Rank one isolated p-minimal subgroups in finite groups

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RANK ONE ISOLATED p-MINIMAL SUBGROUPS IN FINITE GROUPS

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ABSTRACT. This paper studies, for p a prime, rank one isolated p-minimal subgroups P in a finite group G. Such subgroups share many of the features of the minimal parabolic subgroups in groups of Lie type. The structure of Y, the normal closure in G of $O^p(P)$ is determined where $O^p(P)$ is the smallest normal subgroup of P such that $P/O^p(P)$ is a p-group. We find that if $Y \neq O^p(P)$ and $O_p(G) = 1$, then either $Y/Z(Y)$ is a simple group of Lie type in characteristic p or $p \leq 7$ with $Y/Z(Y)$ given by an explicit list. Of particular note is that twenty four out of the twenty six sporadic simple groups arise as possibilities for $Y/Z(Y)$. This may be viewed as giving an overarching framework which brings together the simple groups of Lie type and (most of) the sporadic simple groups.

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1. INTRODUCTION

In group theory, there is frequent interplay between a group and its subgroups, both in studying specific groups and in developing general theories. Of course this "local–global" approach is prevalent in many other areas of mathematics. Equally, some substructures can assume a greater importance, depending on the global structure being studied. In the case of finite groups, p-subgroups, where p is a prime, and closely related subgroups have played a wide ranging and influential role. Undoubtedly, the origin of this can be traced back to the publication of Sylow's theorems [49] in 1872. Initially Sylow's theorems were applied by the likes of Sylow, Hölder, Burnside, Frobenius and Cole to search for possible orders of finite simple groups. Partly as a result of these endeavours, general results began to emerge such as the normal p-complement theorems of Burnside (Thm II, section 243 of [10]) and Frobenius [15]. Here p-local subgroups made their first significant appearance – a p-local subgroup of a group G is a subgroup of the form $N_G(R)$ where R is a non-trivial p-subgroup of G.

With the exception of the important contributions of Brauer, Grün, P. Hall and Wielandt, finite group theory was a quiet backwater in the first half of the twentieth century. It was normal p-complement theorems due to Thompson (see [50], [51]) that lit the touch paper and saw the reinvigoration of finite group theory. Most spectacularly there was the proof of the Odd Order Theorem [14] by Feit and Thompson, to be followed by a myriad of results characterizing various classes of finite simple groups. Of these the most influential were Thompson's papers on N-groups [52]. Firmly in the centre of the action in [52] were p-local

subgroups, a pattern to be repeated in many other papers on finite simple groups. For an encyclopedic discussion of local-global issues in finite group theory, Glauberman [16] is still an excellent source. What had once seemed a quixotic goal became over the ensuing decades a reality – the classification of the finite simple groups, with p-local subgroups playing a major role.

To have a conceptual framework which captures all the finite non-abelian simple groups (and "little" else) is the holy grail of finite simple group theory. Apart from the finite simple groups of Lie type and ${}^2\mathrm{F}_4(2)'$, there are the alternating groups $\mathrm{Alt}(n), n \geq 5$ and the twenty six sporadic simple groups, so the majority of non-abelian simple groups are the groups of Lie type. It is therefore natural to try to emulate the various elegant descriptions for these groups. For groups of Lie type there are a number of different ways of giving a unified treatment, all developed by J. Tits. The concept of a building [53] gives a simplicial complex whose automorphism group delivers these groups. A more group theoretic scenario is provided by groups with a BN - pair [53]. While a reinterpretation of buildings can be given using chamber graphs [55]. There is a substantial literature following this quest. Attempting to extend the notion of buildings to, in particular encompass the sporadic groups, has led to many varied types of geometries – see, for example, Buekenhout [5], [6], [7], Buekenhout and Buset [8], Buekenhout and Cohen [9], Kantor [26] and Tits [54]. Others have taken a more group theoretic approach, much in the spirit of BN - pairs, looking for generalizations of parabolic subgroups. For a selection of such investigations consult Ashbacher and Smith [2], Lempken, Parker and Rowley [33], Ronan and Smith [47] and Ronan and Stroth [48]. We recall that the parabolic subgroups of a finite simple group of Lie type are, in fact, p-local subgroups, where p is the characteristic of the underlying field. Here we also investigate subgroups which generalize these parabolic subgroups.

Suppose that G is a finite group and p is a prime number. Let $S \in \mathrm{Syl}_p(G)$, and let P be a subgroup of G containing S. If S is not normal in P and S is contained in a unique maximal subgroup of P, then we say that P is a p-minimal subgroup (with respect to S). We denote the set of all such p-minimal subgroups by $\mathcal{P}_G(S)$. For $P \in \mathcal{P}_G(S)$, letting

$$L_G(P,S) = \langle \mathcal{P}_G(S) \setminus \{P\}, N_G(S) \rangle$$

we can now introduce isolated p-minimal subgroups.

Definition 1.1. *A p-minimal subgroup P in $\mathcal{P}_G(S)$ is isolated if $O_p(L_G(P,S)) \not\leq O_p(P)$.*

Isolated subgroups in finite groups were introduced and studied in [40]. Building on the results in [40], we analyze at length rank one isolated p-minimal subgroups in groups G. In order to describe what we mean by a rank one group we introduce the following class of groups.

Definition 1.2. *For p a prime, we define the class of group $\mathcal{L}_1(p)$ as follows.*

(i) If $p \geq 5$, then

$$\mathcal{L}_1(p) = \{\mathrm{SL}_2(p^a), \mathrm{PSL}_2(p^a), \mathrm{SU}_3(p^a), \mathrm{PSU}_3(p^a) \mid a \geq 1\}.$$

(ii) If $p = 3$, then

$$\mathcal{L}_1(3) = \{\mathrm{Q}_8, 2^2, {}^2\mathrm{G}_2(3)' \cong \mathrm{SL}_2(8)\} \cup \{\mathrm{SL}_2(3^a), \mathrm{PSL}_2(3^a), \mathrm{SU}_3(3^{a-1}), {}^2\mathrm{G}_2(3^{2a-1}) \mid a \geq 2\}.$$

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(iii) If $p = 2$, then

$$\mathcal{L}_1(2) = \{3, 5, 3^{1+2}_+, 3^2\} \cup \{\mathrm{SL}_2(2^a), \mathrm{SU}_3(2^a), \mathrm{PSU}_3(2^a), {}^2\mathrm{B}_2(2^{2a-1}) \mid a \geq 2\}.$$

Thus in essence the set $\mathcal{L}_1(p)$ captures non-trivial quotients of rank one groups of Lie type and also adds some small soluble groups which are smaller than we might have expected. For example, we include $O^2(\mathrm{SL}_2(2))$ which has order 3 and $O^2({}^2\mathrm{B}_2(2))$ which has order 5. Before defining rank one p-minimal subgroups, we introduce the notion of a narrow p-minimal subgroup.

Definition 1.3. *Suppose that $P \in \mathcal{P}_G(S)$, and let M be the unique maximal subgroup of P containing S. Set $F = \mathrm{core}_P(M)$ and $E = O^p(P)F/F$. If E is a non-abelian simple group or is elementary abelian with P acting primitively on E, then we say P is narrow.*

Definition 1.4. *A p-minimal subgroup P in $\mathcal{P}_G(S)$ is of rank one if it is narrow and $O^p(P/O_p(P)) \in \mathcal{L}_1(p)$.*

Examples of rank one isolated p-minimal subgroups are always to be found in the simple groups of Lie type whose defining characteristic is p. Let G be such a group of Lie type with Borel subgroup $B = N_G(S), S \in \mathrm{Syl}_p(G)$. Suppose that G has Lie rank at least 2, and choose R to be a rank one parabolic subgroup of G containing B. Then $P = O^{p'}(R)$ is a p-minimal subgroup with $L_G(P, S)$ being the maximal parabolic subgroup of G not containing R. Since $O_p(L_G(P, S)) \not\leq O_p(P)$, we have that P is an isolated subgroup. Moreover it is a rank one isolated subgroup. We remark that in the language of buildings, respectively chamber graphs, B corresponds to a maximal simplex, respectively, a chamber. In the case when G has Lie rank one, we have that $P = G$ is a rank one isolated p-minimal subgroup with $L_G(P, S) = B$.

Our main theorems, Theorems 1.5 and 1.6 and their corollaries, are about the list of the finite simple groups as described in the classification. Thus for the remainder of this paper we are considering simple groups which are isomorphic to either a cyclic group of prime order, an alternating group of degree at least 5, a simple group of Lie type, the Tits group or one of the 26 sporadic simple groups. A group in this set is referred to as $\mathcal{K}$-group. Here we follow [20, Definition 2.2.2 and Theorem 2.2.7] and consider a finite group of Lie type of characteristic p to be $O^{p'}(C_{\overline{K}}(\sigma))$ where σ is a Frobenius endomorphism and $\overline{K}$ is an adjoint simple algebraic group defined over the algebraic closure of $\mathrm{GF}(p)$. These groups are simple with the exception of the following $\mathrm{A}_1(2) \cong \mathrm{SL}_2(2)$, $\mathrm{A}_1(3) \cong \mathrm{SL}_2(3)$, ${}^2\mathrm{A}_2(2) \cong \mathrm{SU}_3(2)$, $\mathrm{B}_2(2) \cong \mathrm{Sp}_4(2)$, ${}^2\mathrm{B}_2(2)$, $\mathrm{G}_2(2)$, ${}^2\mathrm{F}_4(2)$ and ${}^2\mathrm{G}_2(3)$. Considering the groups $\mathrm{B}_2(2)$, $\mathrm{G}_2(2)$, ${}^2\mathrm{F}_4(2)$ we have that $\mathrm{B}_2(2)' \cong \mathrm{Alt}(6) \cong \mathrm{PSL}_2(9)$ and so this group will appear in Theorem 1.6 in two of its guises: once as an alternating group and once as a genuine group of Lie type defined in characteristic 3. Similarly, $\mathrm{G}_2(2)' \cong \mathrm{SU}_3(3)$ and so will be considered as a group of Lie type in characteristic 3. In both these cases, as well as with the Tits group ${}^2\mathrm{F}_4(2)'$, they are not considered as groups of Lie type in characteristic 2 even though they inherit many properties from such groups and their appearance in our theorems is due to their characteristic 2 ancestry. Similar remarks apply to ${}^2\mathrm{G}_2(3)' \cong \mathrm{PSL}_2(8)$. There are also isomorphisms between simple groups which identifies

a group in one characteristic with a group in another characteristic. For example we have $\mathrm{Alt}(5) \cong \mathrm{PSL}_2(5) \cong \mathrm{SL}_2(4)$. In this type of instance, we consider such groups to be groups of Lie type in the relevant characteristic. This particular case arises in Table 1 where, in line one, we explicitly remove $G = \mathrm{PGL}_2(5)$ as an example as $O^2(G) \cong \mathrm{SL}_2(4)$ is a simple group of Lie type in characteristic 2. Nonetheless, we have to be alert to this chameleon behavior when we apply results inductively (see, for example, Theorem 3.3).

Remarkably the existence of a rank one isolated p-minimal subgroup almost always forces G to have a section which is a group of Lie type in characteristic p. This is made more precise by our first main theorem which assumes that all non-abelian composition factors of the groups considered are $\mathcal{K}$-groups.

Theorem 1.5. *Suppose that G is a finite group, p is a prime and P is a rank one isolated p-minimal subgroup of G. Set $Y = \langle O^p(P)^G \rangle$. Assume that $O_p(G) = 1$ and $Y \neq O^p(P)$. Then Y is quasisimple and either $Y/Z(Y)$ is a group of Lie type defined in characteristic p or $p \leq 7$ and the possibilities for $Y/Z(Y)$ are explicitly known.*

If, in Theorem 1.5, we assume additionally that $F^*(G)$, the generalized Fitting subgroup of G, is a non-abelian simple group, then with some explicitly described exceptions, which include twenty four of the sporadic simple groups, $F^*(G)$ is a group of Lie type in characteristic p. This is a substantial haul of sporadic groups. The detailed statement of this result is our second theorem.

Theorem 1.6. *Suppose that p is a prime, G is a finite group, $S \in \mathrm{Syl}_p(G)$ and P is a rank one isolated p-minimal subgroup of G in $\mathcal{P}_G(S)$. Set $X = F^*(G)$. If X is a non-abelian simple $\mathcal{K}$-group, then either X is a group of Lie type in characteristic p or*

> (i) $p = 2$ and either
>> (a) $X \cong \mathrm{Alt}(6)$ or $X \cong \mathrm{Alt}(12)$;
>> (b) X is the Tits group, ${}^2\mathrm{F}_4(2)'$;
>> (c) X is a group of Lie type in characteristic r of rank at most 12 with the possibilities for r, XS, P and $L_G(P,S)$ as listed in Table 1; or
>> (d) X is one of M_{12}, M_{22}, J_2, M_{23}, HS, J_3, M_{24}, He, Ru, Suz, O'N, Co_3, Co_2, Fi_{22}, HN, Th, Fi_{23}, Co_1, J_4, Fi'_{24}, $\mathbb{B}$ or $\mathbb{M}$.
> (ii) $p = 3$ and either
>> (a) X is a group of Lie type in characteristic r of rank at most 8 with the possibilities for r, XS, P and $L_G(P,S)$ as listed in Table 2; or
>> (b) X is one of M_{12}, J_2, McL, Suz, Co_3, Co_2, Fi_{22}, Th, Fi_{23}, Co_1, Fi'_{24}, $\mathbb{B}$ or $\mathbb{M}$.
> (iii) $p = 5$ and X is one of HN, Ly, Co_1, $\mathbb{B}$ or $\mathbb{M}$.
> (iv) $p = 7$ and $X \cong \mathbb{M}$.

We emphasise that, for example, though $G = \mathrm{Sp}_4(2) \cong \mathrm{Sym}(6)$ is a group of Lie type in characteristic 2, $X = F^*(G) \cong \mathrm{Alt}(6) \cong \mathrm{PSL}_2(9)$ is not and so, since X is a Lie type group in characteristic 3, according to Theorem 1.6 the examples in both G and X must be listed in Table 1.

In Tables 1 and 2 we have mainly used ATLAS notation to describe group extensions of a group A by a group B. Thus $A{:}B$ denotes a split extension, $A{\cdot}B$ a non-split extension and

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XS and conditions	P	$L_{XS}(P,S)$
1	$\mathrm{PGL}_2(r^a)$, $r^a \equiv 5 \pmod 8$, $r \neq 5$	$\mathrm{Sym}(4)$	$\mathrm{Dih}(2(r^a-1))$
2	$\mathrm{PGL}_2(r^a)$, $r^a \equiv 3 \pmod 8$	$\mathrm{Sym}(4)$	$\mathrm{Dih}(2(r^a+1))$
3	$\mathrm{PGL}_2(11)$	$\mathrm{Dih}(24)$	$\mathrm{Sym}(4)$
4	$\mathrm{PGL}_2(13)$	$\mathrm{Dih}(24)$	$\mathrm{Sym}(4)$
5	$\mathrm{PGL}_2(19)$	$\mathrm{Dih}(40)$	$\mathrm{Sym}(4)$
6	$\mathrm{PSL}_2(9) \cong \mathrm{Alt}(6)$	$\mathrm{Sym}(4)$	$\mathrm{Sym}(4)$
7	$\mathrm{P\Gamma L}_2(9) \cong \mathrm{Sym}(6)$	$2 \times \mathrm{Sym}(4)$	$2 \times \mathrm{Sym}(4)$
8	$\mathrm{PSp}_6(3)$, $2^b \sim G/X \leq F \cong 2$	$\frac{1}{2}(\mathrm{Q}_8 \times (2^{1+4}{:}\mathrm{SL}_2(4))).2^b$	$\frac{1}{2}(\mathrm{SL}_2(3) \wr \mathrm{Sym}(3)).2^b$
9	$\mathrm{PSU}_3(r^a)$, $r^a \equiv 3 \pmod 8$	$2 \cdot \mathrm{Sym}(4) * 4$	$(r^a+1)^2{:}\mathrm{Sym}(3)$
10	$\mathrm{PSU}_3(r^a){:}2$, $r^a \equiv 3 \pmod 8$	$(2 \cdot \mathrm{Sym}(4) * 4).2$	$((r^a+1)^2{:}\mathrm{Sym}(3)).2$
11	$\mathrm{PSU}_3(3) \cong \mathrm{G}_2(2)'$	$4^2{:}\mathrm{Sym}(3)$	$2 \cdot \mathrm{Sym}(4) * 4$
12	$\mathrm{PSL}_3(r^a){:}2$, $r^a - 1 \in \{2^b, 2^b.3\}$, $b \geq 2$, a odd	$(2^b)^2{:}(\mathrm{Sym}(3) \times 2)$	$\mathrm{GL}_2(r^a){:}2$
13	$\mathrm{PSU}_3(r^a){:}2$, $r^a + 1 \in \{2^b, 2^b.3\}$, $b \geq 2$, a odd	$(2^b)^2{:}(\mathrm{Sym}(3) \times 2)$	$\mathrm{GU}_2(r^a){:}2$
14[a]	$\mathrm{PSL}_3(9){:}2^b$, $b=1,2$	$(2^3)^2{:}(\mathrm{Sym}(3) \times 2^b)$	$\mathrm{GL}_2(9){:}2^b$
15[a]	$\mathrm{PSL}_3(25){:}2^b$, $b=1,2$	$(2^3)^2{:}(\mathrm{Sym}(3) \times 2^b)$	$\mathrm{GL}_2(25){:}2^b$
16	$\mathrm{PSL}_3(r^a){:}2$, $r^a \equiv 5 \pmod 8$, $r \neq 5$	$2 \cdot \mathrm{Sym}(4) * \mathrm{Q}_8$	$(r^a-1)^2{:}(2 \times \mathrm{Sym}(3))$
17[b]	$\mathrm{PSU}_4(3)$, $G/X \cong E \leq F \cong \mathrm{Dih}(8)$	$R_2 S \sim 2^{2+2+2}.\mathrm{Sym}(3).E$	$R_{13}S \sim 2^{1+4}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3)).E$
18[b]	$\mathrm{PSU}_4(3)$, $2^b \sim G/X \leq F = (2^2)_{122}$	$R_1 S \sim 2^{1+2+1+2}.\mathrm{Sym}(3).2^b$	$R_{23}S \sim 2^4.\mathrm{Alt}(6).2^b$
19[b]	$\mathrm{PSU}_4(3)$, $2^b \sim G/X \leq F = (2^2)_{122}$	$R_3 S \sim 2^{1+2+1+2}.\mathrm{Sym}(3).2^b$	$R_{12}S \sim 2^4.\mathrm{Alt}(6).2^b$
20	$\mathrm{PSU}_6(3)$, $2^b \sim G/X \leq F \cong 2^2$	$\frac{1}{2}(4^5.(\mathrm{Sym}(4) \times 2) \cap X).2^b$	$\frac{1}{2}(\mathrm{GU}_2(3) \wr \mathrm{Sym}(3) \cap X).2^b$
21	$\mathrm{P\Omega}_7^+(3)$, $2^b \sim G/X \leq F \cong 2$	$2^{1+1+2+1+2+1}.\mathrm{Sym}(3).2^b$	$2^6.\mathrm{Alt}(7).2^b$
22	$\mathrm{P\Omega}_8^+(3)$, $G/X \cong E \leq F \cong \mathrm{Dih}(8)$	$(\frac{1}{2}\mathrm{O}_2^-(3) \wr \mathrm{Sym}(4) \cap X).E$	$(\frac{1}{2}\mathrm{O}_4^+(3) \wr \mathrm{Sym}(2) \cap X).E$
23	$\mathrm{P\Omega}_{12}^+(3)$, $G/X \cong E \leq F \cong \mathrm{Dih}(8)$	$(\frac{1}{2}(\mathrm{O}_2^-(3) \wr \mathrm{Sym}(4) \times \mathrm{O}_2^-(3) \wr 2 \cap X()).E$	$(\frac{1}{2}\mathrm{O}_4^+(3) \wr \mathrm{Sym}(3) \cap X).E$
24	$\mathrm{G}_2(3)$, $2^b \sim G/X \leq F \cong 2$	$4^2{:}\mathrm{Dih}(12).2^b$	$2^{1+4}_+.3^2.2.2^b$
25	${}^3\mathrm{D}_4(3)$	$4^2{:}\mathrm{Dih}(12)$	$(\mathrm{SL}_2(3) * \mathrm{SL}_2(27)).2$
26	$\mathrm{E}_7(3)$, $2^b \sim G/X \leq F \cong 2$,	$4^7.2^5.2^3\mathrm{SL}_2(2).2^b$	$2^3.(\mathrm{PSL}_2(3))^7.2^3.\mathrm{PSL}_3(2).2^b$

TABLE 1. The shape of XS, P and $L_{XS}(P,S)$ for group of Lie type exceptions with $p = 2$ in Theorem 1.6 (i)

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[a]See Lemma 7.6; XS must contain the inverse transpose automorphism.

