau_{r(u)}(x,\tilde{x}) \geq n\}$ with $r(u) = \psi^{-1}(u)$. Hence all three processes $S_n(x), M_n(x), \tau_r(x,\tilde{x})$ are interlinked. This allows us to transfer (almost sure) statistical information from any one of these processes, to the other two. The relation between $M_n(x)$ and $\tau_r(x,\tilde{x})$ is explained in a very general setting and in more detail in Section 2.3. This construction allows us to establish new results (Theorem 2.4 and Proposition 2) on the almost sure growth of M_n for a more general class observables (not related to the distance from a given point) with respect to those considered in e.g. [12, 22, 36] and to systems having an invariant measure which is not absolutely continuous (including the case of measures having a non integer local dimension). In particular these results have relevance to the case where ϕ is a *physical observable*, see [37, 42].

1.2. **Systems preserving an infinite measure.** Based on the findings on the behaviour of Birkhoff sums and maxima of an infinite observable, we are able to address a number of relevant topics relating to systems preserving an infinite measure. We formulate a general framework, and show application to the celebrated family of "intermittent" maps studied by Manneville–Pomeau in [43], and by Liverani–Saussol–Vaienti in [41]. We focus on the case where (f, X, μ) is conservative, ergodic, and μ is σ-finite, with $\mu(X) = \infty$. The main idea here is to analyse a map induced over a finite part of the infinite system. The dynamical behaviour of the finite induced system is then easier to study and the findings can be applied to the original system, which can be seen as a suspension of the induced one (the construction is outlined at the beginning of Section 2.2). The suspension in our case will have an associated infinite observable which plays the role as the "return time function." The results motivated in the previous sections give important information in this construction, such as understanding the Birkhoff sums of this observable.[4] We find that the behaviour of the infinite observable gives a kind of "time rescaling" factor which is important in the behaviour of several quantitative ergodic aspects of the dynamics of such infinite systems. In particular we find this "anomalous scaling" in the following aspects:

The behaviour of the hitting time to small targets, and logarithm laws. Here we are interested in the time needed for a typical trajectory of the system to hit a small target which could be seen as an extreme event. Let A_n be a sequence of targets of measure going to zero and consider the hitting time to the n-th target

$$\tau(f, x, A_n) := \inf\{\, m \geq 0 : f^m(x) \in A_n \,\}.$$

It is proved (see [22, 23, 24, 27, 28, 29] and references therein) that in a wide variety of systems preserving a probability measure, a logarithm law holds

$$\lim_{n\to\infty} \frac{\log(\tau(f, x, A_n))}{-\log(\mu(A_n))} = 1, \tag{5}$$

provided the target sets A_n are regular enough, and the system is sufficiently chaotic (a precise statement of this kind is shown in Proposition 6). For infinite systems having a fast mixing first return map on a finite subspace, we show that the ratio in (5) converges to a number α that depends on the return time associated to the return map. Hence we obtain an anomalous behaviour in a wide class of infinite

[4]We remark that the return time function can be an observable with quite a complicated structure, not necessarily related to the distance from a point. Thus to get information on its Birkhoff sum, the conclusion of Theorem 2.4 is important as it extends to general observables.

systems (see Proposition 3 for a precise statement). We remark that a similar anomalous scaling was already found in quantitative recurrence indicators, in [26].

Almost sure scaling laws for the statistics of extremes in systems preserving an infinite measure. For infinite systems (f, X, μ) which are conservative, ergodic and μ is σ-finite, we consider the behaviour of maxima M_n of a given observable ϕ, see (3). As discussed in the Section 1.1.2 (see (4)), this is naturally related to the hitting times. In Proposition 5 we show a precise, quantitative link between maxima and hitting time behaviour. As a consequence we obtain an estimate for the behaviour of M_n in infinite systems (see Corollary 1). As obtained for the hitting time problems and logarithm laws, we show that the scaling behaviour of M_n depends on the return time statistics associated to the infinite measure system (in a way we make precise in Section 2), as well as on the local regularity of the observable function ϕ. This is unlike the behaviour of M_n in the probability measure preserving case. This will be done in Section 2.5. We apply our theory to a family of intermittent maps in Section 2.6.

Dynamical Borel–Cantelli laws for infinite measure preserving systems. Consider a measure preserving dynamical system (f, X, μ), and let (ϕ_n) be a sequence of observables. Furthermore let $U_n(x) := \sum_{k=0}^{n-1} \phi_k(f^k(x))$, with $\mu(\phi_k) := \int \phi_k \, d\mu$. Now suppose that $\mu(\phi_k) \to 0$, but $\sum_k \mu(\phi_k) = \infty$. A *dynamical Borel–Cantelli problem* is the problem to show existence (or otherwise) of a sequence $a_n \to \infty$ with $U_n(x)/a_n \to 1$, μ-almost surely. In the case $\int \phi_k \, d\mu = c$ for all k, we are just in a strong law of large numbers type of situation. Hence, we aim to generalise this concept in a non-stationary setting, i.e. where the observable ϕ changes with time. We address this problem for the system (f, X, μ), where μ is a σ-finite (infinite) invariant measure. In the probability preserving case, this problem has been widely studied, and forms the basis of *dynamical Borel–Cantelli Lemma* results, see [32, 34, 35, 38]. For such systems it is shown that $a_n = \sum_{k=0}^{n-1} \mu(\phi_k)$ is the typical scaling law, and this is consistent with the corresponding theory for i.i.d. random variables, see [16, 21]. For infinite systems, we show that this scaling sequence is not the appropriate one to use, and we derive the corresponding scaling law. Such a result is new, and we apply our methods to obtain shrinking target (Borel–Cantelli) results for the intermittent map family described in [41] for the σ-finite (infinite) invariant measure case. As a further application we consider infinite systems modelled by Young towers, see Section 9.

Dynamical run-length problems for infinite measure preserving systems. Suppose further that the measure preserving dynamical system (f, X, μ) admits a countable or finite partition $\{I_j\}_{j \in \mathcal{I}}$ on X (with $\mathcal{I} \subset \mathbb{N}$ an index set), and each $x \in X$ is coded with the sequence $(\varepsilon_k(x))_{k=1}^{\infty}$, by $\varepsilon_k(x) = j$ if and only if $f^{k-1}(x) \in I_j$. The *dynamical run length function of digit* j is defined by

$$\xi_n^{(j)}(x) := \max\{\, 0 \le k \le n : \exists 0 \le i \le n-k, \varepsilon_{i+1}(x) = \ldots = \varepsilon_{i+k}(x) = j \,\}. \tag{6}$$

In the setting of successive experiments of coin tossing, ξ_n corresponds to the longest length of consecutive terms of "heads/tails" up to n-times experiments [17, 46]. Thus, the studies of dynamical run length functions is concerned with quantifying the asymptotic growth behaviour of $\xi_n(x)$ for μ-typical x. Such studies admit various applications in DNA sequencing [4], finance and non-parametric statistics [6, 7, 8, 47], reliability theory [47], Diophantine approximation theory to β-expansions of real numbers [11, 18, 49], and Erdős–Rényi strong law of large numbers [14, 15, 17, 33].

We analyse the dynamical run length function in the case where (f, X, μ) is conservative, ergodic, and μ is σ-finite. In particular, we explicitly estimate the growth rate for $\xi_n(x)$ for a family of intermittent maps in the σ-finite measure case in Section 7. In contrast to probability measure preserving systems (e.g. uniformly hyperbolic Gibbs–Markov systems; logistic-like maps satisfying the Collet–Eckmann condition; and families of intermittent maps preserving a.c.i.p. [13, 14, 15, 33, 49]), we show that apart from the local dimension, there is an additional scaling contribution, arising from the asymptotics of the return time function associated to the induced transformation, which needs to be taken into account in the growth rate for ξ_n of infinite systems. As the reader will realize, our proof is based on a natural link between the dynamical run length function, hitting time, and growth of maximum for the return time functions. We are not aware of any such links, previously established in the literature of this subject.

1.3. **Outline of the paper.** We structure the paper as follows. In Section 2 we state the main theoretical results. This includes results on the growth of Birkhoff sums for rapidly mixing systems, on the link between hitting time laws and growth of extremes, and dynamical Borel–Cantelli Lemma results for systems preserving a σ-finite infinite measure. In Sections 3 and 5 we prove these results, and then consider several independent topics which relate to our theory. This includes a result on the almost sure growth rates of extremes and hitting times for infinite systems, see Section 2.5. We then apply our theory to an intermittent map case study in Section 2.6, which includes a study of dynamical run-length problems in Section 2.6.3. We then describe situations in which the Birkhoff sums can wildly oscillate in Section 8. Finally we consider Borel–Cantelli results for general Markov extensions, such as Young towers (Section 9).

2. Statement of main results.

2.1. **Birkhoff sums, maxima, and hitting time statistics.** Let us consider a metric space (X, d), consider on the space X a dynamical system preserving a probability measure (f, X, μ), together with an observable function $\phi\colon X \to [0, \infty)$ with $\int \phi \, d\mu = \infty$. Let us recall the notation used for Birkhoff sums and respectively maxima of an observable ϕ.

$$S_n(x) := \sum_{k=0}^{n-1} \phi(f^k(x)), \qquad M_n(x) := \max_{0 \leq k \leq n-1} \phi\big(f^k(x)\big).$$

In specific contexts, we sometimes emphasize the dependence on ϕ, and write $S_n^{\phi}(x)$ for $S_n(x)$ (and similarly for maxima).

As noted before (see (1)) it is impossible to get precise estimates for the asymptotic behaviour of S_n as n increases. However, under suitable assumptions on ergodicity and on the chaotic properties of the system, coarser estimates on asymptotic growth rates are possible.

We show that we can achieve estimates for the scaling behaviour of both S_n and M_n for systems which are super-polynomially mixing, and for quite a large class of observables having some regularity. The regularity we need is a kind "Lipschitz" regularity of the suplevels of the observable ϕ. This is explained in the next definition. Essentially we ask that the suplevels of ϕ are regular enough that they could be sublevels of a Lipschitz function.

Definition 2.1. Let $\phi\colon X \to [0,\infty]$ be an unbounded function. Consider the suplevels of ϕ, defined by

$$A_n = \{\, x \in X : \phi(x) \geq n \,\}.$$

We say that ϕ has regular suplevels if the following holds:

(i) We have $\lim_{n\to\infty} \mu(A_n) = 0$, there is a constant $c > 0$ satisfying $\mu(A_{n+1}) > c\mu(A_n)$ eventually as n increases, and there is $\alpha_\phi \in [0,\infty)$, such that

$$\alpha_\phi = \lim_{n\to\infty} \frac{\log \mu(A_n)}{-\log(n)}.$$

(ii) There is $\beta \geq 0$ and a Lipschitz function $\tilde{\phi}\colon X \to \mathbb{R}^+$ such that

$$A_n = \{\, x \in X : \tilde{\phi}(x) \leq (\mu(A_n))^\beta \,\}.$$

This assumption is verified by a large class of observables, including observables related to the distance from a point.

Example 1. Suppose X is a Riemannian manifold with boundary, and $d(.,.)$ the Riemannian distance. For $\tilde{x} \in X$ consider an observable of the form $\phi(x) = d(x,\tilde{x})^{-k}$ with $k \geq 0$ in a neighbourhood of $\tilde{x}$, and suppose $d_\mu(\tilde{x})$ exists and $d_\mu(\tilde{x}) > 0$. Such conditions are verified for almost each $\tilde{x}$ in a wide class of uniformly and non-uniformly hyperbolic systems, see [45]. Then we have that A_n is a ball of radius $r_n = \frac{1}{\sqrt[k]{n}}$, and hence a regular set. In this case

$$\alpha_\phi := \lim_{n\to\infty} \frac{\log \mu(\{x \in X : \phi(x) \geq n\})}{-\log(n)} = \frac{d_\mu(\tilde{x})}{-k},$$

and for each $\varepsilon > 0$, eventually $n^{\frac{d_\mu(\tilde{x})+\varepsilon}{-k}} \leq \mu(A_n) \leq n^{\frac{d_\mu(\tilde{x})-\varepsilon}{-k}}$. Consider β such that $\beta d_\mu(\tilde{x}) > 1$, and $\tilde{\phi}(x)$ defined as

$$\tilde{\phi}(x) = \mu(A_i)^\beta + \frac{\mu(A_{i-1})^\beta - \mu(A_i)^\beta}{r_{i-1} - r_i}(d(\tilde{x},x) - r_i),$$

when $d(\tilde{x},x) \in [r_i, r_{i-1})$. Since $\frac{\mu(A_i)^\beta}{r_i} \leq i^{\frac{\beta(d_\mu(\tilde{x})-\varepsilon)-1}{-k}}$ is bounded for our choice of β, the function $\tilde{\phi}$ is Lipschitz.

Other examples include cases where the suplevels correspond to tubes or other sets, see [23, 27] for results about hitting times on targets which are suplevels of a Lipschitz function, applied to the geodesic flow, in which the targets relate to "cylinders" in the tangent bundle instead of balls. Thus the regular sublevels assumption of Definition 2.1 holds for a wide class of dynamical systems and observable geometries. We now consider the notion of decay of correlations.

Definition 2.2. Let $\mathcal{B}$ be a Banach space of functions from X to $\mathbb{R}$. A measure preserving system (f, X, μ) is said to have decay of correlations in $\mathcal{B}$ with rate function $\Theta(n)$, if for $\varphi, \psi \in \mathcal{B}$, we have

$$\left| \int \varphi \circ f^n \psi \, d\mu - \int \varphi \, d\mu \int \psi \, d\mu \right| \leq \|\varphi\| \|\psi\| \Theta(n). \tag{7}$$

Here $\|\cdot\|$ stands for the norm on $\mathcal{B}$.

Usually decay of correlations is proved for a particular space $\mathcal{B}$, and a specified rate $\Phi(n)$.

Definition 2.3 (Condition (SPDCL))**.** We say that (f, X, μ) satisfies condition (SPDCL) (Super-Polynomial Decay of Correlations with respect to Lipschitz functions) if the system satisfies (7) and for all $p > 0$, we have $\lim_{n\to\infty} n^p \Theta(n) = 0$, and $\mathcal{B}$ is the Banach space of Lipschitz continuous functions, considered with the Lipschitz norm. [5]

This condition is quite general, and many systems having some form of piecewise hyperbolic behaviour satisfy it. See [5] for a survey containing a list of classes of examples having exponential or stretched exponential decay of correlations. We remark that if a system has a certain decay of correlations with respect to Hölder observables, then it will have the same or faster speed when smoother observables (such as Lipschitz ones) are considered.

Now, suppose the non-integrable observable ϕ, has regular suplevels as in Definition 2.1. The following theorem concerns the growth of maxima and Birkhoff sums of ϕ.

Theorem 2.4. *Let (X, f, μ) be a probability measure preserving system satisfying condition (SPDCL). Let ϕ, A_n and α_ϕ be as in Definition 2.1, with $\|\phi\|_1 = \infty$. If $\alpha_\phi > 0$, then for each $0 < \varepsilon < \alpha_\phi$ and μ-a.e. $x \in X$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,*

$$n^{\frac{1}{\alpha_\phi+\varepsilon}} \leq M_n(x) < S_n(x) \leq n^{\frac{1}{\alpha_\phi-\varepsilon}}.$$

If $\alpha_\phi = 0$, then for μ-a.e. $x \in X$

$$\lim_{n\to\infty} \frac{\log(S_n(x))}{\log n} = \lim_{n\to\infty} \frac{\log(M_n(x))}{\log n} = \infty.$$

This theorem therefore applies to a wide class of observable geometries. In the case where ϕ is related to the distance from a point, or for particular dynamical systems, see [22, 30] or [36, Theorem 2.5], or [12, Section 2.3]. Particular systems that are captured by the theory include Hénon maps [9], and certain Poincaré return maps for Lorenz attractors [29], to name a few. The proof of Theorem 2.4 can be found in Section 3.

Notation. A statement of the form $v(n) \sim u(n)$ as $n \to \infty$ means that there is a constant $c > 0$ such that

$$c^{-1} \leq \frac{v(n)}{u(n)} \leq c$$

holds for sufficiently large n.