[b]See Example 7.9 for notation

	XS and conditons	P	$L_{XS}(P,S)$
1	$\mathrm{SL}_2(8)$	$^2\mathrm{G}_2(3)' \cong \mathrm{SL}_2(8)$	$\mathrm{Dih}(18)$
2	$\mathrm{SL}_2(8).3$	$^2\mathrm{G}_2(3) \cong \mathrm{SL}_2(8).3$	$\mathrm{Dih}(18).3$
3	$\mathrm{PGL}_3(r^a), r^a \equiv 4,7 \pmod 9, r^a \neq 4$	$3^2{:}\mathrm{SL}_2(3)$	$(r^a-1)^2{:}\mathrm{Sym}(3)$
4	$\mathrm{PGU}_3(r^a), r^a \equiv 2,5 \pmod 9, r^a \neq 2$	$3^2{:}\mathrm{SL}_2(3)$	$(r^a+1)^2{:}\mathrm{Sym}(3)$
5	$\mathrm{PGU}_3(5)$	$(3^2 \times 2^2){:}\mathrm{Sym}(3)$	$3^2{:}\mathrm{SL}_2(3)$
6	$\mathrm{PGL}_3(7)$	$(3^2 \times 2^2){:}\mathrm{Sym}(3)$	$3^2{:}\mathrm{SL}_2(3)$
7	$\mathrm{PSU}_4(r^a), r^a \equiv 2,5 \pmod 9, r^a \neq 2$	$3^{1+2}_+{:}\mathrm{SL}_2(3)$	$\frac{1}{(4,r^a+1)}(r^a+1)^3{:}\mathrm{Sym}(4)$
8	$\mathrm{PSU}_5(2)$	$3 \times 3^{1+2}_+.\mathrm{SL}_2(3)$	$3^4.\mathrm{Sym}(5)$
9	$\mathrm{PSU}_6(2)$	$3^5{:}\mathrm{Alt}(6)$	$3^4.\mathrm{Sym}(3)$
10	$\mathrm{PSU}_6(2){:}3$	$3^6{:}\mathrm{Alt}(6)$	$3^{1+4}_+(\mathrm{Q}_8 \times \mathrm{Q}_8).\mathrm{Sym}(3)$
11	$\mathrm{P\Omega}_8^+(r^a), r^a \equiv 2,5 \pmod 9$	$3^{1+2}_+.\mathrm{SL}_2(3) \times 3$	$3^{1+4}_+(\mathrm{Q}_8 \times \mathrm{Q}_8).\mathrm{Sym}(3).3$
12	$\mathrm{P\Omega}_8^+(r^a).3, r^a \equiv 4,7 \pmod 9, r^a \neq 4$	$3^{1+4}_+.\mathrm{SL}_2(3)$	$\frac{1}{(2,r^a-1)}(\Omega_2^-(r^a)^4.(2,r^a-1)^3.2^3 \wr \mathrm{Sym}(4))$
13	$\mathrm{P\Omega}_8^+(r^a).3, r^a \equiv 2,5 \pmod 9,$	$3^{1+4}_+.\mathrm{SL}_2(3)$	$\frac{1}{(2,r^a-1)}(\Omega_2^+(r^a)^4.(2,r^a-1)^3.2^3 \wr \mathrm{Sym}(4).3$ $\frac{1}{(2,r^a-1)}(\Omega_2^-(r^a)^4.(2,r^a-1)^3.2^3 \wr \mathrm{Sym}(4).3$
14	$\mathrm{G}_2(r^a), r^a \equiv 4,7 \pmod 9$	$\mathrm{SU}_3(3)$	$\mathrm{SL}_3(r^a).2$
15	$\mathrm{G}_2(r^a), r^a \equiv 2,5 \pmod 9$	$\mathrm{SU}_3(3)$	$\mathrm{SU}_3(r^a).2$
16	$^2\mathrm{E}_6(2), 3^b \sim G/X \leq F \cong 3$	$3^{2+1+1+2+2}.\mathrm{SL}_2(3).3^b$	$(\frac{1}{3}\mathrm{PSU}_3(2) \wr \mathrm{Sym}(3).3 \cap X).3^b$
17	$\mathrm{E}_8(2)$	$3^{[12]}.\mathrm{SL}_2(3)$	$3^2.(\mathrm{PSU}_3(2)^4).3^2.\mathrm{GL}_2(3)$

TABLE 2. The shape of XS, P and $L_{XS}(P,S)$ for group of Lie type exceptions with $p=3$ in Theorem 1.6(ii)

$A.B$ an extension of unspecified type. Notation such a $\frac{1}{c}(A \times B)$ means that the "obvious" central subgroup of order c has been factored out.

The remainder of our mostly standard notation will be introduced in Section 2. We remark that more details of the examples which arise in the sporadic simple groups can be found in Section 10 and especially in (10.2.1) to (10.2.26). We point out that just two of the sporadic simple groups, M_{11} and J_1, are bereft of rank one isolated p-minimal subgroups whereas some get more than their fair share. We further remark that in the cross characteristic case of Theorem 1.6, that is when $p \neq r$, frequently the rank one isolated p-minimal subgroup is soluble.

Corollary 1.7. *Suppose that p is a prime, G is a finite group for which $X = F^*(G)$ is a quasisimple group. If G contains a rank one isolated p-minimal subgroup, then $X/Z(X)$ appears in the conclusion of Theorem 1.6.*

Corollary 1.8. *Assume p is a prime, G is a finite group containing a rank one isolated p-minimal subgroup. If $X = F^*(G)$ is a non-abelian quasisimple group and $p \geq 5$, then either $X/Z(X)$ is a simple group of Lie type in characteristic p or is one of the sporadic simple groups* HN, Ly, Co_1, $\mathbb{B}$ *or* $\mathbb{M}$.

We recall from [40, Definition 1.5] that a group G with $S \in \mathrm{Syl}_p(G)$, p a prime, is called completely isolated if P is isolated for all $P \in \mathcal{P}_G(S)$.

Corollary 1.9. *Suppose that G is a finite non-abelian simple group and p is a prime. If G is completely isolated and all its p-minimal subgroups are of rank one, then either G is a group of Lie type of characteristic p or*

(i) *$p = 2$ and G is isomorphic to one of* $\mathrm{Alt}(6)$, $\mathrm{PSU}_3(3)$, $^2F_4(2)'$, $\mathrm{PSU}_4(3)$, M_{12}, J_2, J_3 *and* Suz;

(ii) *$p = 3$ and G is isomorphic to one of* $\mathrm{SL}_2(8)$, M_{12}, Th *and* Co_1;

(iii) *$p = 5$ and G is isomorphic to* Co_1.

Theorems 1.5 and 1.6 may be compared with results on groups with a BN-pair, specifically Tits's theorem [53] which says that a finite simple group with a BN-pair of rank at least 3 is isomorphic to a group of Lie type. Tits's theorem does not need to assume the simple group classification as a BN-pair possesses a Weyl group which gives global information whereas here we have no comparable subquotient.

Isolated p-minimal subgroups, and those of rank one in particular, arise in a subcase of what has become known as the "Third Generation Proof" of the classification of finite simple groups. This is a new approach, pioneered by Meierfrankenfeld, Stroth and Stellmacher, to improve the classification of the finite simple groups. Isolated subgroups appear in this programme in a situation where it has already been proved in [44] that there is a p-local maximal subgroup C of a group G which contains all but exactly one of the p-minimal subgroups of G containing a Sylow p-subgroup S. Let $P \in \mathcal{P}_G(S)$ be this subgroup. In the case that there are two p-minimal subgroups in C containing S which do not normalize $O^p(P)$, then [41] determines the structure of C and P, a desired conclusion of the programme. So either the structure of C and P is known, or C has a

subgroup $L = N_C(O^p(P))$ and a unique p-minimal subgroup P_1 containing S such that $P_1 \nleq L$. It turns out that $P_1/O_p(C)$ is isolated in $C/O_p(C)$. Thus the main results of this article are applicable to name the group $\langle O^p(P_1)^C \rangle/O_p(C)$ and identify $P_1/O_p(C)$ and $L/O_p(C)$. Notice that this uses Theorem 1.6 in the p-local subgroup C of G and so the $\mathcal{K}_p$-hypothesis, which states that composition factors of p-local subgroups are $\mathcal{K}$-groups, is all that is required to use our results which are about $\mathcal{K}$-groups.

We now give an overview of this paper while also outlining our overall strategy in proving the main theorems of this work. Section 2 contains our basic weaponry as well as discussing the notation we shall be using. In this arsenal we have Lemma 2.2 containing elementary properties of isolated p-minimal subgroups and which are used at every turn. As is Lemma 2.4 in which the group structure of a rank one p-minimal subgroup is described. Let G, p, S, P and X be as in the statement of Theorem 1.6. Because of Lemma 2.17 we may suppose that $G = XS$ whilst proving Theorem 1.6. Frequently we move into subquotients of subgroups of G which contain P, and then Lemmas 2.6 and 2.7 are used.

The basic thrust and counter-thrust of our arguments is to locate P and $L_G(P, S)$. Thus we are always interested in subgroups H of X which are normalized by S. For such subgroups, if $HS \nleq L_G(P, S)$, then Lemma 2.2(iii) tells us that then $P \leq HS$. Moreover, P is also a rank one isolated p-minimal subgroup of HS. Clearly this gives us the opportunity to argue inductively, and we take full advantage of this. We will say more on this shortly, after noting another string to our bow in this type of situation. Many of the subgroups HS we encounter have a wreath product type of structure and using the big gun Theorem 2.15 we are usually able to severely restrict the structure of HS. Now, depending on the circumstances, we may be able to deduce that P cannot be in HS (for example, inductively HS possesses no rank one isolated p-minimal subgroups or deducing a contradiction to the structure of H using Theorem 2.15 and other results). As a consequence we will have shown that $HS \leq L_G(P, S)$. Now $L_G(P, S)$ is a maximal subgroup of G (see Lemma 2.2(v)) and $O_p(L_G(P, S)) \neq 1$ and so if HS is known to be a maximal p-local subgroup of G we then deduce that $L_G(P, S) = HS$ and HS is a maximal subgroup of G. A by-product of pinpointing $L_G(P, S)$ is that we often force P to be in some other subgroup(s) that we have also been investigating. One other general remark is that Theorem 2.15 frequently ends up forcing the target subgroup of G to not have any components from which we (usually) conclude that $p \in \{2, 3\}$. Sometimes this will then lead to an impossible configuration or we capture either various cross characteristic examples or the exceptional examples tabulated in Tables 1 and 2.

The proof of Theorem 1.6 occupies Sections 3 through to 10, with Section 3 dealing with the case when X is an alternating group. Apart from examples resulting from the well-known isomorphisms involving symmetric groups, alternating groups and groups of Lie type itemized in Lemma 3.1, there is one further example which breaks cover from the undergrowth in the symmetric group Sym(12). Section 5 begins the lengthy campaign of

analyzing the case when X is a group of Lie type. This section dealing with the case when $X \cong \mathrm{PSL}_2(r^a)$, r a prime. Then Sections 6, 7, 8, 9 and 10 look at the cases when X is, respectively, a projective symplectic group, a projective special linear or unitary group, a projective orthogonal group, an exceptional group and a sporadic simple group. While Section 4, in addition to containing some background material on Sylow p-subgroup orders for groups of Lie type, has two results Lemmas 4.5 and 4.6, which are frequently mentioned in dispatches. Lemma 4.5 has the effect of mowing down any X for which $X \cap S$ is abelian, $G \neq X$ and $p \notin \{2,3\}$, while Lemma 4.6 shows that $X \cap S$ is not contained in any proper parabolic subgroup of X (when X is a group of Lie type, and G/X does not induce any graph type automorphisms on X). These two lemmas foreshadow the fact that there are not many exceptional rank one isolated p-minimal subgroups and when they occur p is small, typically $p \in \{2,3\}$.

The order of battle is very much determined by inductive requirements. Symmetric groups seen in Section 3 reappear in the wreath product type subgroups in many places. While it is no surprise the rank one groups $\mathrm{PSL}_2(r^a)$ emerge when X is a group of Lie type. Projective symplectic groups are covered in Section 6, so as to be used via the centralizer of the transpose inverse automorphism in (7.12.3). We next discuss in more detail the case when X is either a projective special or projective special unitary group. So we have $X \cong \mathrm{PSL}_n^\epsilon(r^a)$ with $X \leq G \leq \mathrm{Aut}(X)$ (where $\epsilon = \pm$, notation chosen so as to simultaneously cover the two classes of groups). We begin in Section 7 looking at $\widehat{X} \cong \mathrm{SL}_n^\epsilon(r^a)$ so as to take advantage of the natural module V for $\widehat{X}$ and the subgroups of $\widehat{X}$ which stabilize various subspaces of V. Of course we then project these subgroups into X. After defining d_ϵ and s we consider

$$\widehat{M^*} = \mathrm{GL}_{d_\epsilon}^\epsilon(r^a) \wr \mathrm{Sym}(s) \times \mathrm{GL}_{n-sd_\epsilon}^\epsilon(r^a),$$

a subgroup of $\mathrm{GL}_n^\epsilon(r^a)$. Set $\widehat{M} = \widehat{M^*} \cap \widehat{X}$ and let M be the image of $\widehat{M}$ in X. Now $\widehat{M}$ is of interest as it contains a Sylow p-subgroup of $\widehat{X}$, and it also has a wreath product structure – such subgroups having been mentioned earlier.

Then battle commences to prove Theorem 7.1, the main result of this section, beginning with Lemmas 7.2 and 7.3. Lemma 7.3 is particularly useful as it gives information about p-local subgroups of X. Because of Theorem 2.16(ii) $L_G(P,S) \cap X$ is a maximal p-local subgroup of X and so p-local data is relevant to locating $L_G(P,S) \cap X$. Lemma 7.4 then looks at the situation when $p > 3$ and n is "small" (meaning $n \leq d_\epsilon p$) and concludes that G has no rank one isolated p-minimal subgroups. Here we see all the results we have mentioned acting in concert – Lemma 4.5 eliminating the possibility that $S \cap X$ is abelian. Then Theorem 2.15 forces either $d_\epsilon = 1$ or $\mathrm{GL}_{d_\epsilon}^\epsilon(r^a)$ to be soluble. These two possibilities lead to contradictions using the structure of P as given in Lemma 2.4. After further skirmishes in Lemmas 7.5, 7.6 and 7.7 when $p \in \{2,3\}$ and $3 \leq n \leq 7$ are closely examined, the final push comes in Theorem 7.12. We remark that all the exceptional examples here arise when $p \in \{2,3\}$ and $3 \leq n \leq 7$. The assumptions (i) - (v) in Theorem 7.12 are

couched so as to avoid these examples. We then choose G to be a minimal counterexample to this theorem. Then, examining $\widehat{M}^*$, and deploying Lemma 4.6 it is shown that $n = d_\epsilon s$ (so $\widehat{M}^* = \mathrm{GL}_{d_\epsilon}^\epsilon(r^a) \wr \mathrm{Sym}(s)$). After observing that $s > p$ in (7.12.2) (unless $d_\epsilon = 2$, $p = 3$ and $n = 6$), (7.12.3) looks at the various consequences of having $d_\epsilon > 1$. Of particular note here is that $MS \leq L_G(P, S)$ (a notational point here M, $\widehat{M}^*$ without the decorations, is the image in X). Here Theorem 2.15 again prevails, quickly reducing to the cases $r^a = 2$ or 3. Then the proof of (7.12.3) is completed with the help of two different subgroups of X normalized by S. When $d_\epsilon = 1$, (7.12.4) also shows that we have $MS \leq L_G(P, S)$. This assertion is proved by contradiction. So we have $MS \not\leq L_G(P, S)$ (and $d_\epsilon = 1$) whence, using Lemma 2.2(iii), $P \leq MS$. Because $d_\epsilon = 1$ (and $d_\epsilon s = n$ by (7.12.1)), we now have

$$\widehat{M}^* = \mathrm{GL}_1^\epsilon(r^a) \wr \mathrm{Sym}(s) = (r^a - \epsilon) \wr \mathrm{Sym}(s)$$

and Theorem 3.3 used on the $\mathrm{Sym}(s)$ quotient of $\widehat{M}^*$ (as well as other similar wreath type groups) serves to restrict M (and like groups), the outcome being that P cannot be a rank one isolated p-minimal subgroup, the desired contradiction. So, by (7.12.3) and (7.12.4), we have that $MS \leq L_G(P, S)$. With the aim of finding more subgroups of X normalized by S we next show in (7.12.5) that p does not divide s. This brings

$$\widehat{M_1}^* = \mathrm{GL}_{d_\epsilon p}^\epsilon(r^a) \wr \mathrm{Sym}(j) \times \mathrm{GL}_{d_\epsilon k}^\epsilon(r^a)$$

and $\widehat{K_1}^* \cong \mathrm{GL}_{d_\epsilon p}^\epsilon(r^a)$ into the fray where $s = jp + k$ (and we have $1 \leq k \leq p - 1$). Together with M, Lemma 2.11 annihilates the possibility that $k > 1$. So $k = 1$ and another salvo from Theorem 2.15 yields $j = 1$. This now corners P in $K_1 S$, and quickly leads to the much sought contradiction.

The expedition against the even dimensional projective orthogonal groups in Theorem 8.11 of Section 8 is broadly similar to that in Theorem 7.12 of Section 7, with similar intermediate strategic aims. There are, as is to be expected, greater complexity and differences in the details. For example the parameter d_ϵ in Section 7 is replaced by d_0, and s has a slightly different definition. Then we have

$$\widehat{M} = \mathrm{O}_{2d_0}^\eta(r^a) \wr \mathrm{Sym}(s) \times \mathrm{O}_{2(n - d_0 s)}^\theta(r^a)$$

contains a Sylow p-subgroup of $\widehat{X}$. For the types η and θ see the beginning of Section 8. And $\widehat{M}^*$ brings with it an underlying decomposition of the orthogonal module. A useful result is Lemma 8.8 which, under certain circumstances, yields a subgroup $\widehat{H}$ of $\widehat{G} = \mathrm{O}_{2n_0}^\epsilon(r^a)$ with $\widehat{S} \leq \widehat{H} \cong \mathrm{GL}_n^\epsilon(r^a)$ and consequently, by Section 7, we know all about the rank one isolated p-minimal subgroups of G which happen to lie in H. The proof of Theorem 8.11 is lengthy, partly because there are two possible types of orthogonal groups (even so it doesn't cover the 8-dimensional case when G/X induces a graph automorphism on X). The case when $X \cong \mathrm{P}\Omega_{2n+1}(r^a)$ is of odd dimension is contained in Theorem 8.12 . Here we can go for a quick knockout since, by looking at the stabilizer of non-singular points in V (the orthogonal module), we can apply Theorem 8.11 to deduce that $\dim V \in \{7, 9, 13\}$ and $p \in \{2, 3\}$. These possibilities, apart from the case $p = 2$ and $X \cong \mathrm{P}\Omega_7(3)$, are quickly put to the sword. The coup de grâce for the projective orthogonal groups is delivered by

Theorem 8.13, when the case with $X \cong P\Omega_8^+(r^a)$ and G/X involves a graph automorphism is dismembered. The proof of Theorem 8.13 relies heavily on the list of maximal subgroups of G given in [27].

Section 9 confronts the case when X is an exceptional group of Lie type. The work in this section is a little different to that in Sections 7 and 8 as we are able to obtain many subgroups containing Sylow p-subgroups from [35]. When X is a sporadic simple group, the style of Section 10 is similar, this time we use [60] (or ATLAS [12]) to supply the ammunition.

The final section of this paper mops up the proofs of Theorems 1.5 and 1.6 and presents proofs of Corollaries 1.7, 1.8 and 1.9.

2. NOTATION AND BACKGROUND RESULTS

The purpose of this section is to gather a war chest of results for the ensuing campaign. First, however, we discuss our notation.

By and large our group theoretic notation is standard, as evinced in the texts [1, 17, 32]. We also employ the ATLAS [12] terminology and conventions to describe (roughly) group structure, and use $\sim$ to indicate that two groups have the same shape. Of course the ATLAS and also [60] supply a vast amount of data, particularly on the sporadic simple groups, which we shall draw upon. Points of divergence from the aforementioned are as follows. We use the ATLAS notation for the sporadic simple groups but not in general for the classical groups. For instance $O_{2n}^\epsilon(r^a)$ denotes the general orthogonal group and not the simple group and a detailed exposition of associated notation is in Section 8. The symmetric groups and alternating groups of degree n will be denoted by, respectively, $\mathrm{Sym}(n)$ and $\mathrm{Alt}(n)$. By $\mathrm{Dih}(n)$, $\mathrm{SDih}(n)$ and $\mathrm{Frob}(n)$ we denote the dihedral, semidihedral and Frobenius group of order n respectively. Another source we use here is [20], and we follow their notation $*$ for central products.

When specifying certain subgroups of matrix groups we shall usually employ an equal sign to indicate that the subgroups have a "canonical" description once a particular basis is used.

Let H be a finite group and p a prime. Then we shall use Q_H to stand for $O_p(H)$ (p being understood from the context) while $F_p^*(H)$ is the inverse image of $F^*(H/Q_H)$. We recall that the generalized Fitting subgroup $F^*(H)$ equals $E(H)F(H)$, $E(H)$ being the product of the components of H and $F(H)$ the Fitting subgroup of H.

Next we marshal results on isolated p-minimal subgroups. For the remainder of this section we assume that G is a finite group, p is a prime and $S \in \mathrm{Syl}_p(G)$. We begin with the elementary and well known observation given in

Lemma 2.1. *We have $G = \langle \mathcal{P}_G(S) \rangle N_G(S)$.*

Proof. See, for example, [40, Lemma 3.1]. □

We call upon the next lemma frequently–note how part (iii) sets us up well for inductive arguments.

Lemma 2.2. *Suppose that $P \in \mathcal{P}_G(S)$ is an isolated subgroup of G. Then*
   (i) *$P \not\leq L_G(P,S)$ and $L_G(P,S) \neq G$;*
   (ii) *$N_G(S) \leq N_G(P)$;*
   (iii) *if $S \leq H \leq G$ and $H \not\leq L_G(P,S)$, then $P \leq H$, $L_H(P,S) = L_G(P,S) \cap H$ is a maximal subgroup of H and P is an isolated subgroup of H;*
   (iv) *$P \cap L_G(P,S)$ is the unique maximal subgroup of P containing S;*
   (v) *$L_G(P,S)$ is a maximal subgroup of G; and*
   (vi) *if $S \leq H \leq G$ and $Q_H \leq Q_P$, then $P \leq H$.*

Proof. For parts (i) and (ii) see [40, Lemma 4.2] and for the remaining parts, except (vi), [40, Lemma 4.3]. While part (vi) is [40, Lemma 4.5 (b)]. □

When seeking the exact location of certain isolated p-minimal subgroups we are frequently able to force the isolated subgroup into a subgroup which is a direct product. Then our next result is used to refine our search.

Lemma 2.3. *Suppose that G has subgroups H_1 and H_2 with $G = H_1 H_2$ and $[H_1, H_2] = 1$. Then*
$$\mathcal{P}_G(S) = \mathcal{P}_{H_1 S}(S) \cup \mathcal{P}_{H_2 S}(S).$$

Proof. See [40, Lemma 3.5]. □

The detailed structure of rank one p-minimal subgroups is laid out in our next result.