2.1.1. *Gibbs–Markov systems.* Theorem 2.4 shows how, with some strong assumptions on the system, we can get information on the scaling behaviour of S_n. We will see (Proposition 2) that if we assume some even stronger assumptions on the system, as the presence of a Gibbs–Markov structure, we can get even more precise estimates.

Consider again a transformation (f, X, μ) and an observable $\phi\colon X \to [0, \infty)$ with $\int \phi \, d\mu = \infty$. We say that (f, X, μ) is a Gibbs–Markov system [2] if we have the following set up.

[5] For each Lipschitz function φ, the Lipschitz norm $\|\varphi\|_{\mathrm{Lip}} = \mathrm{Lip}(\varphi) + \|\varphi\|_\infty$, with $\mathrm{Lip}(\varphi) := \sup\{ \frac{|\varphi(x)-\varphi(y)|}{\|x-y\|} : x \neq y \in X \}$.

(A1) X is an interval and there is a countable Markov partition $\mathcal{P} = \{ X_i : i \in \mathbb{N} \}$ such that $f(X_i)$ contains a union of elements of $\mathcal{P}$, and there exists $c_0 > 0$ such that $|f(X_i)| > c_0$. Let $\mathcal{P}_n = \bigvee_{i=0}^{n} f^{-i}\mathcal{P}$.

(A2) There exists $\lambda > 1$, $C > 0$ such that for all $x \in X$ we have $|(f^n)'(x)| \geq C\lambda^n$ for $n \geq 1$.

(A3) Uniform bounded distortion estimates hold on $\omega \in \mathcal{P}_n$. That is, there exist $0 < \tau < 1$, $C > 0$, such that for all $x, y \in \omega$, and $\forall \omega \in \mathcal{P}_n$,

$$\left| \log\left(\frac{|f'(x)|}{|f'(y)|} \right) \right| \leq C\tau^n.$$

(A4) The measure μ is the unique invariant probability measure which is absolutely continuous with respect to Lebesgue measure.

For a Gibbs–Markov system (f, X, μ) satisfying (A1)–(A4), we will use the following assumptions on the observable ϕ.

(A5) For any $X_i \in \mathcal{P}$ the restriction $\phi|_{X_i}$ of ϕ to X_i is constant.

(A6) The observable ϕ satisfies the following asymptotics: there exists $\beta \in (0, 1)$ such that

$$\mu\{ x \in X : \phi(x) \geq u \} \sim u^{-\beta}, \quad (u \to \infty).$$

Remark 2. Note that assumption (A6) implies that the observable ϕ is non-integrable, $\int \phi(x)\, d\mu = \infty$.

If (f, X, μ) is a Gibbs–Markov maps satisfying assumptions (A1)–(A6), we are able to obtain a result on the asymptotic speed of the typical growth of Birkhoff sums of ϕ, which we will now state. Our result below is similar to a result by Carney and Nicol [12, Theorem 4.1], and the proofs are also similar. Carney and Nicol assumed that the system satisfies a strong Borel–Cantelli lemma, but we do not assume this explicitly.

Proposition 2. *Let (f, X, μ) satisfy assumptions (A1)–(A6) , with β as in assumption (A6). Then each $\varepsilon > 0$, and μ-almost all $x \in X$ there is an N_x such that for all $n > N_x$,*

$$n^{\frac{1}{\beta}} (\log n)^{-\frac{1}{\beta}-\varepsilon} \leq M_n(x) < S_n(x) \leq n^{\frac{1}{\beta}} (\log n)^{\frac{1}{\beta}+\varepsilon}.$$

2.1.2. *Oscillating Birkhoff sums.* Proposition 1 gives us a general upper bound on the increase of S_n. It does not depend on quantitative properties of the dynamical system but appears to be near to an optimal estimate in many strongly chaotic systems, see [12] for a discussion. However, there are chaotic systems for which the bound we obtain from (2) is far from the actual behaviour of S_n, and there are examples in which their Birkhoff sum is *strongly oscillating.* A family of such examples take the form of a skew product map $f\colon [0, 1] \times S^1 \to [0, 1] \times S^1$ defined by

$$f(x, t) = (T(x), t + \theta\eta(x)), \tag{8}$$

where $T(x)$ is a uniformly expanding interval map, S^1 is the circle, $\theta \in [0, 1]$ an irrational number, and $\eta(x)$ a specified "skewing" function. We state the following Theorem, whose proof, and precise form of $f(x, t)$ is described in Section 8.

Theorem 2.5. *There are measure preserving systems (f, X, μ) of the skew product form (8) which preserve a probability measure μ and have polynomial decay of* *correlations on Lipschitz functions. Moreover for a non integrable observable of the kind* $\phi(x) = d(x, \tilde{x})^{-k}$*, for some* $\tilde{x} \in X$ *and* $k > 0$*, we have*

$$\liminf_{n\to\infty} \frac{\log S_n(x)}{\log n} < \limsup_{n\to\infty} \frac{\log S_n(x)}{\log n},$$

for μ*-a.e.* $x \in X$*. Furthermore there are measure preserving systems with polynomial decay of correlations where even the limsup, and power law behaviour of the Birhkhoff sums does not follow the ratio* $\frac{k}{d_\mu(\tilde{x})}$ *suggested by* (2)*. In such systems for* μ*-a.e.* $x \in X$

$$\limsup_{n\to\infty} \frac{\log S_n(x)}{\log n} < \frac{k}{d_\mu(\tilde{x})}.$$

2.2. **Application to extreme events and hitting times in systems having a fast mixing return map.** The estimates on Birkhoff sums of infinite observables are useful to investigate quantitative aspects of the dynamics of systems preserving an infinite measure. Consider an infinite system (f, X, μ), where μ is assumed to be infinite but σ-finite. A classical approach to study such an infinite system is by inducing the dynamics on a subset Y of positive (finite) measure. Let R be the return time function to the domain Y, that is for $x \in Y$,

$$R(x) = \min\{\, n > 0 : f^n(x) \in Y \,\}.$$

Then $f^R \colon x \mapsto f^{R(x)}(x)$ defines a dynamical system (f^R, Y) which preserves the measure $\mu_Y = \mu|_Y$. (We shall now denote f^R by f_Y). The system (f_Y, Y, μ_Y) is called the induced system. It is a finite measure preserving system, and its dynamics gives information on the original infinite system. The original system can be seen as a suspension on the induced system, that is if we define $\hat{f} \colon \hat{X} \to \hat{X}$ by

$$\hat{X} = \{\, (x, n) \in Y \times \mathbb{N} : 0 \leq n < R(x) \,\},$$

and

$$\hat{f}(x, n) = \begin{cases} (x, n+1) & \text{if } n + 1 < R(x), \\ (f_Y(x), 0) & \text{if } n + 1 = R(x), \end{cases} \tag{9}$$

then $(\hat{f}, \hat{X})$ is a suspension of (f_Y, Y) and $(\hat{f}, \hat{X}, \hat{\mu})$ is isomorphic to (f, X, μ) if $\hat{\mu}$ is defined in the natural way, e.g. see [54].

Here a major role is played by the return time function R. In this case R will be a non-integrable observable on (f_Y, Y, μ_Y), and to this situation we can apply the findings of the previous section. We remark that the observable $\phi \equiv R$ is not necessarily related to the distance from a certain point. In particular if we are interested in hitting time or extreme problems then the asymptotic behaviour of the Birkhoff sums S_n^R of the return time in the induced system is particularly important. We show in the following Sections 2.2.1 and 2.3 that the behaviour of S_n^R implies anomalous scaling behaviour for the hitting time to small targets, and for growth of extreme events.

2.2.1. *Logarithm law and the anomalous hitting time behaviour in infinite systems.* We derive a limit (logarithm) law for the hitting time function. First, recall some definitions relating to the hitting time to general targets and logarithm laws in this context. Consider a dynamical system (f, X, μ) on a metric space X. Let $B_n \subseteq X$ be a decreasing sequence of targets; let us consider the hitting time of the orbit starting from $x \in X$ to the target B_n

$$\tau(f, x, B_n) = \min\{\, n \geq 0 : f^n(x) \in B_n \,\}$$

(In case the map considered is obvious from the context, instead of $\tau(f, \cdots)$ we may write $\tau(\cdots)$ for simplicity.)

The classical logarithm law results relates the hitting time scaling behaviour to the measure of the targets. In many cases when f preserves a probability measure μ and the system is chaotic enough, or it has generic arithmetic properties then the following holds for μ-a.e. $x \in X$

$$\lim_{n\to\infty} \frac{\log \tau(f, x, B_n)}{\log n} = \lim_{n\to\infty} \frac{\log(\mu(B_n))}{-\log n}. \tag{10}$$

In words: *the hitting time scales as the inverse of the measure of the targets* (compare with (5)).

We see that *in systems preserving an infinite measure* this law does not hold anymore, but under some chaoticity assumptions we can replace the equality (10), with a rescaled version of it. In fact, the rescaling factor depends on the return time behaviour of the system on some subset $Y \supseteq B_n$ containing the target sets B_n.

Suppose (f, X, μ) preserves an infinite measure μ, $Y \subseteq X$ is such that $\mu(Y) < \infty$, and consider $B_n \subseteq Y$. The following holds.

Proposition 3. *Let (f, X, μ) be a dynamical system preserving an infinite measure μ. Let (f_Y, Y, μ_Y) be the induced system over a domain Y of finite positive measure, preserving a probability measure $\mu_Y = \mu|_Y$, and with return time function $R\colon Y \to \mathbb{N}$. Suppose R has regular suplevels with associated exponent α_R (see Definition 2.1). Suppose that (f_Y, Y, μ_Y) satisfies Condition (SPDCL). Let $B_n \subseteq Y$ be a decreasing sequence of targets also satisfying items (i) and (ii) of Definition 2.1. Consider $\alpha_B \geq 0$, such that $\alpha_B = \lim_{n\to\infty} \frac{\log \mu(B_n)}{-\log n}$. Then for μ-a.e. $x \in X$,*

$$\lim_{n\to\infty} \frac{\log \tau(f, x, B_n)}{\log n} = \frac{\alpha_B}{\alpha_R}.$$

The proof of Proposition 3 is in Section 4.1. In the next section achieve a corresponding statement for the maxima process M_n.

2.3. On the link between almost sure growth of maxima and hitting time laws.

Suppose $X \subset \mathbb{R}^d$, and consider a sequence of functions $H_n\colon X \to \mathbb{R}$. For $x \in X$ consider the following maximum function sequence and corresponding hitting time function sequence defined by

$$\tilde{M}_n(x) := \max_{0\leq k<n} H_k(x), \quad \tau_u(x) = \min\{\, n \geq 0 : H_n(x) \geq u \,\}.$$

Examples include the case where we have a probability space $(X, \mathcal{B}_X, \mu)$, with $\mathcal{B}_X$ the σ-algebra of subsets of X, μ a probability measure, and (H_n) a sequence of random variables. Another case includes that of a measure preserving system (f, X, μ), where we set $H_n(x) = \phi(f^n(x))$, with specified observable function $\phi\colon X \to \mathbb{R}$. In this latter case, $\tilde{M}_n(x)$ coincides with the usual definition of M_n given in (3).

In this section, we derive a precise link between the growth rate of $\tilde{M}_n(x)$ (as $n \to \infty$), and the growth rate of $\tau_u(x)$ (as $u \to \infty$). We'll assume further that either $\tilde{M}_n(x) \to \infty$ as $n \to \infty$, or $\tau_u(x) \to \infty$ as $u \to \infty$.

First we make the basic observation that the event $\{\tilde{M}_n \leq u\}$ is the same as $\{\tau_u \geq n\}$. We state our first elementary result.

Proposition 4. *Suppose $X \subset \mathbb{R}^d$, and consider the sequence of functions $H_n\colon X \to \mathbb{R}$.*

1. *Suppose that* $\ell_1, \ell_2 \colon [0,\infty) \to [0,\infty)$ *are monotone increasing functions, such that* $\ell_1(u)$, $\ell_2(u) \to \infty$ *as* $u \to \infty$. *Suppose for given* $x \in X$, *there exists* $N(x) > 0$, *such that for all* $n \geq N$ *we have* $\ell_1(n) \leq \tilde{M}_n(x) \leq \ell_2(n)$. *Then there exists* $u_0(x)$, *such that for all* $u \geq u_0$ *we have*
$$\ell_2^{-1}(u-1) \leq \tau_u(x) \leq \ell_1^{-1}(u+1).$$
2. *Suppose that* $\hat{\ell}_1, \hat{\ell}_2 \colon [0,\infty) \to [0,\infty)$ *are monotone increasing functions, such that* $\hat{\ell}_1(u)$, $\hat{\ell}_2(u) \to \infty$ *as* $u \to \infty$. *Suppose that for given* $x \in X$, *there exists* $u_0(x) > 0$, *such that for all* $u \geq u_0$, *we have* $\hat{\ell}_1(u) \leq \tau_u(x) \leq \hat{\ell}_2(u)$. *Then there exists* $N(x) > 0$, *such that*
$$\hat{\ell}_2^{-1}(n) \leq \tilde{M}_n(x) \leq \hat{\ell}_1^{-1}(n).$$

Remark 3. In the statement of Proposition 4, we do not assume that X is a measure space. In the case where (H_n) is a stationary process, defined on a suitable measure space, then Proposition 4 asserts that almost sure bounds for $\tilde{M}_n$ imply almost sure bounds for τ_u, and vice versa. We will use this fact in our dynamical systems applications.

Proposition 4 is proved in Section 4.2. We now consider specific applications of this result. For a measure preserving dynamical system (f, X, μ) define
$$\tau_u^\phi(x) = \inf\{\, n : \phi(f^n(x)) \geq u \,\}, \tag{11}$$
and put $\tilde{M}_n(x) = M_n(x)$, where we recall $M_n(x) = \max_{k \leq n-1} \phi(f^k(x))$. Here $\phi \colon X \to \mathbb{R}$ is an observable function. Examples include $\phi(x) = -\log d(x, \tilde{x})$ or $\phi(x) = d(x, \tilde{x})^{-1}$, for a given $\tilde{x} \in X$, but we have seen that our theory allows us to consider much more general cases.

2.3.1. *The logarithm law for hitting times and maxima.* We now consider the *logarithm law*, especially for the hitting time function. We show via Proposition 4 that a logarithm law for hitting time implies a logarithm law for maxima (and conversely). Again this is a pointwise result. See [30, Proposition 11] for a similar statement.

Proposition 5. *Consider a dynamical system* (f, X). *Suppose that* $0 < a_1 < a_2 < \infty$, *and* $x \in X$. *Then we have the following implications.*
$$\limsup_{n\to\infty} \frac{\log[M_n(x)]}{\log n} = a_1 \quad \Longleftrightarrow \quad \liminf_{u\to\infty} \frac{\log[\tau_u^\phi(x)]}{\log u} = \frac{1}{a_1},$$
$$\liminf_{n\to\infty} \frac{\log[M_n(x)]}{\log n} = a_2 \quad \Longleftrightarrow \quad \limsup_{u\to\infty} \frac{\log[\tau_u^\phi(x)]}{\log u} = \frac{1}{a_2}.$$
Moreover, if $a_1 = a_2$, *then*
$$\lim_{u\to\infty} \frac{\log[\tau_u^\phi(x)]}{\log u} = \left(\lim_{n\to\infty} \frac{\log[M_n(x)]}{\log n} \right)^{-1}$$
provided the corresponding limits exist at x.

Remark 4. In Proposition 5, the logarithm function diminishes any behaviour associated to sub-polynomial corrections associated to the growth of M_n^ϕ (as $n \to \infty$), or to that of τ_u^ϕ (as $u \to \infty$). In certain cases, this subpolynomial growth can be further quantified as we discuss below.

Proposition 5 is proved in Section 4.2. For infinite systems, we state the following corollary concerning the almost sure behaviour of the maxima process.

Corollary 1. *Let (f, X, μ) be a dynamical system preserving an infinite measure μ. Let (f_Y, Y, μ_Y) and R be as in Proposition 3. Consider $\phi\colon X \to [0, \infty)$, a function which is not bounded and that $\phi|_{X\setminus Y}$ is bounded. Suppose that also ϕ has regular suplevels. Then*

$$\lim_{n\to\infty} \frac{\log[M_n(x)]}{\log n} = \frac{\alpha_R}{\alpha_\phi}$$

holds for μ-a.e. $x \in X$.

The proof of Corollary 1 can be found in Section 4.2. In Corollary 1, we have assumed (SPDCL) for (f_Y, Y, μ_Y). In the case where (f_Y, Y, μ_Y) satisfies stronger hypotheses, such as being Gibbs–Markov, then we can obtain stronger bounds on almost sure behaviour of the maxima function (as $n \to \infty$), and also the hitting time function via Proposition 4. We remark further that Corollary 1 gives almost sure bounds on the maxima process in the case of observables having general geometries (beyond functions of distance to a distinguished point). Thus if we know (almost sure) bounds on the hitting time function, then we get corresponding bounds for the maxima process via Proposition 4 (or 5). This result allows us to address a question posed in e.g. [36, Section 6] concerning the existence of an almost sure behaviour of maxima for general observables (that are not solely a function of distance to a distinguished point).

2.3.2. *On finding precise asymptotics on the maxima and hitting time functions.* For certain stationary processes (or dynamical systems), the rate functions $\ell_1(n), \ell_2(n)$ as appearing in Proposition 4 can be optimised. For i.i.d. processes (H_n), optimal expressions for these functions are given in e.g. [16, 21]. For dynamical systems having exponential decay of correlations, higher order corrections to the almost sure maxima function growth (beyond that given by a standard logarithm law in Proposition 5) are discussed in e.g. [34, 36]. To translate such results to almost sure behaviour of hitting times, then inversion of the functions $\ell_1(n), \ell_2(n)$ is required. This we now discuss via an explicit example. Generalisations just depend on an analysis of the functional forms of $\ell_1(n)$ and $\ell_2(n)$.

Consider the *tent map* $T_2(x) = 1-|2x-1|$, $x \in [0, 1]$, and the observable function $\phi(x) = -\log d(x, \tilde{x})$. It is shown that there exist explicit constants $c_1, c_2 > 0$ such that for Lebesgue-a.e. $\tilde{x} \in [0, 1]$

$$\log n - c_1 \log\log n \leq M_n(x) \leq \log n + c_2 \log\log n,$$

eventually in n for Lebesgue-a.e. $x \in [0, 1]$, see [36]. We deduce the following asymptotics for the hitting time function. A proof is given in Section 4.2.

Lemma 2.6. *Consider the tent map $T_2\colon [0, 1] \to [0, 1]$, and observable $\phi(x) = -\log d(x, \tilde{x})$. Then for all $\tilde{x} \in [0, 1]$, and μ-a.e. $x \in [0, 1]$, there exists $u_0 > 0$ such that for all $u \geq u_0$*

$$\log \tau_u^\phi(x) = u + O(\log u).$$

Here the $O(\cdot)$ constant depends on $x \in [0, 1]$, and on c_1, c_2.

Remark 5. We immediately deduce a logarithm law for entrance to balls $B(\tilde{x}, r)$ with an error rate. In particular if we let $u = (-\log r)^{-1}$, then $\tau^\phi_{u(r)}(x, \tilde{x}) = \inf\{\, n : d(x, \tilde{x}) \leq r \,\}$. Then by Lemma 2.6, we obtain for μ-a.e. $x \in X$ that

$$\log \tau^\phi_{u(r)}(x) = -\log r + O(\log\log r).$$

In the example above, we used a higher order asymptotic on the growth rate for the maxima to deduce a similar asymptotic for the hitting time function. We note that a converse result applies if we have knowledge of such asymptotics for the almost sure growth of the hitting time function, but no apriori bounds for the growth of the maxima function. For either the hitting time function, or maxima function almost sure growth rates are usually deduced via Borel–Cantelli arguments, see [21, 34, 36] for maxima, and [25] for hitting times. We elaborate in Section 2.5. Thus, Proposition 4 allows us to translate limit laws between maxima and hitting times without too much extra work, except for estimating inverses of the corresponding rate functions. We remark that in the case of distributional limits for maxima and hitting times (as opposed to almost sure bounds), a relation between their limit laws is described in [19, 20].

2.4. **Dynamical Borel–Cantelli Lemmas for infinite measure preserving systems.** For a (probability) measure preserving dynamical system (f, X, μ), a dynamical Borel–Cantelli Lemma result asserts that for a sequence of sets (B_n) with $\sum_n \mu(B_n) = \infty$, we have

$$\mu\Big(\bigcap_{i=1}^{\infty}\bigcup_{n=i}^{\infty}\{\, x : f^n(x) \in B_n \,\}\Big) = 1,$$

i.e. $\mu\{\, x \in X : f^n(x) \in B_n, \text{infinitely often} \,\} = 1$. A quantitative version leads to having the *strong Borel–Cantelli* (SBC) property defined as follows. Given a sequence of sets (B_n) with $\sum_n \mu(B_n) = \infty$, let $E_n = \sum_{k=0}^{n-1} \mu(B_k)$.

Definition 2.7. We say that (B_n) satisfies the strong Borel–Cantelli property (SBC) if for μ-a.e. $x \in X$

$$\lim_{n\to\infty} \frac{S_n(x)}{E_n} = 1,$$

where $S_n(x) = \sum_{k=0}^{n-1} 1_{B_k}(f^k x)$, and $1_{B_k}(x)$ denotes the indicator function on the set B_k.

For dynamical systems preserving a probability measure μ, (SBC) results are now known to hold for various systems, see [34, 35, 36, 38]. Here, we derive corresponding Borel–Cantelli results for infinite systems (f, X, μ), with μ a σ-finite measure, and $\mu(X) = \infty$.

We consider a conservative, ergodic system (f, X, μ), and suppose there exists $Y \subset X$ for which the induced system (f_Y, Y, μ_Y) is Gibbs–Markov (see Sections 2.1, 2.2 for conventions), but now the return time function $R\colon Y \to Y$ is not integrable with respect to μ_Y. In the case of integrable return times, [38, Theorem 3.1] established strong Borel–Cantelli results for the system (f, X, μ) assuming strong Borel–Cantelli results for the induced system (f_Y, Y, μ_Y). Formally, consider a function sequence p_j with $\sum_j \mu(p_j) = \infty$, where $\mu(p_j) = \int p_j(x)\, d\mu$. We say that the strong Borel–Cantelli property holds for this sequence, with respect to (f_Y, Y, μ_Y), if (necessarily) $\sum_{j=1}^{n} \mu(p_{Q(j,x)}) \to \infty$ as $n \to \infty$ and

$$\frac{\sum_{j=1}^{n} p_{Q(j,x)}(f_Y^j(x))}{\sum_{j=1}^{n} \mu(p_{Q(j,x)})} \to 1, \tag{12}$$

where $Q(j, x) = \sum_{i=0}^{j} R(f_Y^i(x))$ is the total clock time associated to the j'th return to the base. We now state the corresponding dynamical Borel–Cantelli result as applicable for infinite systems.

Theorem 2.8. *Suppose that (f, X, μ) is ergodic and conservative, and that the induced system (f_Y, Y, μ_Y) satisfies assumptions (A1)–(A6), (where observable $\phi(x)$ is identified with $R(x)$) and β given by (A6). Let (p_n) be a sequence of non-negative functions which satisfy $p_1 \geq p_2 \geq \dots$, and assume further that $\mathrm{supp}(p_n) \subset Y$. We have the following cases.*

1. *Suppose that there exists $\varepsilon_1 > 0$ such that $\sum_{n\geq 1} \mu(p_{n^{\frac{1}{\beta}+\varepsilon_1}}) = \infty$. If every subsequence p_{n_k} with $\sum \mu(p_{n_k}) = \infty$ is a strong Borel–Cantelli sequence with respect to $f_Y(x)$, then we have for all $\varepsilon \in (0, \varepsilon_1]$, and eventually as $n \to \infty$, that*
$$\sum_{k=1}^{n^{\frac{1}{\beta}-\varepsilon}} \mu\Big(p_{k^{\frac{1}{\beta}+\varepsilon}}\Big) \leq \sum_{k=1}^{n} p_k(f^k(x)) \leq \sum_{k=1}^{n^{\frac{1}{\beta}+\varepsilon}} \mu\Big(p_{k^{\frac{1}{\beta}-\varepsilon}}\Big), \tag{13}$$
for μ-a.e. $x \in X$.
2. *Suppose there exists $\varepsilon_2 \in (0, \frac{1}{\beta})$ such that $\sum_{n\geq 1} \mu(p_{n^{\frac{1}{\beta}-\varepsilon_2}}) < \infty$, then*
$$\lim_{n\to\infty} \sum_{k=1}^{n} p_k(f^k(x)) < \infty, \qquad \textit{for } \mu\textit{-a.e. } x \in X.$$

(In the statement of the theorem, $p_{g(n)}$ should be interpreted as $p_{[g(n)]}$, where $[\cdot]$ denotes the integer part.)

We make several remarks and discuss immediate consequences of Theorem 2.8. Firstly, with slightly more effort, it is possible in Item (1) to replace the correction by $\pm\varepsilon$ in the exponents with corrections by logarithms. In some cases, if Y is a Darling–Kac set, then we can get an even more precise upper bound, see Aaronson–Denker [3] and the proof of Theorem 2.8 for more details.

A condition of Theorem 2.8 is that we must assume that
$$\sum_{n\geq 1} \mu\Big(p_{n^{\frac{1}{\beta}+\varepsilon_1}}\Big) = \infty$$
for some $\varepsilon_1 > 0$. This puts a restriction on the sequence of functions p_n. Indeed it is not difficult to construct sequences with $\sum_{n\geq 1} \mu(p_n) = \infty$, but $\sum_{n\geq 1} \mu(p_{n^{\frac{1}{\beta}+\varepsilon}}) < \infty$. If for example $\mu(p_n) = n^{-\zeta}$, $(\zeta > 0)$, then we require $\zeta < \beta(1 + \varepsilon\beta)^{-1}$. In the case $\beta \to 1$, we find that any $\zeta < 1$ will do. In the case of Item (2), if
$$\sum_{n\geq 1} \mu\Big(p_{n^{\frac{1}{\beta}-\varepsilon_2}}\Big) < \infty,$$
(for some $\varepsilon_2 > 0$), then via an argument using the first Borel–Cantelli Lemma we show that for μ-a.e. $x \in X$ we have
$$\sum_{k=1}^{\infty} p_k(f^k(x)) < \infty.$$
We remark that the bounds given in equation (13) appear mysterious at first glance. In the case where $\mu(p_n)$ is described by a functional sequence $g(n) = \mu(p_n)$, with $g\colon (0, \infty) \to (0, \infty)$ a monotone decreasing real valued function, then a simple change of variable argument implies that the bounds in equation (13) can be written as
$$\sum_{k=1}^{n^{\frac{1}{b}}} \mu(p_{k^b}) \sim \sum_{k=1}^{n} k^{1-\frac{1}{b}} g(k),$$

with $b = \frac{1}{\beta} \pm \varepsilon$ accordingly. Furthermore, in the case μ is a probability measure then, (as established in [38]) the usual strong Borel–Cantelli property holds:

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} p_k(f^k(x))}{\sum_{k=1}^{n} \mu(p_k)} = 1.$$

In this case, the ε in Theorem 2.8 is not needed. The boundary case arises when $\beta = 1$ (and $\int R\, d\mu = \infty$). Here ε is still required in equation (13), in the sense that we have not proved a result without the ε. As in [38], the Gibbs–Markov assumption is then not required in the case μ is a probability measure. We require the Gibbs–Markov assumption to get quantitative (lower) bounds on the maximum growth of the return time function, see Lemma 5.1.

In the case where f_Y is a Gibbs–Markov map, or non-uniformly expanding map with fast decay of correlations, the sequence of functions (p_j) that lead to the strong Borel–Cantelli property include indicator functions of balls. More general classes of functions may also lead to the strong Borel–Cantelli property, see e.g. [32, 34, 38].

A further question that arises is what can be said about dynamical Borel–Cantelli results for functions (p_n) no longer supported on Y? In general, Theorem 2.8 gives no immediate answer. The fact that the return time function R is a *first return time* allows us to track the frequency of visits of typical orbits to the regions where $p_j > 0$. If these functions are not supported on Y, then an orbit can have multiple visits to the regions where $p_j > 0$ before returning to Y, and this visit frequency cannot in general be controlled. However, as we will see for a family of intermittent maps described in Section 2.6, it is sometimes possible to explicitly track orbits once they leave Y, and hence establish dynamical Borel–Cantelli results for functions (p_n) that are no longer assumed to be supported on Y.

As a further application, we also establish dynamical Borel–Cantelli results for infinite systems that are modelled by Young towers, [52]. This is discussed in Section 9. This builds upon the work of [34], where they establish dynamical Borel–Cantelli results for Young towers in the case of the system preserving a probability measure.

2.5. **Refined limit laws for maxima and quantitative results on hitting time laws for the Markov case.** We have seen in Section 2.3, via Proposition 5, that a logarithm law for maxima $M_n(x)$ can be achieved if a logarithm law for the hitting time function $\tau_u^{\phi}(x)$ is known, especially for systems (f, X, μ) built over an induced system (f_Y, Y, μ_Y) satisfying Condition (SPDCL). In this section we establish refined growth rates of maxima M_n for infinite systems via knowledge of a strong Borel–Cantelli result for the induced system (f_Y, Y, μ_Y). Note, via Proposition 5 we achieve almost sure bounds for $\tau_u^{\phi}(x)$. To keep the exposition simple we assume a Gibbs–Markov property for (f_Y, Y, μ_Y). We point out various generalisations below.

We suppose that (f, X, μ) is an ergodic, conservative, and preserving a σ-finite (infinite) measure μ. Let ψ be a monotonically decreasing measureable function, and let $\phi(x) = \psi(d(x, \tilde{x}))$. For the induced system (f_Y, Y, μ_Y) we consider the case where $\tilde{x} \in Y$, so that for sufficiently large u, the set $\{\, x : \phi(x) \geq u \,\}$ is supported in Y. In specific applications, we show that this constraint can be sometimes relaxed. We state the following result. As before, β is defined in (A6).

Theorem 2.9. *Suppose that (f, X, μ) is conservative and ergodic, and that the induced system (f_Y, Y, μ_Y) with return-time $R\colon Y \to \mathbb{N}$ satisfies assumptions (A1)–(A6) (with R playing the role as the observable in (A6)). For a monotonically decreasing function $\psi\colon [0,\infty) \to [0,\infty)$, assume that $\phi(x) = \psi(d(x,\tilde{x}))$ for some point $\tilde{x} \in Y$, and assume the density of μ exists and is positive at $\tilde{x}$. Then for all $\varepsilon > 0$, and μ-a.e. $x \in X$, there exists an N_x such that for all $n \geq N_x$*

$$\psi\left(\frac{(\log n)^{c_1}}{n^\beta}\right) \leq M_n(x) \leq \psi\left(\frac{1}{n^\beta(\log n)^{c_2}}\right), \tag{14}$$

for some constants $c_1, c_2 > 0$.

Using Proposition 4 we can then obtain bounds on the almost sure behaviour of $\tau_u^\phi(x)$ (as $u \to \infty$). For certain forms of ψ these bounds can be made explicit. We state the following, whose proof is similar to that of Lemma 2.6.

Corollary 2. *Let (f, X, μ), and $\tilde{x}$ be as in Theorem 2.9, and let $\phi(x) = -\log d(x, \tilde{x})$. Then for μ-a.e. $x \in [0,1]$, there exists $u_0 > 0$ such that for all $u \geq u_0$,*

$$\log \tau_u^\phi(x) = \frac{1}{\beta}u + O(\log u).$$

Here the $O(\cdot)$ constant depends on $x \in [0,1]$, and on c_1, c_2 as appearing in Theorem 2.9.