Lemma 2.4. *If P is a rank one p-minimal subgroup of G, then setting $\overline{P} = P/Q_P$ exactly one of the following holds.*
   (i) *$p^a \geq 4$ and $\overline{P}$ is isomorphic to $\mathrm{SL}_2(p^a)$ or $\mathrm{PSL}_2(p^a)$ perhaps extended by field automorphisms of order a power of p.*
   (ii) *$p^a \geq 3$ and $\overline{P}$ is isomorphic to $\mathrm{SU}_3(p^a)$ or $\mathrm{PSU}_3(p^a)$ perhaps extended by field automorphisms of order a power of p.*
   (iii) *$p = 2$, $a \geq 2$ and $\overline{P}$ is isomorphic to ${}^2\mathrm{B}_2(2^{2a-1})$.*
   (iv) *$p = 3$, $a \geq 2$ and $\overline{P}$ is isomorphic to ${}^2\mathrm{G}_2(3^{2a-1})$ perhaps extended by field automorphisms of order a power of 3.*
   (v) *$p = 2$, $O^2(\overline{P}) \cong 3$ and $\overline{P} \cong \mathrm{SL}_2(2) \cong \mathrm{Sym}(3)$.*

(vi) $p = 3$, $O^3(\overline{P}) \cong Q_8$ or 2^2 and, respectively, $\overline{P} \cong \mathrm{SL}_2(3) \cong 2\cdot \mathrm{Alt}(4)$ or $\mathrm{PSL}_2(3) \cong$ Alt(4).

(vii) $p = 2$, $O^2(\overline{P}) \cong 3^{1+2}_+$ or 3^2 and $P/O^2(P)Q_P \cong \mathrm{SDih}(16)$, 8 or Q_8.

(viii) $p = 2$, $O^2(\overline{P}) \cong 5$ and $\overline{P} \cong \mathrm{Dih}(10)$ or $^2\mathrm{B}_2(2) \cong \mathrm{Frob}(20)$.

(ix) $p = 3$ and $\overline{P}$ is isomorphic to $^2\mathrm{G}_2(3)' \cong \mathrm{SL}_2(8)$ or $^2\mathrm{G}_2(3) \cong \mathrm{SL}_2(8){:}3$.

Proof. See [40, Lemma 3.6]. □

We observe that the requirement of a rank one p-minimal subgroup to be narrow leads in part (vii) to the exclusion of the possibility that $P/O^2(P)Q_P$ is a subgroup of Dih(8).

Lemma 2.5. *Suppose that $P \in \mathcal{P}_G(S)$ is a rank one isolated p-minimal subgroup of G and $Q_G = 1$. Set $L = L_G(P, S)$. If $Q_L \leq Z(S)$ and $Z(S)$ is cyclic, then $|S| \leq p^3$ or $S \cong \mathrm{SDih}(16)$.*

Proof. Since $[Q_P, Q_L] = 1$ and $\langle Q_L^P \rangle \geq O^p(P)$, $[Q_P, O^p(P)] = 1$. Thus, if $Q_P \neq 1$, $\Omega_1(Q_L) = \Omega_1(Z(S)) \leq Q_P$ is normalized by $\langle L, O^p(P) \rangle = \langle L, P \rangle = G$, a contradiction. Thus $Q_P = 1$. If P is soluble, then, by Lemma 2.4, $O^p(P)$ has order at most 27 and the result follows from the structure of $\mathrm{Aut}(O^p(P))$. Thus we may suppose that P is not soluble. In particular, p divides $|O^p(P)|$ and so $\Omega_1(Q_L) \leq Z(S \cap O^p(P))$. Since $O^p(P) \in \mathcal{L}_1(p)$, $N_{O^p(P)}(S \cap O^p(P))$ acts irreducibly on $\Omega_1(Z(S \cap O^p(P)))$. Because $N_{O^p(P)}(S \cap O^p(P))S$ does not contain P, $N_{O^p(P)}(S \cap O^p(P))S \leq L$. Thus $\Omega_1(Q_L) = \Omega_1(Z(S \cap O^p(P)))$. As $|\Omega_1(Q_L)| = p$, we deduce that $O^p(P)$ is defined over $\mathrm{GF}(p)$. It follows that either $S = S \cap O^p(P)$ or $O^p(P)/Z(O^p(P)) \cong {}^2\mathrm{G}_2(3)'$. In either case $|S| \leq p^3$. □

When analyzing various subquotients of groups which contain an isolated p-minimal subgroup, we need our next result to hand.

Lemma 2.6. *Suppose that $P \in \mathcal{P}_G(S)$ is an isolated p-minimal subgroup of G and N is a normal subgroup of G. Then exactly one of the following holds*

(i) $G = L_G(P, S)N$; or

(ii) PN/N is an isolated p-minimal subgroup of G/N and $L_{G/N}(PN/N, SN/N) = L_G(P, S)N/N$.

Furthermore, if (i) holds, then $O^p(P) \leq N$.

Proof. Set $L = L_G(P, S)$. Suppose that $G \neq LN$. By Lemma 2.2 (v), L is a maximal subgroup of G. Hence $N \leq L$ and $O^p(P) \not\leq N$. Employing [40, Lemma 4.6 (d)] yields that PN/N is an isolated p-minimal subgroup of G/N and, as $L/N \geq S/N$ is a maximal subgroup of G/N, the result follows.

On the other hand, if $G = LN$, then $Q_L N$ is a normal subgroup of G and so $O^p(P) \leq \langle Q_L^P \rangle \leq Q_L N \leq SN$ which means $O^p(P) \leq N$. Hence $PN/N = SN/N$ is not a p-minimal subgroup of G/N. □

Lemma 2.7. *Suppose that N is a normal subgroup of G such that N is p-closed and $O_p(G/N) = 1$. If P is an isolated p-minimal subgroup of G, then PN/N is an isolated p-minimal subgroup of G/N.*

Proof. Assume that the lemma is false, and set $L = L_G(P, S)$. Then, by Lemma 2.6, $G = LN$. Hence

$$Q_L N/N \le O_p(G/N) = 1$$

and so, as N is p-closed,

$$Q_L \le Q_N \le Q_P,$$

which is impossible. $\square$

Lemma 2.8. *Suppose that $F^*(G)$ is quasisimple. If $K \le G$ and $G = Z(F^*(G))K$, then $K = G$.*

Proof. Set $X = F^*(G)$ and assume that $G = Z(X)K$. Then

$$X = X \cap Z(X)K = Z(X)(X \cap K).$$

Since X is quasisimple, $X = X' = (X \cap K)' \le K$ and so $G = Z(X)K = K$. $\square$

Lemma 2.9. *Suppose that $F^*(G)$ is quasisimple and set $Z = Z(F^*(G))$. Then $O^p(P) \not\le Z$. In particular, PZ/Z is a rank one isolated p-minimal subgroup of G/Z.*

Proof. Let $L = L_G(P, S)$. Suppose that $O^p(P) \le Z$. Then, by Lemma 2.1, $G = \langle O^p(P), L \rangle = ZL$. Hence $L = G$ by Lemma 2.8, contrary to $O^p(P) \not\le L$. Therefore $O^p(P) \not\le Z$ and the final statement follows from Lemma 2.6 with $N = Z$. $\square$

Lemma 2.10. *Suppose that A and B are subgroups of G which are normalized by S. If $A \cap B = 1$, then $AS \cap BS = S$.*

Proof. Since S normalizes A and B, we have

$$O^p(AS \cap BS) \le O^p(AS) \cap O^p(BS) \le A \cap B = 1.$$

Hence $AS \cap BS$ is a p-group and so $AS \cap BS = S$.

$\square$

Lemma 2.11. *Suppose that P is an isolated p-minimal subgroup of G. Assume that A, B and C are subgroups of G each normalized by S and that*

$$O_p(\langle A, C \rangle S) = O_p(\langle B, C \rangle S) = 1.$$

If $A \cap B = 1$, then $P \le CS$.

Proof. Set $L = L_G(P, S)$, and assume that $P \not\le CS$. Then $CS \le L$ and, as $Q_L \ne 1$, we deduce that $P \le AS$ and $P \le BS$. However, $AS \cap BS = S$ by Lemma 2.10, which contradicts P being p-minimal. Hence $P \le CS$ as claimed. $\square$

In certain cases, particularly in Section 10 which analyzes the sporadic simple groups, our next lemma easily eliminates various possibilities.

Lemma 2.12. *Suppose that $P \in \mathcal{P}_G(S)$ is an isolated subgroup of G. If either $N_G(S)$ acts irreducibly on S, or $N_G(S) = L_G(P, S)$, or $N_G(S)$ is contained in a unique maximal subgroup of G, then P is normal in G.*

Proof. See [40, Lemma 4.16]. □

Theorem 2.13. *Suppose that G is a finite non-abelian simple group. If S is abelian, then $N_G(S)$ acts irreducibly on $\Omega_1(S)$.*

Proof. See [18, 12-1, page 158] or [20, 7.8.1]. □

Lemma 2.14. *Suppose that G is a non-abelian simple group and S is abelian. If $P \in \mathcal{P}_G(S)$ is a rank one isolated subgroup of G, then $P = G$.*

Proof. Set $L = L_G(P, S)$. Suppose that S is abelian and $P \neq G$. Then, as $Q_L \neq 1$ and $N_G(S) \leq L$, Theorem 2.13 implies that $\Omega_1(S) \leq Q_L$. Hence $\Omega_1(S) = \Omega_1(Q_L)$. Since $N_G(S)$ also normalizes Q_P by Lemma 2.2 (ii), if $Q_P \neq 1$, then $\Omega_1(Q_P) = \Omega_1(Q_L)$. Hence $\Omega_1(Q_L)$ is normal in $\langle L, P \rangle = G$, which is impossible. Therefore $Q_P = 1$. Since $O^p(P/Q_P) \in \mathcal{L}_1(p)$ and S is abelian, by Lemma 2.4, S is either elementary abelian or cyclic. If S is elementary abelian, then Theorem 2.13 shows that $N_G(S)$ acts irreducibly on S and so Lemma 2.12 gives $P = G$, whereas $P \neq G$. So S is cyclic. Since $S \neq Q_L$ by Lemma 2.12, $|S| \geq p^2$. Now $O^p(P) \in \mathcal{L}_1(p)$ and so Lemma 2.4 implies that either $P \cong {}^2\mathrm{G}_2(3)' \cong \mathrm{SL}_2(8)$ or S is a cyclic 2-group. Hence, as G is a simple group, we obtain the former possibility. In particular, $|S| = 9$.

Since the only non-abelian alternating group with cyclic Sylow 3-subgroups is $\mathrm{Alt}(5)$, G is not an alternating group. Also neither the sporadic simple groups nor ${}^2\mathrm{F}_4(2)'$ have cyclic Sylow 3-subgroups of order 9. Thus G is a simple group of Lie type. Since S is cyclic of order 9, G is not a group of Lie type in characteristic 3. From [46, Theorem 9.8], we have that either $X \cong \mathrm{PSL}_2(r^a)$ with $r^a \equiv 8, 10, 17, 19 \pmod{27}$ or $\mathrm{PSL}_3(r^a)$ with $r^a \equiv 8, 17 \pmod{27}$ or $\mathrm{PSU}_3(r^a)$ with $r^a \equiv 10, 19 \pmod{27}$. Let $\widehat{X} = \mathrm{SL}_2(r^a), \mathrm{SL}_3(r^a) = \mathrm{SL}_3^+(r^a)$ and $\mathrm{SU}_3(r^a) = \mathrm{SL}_3^-(r^a)$ respectively. Observe that 3 does not divide $r^a - \epsilon$ in the $\mathrm{SL}_3^\epsilon(r^a)$-case. Hence, in all three cases, 3 does not divide $|Z(\widehat{X})|$ and there exists a unique preimage $\widehat{S}$ of S in $\widehat{X}$ with $|\widehat{S}| = S$. Let $\mathbb{K}$ be an algebraic closure of $\mathrm{GF}(r^a)$ and consider the natural 2-, respectively 3-dimensional natural module $\mathbb{K}\widehat{X}$-module V. Let $1 \neq s \in \widehat{S}$ with $|s| = 9$. In $P \cong \mathrm{SL}_2(8)$ we see that s is inverted by an element of order 2. Also s has determinant 1 on V. It follows that the eigenvalues of s on V are κ, κ^{-1}, and (in the $\mathrm{SL}_3^\epsilon(r^a)$ case) 1, where $\kappa \in \mathbb{K}^\#$ has order 9. Hence s^3 and s have the same eigenspaces on V and $N_{\mathrm{GL}(V)}(\widehat{S}) = N_{\mathrm{GL}(V)}(\Omega_1(\widehat{S}))$. This in turn implies that $N_G(S) = N_G(Q_L)$. As $N_G(S) \leq L$ and L is maximal subgroups of G we get $N_G(S) = L$. But now Lemma 2.12 shows that P is normal in G, so as $P \neq G$ and G is simple, we have a contradiction. □

The main results of [40] which we now state play a significant role in our investigations. We present a version of them in the next two theorems in a form useful for the present work.

Theorem 2.15. *Suppose that P is a rank one isolated p-minimal subgroup of G, and set $Y = \langle O^p(P)^G \rangle$.*

(i) Either $Y = O^p(P)$ or Y/Q_Y is quasisimple.
(ii) $O^p(P) \leq F_p^*(G)$.

Proof. This follows from [40, Theorem 1.3] and [40, Corollary 1.4]. $\qquad\square$

The next innocuous looking statement is surprisingly versatile.

Theorem 2.16. *Suppose that G is a finite group and P is a rank one isolated p-minimal subgroup of G.*

(i) Either $O^p(P) \trianglelefteq G$ or $O_p(Z(L_G(P,S))) \leq O_p(P)$.
(ii) $O_p(L_G(P,S)) \cap O^p(G) \neq 1$; in particular, $L_G(P,S) \cap O^p(G)$ is a p-local subgroup of $O^p(G)$.
(iii) If $p = 2$ and $|O_2(L_G(P,S))| = 2$, then $O^2(P)$ is normal in G.

Proof. For parts (i) and (ii) consult [40, Theorem 1.7] and [40, Corollary 5.2], remembering our presumption concerning finite non-abelian simple groups. For part (iii), if $p = 2$ and $|O_2(L_G(P,S))| = 2$, then $O_2(L_G(P,S)) = O_2(Z(L_G(P,S)))$. Since P is isolated, part (i) implies $O^2(P)$ is normal in G. $\qquad\square$

The next lemma is just an application of Theorem 2.15 (ii). It allows us to suppose in the proof of Theorem 1.6 that $G = F^*(G)S$.

Lemma 2.17. *Suppose that P is a rank one isolated p-minimal subgroup of G and assume that $X = F^*(G)$ is a non-abelian simple group. Then P is a rank one isolated p-minimal subgroup of XS.*

Proof. Since $Q_G = 1$, $F_p^*(G) = F^*(G) = X$. Thus Theorem 2.15 (ii) gives $P \leq XS$ and the result follows from Lemma 2.2 (iii).

$\qquad\square$

Because of Lemma 2.17, to accomplish the proof of Theorem 1.6, it suffices to work under the following

Main Hypothesis 2.18. *Suppose that G is a finite group, p is a prime and $S \in \mathrm{Syl}_p(G)$ for which the following hold:–*

(i) $X = F^*(G)$ is a non-abelian simple;
(ii) $G = XS$; and
(iii) P is a rank one isolated p-minimal subgroup of G.

The final lemma of this section has a myriad of uses. It helps us to restrict the rank of a group of Lie type in cases where the group has an isolated p-minimal subgroup but it is also used to show that certain p-local subgroups in particular imprimitive subgroups of classical groups are maximal p-local subgroups.

Lemma 2.19. *Suppose that $n \geq 3$ is an integer, $H = \mathrm{Sym}(n)$ and t is a prime. Let V be the natural $\mathrm{GF}(t)H$-permutation module with natural $\mathrm{GF}(t)$-basis $\{v_1, \ldots, v_n\}$. Set*

$$V_E = \langle v_i - v_j \mid 1 \leq i < j \leq n \rangle$$

and

$$V_0 = \langle \sum_{i=1}^{n} v_i \rangle.$$

Then the following hold.

(i) $\dim V_0 = 1$ and $\dim V_E = n - 1$.
(ii) If $(t, n) = 1$, $V = V_0 \oplus V_E$ and V_E is an irreducible $\mathrm{GF}(t)H$-module.
(iii) If $(t, n) = t$, then V is uniserial with $V_E > V_0$ and V_E/V_0 is irreducible as a $\mathrm{GF}(t)H$-module. Furthermore, if $n \geq 5$, V_E/V_0 is a faithful $\mathrm{GF}(t)H$-module and if $n = 3$ or 4, then the kernel of the action of H on V_E/V_0 has order 3 or 4, respectively.

In particular, if U is an H-invariant submodule of V of dimension 2, then $n \leq 3$.

Proof. This is well-known and easy to calculate. $\qquad\square$

3. Alternating groups

In this short section we skirmish with the alternating groups which foreshadows the tussles to come. Among the small alternating and symmetric groups there are a number of isomorphisms with the classical groups and these examples always lead to G possessing a rank one isolated p-minimal subgroup for the appropriate prime. Before proving our main result on the symmetric groups we recall some of the isomorphisms which are relevant in this section.

Lemma 3.1. *We have the following isomorphisms*

(i) $\mathrm{Sym}(3) \cong \mathrm{SL}_2(2)$;
(ii) $\mathrm{Alt}(4) \cong \mathrm{PSL}_2(3)$;
(iii) $\mathrm{Alt}(5) \cong \mathrm{SL}_2(4) \cong \mathrm{PSL}_2(5)$;
(iv) $\mathrm{Alt}(6) \cong \mathrm{PSL}_2(9) \cong \mathrm{Sp}_4(2)'$;
(v) $\mathrm{Sym}(6) \cong \mathrm{Sp}_4(2)$;
(vi) $\mathrm{Alt}(8) \cong \mathrm{PSL}_4(2)$; and
(vii) $\mathrm{Sym}(8) \cong \mathrm{O}_6^+(2)$.

Proof. These facts are well-known and may be confirmed by consulting [30, Proposition 2.9.1]. $\qquad\square$

It follows from Lemma 3.1 that the groups listed have rank one isolated p-minimal subgroups for the primes specified on the right hand side of each entry.

Example 3.2. *Suppose that $\Omega = \{1, \ldots, 12\}$ and $G = \mathrm{Sym}(\Omega)$. Let*

$$\mathcal{B} = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \{9, 10, 11, 12\}\}$$

and

$$\mathcal{C} = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}, \{11, 12\}\}.$$

Assume that M is the stabilizer in G of $\mathcal{B}$ and H is the stabilizer in G of $\mathcal{C}$. Then $M = \mathrm{Sym}(4) \wr \mathrm{Sym}(3)$, $H = \mathrm{Sym}(2) \wr \mathrm{Sym}(6)$ and $S \cong \mathrm{Sym}(2) \wr \mathrm{Sym}(2) \wr \mathrm{Sym}(2) \times \mathrm{Sym}(2) \wr \mathrm{Sym}(2)$ is a subgroup of H and M. Let K be the stabilizer in G of $\{1, \ldots, 8\}$. So $K = \mathrm{Sym}(8) \times \mathrm{Sym}(4)$. Importantly both $\mathrm{Sym}(8) \cong \mathrm{O}_6^+(2)$ and $\mathrm{Sym}(6) \cong \mathrm{Sp}_4(2)$ contain rank one isolated 2-minimal subgroups. Finally set $P = K \cap H$. So $P = \mathrm{Sym}(2) \wr \mathrm{Sym}(4) \times \mathrm{Sym}(2) \wr \mathrm{Sym}(2)$. It is easy to check that $M = L_G(P, S)$ and P is a rank one isolated 2-minimal subgroup of G, as is $P \cap G'$ in $G' \cong \mathrm{Alt}(12)$.

Theorem 3.3. *Suppose that Hypothesis 2.18 holds with $X = F^*(G) = O^p(G) \cong \mathrm{Alt}(n)$, $n \geq 5$. Then one of the following holds.*

(i) $P = G$ and either $p = 2$ with $G \cong \mathrm{Alt}(5)$ or $\mathrm{Sym}(5)$ or $p = 3$ and $G \cong \mathrm{Alt}(6)$ or $p = 5$ and $G \cong \mathrm{Alt}(5)$.

(ii) $p = 2$ and $G \cong \mathrm{Alt}(6)$, $\mathrm{Sym}(6)$, $\mathrm{Alt}(8)$ or $\mathrm{Sym}(8)$.

(iii) $p = 2$, $G \cong \mathrm{Alt}(12)$ or $G \cong \mathrm{Sym}(12)$.

Moreover, in the case $n = 12$, P and L, respectively $P \cap X$ and $L \cap X$, are as described in Example 3.2.

Proof. Because $X = F^*(G)$, $X \leq G \leq \mathrm{Aut}(X)$. Recall that $\mathrm{Aut}(X) \cong \mathrm{Sym}(n)$ if $n \neq 6$ and $\mathrm{Aut}(X) \cong \mathrm{P\Gamma L}_2(9)$ if $n = 6$. Assume $p \neq 2$. Then $G = X$ and, using Lemma 2.14, we have that either $n \geq p^p$ or $P = G = X$. As $P/Q_P \in \mathcal{L}_1(p)$, we obtain (i) from Lemmas 2.4 and 3.1. Thus, when p is odd, $n \geq p^p$. Since $n \geq 5 \geq 2^2$, we have $n \geq p^p$.

Suppose for a moment that $n \leq 8$. Then $p = 2$. Now, by Lemma 3.1, $\mathrm{Alt}(n)$ and $\mathrm{Sym}(n)$ for $n = 5, 6$ and 8 satisfy the hypothesis of the theorem. These groups are listed in parts (i) and (ii). If $G \cong \mathrm{P\Gamma L}_2(9)$, $\mathrm{PGL}_2(9)$ or M_{10} (the other candidates with $X \cong \mathrm{Alt}(6)$), $\mathcal{P}_G(S) = \{G\}$ which is ruled out as G is not in $\mathcal{L}_1(2)$. Hence $G = \mathrm{Alt}(7)$ or $\mathrm{Sym}(7)$. For $H \leq \mathrm{Sym}(7)$ let $H^* = H \cap G$. We have $O_2(\mathrm{Sym}(6)^*) = 1$ and so by Lemma 2.2 (vi), $P \leq \mathrm{Sym}(6)^*$. Now the 2-minimal subgroups $(\mathrm{Sym}(5) \times \mathrm{Sym}(2))^*$ and $(\mathrm{Sym}(3) \times \mathrm{Sym}(2) \wr \mathrm{Sym}(2))^*$ are not contained in $\mathrm{Sym}(6)^*$ and so are different from P. Thus they are both contained in L. This is a contradiction, since $(\mathrm{Sym}(5) \times \mathrm{Sym}(2))^*$ is a maximal subgroup of G.