We make the following remarks. The proof of Theorem 2.9 uses explicit almost sure bounds achieved on the maxima process for the Gibbs–Markov map f_Y, see e.g. [36, Proposition 3.4] which utilises a *quantitative strong Borel–Cantelli* (QSBC) property for the system (f_Y, Y, μ_Y). Informally this property is described as follows: if p_k is a decreasing sequence of functions, with $E_n := \sum_{k=1}^n \mu_Y(p_k) \to \infty$, then a QSBC property takes the form

$$\sum_{k=1}^n p_k(f_Y^k(x)) = E_n + O\big(E_n^{\beta'}\big), \qquad \text{for } \mu_Y\text{-a.e. } x \in Y,$$

and for some $\beta' \in (0,1)$. To get the logarithmic correction terms in equation (14), the QSBC property is used. If instead a *standard* SBC property is used, i.e. ignoring the $E_n^{\beta'}$ correction, then we obtain slightly weaker estimates on the bounds for the maxima as given in Theorem 2.9. These latter bounds are still sufficient to obtain a logarithm law for the entrance time via Proposition 4, and commensurate with Corollary 1. The results are in fact consistent with the approaches used in [25], where a logarithm law of the hitting time function is obtained using (standard) strong Borel–Cantelli assumptions. Notice further, that we could have worked directly with the system (f, X, μ), and the Borel–Cantelli property achieved in Theorem 2.8 to deduce results on almost sure growth rates of maxima and hitting times. However if we had done this, we would have lost information in the (logarithmic) asymptotic corrections and obtained suboptimal results.

2.6. **Applications: Hitting times, extremes, and run length for intermittent maps with a σ-finite measure.** Let $S^1 = [0,1)$, and we finally consider application of our results of Sections 2 to the following family of intermittent maps (f_α, S^1, μ), where $f_\alpha\colon S^1 \to S^1$ is defined by

$$f_\alpha(x) = \begin{cases} x(1 + 2^\alpha x^\alpha) & x < 1/2, \\ 2x - 1 & x \geq 1/2. \end{cases} \tag{15}$$

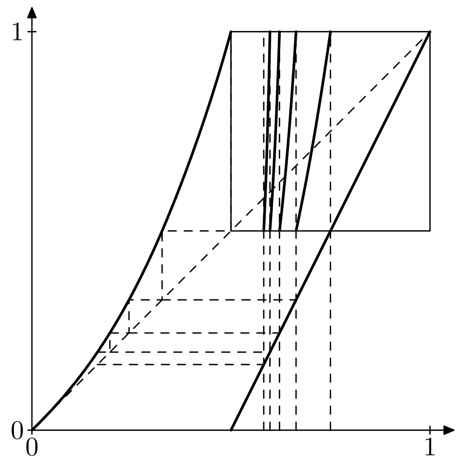

FIGURE 1. An intermittent map, with induced map in top right quadrant.

See Figure 1. We remark that the use of the parameter α here matches the convention of having $\alpha = 1/\beta$, where β appears in (A6). The convention also matches $\alpha = 1/\alpha_R$. (See Theorem 2.10 below for precise statements). In the dynamical systems literature, these maps have been well studied, e.g. for their mixing properties [32, 41, 44, 52, 53], and also their recurrence properties in relation to entrance time statistics, extremes and dynamical Borel–Cantelli results [1, 32, 36, 38, 42], to name a few. We shall focus on the case $\alpha \geq 1$, for which the map preserves a σ-finite measure μ, with $\mu(X) = \infty$ [48]. This measure is absolutely continuous with respect to the Lebesgue measure, but has a non-normalisable density function. One ergodic, invariant probability measure which is physically meaningful for this map is the Dirac measure at $\{0\}$, and thus at first glance the (long-run) statistics appear to be trivial. However, the orbit of Lebesgue almost-every $x \in S^1$ is dense, and thus it is natural to study the asymptotic recurrence behaviour of typical points captured by the statistics of the infinite measure μ. This makes the problem of analysing the statistics of entrance times, extremes and dynamical Borel–Cantelli results an interesting one.

To apply the results already established in the earlier part of Section 2, we consider an induced system (f_Y, Y, μ_Y) (we drop the subscript α) together with a first return time function $R\colon Y \to \mathbb{N}$. We take $Y = (1/2, 1]$, and hence take return-time $R\colon Y \to \mathbb{N}$ defined by

$$R(x) = \min\{\, n \geq 1 : f^n(x) \in Y \,\},$$

with $x \in Y$. As before, we write

$$f_Y(x) = f^{R(x)}(x), \qquad R_j(x) = R(f_Y^j(x)).$$

We summarise key properties of f and f_Y as follows. Define the sequence (x_n) by

$$x_{-1} = 1, \qquad x_0 = 1/2, \qquad f(x_{n+1}) = x_n,$$

keeping $x_n < 1/2$ for all $n \geq 1$. Then we have the following asymptotic relation,

$$x_n \sim (\alpha n)^{-1/\alpha}.$$

The map f_Y is uniformly expanding, and moreover there is a countable Markov partition $\mathcal{P} = \{\, Y_i : i \in \mathbb{N} \,\}$, with $R|_{Y_i}$ constant, and $f_Y(Y_i) = Y$. To be more explicit, for $n \geq 1$, let z_n be the unique $z_n \in [1/2, 1]$ such that $f(z_n) = x_{n-1}$, and let $z_0 = 1$. If we write $Y_n = (z_n, z_{n-1}]$, then $f(Y_n) = (x_{n-1}, x_{n-2}]$, $f^2(Y_n) = (x_{n-2}, x_{n-3}]$ and so on until $f^n(Y_n) = (x_0, x_{-1}] = Y$. Hence $R|_{Y_n} = n$. If $\mathcal{P}_n = \bigvee_{i=1}^{n} f_Y^{-i}\mathcal{P}$, then for all $k \leq n$, the iterate f_Y^k satisfies uniform bounded distortion estimates on all $\omega \in \mathcal{P}_n$. In particular there is an f_Y-invariant probability measure μ_Y on Y, which is equivalent to Lebesgue measure. Thus, this map satisfies (A1)–(A6), and the function R is a first return time to Y. In particular the return time $R\colon Y \to \mathbb{R}$ has a behaviour

$$R\left(\frac{1}{2} + x\right) = c(x)x^{-\alpha}$$

with $m \leq c(x) \leq M$. For $\alpha \geq 1$, R plays the role of an infinite observable, and hence our theoretical results on maxima growth, hitting time laws, and Borel–Cantelli results can be applied. More precisely, to study the system (f, X, μ) with the observable ϕ, we view also R as an observable in order to transfer the study of the system (f, X, μ) to a study of the induced system (f_Y, Y, μ_Y) with a new observable which is related to the observable ϕ.

In the context of this example, we also state the results for Lebesgue measure, rather than the infinite measure μ. On compact subsets in $S^1 \setminus \{0\}$, the restriction of μ is equivalent to Lebesgue measure.

2.6.1. *Results on extremes and entrance time laws.* We first study the almost sure growth rate of the maximum process

$$M_n(x) = \max\{\, \psi(d(f^j x, \tilde{x})) : 0 \leq j < n \,\},$$

where $\psi\colon [0, \infty) \to \mathbb{R}$ is monotone decreasing function, taking its maximum at 0, and $\tilde{x} \in X$ is given.

Theorem 2.10. *Suppose (f_α, S^1, μ) is an intermittent map as defined in equation (15) for $\alpha \geq 1$. Consider the observable function $\phi(x) = \psi(d(x, \tilde{x}))$, where $\psi\colon [0, \infty) \to \mathbb{R}$ is a monotonically decreasing function. Then for all $\varepsilon > 0$ and Lebesgue almost all $x \in S^1$, we have the following cases.*

1. *If $\tilde{x} = 0$, then*
$$\psi\left(\frac{(\log n)^{2+\varepsilon}}{n^{1/\alpha}}\right) \leq M_n(x) \leq \psi\left(\frac{1}{n^{1/\alpha}(\log n)^{2+\varepsilon}}\right),$$
when n is large enough.
2. *If $\tilde{x} \in (0, 1]$ we have*
$$\psi\left(\frac{(\log n)^{4+\varepsilon}}{n^{1/\alpha}}\right) \leq M_n(x) \leq \psi\left(\frac{1}{n^{1/\alpha}(\log n)^{2+\varepsilon}}\right)$$
when n is large enough.

We also obtain the corresponding law for the entrance time.

Corollary 3. *For the intermittent maps given in equation (15), let $\tilde{x} \in (\frac{1}{2}, 1]$. Then the hitting time behaviour in balls around $\tilde{x}$ scales as*

$$\lim_{r \to 0} \frac{\log[\tau_r(x, \tilde{x})]}{-\log r} = \max(1, \alpha),$$

for Lebesgue-a.e. $x \in S^1$.

Remark 6. We remark that for the intermittent maps (f_α, S^1, μ), even when the *natural* invariant measure is infinite, it is still absolutely continuous with respect to Lebesgue measure, with local dimension 1 (except at the origin). In this case, the exponent α plays the role of a rescaling factor to get a logarithm law for this case.

As a comparison, we now consider the growth of maxima in the finite measure case. Using the inducing technique, we can also apply the previous arguments to the family (f_α, S^1, μ) in the case where $\alpha < 1$. This allows us to improve on the result stated in [36, Corollary 4.2], which is primarily based on dynamical Borel–Cantelli estimates for systems with polynomial decay of correlations. For $\alpha < 1$, μ is now a probability measure. Hence for almost every $x \in Y$, there is a $C = C(x)$ such that:

$$n \leq \sum_{j=0}^{n} R_j(x) \leq Cn.$$

That is, we have the asymptotic $S_n^R(x) \in [n, Cn]$, μ-a.e. We therefore have the following, and the proof follows step by step the arguments above.

Corollary 4. *Suppose (f_α, S^1, μ) is the map given in (15), and defined for $\alpha < 1$. Consider the observable function $\phi(x) = \psi(d(x, \tilde{x}))$, where $\psi \colon [0, \infty) \to \mathbb{R}$ is a monotonically decreasing function, and $\tilde{x} \in S^1$. Then for all $\varepsilon > 0$ and Lebesgue almost every $x \in S^1$, we have*

$$\psi\left(\frac{(\log n)^{4+\varepsilon}}{n}\right) \leq M_n(x) \leq \psi\left(\frac{1}{n(\log n)^{2+\varepsilon}}\right),$$

for sufficiently large N.

For example, in the case where $\phi(x) = -\log d(x, \tilde{x})$, we have

$$\lim_{n\to\infty} \frac{M_n(x)}{\log n} = 1.$$

Notice that in the case where $\tilde{x} = 0$, the local dimension $d_\mu(\tilde{x})$ is $1 - \alpha$, and so we get an anomaly in the growth of M_n at this point. However, for systems with superpolynomial decay of correlations we generally expect $\frac{M_n(x)}{\log n}$ to converge to $1/d_\mu(\tilde{x})$.

2.6.2. *Dynamical Borel–Cantelli results.* For the intermittent maps (f_α, S^1, μ), we can apply the techniques and results of Section 2.4 to study dynamical Borel–Cantelli results for shrinking targets. However, using the dynamical features of these maps we can extend such results off the inducing set $Y = (1/2, 1]$.

Corollary 5. *Consider the intermittent maps (f_α, S^1, μ) given in equation (15), for $\alpha \geq 1$. Suppose (B_n) is a decreasing sequence of balls, with*

$$\sum_{n\geq 1} \mu(B_{n^{\alpha+\varepsilon_1}}) = \infty$$

for some $\varepsilon_1 \in (0, \alpha)$, and $\{0\} \notin \cap_n \overline{B}_n$. Then for μ-almost every x, and for all $\varepsilon \in (0, \varepsilon_1]$

$$\sum_{k=1}^{n^{\frac{1}{\alpha+\varepsilon}}} \mu(B_{k^{\alpha+\varepsilon}}) \leq \sum_{k=1}^{n} 1_{B_k}(f^k(x)) \leq \sum_{k=1}^{n^{\frac{1}{\alpha-\varepsilon}}} \mu(B_{k^{\alpha-\varepsilon}})$$

holds eventually as $n \to \infty$.

We remark that Corollary 5 is stated in a basic form so as to highlight the applicability of our results to the family of intermittent maps f_α. It is clear generalisations are possible.

Proof. Since $\{0\} \notin \cap_n \overline{B}_n$, there exists $y > 0$ and $n_0 > 0$ with $[0, y) \cap (\cap_{n>n_0} B_n) = \emptyset$. Following the proof of Theorem 2.10, we can construct a first return map f_Y over an inducing set Y with $[0, y) \cap Y = \emptyset$. This map will satisfy (A1)–(A6). The remainder of the proof now follows step by step the proof of Theorem 2.8, as applied to the induced map. □

2.6.3. *Dynamical run length function and Erdős–Rényi law.* In this section, we establish dynamical run length results for the family of intermittent maps (f_α, S^1, μ) in the case $\alpha \geq 1$, i.e. for systems that admit a σ-finite (infinite) invariant measure. Here, we choose the natural partition $Y^{(0)} = [0, 1/2)$ and $Y^{(1)} = [1/2, 1)$, and the run length functions $\xi_n^{(0)}$ and $\xi_n^{(1)}$ as specified in equation (6) are defined accordingly to this partition. In other words, $\xi_n^{(0)}$ is the maximal number of consecutive visits to $Y^{(0)}$ and $\xi_n^{(1)}$ is the maximal number of consecutive visits to $Y^{(1)}$ up to time n. We state the following result.

Theorem 2.11. *Suppose (f_α, S^1, μ) is an intermittent map as defined in equation (15) for $\alpha \geq 1$. For Lebesgue almost every $x \in S^1$, we have*

1.
$$\lim_{n\to\infty} \frac{\xi_n^{(1)}(x)}{\log_2 n} = \frac{1}{\alpha},$$

2.
$$\lim_{n\to\infty} \frac{\log_2 \xi_n^{(0)}(x)}{\log_2 n} = 1.$$

This result is proved in Section 7. It is interesting to note that the typical growth rate of $\xi_n^{(1)}$ depends on α, while that of $\xi_n^{(0)}$ does not. This is in contrast to the corresponding run length results in the probabilistic cases [13, Theorem 1], due to the additional scaling contribution arising from the asymptotics of the return time function R. The intuitive reason for this is that when α is larger than 1, then the orbit spends very little time away from the neutral fixed point. When α is less than one, then a typical orbit spends a positive proportion of the time in the right half of the interval (Birkhoff's ergodic theorem), but this is not true when α is larger than one.

Initialized by Erdős–Rényi's work [17], it is worth to mention the dynamical run length function is connected to the Erdős–Rényi strong law of large numbers. This relates to the possible limits of the function

$$\begin{aligned}\Upsilon(\varphi(x), n, K(n)) :=& \max_{0\leq i\leq n-K(n)} \big\{S_{i+K(n)}(\varphi)(x) - S_i(\varphi)(x)\big\}\\ =& \max\big\{\, S_{K(n)}(\varphi)\circ T^i(x) : 0 \leq i \leq n - K(n)\,\big\},\end{aligned}$$

as $n \to \infty$, for prescribed (window) function $K(n)$, and typical x.

Based on Theorem 2.11, we can easily obtain the following corollary on Erdős–Rényi strong law for the particular case of a characteristic function observable, and window length.

Corollary 6. *For the intermittent maps given in equation (15), we have the following.*

(1) *For every integer sequence $K(n)$ with*

$$\limsup_{n\to\infty}\frac{\alpha K(n)}{\log_2 n}<1,$$

we have for Lebesgue almost every $x\in S^1$

$$\lim_{n\to\infty}\frac{\Upsilon(1_Y(x),n,K(n))}{K(n)}=1;$$

(2) *For every integer sequence $K(n)$ with $\limsup_{n\to\infty}\frac{\log K(n)}{\log n}<1$, we have for Lebesgue almost every $x\in S^1$.*

$$\lim_{n\to\infty}\frac{\Upsilon(1_{Y^{(c)}}(x),n,K(n))}{K(n)}=1.$$

Proof. Without loss of generality, we only prove item (1) in Corollary 6. Since $K(n)\le\frac{\log_2 n}{\alpha}$, Theorem 2.11 yields that there is at least one $n'<n-K(n)$, such that $\varepsilon_{n'+1}=\ldots=\varepsilon_{n'+K(n)}=1$ (as $n\to\infty$, and Lebesgue almost surely). Therefore, we have $\lim_{n\to\infty}\frac{\Upsilon(1_Y(x),n,K(n))}{K(n)}=1$, Lebesgue almost surely, as was to be proved. □

3. **Proof of Theorem 2.4.**

In this section we prove Theorem 2.4. As shown in Remark 1 the upper bound for $S_n(x)$ as stated in Theorem 2.4 can be recovered from Proposition 1, in the proof we generalize the idea to the larger class of observables we are going to consider. To get estimates from below we begin with the following proposition. Recall that condition (SPDCL) is superpolynomial decay of correlations with respect to Lipschitz observables.