Now assume that $n \geq 9$. Then $\mathrm{Aut}(X) \cong \mathrm{Sym}(n)$ and we may assume that $\mathrm{Alt}(n) \leq G \leq \mathrm{Sym}(n)$. Again, for $H \leq \mathrm{Sym}(n)$, we let $H^* = H \cap G$. Set $\Omega = \{1, \ldots, n\}$ and consider the action of G on Ω. Put $Z = \Omega_1(Z(Q_L))$. Suppose first that L operates primitively on Ω. Then, as $Z \neq 1$, Z operates transitively on Ω and, as Z is abelian, Z acts regularly on Ω. Hence $|\Omega| = |Z| = p^a$ for some integer $a \geq 2$. Select a p-cycle $x \in S$ (or when $p = 2$ a product of two transpositions). Since $n \geq p^2$ ($n \geq 5 > 2^2$ when $p = 2$), we may select

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$\alpha \in \text{Fix}_\Omega(x)$. Then $\Omega = \{\alpha^g \mid g \in Z\}$. Now $\beta = \alpha^g \in \text{Fix}_\Omega(x)$ $(g \in Z)$ if and only if $\alpha^{gx} = \alpha^g = \alpha^{xg}$ which is equivalent to $gxg^{-1}x^{-1} \in G_\alpha \cap Z = 1$, as Z is a normal subgroup of S. Thus $|\text{Fix}_\Omega(x)| = |C_Z(x)|$, and so

$$p^a - p = |\text{Fix}_\Omega(x)| = |C_Z(x)| \leq p^{a-1}$$

($2^a - 4 \leq 2^{a-1}$ when $p = 2$) which has no solution for our values of $n = p^a$. Therefore L does not act primitively on Ω.

Assume that L is not transitive on Ω and let Ω_1, Ω_2 be proper subsets of Ω stabilized by L with $|\Omega_2| = k \leq \frac{n}{2}$. So $L \leq \text{Stab}_G(\Omega_1) \cap \text{Stab}_G(\Omega_2)$. Since L is a maximal subgroup of G we have $L = \text{Stab}_G(\Omega_1) \cap \text{Stab}_G(\Omega_2) = (\text{Sym}(\Omega_1) \times \text{Sym}(\Omega_2))^*$ and $k \neq \frac{n}{2}$ (otherwise L would not be maximal in G). Since $Q_L \neq 1$, we infer that either $O_p(\text{Sym}(\Omega_1)) \neq 1$ or $O_p(\text{Sym}(\Omega_2)) \neq 1$. Thus, as $n \geq 9$ and $k \neq \frac{n}{2}$, we deduce that $k = 2, 3, 4$ and $Q_L = O_p(\text{Sym}(\Omega_2))$. Note that S is a subgroup of a subgroup $R = (\text{Sym}(\Omega_2)) \wr \text{Sym}([n/|\Omega_2|])^*$ of G. Thus $Q_L \leq Q_R$. By Lemma 2.2 (iii), we have that $R \leq L$, which is a contradiction as R plainly does not normalize Q_L.

Thus L operates transitively but not primitively on Ω. Let $\mathcal{B} = \{\Theta_1, \ldots, \Theta_r\}$ be a system of imprimitivity for L on Ω with $n > |\Theta_1| = k > 1$. Then $L \leq \text{Stab}_G(\mathcal{B})$ and, since L is a maximal subgroup of G and L is a p-local subgroup of G, we have $L = \text{Stab}_G(\mathcal{B})$ and $k \in \{2, 3, 4\}$.

Suppose that $k = 2$ or 4. Then $p = 2$. Write the 2-adic decomposition of n as $n = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_\ell}$ with $a_1 > \cdots > a_\ell \geq 1$, and note that k divides n. Moreover, as $n \geq 9$, $a_1 \geq 3$. For $i = 1, \ldots, \ell$, let S_{a_i} represent a Sylow 2-subgroup of $\text{Sym}(2^{a_i})$. Then we take $S = (S_{a_1} \times S_{a_2} \times \cdots \times S_{a_\ell})^*$ as our Sylow 2-subgroup S.

If $k = 2$, we note that the subgroup

$$H = (\text{Sym}(4) \wr \underbrace{\text{Sym}(2) \wr \cdots \wr \text{Sym}(2)}_{a_1 - 2} \times S_{a_2} \times \cdots \times S_{a_\ell})^*$$

is not contained in L and consequently must contain P. Then, as H acts irreducibly on $O_{2,3}(H)/O_2(H)$ by conjugation, H is actually 2-minimal and so we have that $H = P$. But H is not narrow, and thus we have a contradiction in this case.

Next assume that $k = 4$ and set

$$H = (\text{Sym}(2) \wr \text{Sym}(4) \wr \underbrace{\text{Sym}(2) \wr \cdots \wr \text{Sym}(2)}_{a_1 - 3} \times S_{a_2} \times \cdots \times S_{a_\ell})^*.$$

Again we have that $H \not\leq L$ and so $P \leq H$. Furthermore, if $a_1 \geq 4$, H is 2-minimal and so $H = P$ contradicting the fact that P is narrow. Thus $a_1 = 3$ and since $k = 4$ divides n we infer that $n = 12$, and we have the example displayed in Example 3.2.

Finally suppose that $k = 3$. Then $p = 3$, 3 divides n, $G \cong \text{Alt}(n)$ and $L \cong (\text{Sym}(3) \wr \text{Sym}(n/3))^*$. Let $n = b_1 3^{a_1} + \cdots + b_\ell 3^{a_\ell}$ be the 3-adic decomposition of n where $a_1 > \cdots > a_\ell \geq 1$ and $b_i \in \{1, 2\}$. Assume that $S_{a_i}^{b_i}$ is a Sylow 3-subgroup of $\text{Sym}(b_i 3^{a_i})$. Then we may suppose that

$$S = S_{a_1}^{b_1} \times \cdots \times S_{a_\ell}^{b_\ell}.$$

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If $n = 9$, then we have that $G = \mathrm{Alt}(9)$ which is 3-minimal and this is impossible. Therefore, $n > 9$. Set

$$H = (\mathrm{Sym}(9) \wr \underbrace{\mathrm{Sym}(3) \wr \cdots \wr \mathrm{Sym}(3)}_{a_1 - 2} \wr \mathrm{Sym}(b_1) \times (S_{a_2}^{b_2} \times \cdots \times S_{a_\ell}^{b_\ell}))^*.$$

Then H is not contained in L. Therefore H contains P. Now, if either $a_1 \geq 3$ or $b_1 > 1$, we apply Theorem 2.15 (i) to obtain a contradiction. Therefore, as $n > 9$, $b_1 = 1$, $a_1 = 2$ and we have $n = 12$ or 15. But then H is 3-minimal, forcing $H = P$ whereas $H/O_3(H) \notin \mathcal{L}_1(3)$. This contradiction shows that $k \neq 3$ and completes the proof of the theorem. $\qquad\square$

4. Groups of Lie type in characteristic not p

For the remainder of this paper, until we reach Section 11, Hypothesis 2.18 and the notation therein will be assumed to hold sway. Also for P a rank one isolated p-minimal subgroup of G containing S as in Hypothesis 2.18, we write L for $L_G(P, S)$.

In this section we look at the situation when X is a group of Lie type defined in characteristic $r \neq p$ and provide general information which will be valuable in the battles ahead. Before doing that we give two preliminary lemmas concern cyclotomic polynomials, followed by two results which are about Sylow subgroups of groups of Lie type. Recall that for n a natural number, the *cyclotomic polynomial* $\Phi_n(x)$ is the product of all $x - \theta$ where θ runs through the primitive complex nth roots of unity. Thus we have

$$x^n - 1 = \prod_{d \mid n} \Phi_d(x).$$

Suppose that p is a prime number and s is a natural number. Assume that d is the multiplicative order of s modulo p. We write

$$d = \mathrm{ord}_p(s).$$

Then p divides $\Phi_d(s)$ and p does not divide $\Phi_e(s)$ for any $e < d$.

Lemma 4.1. *Suppose that p is a prime number, and n and e are non-negative integers. Then the following hold*

    (i) $\Phi_p(x) = x^{p-1} + \cdots + 1$;
    (ii) if p divides n, then $\Phi_{pn}(x) = \Phi_n(x^p)$;
    (iii) $\Phi_{p^e}(x) = \Phi_p(x^{p^{e-1}})$; and
    (iv) if p and n are coprime, then $\Phi_n(x^p) = \Phi_{pn}(x)\Phi_n(x)$.

Proof. See [57, page 647]. $\qquad\square$

Lemma 4.2. *Suppose that p is an odd prime and w is a natural number. Let $d = \mathrm{ord}_p(w)$. Then, for $n \geq 1$,*

    (i) p divides $\Phi_n(w)$ if and only if $n = p^e d$ for some $e \geq 0$; and
    (ii) if p^2 divides $\Phi_n(w)$, then $n = d$.

Proof. For (i) see [39, Lemma 5] for example. Part (ii) is embedded in the proof of [18, 10-2] as follows. Since p divides $\Phi_n(w)$, we have $n = p^e d$ for some $e \geq 0$ by part (i). Suppose that $e \geq 1$. Then $\Phi_{p^e d}(x)$ divides $\Phi_d(x^{p^e})$ by a combination of Lemma 4.1 (ii) and (iv). Therefore, $\Phi_{p^e d}(x)$ divides $(x^{p^e d}) - 1 = (x^{p^{e-1}d})^p - 1$. Since $p^{e-1}d$ divides $p^e d$ we know that $\gcd\left(\Phi_{p^e d}(x), x^{p^{e-1}d} - 1\right) = 1$. Hence $\Phi_{p^e d}(x)$ divides $\Phi_p(x^{p^{e-1}d})$. Since $\operatorname{ord}_p(w) = d$, we have $w^d \equiv 1 \pmod{p}$ and so $w^{p^{e-1}d} \equiv 1 \mod p$. Now, writing $w^{p^{e-1}d} = 1 + up$, Lemma 4.1 (i) yields

$$
\begin{aligned}
\Phi_p(w^{p^{e-1}d}) &\equiv (1 + up)^{p-1} + \cdots + (1 + up) \pmod{p^2} \\
&\equiv (1 + up(p-1)) + (1 + up(p-2)) + \cdots + (1 + up) \pmod{p^2} \\
&\equiv p + up \sum_{i=1}^{p-1} i \pmod{p^2} \\
&\equiv p + up^2 \frac{p-1}{2} \pmod{p^2} \equiv p \pmod{p^2}.
\end{aligned}
$$

Thus, if p^2 divides $\Phi_{p^e d}(w)$, then we must have $e = 0$ and this proves part (ii). □

We continue our discussion with a description of the order of the groups of Lie type defined in characteristic r. Thus we let $\widehat{X}(r^a)$ be the universal version of the group of Lie type $X(r^a)$ defined over the field of order r^a. We follow [20, page 237] and obtain

$$
|\widehat{X}(r^a)| = (r^a)^N \prod_i \Phi_i(r^a)^{n_i}
$$

where N is the number of positive roots in the root system for $\widehat{X}(r^a)$. The powers n_i appearing in the formula for the order of $\widehat{X}(r^a)$ are important when we come to describe the order and structure of a Sylow p-subgroup of $\widehat{X}(r^a)$. The product $\prod_i \Phi_i(r^a)^{n_i}$ is nicely presented in [18, Tables 10:1 and 10:2] and we display it here in Table 3 for the reader's convenience.

From here on, if p is odd, we define

$$
d = \operatorname{ord}_p(r^a)
$$

and, when $p = 2$, we set

$$
d = \begin{cases} 1 & r^a \equiv 1 \pmod{4} \\ 2 & r^a \equiv 3 \pmod{4} \end{cases}.
$$

Notice that for p odd, d divides $p-1$. The next lemma illustrates how d plays a fundamental role in the determination of the orders of the Sylow p-subgroups of groups of Lie type.

Lemma 4.3. *Let $\widehat{H}$ be a universal group of Lie type defined over $\mathrm{GF}(r^a)$ and $\widehat{T}$ be a Sylow p-subgroup of $\widehat{H}$ with p odd. Then*

$$
|\widehat{T}| = p^b (\Phi_d(r^a)_p)^{n_d}
$$

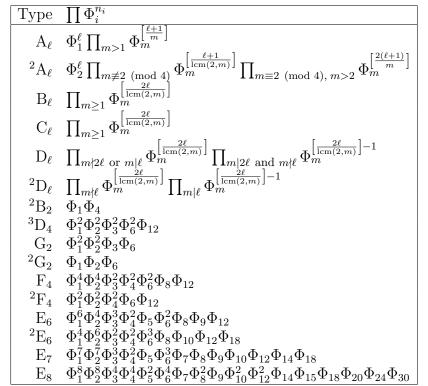| Type	$\prod \Phi_i^{n_i}$
A_ℓ	$\Phi_1^\ell \prod_{m>1} \Phi_m^{\left[\frac{\ell+1}{m}\right]}$
$^2A_\ell$	$\Phi_2^\ell \prod_{m \not\equiv 2 \ (\mathrm{mod}\ 4)} \Phi_m^{\left[\frac{\ell+1}{\mathrm{lcm}(2,m)}\right]} \prod_{m\equiv 2\ (\mathrm{mod}\ 4),\, m>2} \Phi_m^{\left[\frac{2(\ell+1)}{m}\right]}$
B_ℓ	$\prod_{m\geq 1} \Phi_m^{\left[\frac{2\ell}{\mathrm{lcm}(2,m)}\right]}$
C_ℓ	$\prod_{m\geq 1} \Phi_m^{\left[\frac{2\ell}{\mathrm{lcm}(2,m)}\right]}$
D_ℓ	$\prod_{m \nmid 2\ell \text{ or } m\mid\ell} \Phi_m^{\left[\frac{2\ell}{\mathrm{lcm}(2,m)}\right]} \prod_{m\mid 2\ell \text{ and } m\nmid\ell} \Phi_m^{\left[\frac{2\ell}{\mathrm{lcm}(2,m)}\right]-1}$
$^2D_\ell$	$\prod_{m\nmid\ell} \Phi_m^{\left[\frac{2\ell}{\mathrm{lcm}(2,m)}\right]} \prod_{m\mid\ell} \Phi_m^{\left[\frac{2\ell}{\mathrm{lcm}(2,m)}\right]-1}$
$^2B_2$	$\Phi_1\Phi_4$
$^3D_4$	$\Phi_1^2\Phi_2^2\Phi_3^2\Phi_6^2\Phi_{12}$
G_2	$\Phi_1^2\Phi_2^2\Phi_3\Phi_6$
$^2G_2$	$\Phi_1\Phi_2\Phi_6$
F_4	$\Phi_1^4\Phi_2^4\Phi_3^2\Phi_4^2\Phi_6^2\Phi_8\Phi_{12}$
$^2F_4$	$\Phi_1^2\Phi_2^2\Phi_4^2\Phi_6\Phi_{12}$
E_6	$\Phi_1^6\Phi_2^4\Phi_3^3\Phi_4^2\Phi_5\Phi_6^2\Phi_8\Phi_9\Phi_{12}$
$^2E_6$	$\Phi_1^4\Phi_2^6\Phi_3^2\Phi_4^2\Phi_6^3\Phi_8\Phi_{10}\Phi_{12}\Phi_{18}$
E_7	$\Phi_1^7\Phi_2^7\Phi_3^3\Phi_4^2\Phi_5\Phi_6^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$
E_8	$\Phi_1^8\Phi_2^8\Phi_3^4\Phi_4^4\Phi_5^2\Phi_6^4\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$

TABLE 3. Cyclotomic polynomials expressing the r'-part of the orders of universal versions of groups of Lie type of characteristic r, giving the exponents n_j of Φ_j used in Lemma 4.3.

where

$$b = \sum_{c\geq 1} n_{dp^c},$$

with n_j given in Table 3 and $\Phi_d(r^a)_p$ is the p-part of $\Phi_d(r^a)$. Furthermore, T has exponent at least $\Phi_d(r^a)_p$ and, if $b=0$, then the Sylow p-subgroups of $\widehat{H}$ are abelian.

Proof. Consult [20, Theorem 4.10.2 (c)] and its erratum in [21] $\qquad\square$

In the next lemma we use facts about automorphisms of groups of Lie type which can be found in [20, Theorem 2.5.12].

Lemma 4.4. *Suppose that H is an adjoint group of Lie type defined in characteristic r over a field of order r^a and $t \neq r$ is a prime with $t \geq 5$. If t divides $|\mathrm{Out}(H)|$ and $|H|$, then the Sylow t-subgroups of H are not elementary abelian.*

Proof. Identify H as a subgroup of $\mathrm{Aut}(H)$, let $T \in \mathrm{Syl}_t(H)$ and let $\alpha \in \mathrm{Aut}(H)$ be a t-element which is not contained in H and which normalizes T.

Suppose that α is a diagonal automorphism. Then, as $t \geq 5$, either t divides $(n, \Phi_1(r^a))$ and $X \cong \mathrm{PSL}_n(r^a)$ or t divides $(n, \Phi_2(r^a))$ and $X \cong \mathrm{PSU}_n(r^a)$. In the first case we have

that H contains a monomial subgroup of shape $\frac{1}{(n,r^a-1)}(r^a-1)^{n-1}.\mathrm{Sym}(n)$ and in the second case H contains a monomial subgroup of shape $\frac{1}{(n,r^a+1)}(r^a+1)^{n-1}.\mathrm{Sym}(n)$. Since $n \geq t \geq 5$, these subgroups witness the fact that T is non-abelian. Therefore we may assume that H has no diagonal automorphisms of order t. Thus, as there are no graph automorphisms of order $t \geq 5$, α can be chosen to induce a field automorphism of H.

Since α induces a field automorphism of H of order t this means that $r^a = r^{tb}$ for some integer b. Let $d = \mathrm{ord}_t(r^a)$. Then, by Lemma 4.3, T has exponent at least $\Phi_d(r^a)_t$. Now, by Lemma 4.1 (iv)

$$\Phi_d(r^a) = \Phi_d(r^{tb}) = \Phi_{td}(r^b)\Phi_d(r^b).$$

By Fermat's Little Theorem we have $r^a = r^{bt} \equiv r^b \mod t$, so $\mathrm{ord}_t(r^b) = d$. Now t divides $\Phi_d(r^b)$ and $\Phi_{td}(r^b)$ by Lemma 4.2 (i). Thus t^2 divides $\Phi_d(r^a)$, and we have proved that T is not elementary abelian. $\qquad\square$

Lemma 4.5. *Suppose that $X = F^*(G)$ is a group of Lie type defined in characteristic r, $r \neq p$ (with Hypothesis 2.18 holding). If X has abelian Sylow p-subgroups, then either $P = G$ or $G \neq X$ and $p \in \{2,3\}$.*

Proof. Assume that $P \neq G$. Then, by Lemma 2.14, we have $G > X$. Suppose $p \geq 5$ and let us argue for a contradiction. Put $S_0 = S \cap X$ and $K = O^p(P)$. Note that $K \leq X$ and so K and K/Q_K have abelian Sylow p-subgroups. Since $p \geq 5$, Lemma 2.4 now implies that $K/Q_K \cong \mathrm{PSL}_2(p^b)$ or $\mathrm{SL}_2(p^b)$ for some $b \geq 1$. In particular, $S \cap K \not\leq Q_K$ and $K = \langle (S \cap K)^K \rangle$. As $S \cap K = S_0 \cap K$ is abelian this shows that $Q_K \leq Z(K)$. Because K has abelian Sylow p-subgroups and the Schur multiplier of $\mathrm{PSL}_2(p^b)$ has order 2 (or 6 if $p^b = 9$) [23, 25.7], we obtain $K \cong \mathrm{SL}_2(p^b)$ or $\mathrm{PSL}_2(p^b)$. In particular, $[Q_P, K] = 1$ and $P \not\leq N_G(S_0)$.

From Theorem 2.16 (ii) we have $Q_L \cap X > 1$. Hence $1 \neq Q_L \cap X \cap \Omega_1(S_0)$. Now $S \leq N_G(S_0)$ and $P \not\leq N_G(S_0)$. Hence $N_G(S_0) \leq L$ and $\Omega_1(Q_L \cap X)$ is normalized by $N_X(S_0)$. Since $N_X(S_0)$ acts irreducibly on $\Omega_1(S_0)$ by Lemma 2.13, this forces

$$\Omega_1(Q_L \cap X) = \Omega_1(S_0).$$

Hence L normalizes $\Omega_1(S_0)$.

Since S_0 is abelian, S_0 does not induce field automorphisms on K. So $S_0 K = O_p(S_0 K)K$ and thus $S_0 = (S_0 \cap Q_P) \times (S \cap K)$. Since $1 \neq S \cap K$ is elementary abelian, this shows that $\Omega_1(S_0) \not\leq \Phi(S_0)$. As $N_X(S_0)$ acts irreducible on $\Omega_1(S_0)$ we get $\Omega_1(S_0) \cap \Phi(S_0) = 1$. So also $\Phi(S_0) = 1$, that is S_0 is elementary abelian. Now Lemma 4.4 implies that p does not divide $|\mathrm{Out}(X)|$. Since $G = XS$ this gives $G = X$, a contradiction. We have shown that $G \neq X$ and $p \in \{2,3\}$. $\qquad\square$

We remark here that there are examples of groups G which satisfy Hypothesis 2.18 with $X = O^3(G)$ a simple group of Lie type with abelian Sylow 3-subgroups and $|G/X| = 3$ (see Theorem 5.1 and Lemma 7.5).

We close this section with an observation which in part reveals why there are so few examples of rank one isolated p-minimal subgroups in groups of Lie type defined in characteristic $r \neq p$.

Lemma 4.6. *Suppose that $X = F^*(G)$ is a simple group of Lie type defined over a field of characteristic r with $p \neq r$. Assume that R is a parabolic subgroup of X and $G = N_G(R)X$. Then $X \cap S$ is not contained in R. In particular, if G/X is contained in the subgroup generated by diagonal and field automorphisms of X, then $X \cap S$ does not commute with a non-trivial r-element.*

Proof. We follow [20]. Let Φ be a (twisted) root system for X with set of fundamental roots Π and corresponding positive roots Φ^+ and negative roots Φ^-. Set $\mathcal{R} = \{X_\alpha \mid \alpha \in \Phi\}$ to be the root subgroups of X with respect to Φ. Then, by [20, Theorems 2.3.4 (d) and 2.3.8 (e)], $X = \langle X_\alpha \mid \alpha \in \Phi \rangle$ and the subgroups $U = \prod_{\alpha \in \Phi^+} X_\alpha$ and $U_- = \prod_{\alpha \in \Phi^-} X_\alpha$ are Sylow p-subgroups of X. Furthermore, $U \cap U_- = 1$. Let $H \leq X$ be the Cartan subgroup of X which normalizes every subgroup in $\mathcal{R}$ as described in [20, Theorem 2.4.7]. The basic structure of the parabolic subgroups of X is given in [20, Theorem 2.6.5]. In particular, $B = UH = N_X(U)$ is a Borel subgroup of G. The action of carefully chosen automorphisms of X is described in [20, Theorem 2.5.1]. From there we obtain a subgroup $C \geq C \cap X = H$ such that $CX = G$ and C permutes $\mathcal{R}$ leaving $\{X_\alpha \mid \alpha \in \Pi\}$ invariant. In particular, C normalizes U and U_- and we have $N_G(U) = UC$, $N_G(U_-) = U_-C$.

Suppose now that $R \geq B$ is a parabolic subgroup of X such that $G = N_G(R)X$. Then there is a subset J of Π such that $R = U_J M_J H$ where $M_J = \langle X_{\pm\alpha} \mid \alpha \in J \rangle$ and $U_J = O_r(R) \leq U$.

Let $w \in N_G(R)$. Then $w = cx$ where $x \in X$ and $c \in C$. Hence $R = R^w = R^{cx}$. Since R and R^c both contain $B = (N_G(R) \cap X)^c = B^c$, R and R^c are parabolic subgroups of X containing B and they are conjugate by the element $x \in X$. It follows that $R = R^c$ (see [20, Theorem 2.6.5 (c)]). Thus $C \leq N_G(R)$, C normalizes U_J and, as C normalizes R and preserves Π, C normalizes $M_J H$.

Assume for a contradiction that $R \geq S \cap X$. Then $S \cap X$ is conjugate to a subgroup of $M_J H$ and S is conjugate to a subgroup of $M_J C$. Let $\Psi = \Phi^+ \setminus \{\alpha \in \Phi^+ \mid X_\alpha \leq M_J\}$. Then $U_J = \langle X_\alpha \mid \alpha \in \Psi \rangle$ and $U_{-J} = \langle X_{-\alpha} \mid \alpha \in \Psi \rangle$ are both normalized by $M_J C$. Hence $U_J S$ and $U_{-J} S$ are over-groups of S in G. Since $C_{RC}(U_J) = Z(U_J)$ by [20, Theorem 2.6.5 (e)], we have $O_p(SU_J) = O_p(SU_{-J}) = 1$. In particular, as $U_J \cap U_{-J} \leq U \cap U_- = 1$,

$$P \leq SU_J \cap SU_{-J} = S$$

by Lemma 2.10, which is impossible. This contradiction proves the main claim of the lemma.

The final statement follows since any r-local subgroup of X is contained in a parabolic subgroup of X by the Borel Tits Theorem ([20, Theorem 3.1.3]). $\qquad\square$

Lemma 4.7. *Suppose that $p \neq 2$ and $X = F^*(G)$ is isomorphic to one of $\mathrm{Sp}_4(2)'$, $\mathrm{G}_2(2)'$ and $^2\mathrm{F}_4(2)'$. Then $S \leq X$ and every proper over-group of S has even index in G. In particular, $C_X(S)$ has odd order.*

Proof. Since in each case, $\mathrm{Out}(X)$ is a 2-group, $S \leq X$. The maximal subgroups of $\mathrm{Sp}_4(2)' \cong \mathrm{PSL}_2(9)$, $\mathrm{G}_2(2)' \cong \mathrm{PSU}_3(3)$ and ${}^2\mathrm{F}_4(2)'$ can be found in [23, 25.7], [20, Theorem 6.5.3] [58] respectively. The result follows as none of the maximal subgroups contains both a Sylow 2-subgroup and a Sylow p-subgroup. Let X^* be $\mathrm{Sp}_4(2)$, $\mathrm{G}_2(2)$ or ${}^2\mathrm{F}_4(2)$ in the respective cases. If $C_X(S)$ has even order, then $C_{X^*}(S)$ has even order and the Borel Tits Theorem implies $S \leq R$, a parabolic subgroup of X^*. But then $S \leq R \cap X$ has odd index in X, a contradiction. $\qquad\square$

5. THE CASE OF $X \cong \mathrm{PSL}_2(r^a)$

In this section we assume that Hypothesis 2.18 holds and $X = F^*(G) \cong \mathrm{PSL}_2(r^a)$.

Theorem 5.1. *Suppose that $X \cong \mathrm{PSL}_2(r^a)$. Then one of the following holds.*

(i) $p = r$.

(ii) $p = 2$, $r^a \equiv 3,5 \pmod 8$, $r^a > 3$, $r \neq 5$, $G \cong \mathrm{PGL}_2(r^a)$, $L = C_G(\Omega_1(Z(S)))$ and $P \cong \mathrm{Sym}(4)$.

(iii) $p = 2$, $G \cong \mathrm{PGL}_2(11)$ *or* $\mathrm{PGL}_2(13)$, $L \cong \mathrm{Sym}(4)$, $P \cong \mathrm{Dih}(24)$ *with* $P/Q_P \cong \mathrm{SL}_2(2)$.

(iv) $p = 2$, $G \cong \mathrm{PGL}_2(19)$, $L \cong \mathrm{Sym}(4)$ *and* $P \cong \mathrm{Dih}(40)$ *with* $P/Q_P \cong \mathrm{Dih}(10)$.

(v) $p = 2$, $G \cong \mathrm{PSL}_2(7)$ *or* $\mathrm{PSL}_2(9)$ *and* $L \cong P \cong \mathrm{Sym}(4)$ *or* $G \cong \mathrm{P\Sigma L}_2(9) \cong \mathrm{Sym}(6)$ *and* $L \cong P \cong \mathrm{Sym}(4) \times 2$.

(vi) $p = 2$ *and* $P = G$ *and* $X \cong \mathrm{PSL}_2(5)$.

(vii) $p = 3$, $P = G$, $X \cong \mathrm{PSL}_2(8) \cong {}^2\mathrm{G}_2(3)'$ *and* $|G/X| \leq 3$.

Proof. If $p = r$, then (i) holds. So we may assume that $p \neq r$ for the rest of the proof. Set $S_0 = S \cap X$. We first determine the configurations when $p = 2$. In this case r^a is odd and S_0 is a dihedral 2-group of order at least 4.

By Theorem 2.16(ii) we have that $\Omega_1(Z(Q_L \cap X))$ is a non-trivial normal subgroup of L. Let t be a non-trivial element in $\Omega_1(Z(Q_L \cap X))$ and set $D = C_G(t)$. As the 2-rank of S_0 is 2, either $\Omega_1(Z(Q_L \cap X)) = \langle t \rangle$ and $L = D$ or $\Omega_1(Z(Q_L \cap X))$ is a fours group, E, which is normalized by S and, in particular, by S_0. Suppose that the latter case pertains. Then, since E is normal in S_0 and S_0 is a dihedral group, either $S_0 = E$ or $S_0 \cong \mathrm{Dih}(8)$ with S_0 containing a further fours group F such that $S_0 = EF$.

Assume that $S_0 = E$. Then, as L is a maximal subgroup of G, $L = N_G(S_0)$. Suppose that $S = S_0$. Then S is abelian and Lemma 2.12 shows that P is normal in G. Looking at Lemma 2.4 we conclude $P = X \cong \mathrm{SL}_2(4) \cong \mathrm{PSL}_2(5)$ as listed in (vi). Suppose next that $S \neq S_0$. Then $G > X$. Since S_0 has order 4, we have $r^a \equiv 3,5 \pmod 8$ and consequently a is odd and therefore X admits no field automorphisms of order 2. We deduce that $G \cong \mathrm{PGL}_2(r^a)$, $S \cong \mathrm{Dih}(8)$ and $L \cong \mathrm{Sym}(4)$. Furthermore, D is a dihedral group of order $2(r^a - \epsilon)$ where $\epsilon = \pm 1$ and $r^a \equiv \epsilon \pmod 4$. If $D \leq L$, then $D \cong \mathrm{Dih}(8)$ which gives $r^a - \epsilon = 4$. Thus $X \cong \mathrm{PSL}_2(5)$, in particular, $G \cong \mathrm{PGL}_2(5)$ is 2-minimal. So $P = G$ and we have the configuration in (vi) again. So suppose $D \not\leq L$. Then $P \leq D$, whence P is a dihedral group. But then $O^2(P)Q_P/Q_P$ is cyclic and so $P/Q_P \cong \mathrm{Sym}(3)$ or $\mathrm{Dih}(10)$ by Lemma 2.4. Since $|S| = 8$ we have $|Q_P| = 4$ whence from P being a dihedral group we

conclude that $P \cong \mathrm{Dih}(24)$ or $\mathrm{Dih}(40)$. Note that $S = P \cap L$ and by Lemma 2.2 (iii) $L \cap D$ is a maximal subgroup of D. Thus $P = D$. In particular, $r^\alpha - \epsilon = \frac{|D|}{2} = \frac{|P|}{2} \in \{12, 20\}$ and so $r^a = 11, 13$ or 19. This then delivers the examples listed in (iii) and (iv).

Continuing with the supposition that L normalizes E we now assume that $S_0 \cong \mathrm{Dih}(8)$. Then $N_G(E) = N_X(E)S$ and $N_X(E) \cong \mathrm{Sym}(4)$. Furthermore, $N_G(F) = N_X(F)S$ and also $N_X(F) \cong \mathrm{Sym}(4)$. Since, by the maximality of L, $L = N_G(E)$, $P \le N_G(F)$ and, as $P > S$, we have $P = N_G(F)$. Now $P \nleq D$ and so $D \le L$ which means that $D = S$. Consequently, $C_X(t) \cong \mathrm{Dih}(8)$ and it follows that $X \cong \mathrm{PSL}_2(7)$ or $\mathrm{PSL}_2(9)$. Finally, by examining $\mathrm{Aut}(X)$, we see that if $G > X$ then $G \cong \mathrm{P\Sigma L}_2(9)$. Thus we have described the examples in (v) and this completes the analysis of the case $\Omega_1(Z(Q_L \cap X)) = E$ is a fours group.

Now consider the case when $\Omega_1(Z(Q_L \cap X)) = \langle t \rangle$ (still with $p = 2$). So L normalizes $\langle t \rangle$ and hence $L = D$. If $Q_P \cap X > 1$, then, as L does not normalize $\Omega_1(Z(Q_P \cap X))$, $\Omega_1(Z(Q_P \cap X))$ is a fours group normalized by S. Thus once again we have that S_0 is either a fours group or $S_0 \cong \mathrm{Dih}(8)$. If S_0 is a fours group, then $S > S_0$ (as $P \ne N_P(S)$). Therefore, $S \cong \mathrm{Dih}(8)$, $G \cong \mathrm{PGL}_2(r^a)$ with $r^a \equiv 3, 5 \pmod 8$ and $P \cong \mathrm{Sym}(4)$. If $r \ne 5$, this gives the examples in part (ii). So suppose that $r = 5$. Then $P \le H \cong \mathrm{PGL}_2(5) \le \mathrm{PGL}_2(5^a)$. Since H is 2-minimal, we have $P = H$ which is a contradiction as $P \cong \mathrm{Sym}(4)$. If, on the other hand, $S_0 \cong \mathrm{Dih}(8)$, then letting F be the fours subgroup of S_0 with $F \ne \Omega_1(Z(Q_P \cap X))$, we see that $\mathrm{Sym}(4) \cong N_X(F) \le L = D$. Since this is impossible, we conclude that $Q_P \cap X = 1$. Consequently $O_2(O^2(P)) = 1$ and S_0 acts faithfully on $O^2(P)$. Set $R = S_0 \cap O^2(P)$. Then $R \in \mathrm{Syl}_2(O^2(P))$ and R is a subgroup of a dihedral group. Because S_0 acts faithfully on $O^2(P)$, Lemma 2.4 (v), (vii) and (viii) do not occur, whence $|R| = 2^2$ and $N_{O^2(P)}(R) \cong \mathrm{Alt}(4)$. Clearly $P \nleq N_G(R)$ and therefore $N_G(R) \le L$, contrary to $L = D$. This concludes our investigations into P and L when $p = 2$.

Assume that $p \ge 3$. In this case the Sylow p-subgroups of X are cyclic. If $P = G$, then, as $p \ge 3$ and $p \ne r$, Lemma 2.4 (ix) must hold, so giving (vii). So $P \ne G$ and hence, by Lemma 4.5, $X < G$ and $p = 3$. Now $O^3(P) \le X$ has cyclic Sylow 3-subgroups and hence, as $O^3(P/Q_P) \in \mathcal{L}_1(3)$, we see that $O^3(P/Q_P) \cong \mathrm{Q}_8$, 2^2 or $^2\mathrm{G}_2(3)' \cong \mathrm{SL}_2(8)$. Since the Sylow 2-subgroups of X are dihedral groups or elementary abelian (with $r = 2$), we obtain one of $O^3(P) \cong 2^2$ or $O^3(P) \cong {}^2\mathrm{G}_2(3)'$ with $r = 2$ in the latter case.

Suppose $O^3(P) \cong 2^2$. Then $P/Q_P \cong \mathrm{PSL}_2(3)$. Since $Q_P \cap X$ is centralized by $O^3(P)$ and the centralizer of an involution in X is a dihedral group if r is odd or a Sylow 2-subgroup of X if $r = 2$, we find that $Q_P \cap X = 1$. Hence $P \cap X \cong \mathrm{PSL}_2(3)$ and so X has cyclic Sylow 3-subgroups of order 3. However X admits field automorphisms of order 3 which means that $|S \cap X|$ is divisible by 9 (see the argument at the end of Lemma 4.4), a contradiction. Thus $O^3(P) \ncong 2^2$.

It remains to wrangle with the case $O^3(P) \cong {}^2\mathrm{G}_2(3)' \cong \mathrm{SL}_2(8)$. In this case we have $X \cong \mathrm{PSL}_2(2^a)$ and we may assume that $a > 3$ for otherwise (vii) holds. Since $O^3(P) \cong \mathrm{SL}_2(8)$, we infer that 3 divides a and so write $a = 3m$. As $Q_P \cap X$ is centralized by $O^3(P)$ which has even order, $Q_P \cap X = 1$ and $P \cap X = O^3(P)$. Thus X has Sylow 3-subgroups of order 9 and so 3 does not divide m, as otherwise 27 divides $|X|$. Therefore $|G/X| = 3$

and $|S| = 3^3$. If 3 divides $2^a - 1$, then S normalizes a Sylow 2-subgroup T of X and thus $TS \leq L$, contrary to $Q_L \neq 1$. Thus 3 divides $2^a + 1$ and so a is odd.

We now intend to determine $N_G(S)$. Let $\alpha \in S$ act on X as a field automorphism. Then

$$C_G(\alpha) \cong 3 \times \mathrm{SL}_2(2^m)$$

which has a subgroup $K = 3 \times (2^m + 1)$ where the second factor denotes a cyclic subgroup of order $2^m + 1$. Note that $2^m + 1$ is divisible by 3 as a is odd. Now $C_G(S_0 \cap K)$ is cyclic of order $2^a + 1$. Thus S is a normal subgroup of $S_0 K$. Hence $N_G(S) \geq KS_0$ and, as P is a rank one isolated p-minimal subgroup, this group normalizes P by Lemma 2.2 (ii). But then the Frattini Argument shows that a Sylow 2-subgroup of P has normalizer in $N_G(S)P$ of order divisible by $(2^m + 1).7$, which is absurd as Sylow 2-subgroups of P have order 8 and the centralizer of an involution in X is a 2-group. This proves that no examples arise in this situation and completes the proof of the theorem. $\qquad\square$

6. Projective symplectic groups

In this section we continue to work under Hypothesis 2.18, this time with the additional assumption that $X \cong \mathrm{PSp}_{2n}(r^a)'$ and $n \geq 2$. The main result of this section is Theorem 6.3. In proving this theorem, we may suppose that $p \neq r$. Since the graph automorphism of X can only appear when $n = 2 = r$, the fact that G/X is a p-group (by Hypothesis 2.18 (i)) means that we may assume that G does not contain an element which induces a graph automorphism of X. Therefore G/X is a subgroup of $\mathrm{Out}(X)$ which is generated by certain field automorphisms and perhaps the diagonal automorphism of order $(2, r^a - 1)$. Let $\widehat{X} = \mathrm{Sp}_{2n}(r^a)$ and denote by $\widehat{H}$ the preimage in $\widehat{X}$ of a subgroup H of X. Similarly, for $H \leq G$, it will be useful to denote by $\widehat{H}$ the subgroup $\widehat{H \cap X}$.

Recall that when p is odd, $d = \mathrm{ord}_p(r^a)$ represents the order modulo p of r^a and that, when $p = 2$, d is either 1 or 2 depending upon whether 4 divides $r^a - 1$ or not. For the study of the symplectic groups, we set

$$d_s = \mathrm{lcm}(2, \mathrm{ord}_p(r^a)) = \mathrm{lcm}(2, d)$$

and

$$s = \left\lfloor \frac{2n}{d_s} \right\rfloor.$$

Especially, we note that when $p = 2$ or 3, $d_s = 2$. Our upcoming proof focusses on the subgroup

$$\widehat{M} = \mathrm{Sp}_{d_s}(r^a) \wr \mathrm{Sym}(s) \times \mathrm{Sp}_{(2n - sd_s)}(r^a)$$

of $\widehat{X}$ which we note contains a Sylow p-subgroup of $\widehat{X}$ (see [56, Section 3] and [11, Theorem 1]). Since $\widehat{G} = N_{\widehat{G}}(\widehat{M})\widehat{X}$, we may assume that M, the image of $\widehat{M}$ in X, is normalized by S.

Lemma 6.1. *Suppose that $2n = d_s s$. Then the following hold:–*

(i) if either $d_s > 2$ or $r^a > 3$, then $F^*(MS) = E(MS)$ and the components of MS are permuted transitively by conjugation in MS;

(ii) if $d_s = 2$ and $r^a = 2$, then $F^*(MS) = O_3(MS)$; and

(iii) if $d_s = 2$ and $r^a = 3$, then $F^*(MS) = O_2(MS)$.

Proof. These statements are evident from the structure of M. $\qquad\square$

We shall also require the following fact.

Lemma 6.2. *If $H \cong \mathrm{Sp}_6(2)$, then H is 3-minimal.*

Proof. Let $T \in \mathrm{Syl}_3(H)$. From [4, Table 8.28 and 8.29], we read that there is a unique maximal subgroup of H containing T. This provides the fact. $\qquad\square$

The exceptional isomorphisms between groups of Lie type recorded in the statement of the next theorem can be found in [30, Proposition 2.9.1].

Theorem 6.3. *Assume that $X \cong \mathrm{PSp}_{2n}(r^a)'$ with $n \geq 2$ and that P is a rank one isolated p-minimal subgroup of G. Then one of the following holds.*

(i) $p = r$.

(ii) $p = 2$, $X \cong \mathrm{PSp}_4(3) \cong \mathrm{PSU}_4(2)$ and both $L \cap X$ and $O^2(P)N_X(S \cap X)$ are maximal parabolic subgroups of X.

(iii) $p = 2$, $X \cong \mathrm{PSp}_6(3)$, $(\widehat{P \cap X}) \sim \mathrm{Q}_8 \times (2^{1+4}_-{:}\mathrm{SL}_2(4))$ and $(\widehat{L \cap X}) = \widehat{M} \cong \mathrm{SL}_2(3) \wr \mathrm{Sym}(3)$.

(iv) $p = 3$ and $P = G = X \cong \mathrm{PSp}_4(2)' \cong \mathrm{PSL}_2(9)$.

Proof. We suppose that $p \neq r$ and deduce that the exceptional cases given in (ii), (iii) and (iv) must hold. Let V be the natural symplectic space for $\widehat{X}$ and fix a symplectic basis for V. The subgroups that we consider below will be written with respect to this fixed basis. Suppose that $2n \neq d_s s$. Then $S \cap X$ centralizes a non-trivial r-element of X and this contradicts Lemmas 4.6 and 4.7. Therefore $2n = d_s s$.

If $s = 1$, then $d_s = 2n \geq 4$. Hence $p > 3$ and so $S \cap X$ is abelian by Lemma 4.3. Further, by Lemma 2.4, we have $G \neq P$ and this contradicts Lemma 4.5. Therefore

$$d_s s = 2n \text{ and } s > 1.$$

We have exactly two possibilities: either $P \leq MS$ or $L \geq MS$.

Assume first that $P \leq MS$. Then, by Lemma 2.2 (iii), P is an isolated p-minimal subgroup of MS. If either $d_s > 2$ or $r^a > 3$, then, as $s > 1$, Theorem 2.15 (i) and (ii) and Lemma 6.1 (i) imply that r^a is odd and $O^p(P) \leq O_2(M)$. In particular, $p \neq 2$ and $O_2(MS)$ is p-closed. Since $O_p(MS/O_2(MS)) = 1$ and $P \leq O_2(MS)S$, Lemma 2.7 provides a contradiction. Hence

$$d_s = 2, s = n \text{ and } r^a \in \{2, 3\}.$$

Since MS/M is a p-group, $O^p(P) \leq M$. Set $Y = \langle O^p(P)^{MS} \rangle \leq M$. Suppose that Y/Q_Y is quasisimple. Then M is not soluble and, for R the maximal normal soluble subgroup of MS, we have $MS/R \cong \mathrm{Sym}(n)$ and $YR/R \cong \mathrm{Alt}(n)$ where $n \geq 5$. Now YQ_{MS}/Q_{MS} is a component of MS/Q_{MS}. Hence Y centralizes R/Q_{MS} and this contradicts the structure of MS/Q_{MS}. Thus Y/Q_Y is not quasisimple and so Theorem 2.15 implies that $Y = O^p(P)$

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is normal in MS and $O^p(P)$ is nilpotent. In particular, Lemma 2.4 implies that if $r^a = 2$ then $p = 3$ and if $r^a = 3$ then $p = 2$. Because $r^a \in \{2, 3\}$ the exceptional isomorphisms $\mathrm{PSp}_4(2)' \cong \mathrm{PSL}_2(9)$ and $\mathrm{PSp}_4(3) \cong \mathrm{PSU}_4(2)$ yield possibilities (ii) and (iv) if $n = 2$. So we may assume that $n \geq 3$.