Proposition 6. *Suppose the sets $A_n=\{\,x\in X:\phi(x)\ge n\,\}$ are such that ϕ has regular suplevels (Definition 2.1) and that the system (f,X,μ) satisfies the (SPDCL) condition. Then for μ-a.e. x,*

$$\lim_{n\to\infty}\frac{\log(\tau(x,A_n))}{-\log\mu(A_n)}=1.$$

The proof of Proposition 6 is a direct consequence of the main result of [24], which we recall here. Let g be a Borel measurable function such that $g\ge 0$ on X. Consider sublevel sets

$$V_r=\{\,x\in X:g(x)\le r\,\},$$

and let us define indicators for the power law behaviour of the hitting time to the set V_r as $r\to 0$ by

$$\overline{H}(x,g)=\limsup_{r\to 0}\frac{\log\tau(x,V_r)}{-\log(r)}\quad\text{and}\quad\underline{H}(x,g)=\liminf_{r\to 0}\frac{\log\tau(x,V_r)}{-\log(r)}.$$

In this way if $\overline{H}(x,g)=\underline{H}(x,g)=H(x,g)$, then $\tau(x,V_r)$ scales like $r^{-H(x,g)}$ for small r. By analogy with the definition of local dimension of a measure let us consider

$$\overline{d}_\mu(g)=\limsup_{r\to 0}\frac{\log\mu(V_r)}{\log(r)}\qquad\text{and}\qquad\underline{d}_\mu(g)=\liminf_{r\to 0}\frac{\log\mu(V_r)}{\log(r)}.$$

In the following proposition, we deduce that $\overline{H}(x,g)=\underline{H}(x,g)=H(x,g)$ is a typical outcome in the case where (f,X,μ) is rapidly mixing.

Proposition 7 ([24]). *Suppose $g\colon X \to \mathbb{R}^+$ is Lipschitz, the system (f, X, μ) satisfies condition (SPDCL), and $\underline{d}_\mu(g) = \overline{d}_\mu(g) = d_\mu(g) < \infty$. Then for μ-a.e. $x \in X$ it holds that*

$$\overline{H}(x, g) = \underline{H}(x, g) = d_\mu(g).$$

We also use the following elementary fact about real sequences, whose proof is omitted.

Lemma 3.1. *Let r_n be a decreasing sequence such that $r_n \to 0$. Suppose that there is a constant $c > 0$ satisfying $r_{n+1} > c r_n$ eventually as n increases. Let $\tau_r\colon \mathbb{R} \to \mathbb{R}$ be decreasing. Then*

$$\liminf_{n\to\infty} \frac{\log \tau_{r_n}}{-\log r_n} = \liminf_{r\to 0} \frac{\log \tau_r}{-\log r} \quad \text{and} \quad \limsup_{n\to\infty} \frac{\log \tau_{r_n}}{-\log r_n} = \limsup_{r\to 0} \frac{\log \tau_r}{-\log r}.$$

Proof of Proposition 6. Consider $V_r = \{\, x \in X : \tilde{\phi}(x) \le r \,\}$, where $\tilde{\phi}$ is related to ϕ (and hence the sets A_n) by (ii) of Definition 2.1.

By the definition of V_r and the assumption on $\tilde{\phi}$, we have $V_{\mu(A_n)^\beta} = A_n$. Hence, Proposition 7 and Lemma 3.1 imply that

$$d_\mu(\tilde{\phi}) = \lim_{r\to 0} \frac{\log \mu(V_r)}{\log(r)} = \lim_{n\to\infty} \frac{\log \mu(V_{\mu(A_n)^\beta})}{\log(\mu(A_n)^\beta)} = \lim_{n\to\infty} \frac{\log \mu(A_n)}{\log(\mu(A_n)^\beta)} = \frac{1}{\beta}.$$

By Proposition 7, we know that

$$\lim_{n\to\infty} \frac{\log(\tau(x, V_{\mu(A_n)^\beta}))}{\log \mu(A_n)^\beta} = d_\mu(\tilde{\phi}) = \frac{1}{\beta}$$

and hence

$$\lim_{n\to\infty} \frac{\log(\tau(x, A_n))}{-\log \mu(A_n)} = 1. \qquad \square$$

We now complete the proof of Theorem 2.4.

Proof of Theorem 2.4. First we prove a lower bound to S_n. Note that the non-integrability assumption on ϕ implies that $\alpha_\phi \le 1$. Since ϕ is non-negative, we have

$$\lim_{n\to\infty} \frac{\log(S_n(x))}{\log n} \ge \lim_{n\to\infty} \frac{\log(\max_{1\le i\le n} \phi(f^i(x)))}{\log n},$$

and hence from $\frac{\log(\tau(x,A_n))}{-\log \mu(A_n)} \to 1$, we have $\frac{\log(\tau(x,A_n))}{-\log \mu(A_n)} \frac{-\log \mu(A_n)}{\log n} \to \alpha_\phi$. So $\forall \varepsilon \ge 0$, we have eventually $n^{\alpha_\phi - \varepsilon} \le \tau(x, A_n) \le n^{\alpha_\phi + \varepsilon}$. Furthermore eventually with respect to n

$$\begin{aligned}
\max_{1\le i\le n} \phi(f^i(x)) &\ge \max(\{\, i : \tau(x, A_i) \le n \,\}) \\
&\ge \max(\{\, i : i^{\alpha_\phi + \varepsilon} \le n \,\}) \\
&\ge n^{\frac{1}{\alpha_\phi + \varepsilon}} - 1.
\end{aligned}$$

In particular, this lower bound estimation on M_n automatically implies that

$$\lim_{n\to\infty} \frac{\log(S_n)(x)}{\log n} = \lim_{n\to\infty} \frac{\log M_n(x)}{\log n} = \infty$$

for μ-a.e. $x \in X$, whenever $\alpha_\phi = 0$.

To get an upper bound on S_n, let us suppose $\alpha_\phi \ne 0$ and $0 < \varepsilon < \alpha_\phi$.

Consider $a(x) = x^{\alpha_\phi - \varepsilon}$. Then by the definition of α_ϕ, and A_i, we have

$$\int a(\phi)\, d\mu = \int \phi^{\alpha_\phi - \varepsilon} d\mu \leq \sum_{n=0}^{\infty} [(n+1)^{\alpha_\phi - \varepsilon} - n^{\alpha_\phi - \varepsilon}]\mu(A_n).$$

Here the last estimate is made decomposing the integral of $\phi^{\alpha_\phi - \varepsilon}$ in a Lebesgue way, considering that by definition

$$A_n = \{x\ s.t.\ \phi^{\alpha_\phi - \varepsilon}(x) \geq n^{\alpha_\phi - \varepsilon}\}.$$

Recalling that by the definition of α_ϕ, we have that for each ε

$$\mu(A_n) \leq n^{-\alpha_\phi + \frac{\varepsilon}{2}}$$

eventually as n increases we then have that there is $C \geq 0$ such that

$$\begin{aligned}\sum_{n=0}^{\infty} [(n+1)^{\alpha_\phi - \varepsilon} - n^{\alpha_\phi - \varepsilon}]\mu(A_n) &\leq \sum_{n=0}^{\infty} C n^{\alpha_\phi - \varepsilon - 1} \mu(A_n) \\ &\leq \sum_{n=0}^{\infty} C n^{\alpha_\phi - \varepsilon - 1} n^{-\alpha_\phi + \frac{\varepsilon}{2}} \\ &= \sum_{n=0}^{\infty} C n^{-1-\frac{\varepsilon}{2}} < \infty.\end{aligned}$$

Therefore, Proposition 1 implies that $a(S_n(x))/n \to 0$ for almost every x, and hence that

$$\limsup_{n\to\infty} \frac{\log S_n(x)}{\log n} \leq \frac{1}{\alpha_\phi - \frac{\varepsilon}{2}}.$$

Since ε can be taken arbitrarily small, this finishes the proof. $\square$

4. **Proofs of the statements of Sections 2.2 and 2.3.** In this section, we give the proof of results in Section 2.2, namely that of Proposition 3 on the logarithm law of the hitting time for infinite systems. We also prove results stated in Section 2.3, namely those that link the hitting time function with the maxima function.

4.1. **Proofs of results in Sections 2.2.1.**

Proof of Proposition 3. By Proposition 6, for the induced system it holds that for μ_Y-a.e. x

$$\lim_{n\to\infty} \frac{\log \tau(f_Y, x, B_n)}{\log n} = \alpha. \tag{16}$$

For the original map f it holds

$$\tau(f, x, B_n) = \sum_{i=0}^{\tau(f_Y, x, B_n)} R((f_Y)^i(x)).$$

Hence $\tau(f, x, B_n)$ is a Birkhoff sum of the observable R on the system (f_Y, Y, μ_Y), applying Theorem 2.4 we get

$$\lim_{n\to\infty} \frac{\log(\tau(f, x, B_n))}{\log[\tau(f_Y, x, B_n)]} = \frac{1}{\alpha_R}$$

from which applying (16) we get the statement. $\square$

4.2. **Proofs of results in Section 2.3.**

Proof of Proposition 4. First, we suppose there exist $\ell_1(n), \ell_2(n)$ as described in the proposition for which

$$\ell_1(n) \leq \tilde{M}_n \leq \ell_2(n),$$

(eventually, for all large n). Since $\tilde{M}_n(x) \leq \ell_2(n)$ implies $\tau_{\ell_2(n)}(x) \geq n$, it follows that $\tau_n(x) \geq \ell_2^{-1}(n)$. Now fix $n \geq N$, and take $u \in [n, n+1]$. It follows that $\tau_u(x) \geq \ell_2^{-1}(n) \geq \ell_2^{-1}(u-1)$, as $u \to \infty$. A similar estimate is achieved for the upper bound leading to $\tau_u(x) \leq \ell_1^{-1}(u+1)$. This proves the first item.

For the second item of the proposition, set $u = \hat{\ell}_1^{-1}(n), \hat{\ell}_2^{-1}(n)$ accordingly. If $n \to \infty$ then $u \to \infty$. Hence by using the basic observation between maxima and hitting times it follows that $\hat{\ell}_2^{-1}(n) \leq \tilde{M}_n \leq \hat{\ell}_1^{-1}(n)$ for all n sufficiently large. This completes the proof. □

Proof of Proposition 5. We shall apply Proposition 4. As before, note that $M_n(x) \leq u$ if and only if $\tau_u^{\phi}(x) \geq n$. Suppose that $\limsup_{n\to\infty} \frac{\log M_n}{\log n} = a_1$, and let $\varepsilon > 0$. Then there exists an integer N_1 such that for all $n \geq N_1$ we have $\frac{\log M_n}{\log n} \leq a_1 + \varepsilon$. It follows that for all such n, $M_n \leq \ell(n) := n^{a_1+\varepsilon}$. Hence, applying Proposition 4, we have for all sufficiently large u: $\tau_u \geq \ell^{-1}(u-1) = (u-1)^{\frac{1}{a_1+\varepsilon}}$. By applying a similar estimate to get the lower bound we achieve (for sufficiently large u) that

$$(u-1)^{\frac{1}{a_1+\varepsilon}} \leq \tau_u^{\phi} \leq (u+1)^{\frac{1}{a_2+\varepsilon}}.$$

Hence, taking logarithms we get

$$\liminf_{u\to\infty} \frac{\log \tau_u}{\log u} \geq \frac{1}{a_1}, \text{ and } \limsup_{u\to\infty} \frac{\log \tau_u}{\log u} \leq \frac{1}{a_2}.$$

To get equality for the lim inf, we know that for all $\varepsilon > 0$, we have $M_n \geq n^{a_1-\varepsilon}$ infinitely often. Hence $\tau_{u_n} \leq (u_n)^{\frac{1}{a_1-\varepsilon}}$ infinitely often along the sequence $u_n = n^{a_1-\varepsilon}$. Thus by taking logarithms we obtain $\liminf_{u\to\infty} \frac{\log \tau_u^{\phi}}{\log u} \leq \frac{1}{a_1}$. This establishes the implication

$$\limsup_{n\to\infty} \frac{\log[M_n(x)]}{\log n} = a_1 \implies \liminf_{u\to\infty} \frac{\log[\tau_u^{\phi}(x)]}{\log u} = \frac{1}{a_1}.$$

By a symmetric argument, we also establish that for given $a_2 > 0$, and $x \in X$,

$$\liminf_{n\to\infty} \frac{\log[M_n(x)]}{\log n} = a_2 \implies \limsup_{u\to\infty} \frac{\log[\tau_u^{\phi}(x)]}{\log u} = \frac{1}{a_2}.$$

The converse implications follow in a similar way following the proof of Proposition 4. Hence in the case $a_1 = a_2$, and when either limit exists, we obtain the final limit statement in Proposition 5. □

Proof of Corollary 1. Consider the observable ϕ and the hitting time scaling behaviour of suplevels $B_n = \{x \in X : \phi(x) \geq n\}$. Restricting to countably many radii and considering that $\tau_u^{\phi}(x)$ is increasing in u,

$$\lim_{u\to\infty} \frac{\log[\tau_u^{\phi}(x)]}{\log u} = \lim_{n\to\infty} \frac{\log[\tau(f, x, B_n)]}{\log n}.$$

By Proposition 3 we then get that

$$\lim_{u\to\infty} \frac{\log[\tau_u^{\phi}(x)]}{\log u} = \lim_{n\to\infty} \frac{\log[\tau(f, x, B_n)]}{\log n} = \frac{\alpha_{\phi}}{\alpha_R}$$

holds for μ-a.e. $x \in X$. Applying Proposition 5 we directly get the statement. $\square$

Proof of Lemma 2.6. From Proposition 4 it suffices to estimate an expression for the asymptotic inverse function of

$$g(x) := \log x + c \log \log x, \quad c > 0.$$

Consider the function $\tilde{g}(x) = x^{-a} e^x$, for some $a > 0$. If we compute $(g \circ \tilde{g})(x)$, we obtain

$$g(\tilde{g}(x)) = x - a \log x + c \log x + c \log\Bigl(1 - \frac{a \log x}{x}\Bigr).$$

If $a > c$, then $g(\tilde{g}(x)) < x$, as $x \to \infty$, and similarly if $a < c$, then $g(\tilde{g}(x)) > x$, as $x \to \infty$. Hence for all $\varepsilon > 0$, and all sufficiently large x, we have

$$\frac{e^x}{x^{c+\varepsilon}} \leq g^{-1}(x) \leq \frac{e^x}{x^{c-\varepsilon}}.$$

Applying Proposition 4, and then taking logarithms gives the result. $\square$

5. **Proof of Theorem 2.8 and Proposition 2.** In this section we prove Theorem 2.8 on SBC results for infinite systems. We begin with a proof of Proposition 2.

Proof of Proposition 2. Recall that by (A6), we have $\mu\{\phi(x) \geq u\} \sim u^{-\beta}$. For the proof of the upper bound let $a(t) = t^\beta (\log t)^{-1-\varepsilon}$. Then

$$\int a(\phi) \, \mathrm{d}\mu < \infty,$$

since

$$\begin{aligned} \int a(\phi) \, \mathrm{d}\mu &\leq \sum_{k=1}^{\infty} (a(k+1) - a(k)) \mu\{\, x : \phi(x) \geq k \,\} \\ &\sim \sum_{k=1}^{\infty} (a(k+1) - a(k)) k^{-\beta} = \sum_{k=1}^{\infty} \frac{1}{(\log k)^{1+\varepsilon}} \Bigl(\frac{(k+1)^\beta}{k^\beta} - 1 \Bigr) < \infty. \end{aligned}$$

By Proposition 1, we have for almost all x that

$$\frac{a(S_n(x))}{n} \to 0.$$

In particular $a(S_n) < n$ if n is large. Then almost surely,

$$(S_n)^\beta (\log S_n)^{-1-\varepsilon} < n.$$

By asymptotic inversion, we therefore have for almost all x that

$$S_n(x) \leq C n^{1/\beta} (\log n)^{1/\beta + \varepsilon/\beta}$$

for all large n. As ε is arbitrary, we may take $C = 1$ and replace $\frac{\varepsilon}{\beta}$ by ε, which proves the upper bound.