Suppose that $r^a = 2$ and $p = 3$. Then $G = X \cong \widehat{X} = \mathrm{Sp}_{2n}(2)$ and we have

$$M \cong \mathrm{SL}_2(2) \wr \mathrm{Sym}(n).$$

Since $\mathrm{Sp}_6(2)$ is 3-minimal by Lemma 6.2, we may suppose that $n \geq 4$. Let B be the base group of M. Then $M/B \cong \mathrm{Sym}(n)$ and, as an M/B module, B/Q_M is isomorphic to the n-dimensional permutation module defined over $\mathrm{GF}(2)$. Notice that either $F_3^*(M) = B$ or $n = 4$ and $|F_3^*(M)/B| = 4$. Since $O^3(P) \leq F_3^*(M)$ by Theorem 2.15 (ii), we either have $YQ_M/Q_M = O^3(P)Q_M/Q_M$ is elementary abelian of order 4 or $n = 4$ and $YQ_M/Q_M = O^3(P)Q_M/Q_M \cong \mathrm{Q}_8$ by Lemma 2.4 (vi). If $n > 4$, then $O^3(P)Q_M/Q_M$ is a normal subgroup of M of order 4, and Lemma 2.19 provides a contradiction. Hence $n = 4$. If $Y \leq B$, we may apply Lemma 2.19 again to obtain a contradiction. Thus $Y \not\leq B$ and $F_3^*(M) = BY$. If YQ_M/Q_M has order 4, then $[B, Y] \leq B \cap YQ_M = Q_M$ and this contradicts the action of M/B on B/Q_M. Hence $YQ_M/Q_M \cong \mathrm{Q}_8$ and $[Y, B]Q_M/Q_M \leq Z(YQ_M/Q_M)$ and this also contradicts the action of M/B on B/Q_M. This concludes the investigation of the possibilities with $r^a = 2$ and $p = 3$.

Suppose that $r^a = 3$ and $p = 2$. Then $\widehat{X} = \mathrm{Sp}_{2n}(3)$ with $s = n \geq 3$ and

$$\widehat{M} = \mathrm{SL}_2(3) \wr \mathrm{Sym}(n).$$

In this case Q_M is a 2-group and $O^2(P)Q_M/Q_M$ is a normal 3-subgroup of M/Q_M. Suppose that n is even. Then

$$\widehat{K} = \mathrm{Sp}_4(3) \wr \mathrm{Sym}(n/2)$$

is also normalized by S. Since $O_2(KS) \leq O_2(MS) \leq Q_P$, $P \leq KS$ by Lemma 2.2 (vi). But then, as $n/2 \neq 1$, Theorem 2.15 provides a contradiction. Hence n is odd. This time take

$$\widehat{K} = \mathrm{Sp}_{2n-2}(3) \times \mathrm{Sp}_2(3).$$

Again we have $P \leq KS$ as $O_2(KS) \leq O_2(MS) \leq Q_P$. Since $O^2(P)Q_M$ is normalized by MS, Lemma 2.4 (v), (vii) and (viii) imply that $O^2(P)Q_M/Q_M$ has order 3, 3^2 or 3^3 where the latter case can only occur if $n = 3$ (as in this case $\mathrm{Sym}(3)$ has a normal 3-subgroup).

Suppose that $|O^2(P)Q_M/Q_M| = 3$. Let $\widehat{D}$ be the direct factor of the base group of $\widehat{M}$ which is normalized by $\widehat{K}$ (the right hand factor of $\widehat{K}$). Then, as $O^2(P)Q_M/Q_M$ is centralized by $O^2(MS)$, we deduce that $O^2(P)$ acts non-trivially on D/Q_D. It follows that $\langle O^2(P)^{KS} \rangle$ acts non-trivially on D. Hence $\langle O^2(P)^{KS} \rangle Q_{KS}/Q_{KS}$ is not quasisimple. Thus $O^2(P)$ is normalized by KS by Theorem 2.15(i). Now we have $O^2(P) = D$, but D is not normalized by MS. Hence $O^2(P)Q_M/Q_M$ has order 3^2 or 3^3 with $n = 3$ in the latter case. Employing Lemma 2.19, we find that $n = 3$ also when $|O^2(P)Q_M/Q_M| = 9$. Now, on the one hand $|MS/Q_{MS}|_2 \leq 2^2$ while, on the other, Lemma 2.4 (vii) gives $|P/Q_P|_2 \geq 2^3$. This contradiction proves that either $P \not\leq MS$ or possibilities (ii) or (iv) hold.

We may now assume that $L \geq MS$. Since $Q_L \cap X \neq 1$ by Theorem 2.16 (ii), we have $Q_M \neq 1$. Hence, either $p = 2$ and r is odd or $p = 3$ and $r^a = 2$. In each case we also have that $d_s = 2$ and $s = n$. So

$$\widehat{M} = \mathrm{Sp}_2(r^a) \wr \mathrm{Sym}(n).$$

Suppose first that $2n = 4$. Then, as $\mathrm{PSp}_4(3)$ or $\mathrm{PSp}_4(2)'$ are listed in (ii) and (iv) respectively, we may assume $r^a > 3$. In particular, $p = 2$ and r is odd. In addition, as $2n = 4$, $Q_{MS} = Z(MS)$ has order 2 and this contradicts Theorem 2.16 (iii). Thus $2n > 4$.

Recalling that $p \in \{2, 3\}$, we write $n = pk + \ell$ with $0 \leq \ell \leq p - 1$ and select the following subgroup whose image in G is normalized by S

$$\widehat{M_1} = \mathrm{Sp}_{2p}(r^a) \wr \mathrm{Sym}(k) \times \mathrm{Sp}_2(r^a) \wr \mathrm{Sym}(\ell).$$

Then factor $\widehat{M_1}$ as $\widehat{K_1} \times \widehat{K_2}$ with $\widehat{K_2} \cong \mathrm{Sp}_2(r^a) \wr \mathrm{Sym}(\ell)$ (which may be trivial). Notice that $O_p(\langle K_1 S, MS \rangle) = 1$ and $K_1 S$ contains P, an application of Theorem 2.15(i) yields $k = 1$ and so $n = p + \ell \leq 2p - 1$.

If $p = 3$ and $r^a = 2$, then $K_1 \cong \mathrm{Sp}_6(2)$. Since $K_1 S$ is 3-minimal by Lemma 6.2, we deduce that $P = K_1 S$ and this contradicts P being of rank one.

So suppose that $p = 2$. Then $n = 3$, $\widehat{K_1} \cong \mathrm{Sp}_4(r^a)$ with r^a odd and P is a rank one isolated 2-minimal subgroup of $K_1 S$. Hence, noting that $\widehat{L} \cap \widehat{K_1} \geq \widehat{M} \cap \widehat{K_1} = \mathrm{Sp}_2(r^a) \wr 2$, and using $|O_2((\widehat{L} \cap \widehat{K_1})/Z(\widehat{K_1})| > 2$ by Theorem 2.16 (iii) applied to $\widehat{K_1}/Z(\widehat{K_1}) \cong \mathrm{PSp}_4(r^a)$, yields $r^a = 3$. Since $L \geq MS$ and MS is a maximal subgroup of G (see [4, Table 8.28]), we deduce that $L = MS$ and $P \cap K_1 \cong 2_-^{1+4}.\mathrm{SL}_2(4)$. This provides the description of M and P given in part (iii) of the theorem. $\qquad\square$

7. Projective linear and unitary groups in dimension at least 3

In this section we determine the linear and unitary groups in dimension at least 3 which satisfy Hypothesis 2.18. To simplify our notation we sometimes use $\mathrm{GL}_n^+(r^a)$ and $\mathrm{GL}_n^-(r^a)$ to represent the general linear and general unitary groups respectively. The notation $\mathrm{GL}_n^\epsilon(r^a)$ denotes either of these groups. So here $\epsilon = \pm$. We also extend this notation to subgroups where appropriate. Throughout this section $\widehat{X} \cong \mathrm{SL}_n^\epsilon(r^a)$ with r a prime. Recall that $\Gamma\mathrm{L}_n^\epsilon(r^a)$ has $\widehat{X}$ as a normal subgroup and includes all the diagonal and field automorphisms of $\widehat{X}$. The quotient of $\Gamma\mathrm{L}_n^\epsilon(r^a)$ modulo $\widehat{X}$ has order $a(r^a - 1)$ when $\epsilon = +$ and $2a(r^a + 1)$ when $\epsilon = -$. We frequently write $r^a - \epsilon$ where here we regard ϵ as ± 1 according as $\epsilon = \pm$. In the event that $\widehat{X} \cong \mathrm{SL}_n(r^a)$, we denote the inverse transpose automorphism by ι and, for ease of notation, when $\widehat{X} \cong \mathrm{SU}_n(r^a)$ we take ι to be the field automorphism of order 2. Notice ι normalizes block type subgroups and that it is an outer automorphism as $n \geq 3$. This brings us to a subtle point that will come up from time to time in our explanations. If ι restricts to an automorphism of a subgroup K isomorphic to $\mathrm{SL}_2(r^a)$ on which it also induces ι, then it induces an inner automorphism. Indeed the product of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and ι centralizes K.

Let V represent the natural linear space when $\widehat{X} \cong \mathrm{SL}_n(r^a)$ and the natural non-degenerate unitary space when $\widehat{X} \cong \mathrm{SU}_n(r^a)$. For the remainder of this section we take

$$\widehat{X} \le \widehat{G} \le \langle \mathrm{\Gamma L}_n^{\epsilon}(r^a), \iota \rangle$$

with $\widehat{G}/\widehat{X}$ a p-group and note that $\iota \in \mathrm{\Gamma L}_n^{-}(r^a)$.

Finally set $G = \widehat{G}/F(\widehat{G})$, $X = \widehat{X}F(\widehat{G})/F(\widehat{G})$ and for $\widehat{S} \in \mathrm{Syl}_p(\widehat{G})$, $S = \widehat{S}F(\widehat{G})/F(\widehat{G})$. For $\widehat{H}^*$ an $\widehat{S}$-invariant subgroup of $\mathrm{GL}_n^{\epsilon}(r^a)$, we set

$$\widehat{H} = \widehat{H}^* \cap \widehat{G}$$

and $H = \widehat{H}F(\widehat{G})/F(\widehat{G})$. For example, if $G = \mathrm{PSL}_n(r^a)$ with p dividing $r^a - 1$, then $\widehat{G} = \mathrm{SL}_n(r^a)$ and the subgroup $\widehat{H}^* = \mathrm{GL}_1(r^a) \wr \mathrm{Sym}(n) = (r^a-1) \wr \mathrm{Sym}(n)$ of $\mathrm{GL}_n(r^a)$ can be assumed to contain $\widehat{S}$ and has $\widehat{H} \sim (r^a-1)^{n-1}. \mathrm{Sym}(n)$ and $H \sim ((r^a-1)^{n-1}. \mathrm{Sym}(n))/(r^a-1, n)$. Thus the $*$-notation gives us a recognisable way to describe explicit subgroups of $\widehat{G}$ without having to resort to giving approximate similarity type descriptions.

Our objective is to determine for which values of n and r^a, G has a rank one isolated p-minimal subgroup and, in those instances when it does, determine P and L.

Let $d_{\epsilon} \ge 1$ be minimal such that

$$(\epsilon r^a)^{d_\epsilon} \equiv \begin{cases} 1 \pmod{p} & \text{if } p \text{ is odd} \\ 1 \pmod{4} & \text{if } p = 2. \end{cases}$$

We note here that $d_+ = d$ is what it always has been; the order of r^a mod p when p is odd and when $p = 2$, d is either 1 or 2 depending upon whether 4 divides $r^a - 1$ or not. Otherwise, the relationship between d and d_- is as follows

$$d_- = \begin{cases} 2d & d \equiv 1, 3 \pmod 4 \\ d & d \equiv 0 \pmod 4 \\ d/2 & d \equiv 2 \pmod 4. \end{cases}$$

In particular, if $p \in \{2, 3\}$, $d_- = 3 - d_+ = 3 - d \in \{1, 2\}$.

Write $s = \lfloor \frac{n}{d_\epsilon} \rfloor$ and then define

$$\widehat{M}^* = \mathrm{GL}_{d_\epsilon}^{\epsilon}(r^a) \wr \mathrm{Sym}(s) \times \mathrm{GL}_{n-sd_\epsilon}^{\epsilon}(r^a).$$

Observe that this group contains a Sylow p-subgroup of $\widehat{X}$ and is invariant under the standard field automorphisms of $\widehat{X}$ and under ι (see [11, 18, 56]). Thus $\widehat{M}^*$ can be assumed to be normalized by $\widehat{S}$ (in accordance with our notational convention).

Furthermore, we note that the p-adic decomposition of s

$$s = \ell_0 + \ell_1 p + \ell_2 p^2 + \cdots + \ell_k p^k$$

determines a number of further $\widehat{S}$-invariant subgroups such as, for example,

$$\widehat{M}_1^* = \mathrm{GL}_{d_\epsilon}^{\epsilon}(r^a) \wr \mathrm{Sym}(\ell_0) \times \mathrm{GL}_{d_\epsilon p}^{\epsilon}(r^a) \wr \mathrm{Sym}(\ell_1) \times \cdots \times \mathrm{GL}_{d_\epsilon p^k}^{\epsilon}(r^a) \wr \mathrm{Sym}(\ell_k) \le \mathrm{GL}_{d_\epsilon}^{\epsilon}(r^a) \wr \mathrm{Sym}(s)$$

and we shall make regular use of such subgroups.

Our main theorem in this section is:

Theorem 7.1. *Suppose that Hypothesis 2.18 holds with $X \cong \mathrm{PSL}_n^\epsilon(r^a)$ and $n \geq 3$. Then either $p = r$ or $p \in \{2,3\}$ and G, P and L are as described in lines 9 to 20 of Table 1 and lines 3 to 10 of Table 2.*

Throughout the proof of Theorem 7.1, we assume that P is a rank one isolated p-minimal subgroup of G as in Hypothesis 2.18 and we further assume that $p \neq r$. The proof of Theorem 7.1 will be by induction on n. It is worth highlighting one case which is omitted from the tables in Section 1, but needs to be remembered for the induction. The isomorphism $\mathrm{PSL}_3(2) \cong \mathrm{PSL}_2(7)$ means that $\mathrm{PSL}_3(2)$ is 7-minimal and so $P = G$ is an example of a rank one isolated 7-minimal subgroup. We begin with two lemmas which help us discover L in some of the base cases of the induction. Notice that, if $X \cong \mathrm{PSL}_3(4)$ or $\mathrm{PSU}_3(8)$ and $d_\epsilon = 1$, then $Q_{\widehat{M}} \cap \widehat{X}$ is not contained in the base group of $\widehat{M}$ as in exactly these cases $r^a - \epsilon$ is a power of 3 (use Zsigmondy's Theorem for example). This is why these groups are excluded from the next lemma.

Lemma 7.2. *Suppose that $p \geq 3$, $d_\epsilon = 1$, $X \cong \mathrm{PSL}_p^\epsilon(r^a)$ and $X \not\cong \mathrm{PSL}_3(4)$ or $\mathrm{PSU}_3(8)$. Let $Q = Q_M \cap X$. Then*
   (i) $|Q| = \frac{1}{p}(r^a - \epsilon)_p^{p-1}$;
   (ii) $|C_Q(S \cap X)| = p$;
   (iii) if either $p > 3$ or $p = 3$ and $(r^a - \epsilon)_3 > 3$, then $C_{Q/C_Q(S\cap X)}(S \cap X)$ has order p and the preimage of $C_{Q/C_Q(S\cap X)}(S \cap X)$ in Q is elementary abelian; and
   (iv) if $p = 3$ and $(r^a - \epsilon)_3 = 3$, then $\mathrm{PGL}_3^\epsilon(r^a)$ has Sylow 3-subgroups which are extraspecial of order 27.

Proof. Since we have assumed $X \not\cong \mathrm{PSL}_3(4)$ or $\mathrm{PSU}_3(8)$ and $d_\epsilon = 1$, part (i) is easy to verify as $|Z(\widehat{X})| = (p, r^a - \epsilon) = p$, and $|\widehat{Q} \cap \widehat{X}| = (r^a - \epsilon)_p^{p-1}$.

Let $\mathcal{P} = \{V_1, \ldots, V_p\}$ be the 1-dimensional eigenspaces of $Q_{\widehat{M}}$ on V and write the matrices of $\widehat{X}$ with respect to a basis $\{v_1, \ldots, v_p\}$, with $v_i \in V_i$. Then $\widehat{M}$ is the stabilizer of $\mathcal{P}$ and

$$\widehat{M} = \mathrm{GL}_1^\epsilon(r^a) \wr \mathrm{Sym}(p) = (r^a - \epsilon) \wr \mathrm{Sym}(p).$$

Let $\widehat{\pi} \in \widehat{M}$ permute $\mathcal{P}$ as the permutation $(1, \ldots, p)$ and π be its image in XS. Then, as $p > 2$, $\pi \in M' \leq M \cap X$ and $S \cap X = Q\langle\pi\rangle$. Since $\widehat{Q}$ is abelian, we just need to determine $C_Q(\pi) = C_Q(S \cap X)$. For

$$\widehat{e} = \mathrm{diag}(a_1, \ldots, a_p) \in \widehat{Q},$$

we calculate that

$$[\widehat{e}, \widehat{\pi}] = \mathrm{diag}(a_1^{-1}a_2, \ldots, a_p^{-1}a_1).$$

Hence, as $|Z(\widehat{X})| = p$, the preimage of $C_Q(S \cap X)$ in $\widehat{X}$ is the subgroup

$$\widehat{C} = \{\mathrm{diag}(\alpha, \alpha\lambda, \ldots, \alpha\lambda^{p-1}) \mid \lambda^p = 1, \alpha, \lambda \in \mathrm{GL}_1^\epsilon(r^a)\}.$$

Since the determinant of the elements in $\widehat{C}$ is 1, we have $\alpha^p = 1$. Thus $|C| = p$. This proves (ii).

For (iii) and (iv), we note that the set of elements $\widehat{e} \in \widehat{Q}$ such that $[\widehat{e}, \widehat{\pi}] \in C$ is

$$\widehat{D} = \{\mathrm{diag}(\beta, \beta\alpha, \beta\alpha^2\lambda, \ldots, \beta\alpha^{p-1}\lambda^{\frac{(p-1)(p-2)}{2}}) \mid \lambda^p = \alpha^p = 1, \alpha, \beta, \lambda \in \mathrm{GL}_1^\epsilon(r^a)\}$$

subject to $\beta^p\alpha^{p(p-1)/2}\lambda^{(p-1)(p-2)p/6} = 1$ (the power of λ in the determinant being $(p-1)(p-2)p/6$, the sum of the first $p-2$ triangular numbers). Thus, if $p > 3$, we have that $\widehat{D}$ is elementary abelian of order p^3 and, if $p = 3$, then, assuming that 9 divides $r^a - \epsilon$, β has order dividing 9 and $\widehat{D} \cong 9 \times 3$. In both of these cases D is elementary abelian of order 9. The final possibility is that $(r^a - \epsilon)_3 = 3$ and in this case we have that $\mathrm{PGL}_3(r^a)$ has extra special Sylow 3-subgroups of order 27, Q is cyclic of order 3 and $S \cap X$ is elementary abelian of order 9. These outcomes are listed in (iii) and (iv). $\qquad\square$

Lemma 7.3. *Suppose that $p \geq 3$, $d_\epsilon = 1$, $X \cong \mathrm{PSL}_p^\epsilon(r^a)$ and $X \ncong \mathrm{PSL}_3(4)$ or $\mathrm{PSU}_3(8)$. Assume that R is a non-trivial normal subgroup of $S \cap X$. Then either*

(i) $N_G(R) \leq MS$; or

(ii) $p = 3 = (r^a - \epsilon)_3$, $G \cong \mathrm{PGL}_3^\epsilon(r^a)$, $S \cong 3_+^{1+2}$ and $N_G(R) \sim 3^2{:}\mathrm{SL}_2(3)$ or $G = X$, $S = R$ is elementary abelian of order 9 and $N_G(S) \cong 3^2{:}\mathrm{Q}_8$.

Proof. Let $\mathcal{P} = \{V_1, \ldots, V_p\}$ be the $Q_{\widehat{M}} \cap \widehat{X}$ eigenspaces as in Lemma 7.2. If the set of all 1-spaces of V which are left invariant by $\widehat{R}$ coincides with $\mathcal{P}$, then $N_{\widehat{G}}(\widehat{R})$ permutes $\mathcal{P}$ and consequently $N_G(R) \leq MS$.

Suppose that $\widehat{R} \leq Q_{\widehat{M}}$ and that $N_G(R) \nleq MS$. Then $\widehat{R}$ leaves each 1-space in $\mathcal{P}$ invariant and, as $\widehat{R}$ leaves at least one further 1-space invariant, it follows that $V_i|_{\widehat{R}} \cong V_j|_{\widehat{R}}$ for some $i \neq j$. The primitive action of $\widehat{S \cap X}$ on $\mathcal{P}$ and the fact that $S \cap X$ normalizes R, ensures that

$$V_1|_{\widehat{R}} \cong V_2|_{\widehat{R}} \cong \ldots \cong V_p|_{\widehat{R}}.$$

As a consequence, $\widehat{R}$ acts as scalar matrices on V and this contradicts the fact that $R \neq 1$. Thus, if $R \leq Q_M$, then part (i) holds.

Assume that $R \nleq Q_M$. Then, as S/Q_M has order p, $\Phi(R) \leq Q_M$. If $\Phi(R) > 1$, then our earlier observation implies that $N_G(R) \leq N_G(\Phi(R)) \leq MS$. So if (i) fails to hold, we must have that R is elementary abelian. By Lemma 7.2 (ii), $Q_M \cap R = C_{Q_M}(S \cap X)$ has order p. Thus $|R| = p^2$, $C_{S \cap X}(R) = R$ and $\widehat{R}$ is extraspecial of order p^3. Furthermore, as R is normalized by $S \cap X$, $[Q_M \cap X, R] \leq Q_M \cap R$. Suppose that $p > 3$ or $p = 3$ and $(r^a - \epsilon)_3 > 3$. Then we have $|Q_M \cap X| = p^2$ by Lemma 7.2 (iii). The order of $Q_M \cap X$ is also presented in Lemma 7.2 (i) and this gives a contradiction. Hence $(r^a - \epsilon)_3 = 3$ and so a is coprime to 3 and G is isomorphic to a subgroup of $\mathrm{PGL}_3(r^a)$. Thus $27 \geq |S| \geq 9$. Since $\mathrm{PGL}_3^\epsilon(r^a)$ contains maximal subgroups of shape $3^2{:}\mathrm{SL}_2(3)$ by [4, Tables 8.3 and 8.5], we have that (ii) holds. $\qquad\square$

Note that the example appearing in the conclusion of the next lemma is not included as an exceptional case in Theorem 1.6 because of the exceptional isomorphism $\mathrm{SL}_3(2) \cong \mathrm{PSL}_2(7)$ [30, Proposition 2.9.1].