For the proof of the lower bound we will use that (f_Y, Y, μ_Y) is a Gibbs–Markov map satisfying assumptions (A1)–(A6). We will use the following lemma, which we prove in the Appendix.

Lemma 5.1. *Assume that (A1)–(A6) hold. Suppose $\gamma_n \to \infty$ is a monotone sequence. Let*

$$P_n := \mu\{\, x : \phi(f^j(x)) < \gamma_n \text{ for all } j < n \,\}.$$

Then there exists $D_0, D_1 > 0$ such that

$$P_n \leq D_1 \bigl(1 - D_0 \gamma_n^{-\beta}\bigr)^n.$$

Using Lemma 5.1, we now put $\gamma_n = n^{\frac{1}{\beta}}(\log n)^{-\frac{1}{\beta}-\varepsilon}$. It follows that

$$P_n \le D_1(1 - D_0 n^{-1}(\log n)^{1+\varepsilon\beta})^n,$$

For a sequence a_n such that $na_n \to 0$, an elementary estimate gives

$$(1 - a_n)^n = \exp\{n \log(1 - a_n)\} \le \exp\{-a_n n\}.$$

In the case $a_n = D_0\gamma_n^\beta$ we obtain

$$P_n < D_1 \exp(-D_0(\log n)^{1+\varepsilon\beta}) = O(n^{-2}),$$

for large n. Since P_n is summable, it follows by the First Borel–Cantelli Lemma that if n is large, there is always a $j < n$ with

$$\phi(f^j(x)) > n^{\frac{1}{\beta}}(\log n)^{-\frac{1}{\beta}-\varepsilon}.$$

This proves the lower bound for $\max\{\, \phi(f^j(x)) : 0 \le j < n \,\}$, concluding the proof of Proposition 2. □

We now prove Theorem 2.8.

Proof of Theorem 2.8. The proof consists of the following steps. First we obtain an almost sure asymptotic between the inducing time n, and the clock time $S_n^R(x) = \sum_{i=1}^n R_i(x)$, in the limit $n \to \infty$. For systems preserving an infinite measure, the asymptotics of the return function $R(x)$ are important in this step. In the second step, we then use the strong Borel–Cantelli property of the induced system to get quantitative bounds on the recurrence statistics, namely equation (13).

Let

$$Q(n,x) = S_n(R)(x) = \sum_{i=0}^{n-1} R(f_Y^i(x)).$$

Then $f^{Q(n,x)}(x) = f_Y^n(x)$. We begin by using Proposition 2, which tells us that

$$Q(n,x) \le n^{\frac{1}{\beta}}(\log n)^{\frac{1}{\beta}+\varepsilon}, \tag{17}$$

and

$$n^{\frac{1}{\beta}}(\log n)^{-\frac{1}{\beta}-\varepsilon} \le \max\{\, R(f^j(x)) : 0 \le j < n \,\} < Q(n,x)$$

for all large n.

We consider the first item of the theorem. Let $q_n = p_{\left[n^{\frac{1}{\beta}+\varepsilon}\right]}$ with $\varepsilon \in (0, \varepsilon_1]$ and put $n_k = \left[k^{\frac{1}{\beta}+\varepsilon}\right]$. Since $\varepsilon \le \varepsilon_1$ we have $\sum_n \mu(q_n) = \infty$, and so $\sum_k \mu(p_{n_k}) = \infty$. Hence $\{p_{n_k}\}$ forms a strong Borel–Cantelli sequence with respect to f_Y. We obtain

$$\lim_{N\to\infty} \frac{\sum_{k=1}^N q_n(f_Y^k(x))}{\sum_{k=1}^N \mu(q_n)} = 1, \qquad \text{for } \mu_Y\text{-almost every } x \in Y.$$

Consider ℓ such that $Q(n,x) \le \ell < Q(n+1,x)$. By (17), when n is large enough, ℓ satisfies

$$n^{\frac{1}{\beta}}(\log n)^{-\frac{1}{\beta}-\varepsilon} < \ell < (n+1)^{\frac{1}{\beta}}(\log(n+1))^{\frac{1}{\beta}+\varepsilon}. \tag{18}$$

We remark that in some cases, when Y is a Darling–Kac set, then we get an improved lower bound on ℓ [3, Theorem 4], which in turn leads to an improved upper bound in (13) of Theorem 2.8.

By monotonicity of p_n, and noting that $p_k(f^k(x)) = 0$ when $k \ne Q(n,x)$, we have

$$\sum_{k=1}^n q_k(f_Y^k(x)) \le \sum_{k=1}^{\ell} p_k(f^k(x)) + \sum_{k=1}^{N_0} p_1(f_Y^k(x)), \tag{19}$$

for some $N_0 = N_0(x)$. By a rearrangement and division by $\sum_{k=1}^{n} \mu(q_k)$, we obtain

$$\frac{\sum_{k=1}^{\ell} p_k(f^k(x))}{\sum_{k=1}^{n} \mu(q_k)} \geq \frac{\sum_{k=1}^{n} q_k(f_Y^k(x)) - \sum_{k=1}^{N_0} p_1(f_Y^k(x))}{\sum_{k=1}^{n} \mu(q_k)}.$$

As $n \to \infty$ (and $q \to \infty$) the right-hand bracket is $1 + o(1)$ due the strong Borel–Cantelli property of the sequence q_k with respect to f_Y. Using the bounds on ℓ in (18), and the monotonicity of the sequence $\{p_n\}$ we obtain, for infinitely many n that

$$\sum_{k=1}^{\ell} p_k(f^k(x)) = (1 + o(1)) \sum_{k=1}^{n} \mu(q_k) \geq \sum_{k=1}^{\ell^{\frac{1}{\beta}-\varepsilon}} \mu(p_{k^{\frac{1}{\beta}+\varepsilon}}).$$

This leads to the conclusion that

$$\liminf_{n\to\infty} \frac{\sum_{k=1}^{n} p_k(f^k(x))}{\sum_{k=1}^{n^{\frac{1}{\beta}-\varepsilon}} \mu(p_{k^{\frac{1}{\beta}+\varepsilon}})} \geq 1$$

holds for μ_Y-almost every $x \in Y$. Clearly, this estimate then also holds for μ-almost every $x \in X$, since μ-almost every x has $f^k(x) \in Y$ for some k and for all $A \subset X$

$$\mu(A) = \sum_{n=0}^{\infty} \mu_Y((f^{-n}A) \cap \{R > n\})).$$

To get an upper bound, similar to (19), we write

$$\sum_{k=1}^{\ell} p_k(f^k(x)) = \sum_{j=1}^{n} p_{Q(j,x)}(f^{Q(j,x)}(x)) = \sum_{j=1}^{n} p_{Q(j,x)}(f_Y^j(x)).$$

Hence

$$\frac{\sum_{k=1}^{\ell} p_k(f_k(x))}{(\sum_{k=1}^{n} \mu(q_k))} = \frac{\sum_{j=1}^{n} p_{Q(j,x)}(f_Y^j(x))}{\sum_{j=1}^{n} \mu(p_{Q(j,x)})}.$$

By the strong Borel–Cantelli property for (f_Y, Y, μ_Y), we have

$$\frac{\sum_{j=1}^{n} p_{Q(j,x)}(f_Y^j(x))}{\sum_{j=1}^{n} \mu(p_{Q(j,x)})} \to 1$$

for almost every x. Using again the bounds on ℓ in (18), and the monotonicity of the sequence $\{p_n\}$, we obtain, for large enough n that

$$\sum_{k=1}^{n} \mu(q_k) \leq \sum_{k=1}^{\ell^{\frac{1}{\beta}+\varepsilon}} \mu\Big(p_{k^{\frac{1}{\beta}-\varepsilon}}\Big). \tag{20}$$

This leads to the estimate that

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^{n} p_k(f^k(x))}{\sum_{k=1}^{n^{\frac{1}{\beta}+\varepsilon}} \mu(p_{k^{\frac{1}{\beta}-\varepsilon}})} \leq 1$$

holds for μ_Y-almost every $x \in Y$, and therefore also for μ-almost every $x \in X$. This proves the first item.

To prove the second item of Theorem 2.8 we repeat the estimates above. This time we use the First Borel–Cantelli Lemma to deduce first of all that if $\sum_k \mu(q_k) <$

∞ then $\sum_{k=1}^{n} q_k(f_Y^k(x)) < \infty$. Using equation (20), and for all $\varepsilon > 0$ we obtain the eventual bound (in n),

$$\sum_{k=1}^{n} p_k(f^k(x)) \leq \sum_{k=1}^{n^{\frac{1}{\beta}+\varepsilon}} \mu\Big(p_{k^{\frac{1}{\beta}-\varepsilon}}\Big).$$

However by the assumption of Item (2), the right hand sum is uniformly bounded, and hence the First Borel–Cantelli Lemma implies that for μ-a.e. $x \in X$

$$\sum_{k=1}^{n} p_k(f^k(x)) < \infty. \qquad \square$$

6. **Proof of limit laws for maxima and hitting times.** Regarding the almost sure growth of M_n for infinite systems, in this section we prove Theorem 2.9. The main idea is to use directly Proposition 2, and the structure of the induced system (f_Y, Y, μ_Y).

Proof of Theorem 2.9. We use Proposition 2, and note that f_Y and the return time function $R\colon Y \to \mathbb{N}$ satisfy the assumptions (A1)–(A6) (i.e. with ϕ in place of R in (A6)). As before, we let $R_j = R \circ f_Y^j$.

We get that for all $\varepsilon > 0$, and μ_Y-almost all $x \in Y$ there is an n_0 such that

$$n^{\frac{1}{\beta}}(\log n)^{-\frac{1}{\beta}-\varepsilon} \leq \max\{\, R_j(x) : 0 \leq j < n \,\} < \sum_{j=0}^{n} R_j(x) \leq n^{\frac{1}{\beta}}(\log n)^{\frac{1}{\beta}+\varepsilon}$$

holds for all $n > n_0$. Now, let $\psi\colon (0,\infty) \to [0,\infty)$ be a decreasing function, and put

$$M_n(x) = \max\{\, \psi(d(f^j x, \tilde{x})) : 0 \leq j < n \,\}.$$

Suppose now that n is fixed. In the case $\tilde{x} \in Y$, we have $M_n(x) = \hat{M}_k(x)$, where

$$\hat{M}_k(x) := \max_{j \leq k(x)} \psi(d(f_Y^j(x), \tilde{x})),$$

and $k(x)$ is the largest such k for which $n \geq \sum_{j=0}^{k-1} R_j(x)$. By above, we have for μ_Y-almost all x that

$$k^{\frac{1}{\beta}}(\log k)^{-\frac{1}{\beta}-\varepsilon} \leq \max\{R_0, \ldots, R_{k-1}\} < n \leq k^{\frac{1}{\beta}}(\log k)^{\frac{1}{\beta}+\varepsilon} \tag{21}$$

when both n and $k = k(x)$ are large. This implies that

$$n^{\beta}(\log k)^{-1-\varepsilon\beta} \leq k \leq n^{\beta}(\log k)^{1+\varepsilon\beta},$$

and using that $k \leq n$, we obtain

$$n^{\beta}(\log n)^{-1-\varepsilon\beta} \leq k \leq n^{\beta}(\log n)^{1+\varepsilon\beta}. \tag{22}$$

Since the system (f_Y, Y, μ_Y) has exponential decay of correlations, we can use [36, Proposition 3.4] to get refined bounds on the almost sure growth of the maximum function $\hat{M}_k(x)$. That is,

$$\psi\left(\frac{(\log k)^3}{k}\right) \leq \hat{M}_k(x) \leq \psi\left(\frac{1}{k(\log k)^{1+\varepsilon}}\right),$$

for μ_Y-a.e. $x \in Y$, and for all $\varepsilon > 0$. Combining this with (22) and using that $M_n(x) = \hat{M}_k(x)$, we obtain

$$\psi\left(\frac{(\log n)^{4+\varepsilon}}{n^{\beta}}\right) \leq M_n(x) \leq \psi\left(\frac{1}{n^{\beta}(\log n)^{2+\varepsilon}}\right).$$

Since μ-a.e. x has $f^k(x) \in Y$ for some k, and the density of μ is positive at $\tilde{x}$, these bounds pass on to μ-a.e. $x \in X$. Therefore, we complete the proof. □

We are now ready to prove Theorem 2.10 for the explicit family of intermittent maps (f_α, X, μ).

Proof of Theorem 2.10. To prove this result, we start with case (2), and take $\tilde{x} \in (0,1]$, We split into the two subcases $\tilde{x} \in Y$, and $\tilde{x} \notin Y \cup \{0\}$. For case $\tilde{x} \in Y$, we notice that contributions to successive maxima only occur once orbits return to Y, and hence the statistics of the induced system (f_Y, Y, μ_Y) apply to obtain growth rates for M_n. In the case $\tilde{x} \notin Y \cup \{0\}$, we show that the inducing set Y can be enlarged to a new set $\tilde{Y} \supset Y$, with $\tilde{x} \in \tilde{Y}$, and that the corresponding induced system satisfies (A1)–(A6).

Consider first $\tilde{x} \in Y$. Here, we just apply Theorem 2.9 with $\beta = \frac{1}{\alpha}$ directly to this system, and obtain immediately

$$\psi\left(\frac{(\log n)^{4+\varepsilon}}{n^{1/\alpha}}\right) \leq M_n(x) \leq \psi\left(\frac{1}{n^{1/\alpha}(\log n)^{2+\varepsilon}}\right),$$

for μ-a.e. $x \in [0,1]$, and $\tilde{x} \in Y$.

So suppose now that $\tilde{x} \in (0, 1/2)$. We now enlarge the inducing set Y to the set

$$\tilde{Y} = Y \cup \left(\bigcup_{j=1}^{m} (x_j, x_{j-1}]\right),$$

with m the smallest integer so that $\tilde{x}$ lies in the interior of $\tilde{Y}$. Recall previously that Y admits a partition into elements $Y_i = (z_i, z_{i-1}]$, with $R\,|_{Y_n} = n$, and $f(z_n) = x_{n-1}$, with $x_n \sim (\alpha n)^{-1/\alpha}$. Let $W_j = (x_j, x_{j-1}]$, and define a new (first) return time function $\tilde{R}$ via $\tilde{R}|_{Y_i} = 1$ with $i \leq m+1$, so that $\tilde{R}|_{Y_i} = i - (m+1)$ for $i > m+1$, and $\tilde{R}\,|_{W_i} = 1$. Indeed we see that $f(Y_i) = W_{i-1}$, for all $i \leq m+1$. The corresponding induced map $\tilde{f}(x) = f^{\tilde{R}}(x)$ satisfies (A1)–(A6), with respect to the partition $\{Y_i,\, i \geq 1\} \cup \{W_i,\, i \leq m\}$. For any x and j, we have

$$\tilde{R}_j(x) \leq R_j(x) \leq \tilde{R}_j(x) + m,$$

where, as before, $R_j(x)$ denotes a return time with respect to $(1/2, 1]$. Hence, Lemma 2 holds for $\tilde{R}$ as well (this also follows by Hopf's ergodic theorem), and this lets us prove the result in the same way as for the case $\tilde{x} \in (1/2, 1]$. This completes the proof of case (2) of Theorem 2.10.

In the case $\tilde{x} = 0$ we cannot immediately apply Theorem 2.9 as the Gibbs–Markov construction above cannot be extended (uniformly) to include $\{0\}$. We proceed as follows. Given $x \in (0,1]$, let $\hat{x} = f^{n(x)}(x)$, where $n(x) = \inf\{k \geq 0 : f^k(x) \in Y\}$. Then the main contribution to the maxima $M_n(x)$ is then given by the sequence $\{f(f_Y^j(\hat{x})), j \geq 0\}$. Such iterates correspond to close returns to the point $\tilde{x} = 0$. In particular,

$$M_n(x) = \max\bigl\{\psi(x), \max\{\, \psi(f(f_Y^j(\hat{x}))) : 0 \leq j \leq k \,\}\bigr\},$$

where $k = k(x)$ is the largest such k for which $n \geq n(x) + \sum_{j=0}^{k} R_j(x)$. In the case where $R_j \neq 1$, we have $\psi(f(f_Y^j(\hat{x}))) \in (x_{R_j-1}, x_{R_j-2}]$, with $x_{R_j} \sim (\alpha R_j)^{-\frac{1}{\alpha}}$ when R_j is large. We now apply equation (21), which implies

$$k^\alpha (\log k)^{-\alpha-\varepsilon} \leq \max\{R_0, \ldots, R_{k-1}\} \leq k^\alpha (\log k)^{\alpha+\varepsilon} \tag{23}$$

Then we bound k in terms of n via equation (22). This leads to the almost sure bounds on $M_n(x)$ as stated in case (1) of Theorem 2.10. $\square$

7. Dynamical run length problems—proof of Theorem 2.11.

In this section, we prove Theorem 2.11 for the family of intermittent maps (f_α, X, μ). There is a natural link between hitting times and the run length function as we now make concrete. Namely, consider a target point $\tilde{x}$ with a target ball $B_\varepsilon(\tilde{x})$, and recall that the hitting time of a point $x \in S^1$ is defined by

$$\tau_\varepsilon(x, \tilde{x}) = \min\{\, n \geq 1 : f_\alpha^n(x) \in B_\varepsilon(\tilde{x})\,\}.$$

Meanwhile, let $T(x) = 2x \mod 1$ on S^1, and for every $x \in S^1$, denote $x = \sum_{i=1}^{\infty} \frac{x_i}{2^i}$ with $x_i = 0$ (resp. 1) if and only if $T^{i-1}(x) \in [0, 1/2)$ (resp. $[1/2, 1)$). Then we define the *binary symbolic coding distance*

$$\tilde{d}(x, y) = 2^{-n^*(x,y)},\ \forall x, y \in S^1,$$

where $n^*(x, y) := \min\{i \in \mathbb{N},\ x_i \neq y_i\}$.

With these conventions, we commence with the following lemma.

Lemma 7.1. *For every $x \in S^1$ and every $n \in \mathbb{N}$, we have*

(i) $2^{-\xi_n^{(1)}(x)} = \min_{1\leq i\leq n} \max\{\tilde{d}(f_\alpha^i(x), 1), 2^{-(n-i)}\}$;

(ii) $\min_{1\leq i\leq \tau_{2^{-n}}(x,1)} \tilde{d}(f_\alpha^i(x), 1) \leq 2^{-n}$;

(iii) $\min_{1\leq i\leq \tau_{2^{-n}}(x,1)-1} \tilde{d}(f_\alpha^i(x), 1) \geq 2^{-n}$,

where $\tilde{d}(\cdot,\cdot)$ is the (binary) symbolic coding distance.

Proof. The lemma follows directly from the definitions of $\tau_{2^{-n}}(x, 1)$, and binary symbolic coding distance $\tilde{d}(\cdot, \cdot)$. $\square$

Recall that $Y = [1/2, 1)$ and for each $\tilde{x} \in Y$, we analogously define the hitting time on the induced map f_Y by

$$\hat{\tau}_\varepsilon(x, \tilde{x}) := \min\{\, n \geq 1 : f_Y^n(x) \in B_\varepsilon(\tilde{x})\,\}.$$

There is a relationship between τ and $\hat{\tau}$, that is

$$\tau_{2^{-n}}(y, 1) = \sum_{j=0}^{\hat{\tau}_{2^{-n}}(y,1)} R\big(f_Y^j(y)\big), \qquad \text{for all } y \in Y.$$

We are now ready to prove Theorem 2.11.

Proof of Theorem 2.11. We will first prove Item (1) of Theorem 2.11. For any $\varepsilon > 0$ and Lebesgue almost every $y \in Y$, by Proposition 2, we have

$$(\hat{\tau}_{2^{-n}}(y, 1))^{\alpha-\varepsilon} \leq \tau_{2^{-n}}(y, 1) = \sum_{j=0}^{\hat{\tau}_{2^{-n}}(y,1)} R \circ f_Y^j(y) \leq (\hat{\tau}_{2^{-n}}(y, 1))^{\alpha+\varepsilon}$$

if n is large enough. Together with Corollary 3 and Remark 6, we conclude that

$$\lim_{n\to\infty} \frac{\log \tau_{2^{-n}}(y, 1)}{-\alpha \log 2^{-n}} = 1, \qquad \text{for Lebesgue almost every } y \in [1/2, 1).$$

Note also that for Lebesgue almost every $x \in S^1$, there is always a $y \in [1/2, 1)$ such that $\tau_{2^{-n}}(x, 1) = \tau_{2^{-n}}(y, 1)$. This implies that

$$\lim_{n\to\infty} \frac{\log \tau_{2^{-n}}(x,1)}{-\alpha \log 2^{-n}} = 1, \qquad \text{for Lebesgue almost every } x \in S^1.$$

Hence, we have

$$n \sim \frac{1}{\alpha} \log_2^{\tau_{2^{-n}}(x,1)}, \qquad \text{as } n \to \infty.$$

Together with assertions (ii) and (iii) in Lemma 7.1, this implies that

$$n^{-1/\alpha-\varepsilon} \le \min_{1\le i\le n} \tilde{d}(f_\alpha^i(x), 1) \le n^{-1/\alpha+\varepsilon},$$

if n is large enough.

Finally, for any $\varepsilon > 0$, and sufficiently large n, we have

$$\begin{aligned} n^{-1/\alpha-\varepsilon} \le \min_{1\le i\le n} \tilde{d}(f_\alpha^i(x), 1) &\le \min_{1\le i\le n} \max\{\tilde{d}(f_\alpha^i(x), 1), 2^{-(n-i)}\} \\ &\le \min_{1\le i\le n^{1-\varepsilon}} \max\{\tilde{d}(f_\alpha^i(x, 1)), 2^{-(n-i)}\} \\ &\le \min_{1\le i\le n^{1-\varepsilon}} \max\{\tilde{d}(f_\alpha^i(x, 1)), 2^{-n+n^{1-\varepsilon}}\} \\ &\le \min_{1\le i\le n^{1-\varepsilon}} \max\{\tilde{d}(f_\alpha^i(x, 1)), 2^{-n/2}\} \\ &= \max\{\min_{1\le i\le n^{1-\varepsilon}}\{\tilde{d}(f_\alpha^i(x), 1), 2^{-n/2}\}\} \\ &\le \max\{(n^{(1-\varepsilon)})^{-1/\alpha+\varepsilon}, 2^{-n/2}\} \le n^{-(1-\varepsilon)/\alpha+\varepsilon}. \end{aligned}$$

Since ε is arbitrary, this implies that

$$\lim_{n\to\infty} \frac{\log_2\big(\min_{1\le i\le n}\big(\max\{\tilde{d}(f_\alpha^i(x), 1), 2^{-(n-i)}\}\big)\big)}{\log_2 n} = -\frac{1}{\alpha}.$$

Together with Item (i) of Lemma 7.1, we have

$$\lim_{n\to\infty} \frac{\xi_n^{(1)}(x)}{\log_2 n} = \frac{1}{\alpha}, \qquad \text{for Lebesgue almost every } x \in S^1,$$

which is Item (1).

We will now prove Item (2) of Theorem 2.11. As before, we put $x_0 = 1/2$, and $x_{n+1} = f_\alpha^{-1}(x_n) \cap [0, 1/2)$. For $x \in [0, 1/2)$ we have $R(x) = n$ if and only if $x \in [x_n, x_{n-1})$. Moreover, we have $x_n \sim n^{-\frac{1}{\alpha}}$, so $R(x) \sim |x|^{-\alpha}$. It is now clear that

$$\xi_n^{(0)}(x) = \max_{0\le i\le n-1}\{\min\{R(f_\alpha^i(x)), n-i\}\}. \tag{24}$$

Let

$$M_n := \max_{0\le i\le n-1}\{R(f_\alpha^i(x))\}.$$

By Item (1) of Theorem 2.10, with $R = \phi$ and using that $f_\alpha^i = d(f_\alpha^i, 0)$, for any $\varepsilon > 0$, and almost every $x \in S^1$, we have

$$n^{1-\varepsilon} \le M_n(x) \le n^{1+\varepsilon}.$$

if n is large enough. Therefore, we have

$$\begin{aligned} n^{1+\varepsilon} \geq M_n(x) &\geq \max_{0\leq i\leq n-1}\{\min\{R(f_\alpha^i(x)), n-i\}\} \\ &\geq \max_{0\leq i\leq \frac{n-1}{1+\varepsilon}}\{\min\{R(f_\alpha^i(x)), n-i\}\} \\ &\geq \max_{0\leq i\leq \frac{n-1}{1+\varepsilon}}\Big\{\min\Big\{R(f_\alpha^i(x)), \frac{\varepsilon n}{1+\varepsilon}\Big\}\Big\} \\ &\geq \min\Big\{\max_{0\leq i\leq \frac{n-1}{1+\varepsilon}}\{R(f_\alpha^i(x))\}, \frac{\varepsilon n}{1+\varepsilon}\Big\} \\ &\geq \min\Big\{\Big(\frac{n}{1+\varepsilon}\Big)^{1-\varepsilon}, \frac{\varepsilon n}{1+\varepsilon}\Big\} \\ &= \Big(\frac{n}{1+\varepsilon}\Big)^{1-\varepsilon}, \end{aligned}$$

when n is large enough. Together with (24), this implies that for Lebesgue almost every $x \in S^1$,

$$\lim_{n\to\infty} \frac{\log_2 \xi_n^{(0)}(x)}{\log_2 n} = 1,$$

as was to be proved. □

8. **Birkhoff sums of non-integrable observables, systems with strongly oscillating behaviour.**

In this section we exhibit dynamical systems whose Birkhoff sums S_n of an infinite observable have a strongly oscillating bevaviour in the sense that

$$\liminf_{n\to\infty} \frac{\log S_n(x)}{\log n} < \limsup_{n\to\infty} \frac{\log S_n(x)}{\log n}.$$

The behaviour oscillates between two different power laws. We will also see examples with the same behaviour having polynomial decay of correlations (SPDCL). In particular the examples we use allow us to prove Theorem 2.5 in Section 2.1.2. In Theorem 2.4, we showed that the correlation decay assumtion in Theorem 2.5 is optimal in some sense: systems with decay of correlations faster than any polynomial and sufficiently regular observables cannot have such a strongly oscillating behaviour for observables diverging to ∞ at a power law speed.

8.1. **Birkhoff sums of infinite observables and rotations.**

Let us recall the definition of Diophantine type of an irrational number.

Definition 8.1. Given an irrational number $\theta \in \mathbb{R}$ we define the Diophantine *type* of θ as the following (possibly infinite) number.

$$\gamma(\theta) = \inf\{\,\beta \in \mathbb{R} : \liminf_{q\in\mathbb{N}, q\to\infty} q^\beta \|q\theta\| > 0\,\}.$$

Every real number θ has Diophantine type $\gamma(\theta) \geq 1$. The set of numbers θ of type $\gamma(\theta) = 1$ is of full measure; the set of numbers θ of type $\gamma(\theta) = \gamma$ has Hausdorff dimension $\frac{2}{\gamma+1}$. There exist numbers of infinite type, called *Liouville numbers*; the set of which is dense, uncountable and has zero Hausdorff dimension.

Proposition 8. *Let the system* $(f_\theta, S^1, \mathrm{Leb})$ *be the rotation of the circle* S^1 *by the angle* θ *of Diophantine type* γ*. Consider* $\beta \geq 1$ *and the non-integrable observable*

$\psi(x) = [d(x,0)]^{-\beta}$. *When* $\beta > 1$, *and* $\frac{1}{\gamma} < 1 - \frac{1}{\beta}$ *we have*

$$\liminf_{n\to\infty} \frac{\log S_n^\psi(x)}{\log n} \leq 1 + \frac{\beta}{\gamma} < \beta \leq \limsup_{n\to\infty} \frac{\log S_n^\psi(x)}{\log n},$$

for Lebesgue almost every $x \in S^1$.

Before the proof we recall some hitting time results on circle rotations and relation with minimal distance iterations: the behaviour of hitting time in small targets for circle rotations with angle θ depends on the Diophantine type of irrational θ. We state the following, where we recall that $\overline{H}, \underline{H}$ are defined in Section 1.

Lemma 8.2 ([39]). *If* f_θ *is a rotation of the circle,* y *a point on the circle and* γ *is the Diophantine type of* θ *then for Lebesgue almost every* x

$$\overline{H}(x,y) = \gamma, \qquad \underline{H}(x,y) = 1.$$

In [30], the following lemma is proved.

Lemma 8.3 ([30, Proposition 11]). *Given any system* f *on a metric space* (X,d) *let us define* $d_n(x,y) = \min_{0\leq i\leq n} d(f^i(x),y)$. *Then*

$$\underline{H}(x,\tilde{x}) = \left(\limsup_{n\to\infty} \frac{-\log d_n(x,\tilde{x})}{\log n}\right)^{-1}$$

and

$$\overline{H}(x,\tilde{x}) = \left(\liminf_{n\to\infty} \frac{-\log d_n(x,\tilde{x})}{\log n}\right)^{-1}.$$

Proof of Proposition 8. We remark that

$$d_n(x,0)^{-\beta} \leq S_n^\psi(x) \leq n d_n(x,0)^{-\beta}$$

and

$$\frac{-\beta\log d_n(x,0)}{\log n} \leq \frac{\log S_n^\psi(x)}{\log n} \leq \frac{\log n - \beta \log d_n(x,0)}{\log n}.$$

By Lemma 8.2 and 8.3 we get

$$\begin{aligned}
\liminf_{n\to\infty} \frac{\log S_n^\psi(x)}{\log n} &\leq \liminf_{n\to\infty} \frac{\log n - \beta\log d_n(x,0)}{\log n} \\
&\leq 1 + \frac{\beta}{\gamma}, \\
\limsup_{n\to\infty} \frac{\log S_n^\psi(x)}{\log n} &\geq \beta,
\end{aligned}$$

for Lebesgue almost every x. When $\beta > 1$, and $\frac{1}{\gamma} < 1 - \frac{1}{\beta}$ we have

$$\liminf_{n\to\infty} \frac{\log S_n^\psi(x)}{\log n} < \limsup_{n\to\infty} \frac{\log S_n^\psi(x)}{\log n},$$

for Lebesgue almost every x. □

8.2. **Power law mixing examples with oscillating behaviour.** In this section we consider examples of mixing systems also having a strongly oscillating behaviour. Consider a class of skew products $F_\theta\colon [0,1]\times S^1 \to [0,1]\times S^1$ defined by

$$F_\theta(x,t) = (T(x), t+\theta\eta), \tag{25}$$

where

$$T(x) = 2x \mod 1,$$

and $\eta = 1_{[\frac{1}{2},1]}$ is the characteristic function of the interval $[\frac{1}{2},1]$. These maps are piecewise constant toral extensions. In these kind of systems, the second coordinate is rotated by θ if the first coordinate belongs to $[\frac{1}{2},1]$. We now consider the observable $\tilde{\psi}\colon [0,1]\times S^1 \to \mathbb{R}$ depending only on the second coordinate, an example being $\tilde{\psi}(x,t) = [d(t,0)]^{-\beta}$, where d is the distance on S^1. We have the following result.

Proposition 9. *Let $F(x,t)$ be the skew product defined by equation (25), and suppose that $\tilde{\psi}\colon [0,1]\times S^1 \to \mathbb{R}$ is given by $\tilde{\psi}(x,t) = [d(t,0)]^{-\beta}$ for some $\beta > 0$. Then*

$$\liminf_{n\to\infty} \frac{\log S_n^{\tilde{\psi}}(x,t)}{\log n} \le 2 + \frac{\beta}{\gamma(\theta)},$$

$$\limsup_{n\to\infty} \frac{\log S_n^{\tilde{\psi}}(x,t)}{\log n} \ge \beta.$$

Proof. We remark that on the system $(F, [0,1]\times S^1)$ the Lebesgue measure is invariant. Let $S_n^{\varphi}(x)$ be the Birkhoff sum of φ for $(T,[0,1])$. Let $S_n^{\psi}(t)$ be the sum of the observable ψ on S^1 as in Proposition 8. Since ψ is positive and $\int \eta\, dm = \frac{1}{2}$, we have by the pointwise ergodic theorem applied to $(T,[0,1])$, that for a.e. x

$$S_{\frac{n}{4}}^{\psi}(t) \le S_n^{\tilde{\psi}}(x,t) \le n S_{S_n^{\eta}(x,t)}^{\psi}(t) \le n S_n^{\psi}(t)$$

holds eventually in n. By this and Proposition 8, we get

$$\beta \le \limsup_{n\to\infty} \frac{\log S_{\frac{n}{4}}^{\psi}(t)}{\log n} \le \limsup_{n\to\infty} \frac{\log S_n^{\tilde{\psi}}(x,t)}{\log n}$$

and

$$\liminf_{n\to\infty} \frac{\log S_n^{\tilde{\psi}}(x,t)}{\log n} \le 2 + \frac{\beta}{\gamma}. \qquad \square$$

On the above skew products it is possible to establish power law bounds for the rate of decay of correlations on Lipschitz observables. The power law exponent depend on the Diophantine type of the translation angle θ. In [31, Lemma 11 and Section 5], the following is proved.