Lemma 7.4. *Suppose that $X \cong \mathrm{PSL}_n^\epsilon(r^a)$ and P is a rank one isolated p-minimal subgroup of G. If $p > 3$ and $n \leq d_\epsilon p$, then $p = 7$, $G = P \cong \mathrm{SL}_3(2) \cong \mathrm{PSL}_2(7)$.*

Proof. If $n < d_\epsilon p$, then $S \cap X$ is abelian by Lemma 4.3 and applying Lemma 4.5 implies $P = G$. Using Lemma 2.4 then gives $P \cong \mathrm{PSL}_2(7) \cong \mathrm{PSL}_3(2)$. So we have $r^a = 2$, $p = 7$ and $\epsilon = +$. But then, accidentally, X is group of Lie type defined in characteristic 7 and we have included this case in the statement. Therefore, $n = d_\epsilon p$.

Assume that $d_\epsilon \geq 2$. Then, as $n = d_\epsilon p$,

$$\widehat{M}^* = \mathrm{GL}_{d_\epsilon}^\epsilon(r^a) \wr \mathrm{Sym}(p).$$

Since $d_\epsilon \geq 2$ and $p > 3$, $Q_M = 1$. Therefore $P \leq MS$ by Theorem 2.16 (ii). Notice that by Lemma 2.4, as $p > 3$, $O^p(P)$ is not soluble and so $E(M)$ is not soluble and, in particular, $\mathrm{GL}_{d_\epsilon}^\epsilon(r^a)$ is not soluble. Therefore Theorem 2.15 (i) implies that $Y = \langle O^p(P)^M \rangle$ is a normal component of M, which is ridiculous as $p > 3$. This proves $d_\epsilon = 1$ and so $n = p$.

As $Q_L \cap X > 1$ by Theorem 2.16 (ii) and $p > 3$, Lemma 7.3 gives $L \leq MS$. If $Q_P \cap X > 1$, then Lemma 7.3 also forces $P \leq MS$ which is a contradiction as $\langle L, P \rangle = G$. Thus $Q_P \cap X = 1$ and consequently $Q_P = 1$ by [22, Theorem B].

Since $Q_P = 1$ and $p > 3$, we have $X \geq O^p(P) \cong \mathrm{SL}_2(p^b), \mathrm{PSL}_2(p^b), \mathrm{SU}_3(p^b)$ or $\mathrm{PSU}_3(p^b)$ for some $b \geq 1$. As the p-part of the Schur multipliers of these groups is trivial, we have $O^p(\widehat{P}) \cong O^p(P)$. It follows that

$$\Omega_1(Z(\widehat{S} \cap \widehat{X})) \geq \langle \Omega_1(Z(\widehat{S} \cap \widehat{X})) \cap O^p(\widehat{P}), Z(\widehat{X}) \rangle$$

has order at least p^2. This is impossible as $\widehat{S} \cap \widehat{X}$ acts irreducibly on V and so has cyclic centre. This proves Lemma 7.4 $\qquad\qquad\square$

We now hunt down the examples in small dimensions when $p = 2$ and 3 as well as set the foundation for our inductive argument in the proof of Theorem 7.12. The examples in the next lemma occupy lines 3, 4, 5 and 6 of Table 2.

Lemma 7.5. *Suppose that $p = 3$ and $X \cong \mathrm{PSL}_3^\epsilon(r^a)$. Then $G \cong \mathrm{PGL}_3^\epsilon(r^a)$, $r^a - \epsilon \equiv 3, 6$ (mod 9) with $r^a \neq 4$ when $\epsilon = 1$ and either*

> *(i) $P \sim 3^2{:}\mathrm{SL}_2(3)$ and $L \sim (r^a - \epsilon)^2.\mathrm{Sym}(3)$; or*
> *(ii) $G \cong \mathrm{PGL}_3(7)$ or $\mathrm{PGU}_3(5)$, $P \sim (3^2 \times 2^2){:}\mathrm{Sym}(3)$ and $L \sim 3^2{:}\mathrm{SL}_2(3)$.*

Proof. By Lemma 2.14, we may suppose that $X \not\cong \mathrm{PSL}_3(2)$. So, as $\mathrm{PSU}_3(2)$ is soluble, we have $r^a \neq 2$. In addition, when $r^a = 4$ and $X \cong \mathrm{PSL}_3(4)$, we have $\mathrm{PGL}_3(4)$ is 3-minimal and X has abelian Sylow 3-subgroups. So appealing to Lemma 4.5 we see this case cannot occur either. We also observe that $\mathrm{PSU}_3(8)$ is 3-minimal by [4, Table 8.5] or [12], and so we conclude that the omitted cases of Lemma 7.3 cause no problems.

Suppose that $d_\epsilon = 2$. Then

$$\widehat{M}^* = \mathrm{GL}_2^\epsilon(r^a) \times \mathrm{GL}_1^\epsilon(r^a),$$

and, as $d_\epsilon = 2$, $M \cong \mathrm{GL}_2^\epsilon(r^a)$ and $Q_M = 1$. Since $Q_L \cap X \neq 1$, we infer that P is a rank one isolated 3-minimal subgroup of MS. Then calling upon Theorem 5.1, as $p \neq r$, yields

that $P = MS$ with $O^3(P) \cong \mathrm{PSL}_2(8) \cong {}^2\mathrm{G}_2(3)'$. So, as $p \neq r$, we have $r^a = 8$. Thus, as $d_\epsilon \neq 1$, $\epsilon = +$ and $X \cong \mathrm{PSL}_3(8)$. An application of Lemma 4.6 provides a contradiction as ι is not present because $p = 3$ and the parabolic subgroups of $\mathrm{PSL}_3(8)$ contain a Sylow 3-subgroup of $\mathrm{PSL}_3(8)$. Therefore $d_\epsilon = 1$ and so

$$\widehat{M}^* = \mathrm{GL}_1^\epsilon(r^a) \wr \mathrm{Sym}(3) = (r^a - \epsilon) \wr \mathrm{Sym}(3).$$

Since $Q_L \cap X > 1$ by Theorem 2.16 (ii), we either have $L \leq MS$ or $L \sim 3^2{:}\mathrm{SL}_2(3)$ by Lemma 7.3.

Assume that $L \sim 3^2{:}\mathrm{SL}_2(3)$. Then, by Lemma 7.3 again, $G \cong \mathrm{PGL}_3^\epsilon(r^a)$, $(r^a - \epsilon)_3 = 3$ and $P \leq MS$. If $r^a - \epsilon = 3$, then $(r^a, \epsilon) = (2, -)$ or $(4, +)$, a contradiction. Hence there is a prime t dividing $(r^a - \epsilon)$ with $t \neq 3$. Let $T = O_t(M)$. Then TS is a group containing S. Thus either $TS \leq L$ or $P \leq TS$. If $P \leq TS$, then $O^3(P) \leq T$ and so by Lemma 2.4, T is a 2-group. On the other hand, if $TS \leq L$, then, as $L \sim 3^2{:}\mathrm{SL}_2(3)$ is a $\{2, 3\}$-group, we also have that T is a 2-group. Hence M is also a $\{2, 3\}$-group. In particular T has 2-rank 2 and so $T \not\leq L$ (which has Sylow 2-subgroups isomorphic to Q_8). Hence $P \leq TS$. Since TS is 3-minimal and $TS \not\leq L$, we have $P = TS$. Then $P/Q_P \in \mathcal{L}_1(3)$ implies that T has order 4. Thus $(r^a - \epsilon)_2 = 2$ and, as we have already noted that $(r^a - \epsilon)_3 = 3$, this gives case (ii).

We now move onto the case when $L \leq MS$. Then $P \not\leq MS$ and, since $p = 3$, [22, Theorem B] implies that $Q_P \cap X > 1$ or $Q_P = 1$. In the former case, Lemma 7.3 yields that either $P \cap X \leq M \cap X \leq L$ or $P \sim 3^2{:}\mathrm{SL}_2(3)$ and $(r^a - \epsilon)_3 = 3$. Since $P = (P \cap X)S \not\leq MS$, the latter occurs and this means that $r^a - \epsilon \equiv 3, 6 \pmod 9$. As $(r^a, \epsilon) \neq (4, +)$, this case is excluded and we obtain the structure of P and $L \cap X$ as listed in (i). So suppose that $Q_P = 1$. Since

$$|S \cap X| = (r^a - \epsilon)_3^2 \geq 3^2$$

and $S \cap X$ acts faithfully on $O^3(P)$, we have that $O^3(P) \cong \mathrm{SL}_2(3^b)$ ($b > 1$), $\mathrm{PSL}_2(3^b)$ ($b > 1$), $\mathrm{PSU}_3(3^b)$ ($b \geq 1$) or ${}^2\mathrm{G}_2(3^b)'$ ($b \geq 1$). Now the 3-rank of X is 2 and so we may tighten this to $O^3(P) \cong \mathrm{SL}_2(9)$, $\mathrm{PSL}_2(9)$, $\mathrm{PSU}_3(3)$ or ${}^2\mathrm{G}_2(3)'$. As S acts faithfully on $O^3(P)$, we deduce that $P = O^3(P)$ or $P \cong {}^2\mathrm{G}_2(3)$. Furthermore, in the latter case, as $S \cap X$ is not cyclic, we have $P \leq X$ and, in particular, $S \leq X$ in all cases. Since 3 divides $r^a - \epsilon$, X has an outer automorphism δ of order 3 which we may assume normalizes S. As P has no such automorphism, $P^\delta \neq P$ but $P^\delta \geq S$. Therefore $P^\delta \leq L$ and this is impossible as $Q_L \neq 1$ and $Q_P = 1$. $\qquad \square$

The next lemma populates lines 9 to 16 of Table 1. Before we state it, we give some details which better describes the configuration listed in (iv), and highlights the subtlety around the action of ι mentioned at the beginning of this section. In the situation of interest, $p = 2$, $r^a - \epsilon \equiv 0 \pmod 4$, $r^a > 3$ and $X \cong \mathrm{PSL}_3^\epsilon(r^a)$. We have $\widehat{M}^* \cong \mathrm{GL}_1^\epsilon(r^a) \wr \mathrm{Sym}(3)$ and $\widehat{M_1^*} \cong \mathrm{GL}_2^\epsilon(r^a) \times \mathrm{GL}_1^\epsilon(r^a)$, $L = M_1 S$ and $P \leq MS$. Since $p = 2$, the diagonal automorphism of order $(3, r^a - \epsilon)$ plays no role. Assume that XS does not contain ι (identified with its image in G). Let $\widehat{K}_1$ be the component of $\widehat{M_1^*}$. Then $[K_1, Q_L] = 1$.

Since field automorphisms or field automorphisms multiplied by ι do not induce inner automorphisms of K_1, we see that $Q_L \leq X$. Hence $Q_L \leq Q_{M_1} \cap X$ and

$$Q_{\widehat{M_1}^*} = \{\mathrm{diag}(\lambda, \lambda, \mu) \mid \lambda, \mu \in \mathrm{Syl}_2(\mathrm{GL}_1^\epsilon(r^a))\} \leq Q_{\widehat{M}^*}.$$

Hence $Q_L \leq Q_M \leq Q_P$, which is impossible. Now suppose that $\iota \in XS$. Let $\widehat{T} = \left\langle \left(\begin{smallmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix} \right), \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{smallmatrix} \right) \right\rangle$. Then T is centralized by ι. Since $\widehat{T} \leq \widehat{M}^* \cap \widehat{X}$, $T \leq M$ and $TQ_M/Q_M \cong \mathrm{Sym}(3)$. So if we set $P^* = \langle \iota \rangle TQ_M$, we have $\iota \in Q_{P^*}$. On the other hand, ι plainly does not centralize $\widehat{K_1}$ and so $\iota \notin O_2(\widehat{K_1 S})$ and yet ι acts on $\widehat{K_1}$ as the inner automorphism obtain by conjugating by $\left(\begin{smallmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix} \right)$. Thus the product β of ι and $\left(\begin{smallmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix} \right)$ centralizes $\widehat{K_1}$. This is not sufficient for its image to be in Q_{MS} as it must also centralize $O_t(\widehat{M}) \bmod Z(X)$, where t is an odd prime. Now $F(\widehat{M_1}^*) = \{\mathrm{diag}(\lambda, \lambda, \mu) \mid \lambda, \mu \in \mathrm{GL}_1^\epsilon(r^a)\}$ and assume that

$$\mathrm{diag}(\lambda, \lambda, \lambda^{-2}) = \mathrm{diag}(\lambda, \lambda, \lambda^2)^\beta = \mathrm{diag}(\lambda^{-1}\gamma, \lambda^{-1}\gamma, \lambda^2\gamma)$$

for some $\gamma \in \mathrm{GL}_1^\epsilon(r^a)$. Then $\lambda^6 = 1$. Thus, $\beta \in O_2(\widehat{M_1}\widehat{S})$ if and only if $|\mathrm{GL}_1^\epsilon(r^a)| = r^a - \epsilon = 2^b.3^c$ where $b \geq 2$ and $c \in \{0, 1\}$. Assuming this is the case and setting $P = P^*S = \langle T, S \rangle$, we have that the image of β in G is not in Q_P and hence $Q_L \not\leq Q_P$. Finally, we know that P is 2-minimal and we just need to observe that every 2-minimal subgroup of MS is either in L or is equal to P. If $r^a - \epsilon = 2^b$, this is trivial as $P = MS$ and $P \cap M_1 S = P \cap L = S$. So assume that $r^a - \epsilon = 2^b.3$. Then $|MS|_{2'} = 3^2$ and MS/Q_{MS} has a normal subgroup of order 3^2. Since P has index 3 in MS and $\iota \in Q_P$ inverts diagonal matrices, we see that $MS/Q_{MS} \cong \mathrm{Sym}(3) \times \mathrm{Sym}(3)$ and this group has exactly two 2-minimal subgroups, $L \cap MS$ and P. Finally observe that the condition that $r^a - \epsilon = 2^b$ or $2^b.3$ implies that a is odd or $\epsilon = +$ with $r^a \in \{9, 25\}$.

Lemma 7.6. *Suppose that $p = 2$ and $X \cong \mathrm{PSL}_3^\epsilon(r^a)$. Then one of the following holds.*

   (i) $G \cong \mathrm{PSU}_3(r^a)$ with $r^a \equiv 3 \pmod 8$, $P \sim 2\cdot\mathrm{Sym}(4) * 4$ and $L \sim (r^a + 1)^2 {:}\mathrm{Sym}(3)$.

   (ii) $G \cong \mathrm{PSU}_3(r^a){:}\langle\iota\rangle$ with $r^a \equiv 3 \pmod 8$, $P \sim 2\cdot\mathrm{Sym}(4) * \mathrm{Q}_8$ and $L \sim (r^a + 1)^2{:}(2 \times \mathrm{Sym}(3))$.

   (iii) $G \cong \mathrm{PSU}_3(3) \cong \mathrm{G}_2(2)'$, $P \sim 4^2{:}\mathrm{Sym}(3)$ and $L \sim 2\cdot\mathrm{Sym}(4) * 4$.

   (iv) $G \cong \mathrm{PSL}_3(r^a){:}\langle\iota\rangle$ with $r^a \equiv 5 \pmod 8$, $r \neq 5$, $P \sim 2\cdot\mathrm{Sym}(4) * \mathrm{Q}_8$ and $L \sim (r^a - 1)^2{:}(2 \times \mathrm{Sym}(3))$.

   (v) $G \cong \mathrm{PSL}_3^\epsilon(r^a){:}\langle\iota\rangle$, $r^a - \epsilon = 2^b$ or $2^b.3$ with $b \geq 2$ and a odd, $P \sim (2^b)^2{:}(2 \times \mathrm{Sym}(3))$ and $L \sim \mathrm{GL}_2^\epsilon(r^a){:}\langle\iota\rangle$.

   (vi) $H \cong \mathrm{PSL}_3(9){:}\langle\iota\rangle$, $H \leq G \leq \mathrm{Aut}(X)$, $P \cap H \sim (2^3)^2{:}(2 \times \mathrm{Sym}(3))$ and $L \cap H \sim \mathrm{GL}_2^\epsilon(9){:}\langle\iota\rangle$.

   (vii) $H \cong \mathrm{PSL}_3(25){:}\langle\iota\rangle$, $H \leq G \leq \mathrm{Aut}(X)$, $P \cap H \sim (2^3)^2{:}(2 \times \mathrm{Sym}(3))$ and $L \cap H \sim \mathrm{GL}_2^\epsilon(25){:}\langle\iota\rangle$.

Proof. Since $p = 2$, we have $d_\epsilon = 1$ if and only if $r^a \equiv \epsilon \pmod 4$.

Suppose that $d_\epsilon = 2$. In this case,

$$\widehat{M}^* = \mathrm{GL}_2^\epsilon(r^a) \times \mathrm{GL}_1^\epsilon(r^a)$$

and $M \cong \mathrm{GL}_2^\epsilon(r^a)$. Furthermore, $M \cap X \geq S \cap X$ and $S \cap X$ is a semidihedral group of order at least 16. In particular, if R is a normal subgroup of $S \cap X$, then R is either cyclic, (non-abelian) dihedral or quaternion. Thus any 2-local subgroup of X which contains $S \cap X$ centralizes $Z(S \cap X)$ and consequently is contained in $M \cap X = C_X(Z(S \cap X))$. Because $Q_L \cap X \neq 1$, it follows that $L \cap X = M \cap X$ and so $L = MS$. Moreover, $P \not\leq L$ and so $Q_P \cap X = 1$. Now L centralizes $Z(S \cap X)$ and so $O_2(Z(L)) \not\leq Q_P$. Therefore, Theorem 2.16 (i) implies that $O^2(P)$ is normal in G, which is of course impossible. Hence $d_\epsilon = 1$.

Since $d_\epsilon = 1$, we now have

$$\widehat{M}^* = \mathrm{GL}_1^\epsilon(r^a) \wr \mathrm{Sym}(3) = (r^a - \epsilon) \wr \mathrm{Sym}(3).$$

Define

$$\widehat{M_1^*} = \mathrm{GL}_2^\epsilon(r^a) \times \mathrm{GL}_1^\epsilon(r^a).$$

Notice that $O_2(\langle M_1, M \rangle) = 1$. If $r^a = 3$, then $\epsilon = -$ and we use [4, Tables 8.5 and 8.6] to see that statements (i), (ii), (iii) and (iv) hold. So we may assume that $r^a > 3$. Let K_1 be the component of M_1. Then K_1 is S-invariant. If $L \geq K_1 S$, then $P \leq MS$ and the argument in the paragraph before the lemma provides the possibilities for L and P listed in (v), (vi) and (vii). So suppose that P is a rank one isolated 2-minimal subgroup of $K_1 S$.

After noting that $K_1 S / Q_{K_1 S}$ contains a section isomorphic to $\mathrm{PGL}_2(r^a)$ and $r^a \equiv \epsilon$ (mod 4), Theorem 5.1 supplies one of the following possibilities:

$r^a \equiv 5\epsilon$ (mod 8), $r \neq 5$ and $P \not\leq MS \cap K_1 S$;
$r^a = 5$, $P = K_1 S$, $O^2(P) \cong \mathrm{SL}_2(5)$ and $P \not\leq MS \cap K_1 S$; or
$P \leq MS \cap K_1 S$ and $r^a \in \{11, 13, 19\}$.

In the first two cases we have $L \geq MS$. Noting also that when $\epsilon = +$, $\widehat{S} \cap \widehat{X}$ does not operate irreducibly on V, Lemma 4.6 implies that G contains an element which acts like ι on X. Thus in the first case, we have the examples given in parts (i), (ii) and (v). When $r^a = 5$, we have $L \cap K_1 S \leq P$ and $(L \cap K_1 S)Q_P/Q_P \cong \mathrm{Sym}(4)$. Since $K_1 S \cap M_1 S$ is a 2-group when $r^a = 5$, we see that $L \geq \langle MS, L \cap K_1 S \rangle > MS$ and using [4, Tables 8.3 and 8.4] we obtain $L = G$, a contradiction. So we are left to deal with the case when $P \leq MS$ and $r^a = 11$, 13 or 19. Then $O^2(P)$ has order 3, 3 and 5 in the respective cases. Since M is soluble, $O^2(P)Q_{MS}/Q_{MS}$ is normal in MS/Q_{MS} by Theorem 2.15 (i). Notice that MS has a further 2-minimal subgroup and this one has shape $4^2{:}\mathrm{Sym}(3)$ and must be contained in L. However this subgroup together with $L \cap K_1 S$ generate G (see [45] for example). Hence this case does not survive. This completes the proof of the lemma. $\qquad\square$

We next find the examples in lines 7 and 8 of Table 2.

Lemma 7.7. *Suppose that $p = 3$ and $X \cong \mathrm{PSL}_n^\epsilon(r^a)$ with $4 \leq n \leq 5$. Then either*

(i) $G = X \cong \mathrm{PSU}_4(r^a)$ with $r^a \equiv 2, 5$ (mod 9), $P \sim 3_+^{1+2}{:}\mathrm{SL}_2(3)$ and $L \sim \frac{1}{(4, r^a + 1)}(r^a + 1)^3{:}\mathrm{Sym}(4)$;

(ii) $G = X \cong \mathrm{PSU}_4(2) \cong \mathrm{PSp}_4(3)$, $P \sim 3^3{:}\mathrm{Sym}(4)$ and $L \sim 3_+^{1+2}{:}\mathrm{SL}_2(3)$; or

(iii) $G = X \cong \mathrm{PSU}_5(2)$, $P \sim 3 \times 3^{1+2}_+.\mathrm{SL}_2(3)$ *and* $L \sim 3^4.\mathrm{Sym}(5)$.

Proof. If $X \cong \mathrm{PSL}_n(r^a)$ with $n \in \{4, 5\}$, then, as $\widehat{S} \cap \widehat{X}$ cannot act irreducibly on V, $S \cap X$ is contained in a parabolic subgroup of X. This contradicts Lemma 4.6 as $\iota \notin G$ for odd p. So we must have $X \cong \mathrm{PSU}_n(r^a)$.