Proposition 10. *Let p be the exponent of power law decay with respect to Lipschitz observables defined by*

$$p = \liminf_{n\to\infty} \frac{-\log \Theta(n)}{\log n},$$

where $\Theta(n)$ is the correlation decay rate function. For the map defined by (25) the exponent p satisfies

$$\frac{1}{2\gamma(\theta)} \le p \le \frac{6}{\max(2,\gamma(\theta)) - 2}.$$

Combining Propositions 9 and 10, we have found systems with polynomial decay of correlations, for which Birkhoff sums show the same strongly oscillating behaviour as in the case of rotations. This result establishes the first result stated in Theorem 2.5.

8.3. **Mixing systems with slowly increasing time averages.** In this section we see examples where $\limsup_{n\to\infty} \frac{\log S_n^{\psi}(x)}{\log n}$ is bounded from above by the arithmetical properties of the system and can have a very slow increase. To construct these examples we consider two dimensional rotations with suitable angles.

Let us consider a rotation f_θ of the torus $\mathbb{T}^2 \cong \mathbb{R}^2/\mathbb{Z}^2$ by an angle with components (θ, θ'). Suppose that (γ, γ') are respectively the types of θ and θ'. Denote by q_n and q'_n the partial convergent denominators of θ and θ'. Let us consider $\xi > 1$ and let $Y_\xi \subset \mathbb{R}^2$ be the class of couples of irrationals (θ, θ') given by the following conditions on their convergents to be satisfied eventually

$$q'_n \geq q_n^{\xi},$$
$$q_{n+1} \geq {q'_n}^{\xi}.$$

The set Y_ξ is uncountable, dense in $[0,1] \times [0,1]$ and there are points in Y_ξ having finite Diophantine type coordinates. If we take angles in Y_ξ the lower hitting time indicator is bounded from below by ξ. In [31, Section 6] the following is proved.

Proposition 11. *Consider the class of skew products $F_\theta \colon [0,1] \times S^1 \to [0,1] \times S^1$ defined by*

$$F_\theta(x,t) = (T(x), t + \theta\eta(x)),$$

where $\theta \in Y_\xi$,

$$T(x) = 2x \mod 1,$$

and $\eta = 1_{[\frac{1}{2},1]}$ is the characteristic function of the interval $[\frac{1}{2}, 1]$. For each $y \in [0,1] \times S^1 \times S^1$ it holds

$$\underline{H}(x,y) \geq \max(3, \xi)$$

for a.e. x. Furthermore, there are infinitely many $\theta \in Y_4$ such that F_θ is polynomially mixing with respect to Lipschitz observables.

In [22], the following is proved.

Proposition 12 ([22, Theorem 11])**.** *Consider a dynamical system (f, X, μ) where X is a metric space and f a Borel map. Let $\tilde{x} \in X$ and let us consider the observable $\phi(x) = d(x, \tilde{x})^{-k}$, where $k \geq 0$. Let $S_n^{\phi}(x)$ be the usual Birkhoff sum. Then it holds for μ-a.e. $x \in X$*

$$\limsup_{n\to\infty} \frac{\log S_n^{\phi}(x)}{\log n} \leq \frac{k}{\underline{H}(x, \tilde{x})} + 1.$$

By this we easily get the following proposition.

Proposition 13. *Consider the system as described in Proposition 11. Then*

$$\limsup_{n\to\infty} \frac{\log S_n^{\phi}(x)}{\log n} \leq \frac{k}{\max(3, \xi)} + 1.$$

Then in these kind of systems when ξ is large, the growth of Birkhoff sums can be slow even for large k (while the dimension of the invariant measure is 3 for each choice of θ). We remark that already when $\xi = 4$, if $k > 12$, then we can find systems having power law decay of correlations and for which

$$\limsup_{n\to\infty} \frac{\log S_n(x)}{\log n} < \frac{k}{d_\mu(\tilde{x})}$$

as stated in the second part of Theorem 2.5.

9. **Application to Markov extensions, inlcuding Young towers.** In this final section we consider application of Theorem 2.8 (Section 2.4) to infinite systems (f, X, μ) modelled by Markov extensions. Examples of Markov suspensions include Young towers [52, 53]. To keep the exposition simple, we focus on dynamical Borel–Cantelli results for these systems. In a natural way, the analysis can be further extended to study extremes, and dynamical run length problems. A motivation to study (in partcular) dynamical Borel–Cantelli properties for infinite Markov extended systems is based upon results already obtained in the probability measure case, see [34]. Thus we contrast the infinite measure scenario with the probability preserving case.

We elaborate on the suspension construction outlined in Section 2.2). Consider a measure preserving (one-dimensional) system (f, X, μ), and suppose there exists a subinterval $Y \subset X$, and a countable partition $\{Y_i\}_{i\in\mathbb{N}}$ of Y into sub-intervals, together with a function $R\colon Y \to \mathbb{N}$ defined by $R|_{\Lambda_i} = Y_i$ if $f^{Y_i}\colon Y_i \to Y$ is a bijection. The set $\hat{X}$ is defined by

$$\hat{X} = \bigcup_{k=1}^{\infty} \bigcup_{\ell=0}^{R_k} Y_{k,\ell},$$

where $Y_{k,\ell} := Y_k \times \{\ell\}$ is a subset of *level* ℓ. The set $Y_0 = \bigcup_k Y_{k,0}$ is identified with Y. The map $\hat{f}\colon \hat{X} \to \hat{X}$ is precisely equation (9), and the map $\hat{f}^R\colon Y_0 \to Y_0$ is identified naturally with the (Gibbs–Markov) map $f_Y\colon Y \to Y$. The map $\hat{f}^R$ admits an absolutely continuous invariant measure $\hat{\mu}_R$, and this measure lifts to a measure $\hat{\mu}$ for $\hat{f}$ on $\hat{X}$ by defining $\hat{\mu}(A) = \hat{\mu}_R(\hat{f}^{-\ell}(A))$ for $A \subset Y_{k,\ell}$.

In the case where $\langle R\rangle = \int_\Lambda R\, d\hat{\mu} < \infty$, we can normalise $\hat{\mu}$ to a probability measure. There is a (surjective) factor map $\pi\colon \hat{X} \to X$ satisfying $\pi \circ \hat{f} = f \circ \pi$, and for $(x, \ell) \in \hat{X}$ we have explicitly $\pi(x, \ell) = f^\ell(x)$. The measure $\mu = \pi_*\hat{\mu}$ is f-invariant, and is a probability measure in the case where $\hat{\mu}$ is a probability measure. Here, for $A \subset X$ we have $\pi_*\hat{\mu}(A) = \hat{\mu}(\pi^{-1}(A))$. We call $(\hat{f}, \hat{X}, \hat{\mu})$ a Markov extension over (f, X, μ). Examples include Young towers. Under further regularity assumptions on the system $(\hat{f}, \hat{X}, \hat{\mu})$ the measures $\hat{\mu}$ and μ can be shown to be absolutely continuous with respect to m, see [52, 53].

In the present situation we consider the case $\langle R\rangle = \infty$, and hence $\hat{\mu}(\hat{X}) = \infty$. Consider now (f, X, μ) ergodic, conservative and that μ is absolutely continuous with respect to Lebesgue measure. We can again build the suspension $(\hat{f}, \hat{X}, \hat{\mu})$ over (f, X, μ). However, unlike the case $\langle R\rangle < \infty$, the measure $\mu' := \pi_*\hat{\mu}$ need not be σ-finite unless additional conditions are satisfied on the return time function R, see [10]. We assume that $\pi_*\hat{\mu}$ is σ-finite. We have the following result.

Theorem 9.1. *Suppose that (f, X, μ) is an ergodic, conservative measure preserving map on an interval $X \subset \mathbb{R}$, and furthermore (f, X, μ) is modelled by a (one-dimensional) Markov extension $(\hat{f}, \hat{X}, \hat{\mu})$, with return time function satisfying $\hat{\mu}_R\{R = n\} \sim n^{-\beta-1}$, for some $\beta \in (0, 1)$, and that the measure $\mu = \pi_*\hat{\mu}$ is σ-finite.*

Then there exists a set X' with $\mu(X \setminus X') = 0$ with the following property. If $\{B_n\}$ is a nested sequence of intervals with $\cap_n B_n = \{\tilde{x}\}$, $\tilde{x} \in X'$, and $\sum_{k=1}^{\infty} \mu(B_{k^{\frac{1}{\beta}+\varepsilon_1}}) = \infty$ (for some $\varepsilon_1 > 0$), then for all $\varepsilon \in (0, \varepsilon_1]$

$$\liminf_{n\to\infty} \frac{\sum_{k=1}^{n} 1_{B_k}(f^k(x))}{\sum_{k=1}^{n^{\beta-\varepsilon}} \mu(B_{k^{\frac{1}{\beta}+\varepsilon}})} \geq 1 \tag{26}$$

for μ-a.e. x.

The proof of this theorem follows step by step the proof of Theorem 2.8, together with the method of proof of [34, Theorem 2] for Young tower models. We do not repeat the details, but make the following remarks. First of all, the set X' consists of points where the density of μ is bounded away from $\{0, \infty\}$, and where $\pi^{-1}(\tilde{x})$ is non-empty and consists only of interior points within each $Y_{k,\ell}$. Here, the partition sets $Y_{k,\ell}$ are identified with intervals in X under the projection π. For more general Markov extension constructions, even in one dimension, it is possible for the sets $Y_{k,\ell}$ to have more general (Cantor set) geometries. We do not consider these latter situations.

In the case $\langle R \rangle < \infty$, a corresponding result is established in [34], where a *dense Borel–Cantelli* property is achieved (i.e. a lower bound on the lim inf analogous to equation (26)). Their result does not require the assumption $\hat{\mu}_R\{R = n\} \sim n^{-\beta-1}$. In the case $\langle R \rangle = \infty$, the asymptotics of R play a role in the statement on the Borel–Cantelli result via the constant β.

Relative to equation (13) in Theorem 2.8, equation (26) only gives an almost sure lower bound on $\sum_{k=1}^{n} 1_{B_k}(f^k(x))$. (This lower bound can be further refined following the remarks given immediately after Theorem 2.8). For upper bounds, the issue arising here is that $\pi^{-1}(B_n)$ can have non-empty intersection with a countably infinite number of sets of the form $Y_{k,\ell}$, and dynamical Borel–Cantelli results do not ingeneral carry over to countable unions of (shrinking target) sets. However, in certain situations it is possible to re-arrange the Markov extension so that $\pi^{-1}(B_n)$ is contained in a finite number of $Y_{k,\ell}$, thus bypassing the problem. Indeed it is not difficult to show that such a tower can be constructed for the intermittent map family of maps (15) given in Section 2.6. This would apply in the case where the sets (B_n) do not accumulate at the neutral fixed point $\tilde{x} = 0$. To see how to construct such a tower, see the proof of Theorem 2.10 and Corollary 5. This leads us to conclude with the following result.

Corollary 7. *Under the assumptions of Theorem 9.1, suppose that for all n sufficiently large we have $\pi^{-1}(B_n)$ contained in a finite number of $Y_{k,\ell}$. Then for μ-a.e. $x \in X$, and all $\varepsilon \in (0, \varepsilon_1]$ we have eventually in n (as $n \to \infty$)*

$$\sum_{k=1}^{n^{\beta-\varepsilon}} \mu(B_{k^{\frac{1}{\beta}+\varepsilon}}) \leq \sum_{k=1}^{n} 1_{B_n}(f^k(x)) \leq \sum_{k=1}^{n^{\beta+\varepsilon}} \mu(B_{k^{\frac{1}{\beta}-\varepsilon}}).$$

10. Appendix.

Proof of Lemma 5.1. We have $\mathcal{P} = \{X_i\}_{i\in\mathcal{I}}$, with $\mathcal{I}$ an index set $\subset \mathbb{N}$. There is a one-to-one correspondence between elements of the partition $\mathcal{P}_n$ and sequences $(i_0, i_1, \ldots, i_{n-1}) \in \mathcal{I}^n$, which is characterised by the property that for $X_{i_0,i_1,\ldots,i_{n-1}} \in \mathcal{P}_n$ we have

$$f^k(X_{i_0,i_1,\ldots,i_{n-1}}) \subset X_{i_k} \qquad \text{for } k = 0, 1, \ldots, n-1.$$

Note that some sequences $(i_0, i_1, \dots, i_{n-1})$ are not admissible, unless we assume $f(X_i) = X$ (since we only require big images from assumption (A1)). However, this fact is of no consequence to what follows. We may assume that the diameter of the interval X is one. The bounded distortion assumption (A3) then implies that

$$|X_{i_0,i_1,\dots,i_{n-1}}| \leq e^{C\tau^n} |X_{i_0,i_1\dots i_{n-2}}| \cdot |X_{i_{n-1}}|.$$

The measure μ is absolutely continuous with respect to the Lebesgue measure, and there exists a constant c such that the density of μ is bounded by c and bounded away from zero by c^{-1}. Consider

$$Q_n := |\{\, x \in X : \phi(f^j(x)) < \gamma_n \text{ for all } j < n \,\}|.$$

Since the density of μ is bounded, we have $P_n \leq cQ_n$.

We proceed by induction to estimate Q_n. Let n be fixed and define

$$Q_{n,k} = |\{\, x \in X : \phi(f^j(x)) < \gamma_n \text{ for all } j < k \,\}|.$$

In particular, we have $Q_n = Q_{n,n}$.

Clearly, we have

$$Q_{n,1} = \mu\{\, x \in X : \phi(x) < \gamma_n \,\} = 1 - \mu\{\, x \in X : \phi(x) \geq \gamma_n \,\} \leq (1 - D_0 \gamma_n^{-\beta}),$$

for some constant D_0. Suppose that we have

$$Q_{n,k-1} \leq e^{\sum_{j=0}^{k-1} C\tau^j} (1 - D_0 \gamma_n^{-\beta})^{k-1},$$

for some k. Then

$$\begin{aligned}
Q_{n,k} &= \sum_{\substack{i_0,i_1,\dots,i_{k-1} \\ \phi(i_j)<\gamma_n}} |X_{i_0,i_1,\dots,i_{k-1}}| \\
&\leq \sum_{\substack{i_0,i_1,\dots,i_{k-1} \\ \phi(X_{i_j})<\gamma_n}} e^{C\tau^k} |X_{i_0,i_1,\dots,i_{k-2}}| \cdot |X_{i_{k-1}}| \\
&\leq e^{C\tau^k} Q_{n,k-1} \sum_{i_{k-1}:\phi(X_{i_{k-1}})<\gamma_n} |X_{i_k}| = e^{C\tau^k} Q_{n,k-1} Q_{n,1} \\
&\leq e^{\sum_{j=0}^{k} C\tau^j} (1 - D_0 \gamma_n^{-\beta})^k.
\end{aligned}$$

In particular, we have

$$Q_n = Q_{n,n} \leq e^{\frac{C}{1-\tau}} (1 - D_0 \gamma_n^{-\beta})^n$$

and

$$P_n \leq cQ_n \leq ce^{\frac{C}{1-\tau}} (1 - D_0 \gamma_n^{-\beta})^n,$$

which finishes the proof. □

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Received xxxx 20xx; revised xxxx 20xx.

E-mail address: stefano.galatolo@unipi.it
E-mail address: m.p.holland@exeter.ac.uk
E-mail address: tomasp@maths.lth.se
E-mail address: yiweizhang@hust.edu.cn