We consider the case $n = 4$ first. If $r^a = 2$, then $X \cong \mathrm{PSp}_4(3)$ ([30, Proposition 2.9.1]) and we include this case in parts (i) and (ii). So we may assume $r^a > 3$.

Suppose that $d_- = 2$. Then

$$\widehat{M^*} = \mathrm{GU}_2(r^a) \wr \mathrm{Sym}(2)$$

and, since $r^a > 3$, Theorem 2.15 (i) implies that $L \geq E(M)S$. But then from Theorem 2.16 (ii)

$$1 \neq Q_L \cap X \leq F(M)$$

and so 3 divides $|F(M)| = (r^a + 1)^2$, which is a contradiction. So $d_- = 1$ and we have that

$$\widehat{M^*} = \mathrm{GU}_1(r^a) \wr \mathrm{Sym}(4) = (r^a + 1) \wr \mathrm{Sym}(4).$$

Set

$$\widehat{M_1^*} = \mathrm{GU}_3(r^a) \times \mathrm{GU}_1(r^a),$$

let $\widehat{K_1}$ be a component of $\widehat{M_1^*}$ and notice that S normalizes M_1 and K_1 and $O_3(\langle MS, K_1 S \rangle) = 1$. Assume that $L \geq K_1 S$. Then $P \leq MS$ and we obtain $Q_L \leq Q_{K_1 S} \leq Q_{MS} \leq Q_P$, a contradiction. So we have $P \leq K_1 S$. Then Lemma 7.5(i) gives $r^a + 1 \equiv 3, 6 \pmod 9$, $L = MS$ and $P \sim 3^{1+2}_+:\mathrm{SL}_2(3)$ as claimed in (i).

Assume now that $n = 5$. If $d_- = 2$, then $r^a > 3$. This time

$$\widehat{M^*} = \mathrm{GU}_2(r^a) \wr \mathrm{Sym}(2) \times \mathrm{GU}_1(r^a)$$

and we argue that $P \not\leq E(M)S$ as in the $n = 4$ case. Thus $L \geq E(M)S$ and then, as $d_- = 2$, $Q_L \cap X \leq O_3(E(M)S) = 1$, which is a contradiction. Hence $d_- = 1$ and

$$\widehat{M^*} = \mathrm{GU}_1(r^a) \wr \mathrm{Sym}(5) = (r^a + 1) \wr \mathrm{Sym}(5).$$

Set

$$\widehat{M_1^*} = \mathrm{GU}_3(r^a) \times \mathrm{GU}_2(r^a)$$

and assume that $r^a \neq 2$. Let $\widehat{K_1} \cong \mathrm{SU}_3(r^a)$ and $\widehat{K_2} \cong \mathrm{SL}_2(r^a)$ be the components of $\widehat{M_1}$. Since $O_3(\langle MS, K_j \rangle) = 1$ for $j = 1$ and 2, $K_1 \cap K_2 = 1$ and $K_1 K_2$ is a group, Lemma 2.11 implies that $P \leq MS$. Now applying Lemma 2.6 and Theorem 3.3 to $MS/N \cong \mathrm{Sym}(5)$ where N largest soluble normal subgroup of MS, we obtain $P \leq NS$. Now $NS = F(M)S$ is soluble and so Theorem 2.15 implies that $O^3(P)$ is normal in MS. Since $O^3(P)$ is abelian, Lemma 2.4(iv) implies $|O^3(P)| = 2^2$ and now Lemma 2.19 applied to the action of M on $\Omega_1(Q_M)$ delivers a contradiction.

So $r^a = 2$ and we have $X \cong \mathrm{PSU}_5(2)$. In this case we have $G = XS = X$. According to [4, Table 8.20], the maximal subgroups containing S are isomorphic to

$$A_1 = \mathrm{GU}_4(2) \cong 3 \times \mathrm{SU}_4(2),$$

$$A_2 = (\mathrm{SU}_3(2) \times \mathrm{SU}_2(2)):3 \sim (3^{1+2}_+:Q_8 \times \mathrm{Sym}(3)).3 \text{ and } A_3 = 3^4:\mathrm{Sym}(5).$$

By Lemma 2.7 and Theorem 3.3, $L = A_3$. Hence $P \leq A_1 \cap A_2 = O^2(A_2)$. This locates the example described in (iii). $\qquad\square$

The constellation of subgroups in $X = \mathrm{SU}_4(r^a)$ with $r^a \equiv 2, 5 \pmod 9$ is more exotic than a first look suggests. In the case when $r^a = 2$, we have that $\mathrm{PSU}_4(2) \cong \mathrm{PSp}_4(3)$. So suppose that $r^a > 2$. Then in the monomial type subgroup $(r^a + 1)^3{:}\mathrm{Sym}(4)$, there are three subgroups isomorphic to $3^3{:}\mathrm{Sym}(4)$ each containing $N_G(S)$ call them P_1, P_2, P_3 and note that $P_i \cap P_j = N_G(S)$ whenever $i \neq j$. Let $P \sim 3_+^{1+2}.\mathrm{SL}_2(3)$. Then, up to change of notation, we have $\langle P, P_1 \rangle \cong \langle P, P_2 \rangle \cong \mathrm{PSp}_4(3)$ and $\langle P, P_3 \rangle \cong \mathrm{GU}_3(r^a)$. The embedding of $\mathrm{PSp}_4(3)$ into $\mathrm{SU}_4(r^a)$ stems from the fact that $\mathrm{SU}_4(r^a) \cong \Omega_6^-(r^a)$ and $\mathrm{PSp}_4(3)$ has index 2 in the Weyl group of type E_6. Finally we note that the two subgroups $\langle P, P_1 \rangle$ and $\langle P, P_2 \rangle$ are conjugate in $\mathrm{GU}_4(r^a)$.

Because of the configuration for $\mathrm{PSU}_5(2)$ described in Lemma 7.7 (ii), we need to individually inspect the 3-structure of $\mathrm{PSU}_6(2)$ and $\mathrm{PSU}_7(2)$. We obtain the examples written in lines 9 and 10 of Table 2.

Lemma 7.8. *Suppose that $p = 3$, $X \cong \mathrm{PSU}_6(2)$ or $\mathrm{PSU}_7(2)$ and P is a rank one isolated 3-minimal subgroup in G. Then $X \cong \mathrm{PSU}_6(2)$, $|G/X| \leq 3$, $P \cap X \sim 3^4{:}\mathrm{PSL}_2(9) \sim 3^4{:}\mathrm{Alt}(6)$ and $L \cap X \sim 3_+^{1+4}.(\mathrm{Q}_8 \times \mathrm{Q}_8){:}\mathrm{Sym}(3)$.*

Proof. For $X \cong \mathrm{PSU}_6(2)$, we take the maximal subgroups of X from [4, Table 8.26 and 8.27]. We work in $\widehat{X}$ to add clarity. The maximal subgroups containing $\widehat{S \cap X}$ are $\widehat{A_1} = (\mathrm{SU}_3(2) \wr 2).3$ and three copies of $3{\cdot}\mathrm{PSU}_4(3).2$ which we call $\widehat{A_2}$, $\widehat{A_3}$ and $\widehat{A_4}$. Only A_1 is a 3-local subgroup and so we must have $L \cap X = A_1$ and this is the normalizer of the centre of $S \cap X$. Now from the parabolic subgroups of $\mathrm{PSU}_4(3)$ in characteristic 3 we have one of shape $3^4{:}\mathrm{PSL}_2(9)$ and the other normalizes the centre of a Sylow 3-subgroup and so is contained in L (see [4, Table 8.10]). Thus $P \cap X = 3^4{:}\mathrm{PSL}_2(9)$ and we have the described example. So suppose that $\widehat{X} \cong \mathrm{SU}_7(2)$. Then, as $d_- = 1$, $\widehat{M^*} = \mathrm{GU}_1(2) \wr \mathrm{Sym}(7) = 3 \wr \mathrm{Sym}(7)$ and so $M \sim 3^6{:}\mathrm{Sym}(7)$. Since $\mathrm{Sym}(7)$ has no rank one isolated 3-minimal subgroups by Theorem 3.3, we have $L \geq MS$. Set

$$\widehat{M_1^*} = \mathrm{GU}_6(2) \times \mathrm{GU}_1(2).$$

Then M_1 is normalized by S and $O_3(\langle MS, M_1 \rangle) = 1$. Thus $P \leq M_1 S$. From the $\mathrm{PSU}_6(2)$ example and remembering $Z(E(M_1))$ has order 3, we now read that

$$P \cap E(M_1) \sim 3^5{:}\mathrm{Alt}(6) \leq L,$$

which is a contradiction. $\qquad\square$

The situation when $X \cong \mathrm{PSU}_4(3)$ and $p = 2$ is very special indeed. This group behaves very much like a group of Lie type defined in characteristic 2. Let's consider this group for a moment.

Example 7.9. *Let* $p = 2$, $X \cong \mathrm{PSU}_4(3)$ *and set* $S_0 = S \cap X$. *We refer to* [4, *Tables* 8.10 *and* 8.11] *for the subgroup structure of* X. *So* S_0 *has order* 2^7, $N_G(S_0) = S_0$ *and* $\mathcal{P}_X(S_0) = \{R_1, R_2, R_3\}$ *where* $R_i/Q_{R_i} \cong \mathrm{SL}_2(2)$. *For* $1 \le i < j \le 3$, *set* $R_{ij} = \langle R_i, R_j \rangle$. *Then we may choose notation so that*

$$R_{12} \cong R_{23} \sim 2^4 : \mathrm{Alt}(6)$$

and

$$R_{13} = R_1 R_3 = C_X(Z(S)) \sim 2^{1+4}_+ . (\mathrm{Sym}(3) \times \mathrm{Sym}(3)).$$

The entire lattice of over-groups of S_0 *in* X *is* X, R_{12}, R_{13}, R_{23}, R_1, R_2, R_3 *and* S_0 *just as it would be if* X *was a group of Lie type in characteristic* 2. *Therefore we can select any element of* $\mathcal{P}_X(S_0)$ *to be a rank one isolated* 2-*minimal subgroup of* X.

Now, by the Frattini Argument, $\mathrm{Out}(X) \cong \mathrm{Dih}(8)$ *permutes* $\mathcal{P}_G(S)$. *This action swaps* R_1 *and* R_3 *and so the kernel,* F, *of this action has order* 4. *The subgroup* F *is denoted* $(2^2)_{122}$ *in the* Atlas [12]. *In particular, if* G/X *is a subgroup of* F, *then* $\mathcal{P}_G(S) = \{R_1 S, R_2 S, R_3 S\}$ *and each element of* $P_G(S)$ *is rank one and isolated in* G. *If* G/X *is not a subgroup of* F, *then* $P_G(S) = \{R_{13}S, R_2 S\}$ *and* $R_{13}S$ *is not rank one as* $S/O_2(R_{13}S) \cong \mathrm{Dih}(8)$ *(see Lemma 2.4). Thus in this case there is a unique choice for a rank one isolated* 2-*minimal subgroup,* $R_2 S$, *and a corresponding unique choice for* $L = R_{13}S$. *Finally, we mention that if* $G \cong \mathrm{PGU}_4(3)$, *then* G/X *is cyclic of order* 4 *and thus is not contained in* F. *The coset geometry determined by* $\mathcal{P}_X(S_0)$ *is an example of a GAB* [25].

We have included the possibilities discussed in Example 7.9 in lines 17, 18 and 19 of Table 1.

Lemma 7.10. *Suppose that* $p = 2$, $X \cong \mathrm{PSL}_4^\epsilon(r^a)$ *and* P *is a rank one isolated* 2-*minimal subgroup of* G. *Then* $X \cong \mathrm{PSU}_4(3)$ *and the possibilities for* P, L *and* G *are as described in Example 7.9. In particular, if* G *contains* $\mathrm{PGU}_4(3)$, *then* $P \le MS$.

Proof. Let $X \cong \mathrm{PSL}_4^\epsilon(r^a)$. Suppose that $r^a = 3$. If $X \cong \mathrm{PSL}_4(3)$, then, as $Q_L \cap X \ne 1$ using [4, Tables 8.8 and 8.9], we see that either G is 2-minimal or there is no maximal subgroup of G containing S which is a candidate for L, a contradiction. Thus $X \cong \mathrm{PSU}_4(3)$ and the possibilities for L and P have been displayed in Example 7.9. So we assume that $r^a > 3$.

Assume that $d_\epsilon = 1$. Then

$$\widehat{M}^* = \mathrm{GL}_1^\epsilon(r^a) \wr \mathrm{Sym}(4) = (r^a - \epsilon) \wr \mathrm{Sym}(4).$$

Set

$$\widehat{M_1}^* = \mathrm{GL}_2^\epsilon(r^a) \wr \mathrm{Sym}(2)$$

is normalized by $\widehat{S}$. Using Theorem 2.15 and $r^a > 3$, we obtain $P \not\le E(M_1)S$ and so $L \ge E(M_1)S$.

Since $O_2(\langle MS, E(M_1)S \rangle) = 1$, L does not contain MS and so $P \le MS$. If $\iota \notin S$, then $Q_L \le Q_{M_1} \le Q_{MS} \le Q_P$, which is impossible. Hence $\iota \in S$ and we can find $\iota k \in Q_L$ where k corresponding to $\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \in \widehat{M_1}$. Let N be the normal subgroup of MS with $MS/N \cong \mathrm{Sym}(4)$ and notice $\iota k F(M) \in O_2(MS/N)$ as kN maps to a product of two

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transpositions. Since MS is soluble and $P \leq MS$, we have $O^p(P)$ is a normal subgroup of MS by Theorem 2.15. If $O^p(P) \leq N$, then $N \leq F(M)$ and so $O^p(P) \leq M_1 S$, which is impossible. Hence $PN/N = MS/N \cong \mathrm{Sym}(4)$. Since P is rank 1, it follows that $P/Q_P \cong \mathrm{Sym}(3)$. Because $\iota k \in S \cap Q_P F(M) = Q_P$, we have a contradiction. Therefore $d_\epsilon = 2$.

Because $d_\epsilon = 2$,

$$\widehat{M}^* = \mathrm{GL}_2^\epsilon(r^a) \wr \mathrm{Sym}(2)$$

and, as $r^a > 3$ and $L \geq E(M)S$ by Theorem 2.15. Since $d_\epsilon = 2$, 4 does not divide $r^a - \epsilon$. In particular, a is odd and so $\mathrm{Out}(X)$ has order 4. Notice that $1 \neq Q_L \cap X \leq O_2(E(M)S) \cap X$ which has order 2 and so $L \leq C_G(O_2(E(M)S) \cap X) = MS$. Hence $L = MS$. If $\iota \notin S$, then $Q_L = O_2(Z(MS))$ has order 2 and this contradicts Theorem 2.16 (iii). Hence $\iota \in S$ and, taking $k \in \widehat{M}$ as above, we see that Q_L has order 4 and contains the image of ιk. However,

$$F(\widehat{L}) \geq F(\widehat{M} \cap \widehat{X}) = \{\mathrm{diag}(\lambda, \lambda, \mu, \mu) \mid \lambda, \mu \in \mathrm{GL}_1^\epsilon(r^a), \lambda^2 \mu^2 = 1\}.$$

As conjugation by ιk inverts every element of this subgroup, we deduce that $F(M \cap X)$ is a 2-group, since $d_\epsilon = 2$ this means that $r^a - \epsilon = 2$ which is impossible as $r^a > 3$. This completes the proof of Lemma 7.10. $\qquad\square$

The next lemma provides the final example of an isolated p-minimal subgroups of a linear or unitary group and appears in line 20 of Table 1.

Lemma 7.11. *Suppose that $p = 2$, $X \cong \mathrm{PSU}_n(3)$ with $n \in \{5, 6, 7\}$ and P is a rank one isolated 2-minimal subgroup of G. Then $X \cong \mathrm{PSU}_6(3)$, $\widehat{L}$ contains $\mathrm{GU}_2(3) \wr \mathrm{Sym}(3) \cap \widehat{X}$ and $P \leq MS$. Furthermore, $\widehat{P} \cap \widehat{X} \sim 4^5.(\mathrm{Sym}(4) \times 2)$.*

Proof. First suppose that $X \cong \mathrm{PSU}_5(3)$. Then

$$\widehat{M}^* = \mathrm{GU}_1(3) \wr \mathrm{Sym}(5) = 4 \wr \mathrm{Sym}(5)$$

which is itself a 2-minimal group. Thus either $MS \leq L$ or $P = MS$. Let

$$\widehat{M_1}^* = \mathrm{GU}_4(3) \times \mathrm{GU}_1(3) = \mathrm{GU}_4(3) \times 4.$$

Then $M_1 S \cap X \cong \mathrm{GU}_4(3)$. If $L \geq M_1$, then, as $O_2(\langle MS, M_1 S \rangle) = 1$, $P = MS$ and we obtain $Q_L \leq Q_M \leq Q_P$, a contradiction. Therefore, $P \leq M_1 S$ and $L \geq M$. However, since $M_1/O_2(M_1) \cong \mathrm{PGU}_4(3)$, applying Lemma 7.10 we obtain $P \leq MS \leq L$, a contradiction.

Next consider $X \cong \mathrm{PSU}_6(3)$. Then

$$\widehat{M}^* = \mathrm{GU}_1(3) \wr \mathrm{Sym}(6) = 4 \wr \mathrm{Sym}(6).$$

We set

$$\widehat{M_1}^* = \mathrm{GU}_4(3) \times \mathrm{GU}_2(3),$$
$$\widehat{M_2}^* = \mathrm{GU}_2(3) \wr \mathrm{Sym}(3)$$

and note that both M_1 and M_2 are S-invariant. If $P \leq M_2S$, then P is soluble and Theorem 2.15 implies that $O^2(P)$ is normal in M_2S and $O^2(P) \leq F_2^*(M_2S)$. We have $F_2^*(M_2S)/Q_{M_2S}$ is elementary abelian of order 3^3 and is a minimal normal subgroup of M_2S/Q_{MS}. Therefore $O^2(P) = F_2^*(M_2S)$ which is a contradiction as $O^2(P)$ has 3-rank at most 2 by Lemma 2.4. Hence $P \not\leq M_2S$. Since $O_2(\langle M_2S, MS\rangle) = 1$, we have $P \leq MS$ and similarly $P \leq M_1S$. This gives the configuration described in the lemma.

Suppose finally that $X \cong \mathrm{PSU}_7(3)$. Then

$$\widehat{M}^* = \mathrm{GU}_1(3) \wr \mathrm{Sym}(7) = 4 \wr \mathrm{Sym}(7).$$

Since $\mathrm{Sym}(7)$ has no rank one isolated 2-minimal subgroups, using Lemma 2.7 gives $L \geq MS$. Defining

$$\widehat{M_1}^* = \mathrm{GU}_6(3) \times \mathrm{GU}_1(3),$$

and noting that $O_2(\langle M_1S, MS\rangle) = 1$ yields that P is an isolated p-minimal subgroup of M_1S and we may apply our results for $\mathrm{PSU}_6(3)$. Therefore, $P \leq (M_1 \cap M)S \leq L$, a contradiction. $\qquad\square$

Theorem 7.12. *Suppose that $X \cong \mathrm{PSL}_n^\epsilon(r^a)$ and one of the following conditions hold.*

    (i) $p = 2$, $n \geq 5$ and either $X \cong \mathrm{PSL}_n(r^a)$ or $X \cong \mathrm{PSU}_n(r^a)$ with $r^a \neq 3$;
    (ii) $p = 2$, $n \geq 8$ and $X \cong \mathrm{PSU}_n(3)$;
    (iii) $p = 3$, $n \geq 6$ and either $X \cong \mathrm{PSL}_n(r^a)$ or $X \cong \mathrm{PSU}_n(r^a)$ with $r^a \neq 2$;
    (iv) $p = 3$, $n \geq 8$ and $X \cong \mathrm{PSU}_n(2)$; or
    (v) $p \geq 5$ and $n > d_\epsilon p$.

Then G does not contain a rank one isolated p-minimal subgroup.

Proof. Suppose that the theorem is false and let $P \leq G$ be a rank one isolated p-minimal subgroup containing S. Furthermore, assume that n is defined as in the statement of the theorem and then is chosen minimally so that G is a counter example to the theorem. Because of Lemmas 7.4, 7.7, 7.8 7.10, and 7.11, we always have that groups $\mathrm{PGL}_{n-1}^\epsilon(r^a)$ have no rank one isolated p-minimal subgroups.

Recall that

$$\widehat{M}^* = \mathrm{GL}_{d_\epsilon}^\epsilon(r^a) \wr \mathrm{Sym}(s) \times \mathrm{GL}_{n-d_\epsilon s}^\epsilon(r^a)$$

where $s = \lfloor \frac{n}{d_\epsilon} \rfloor$ and define

$$\widehat{K}^* = \mathrm{SL}_{d_\epsilon}^\epsilon(r^a)T \wr \mathrm{Sym}(s)$$

where $T \in \mathrm{Syl}_2(\mathrm{GL}_{d_\epsilon}^\epsilon(r^a))$. Then both M and K are normalized by S.

(7.12.1) *We have $d_\epsilon s = n$.*

Assume that $d_\epsilon s < n$. Then plainly $d_\epsilon > 1$ and so $\widehat{S} \cap \widehat{X}$ leaves invariant an $(n - d_\epsilon s)$-dimensional subspace of V. In particular, if p is odd it centralizes a 1-dimensional subspace W of V and if $p = 2$ it negates such a subspace.

Suppose for a moment that $\epsilon = -$ and W contains an isotropic vector, then $S \cap X$ is contained in a parabolic subgroup of X and we have a contradiction via Lemma 4.6. Therefore W contains no isotropic vectors when $\epsilon = -$. Setting

$$\widehat{M_1^*} = \mathrm{GL}_{n-1}^\epsilon(r^a) \times \mathrm{GL}_1^\epsilon(r^a) = \mathrm{GL}_{n-1}^\epsilon(r^a) \times (r^a - \epsilon)$$

(fixing W), $S \cap X \leq M_1$ and M_1 is normalized by S. Because of the choice of $n, E(M_1)S$ does not possess a rank one isolated p-minimal subgroup. Hence $L \geq E(M_1)S$. Since $Q_L \cap X > 1$, we infer that when p is odd, p divides $r^a - \epsilon$ contrary to $d_\epsilon > 1$. Hence $p = 2$ and, as $d_\epsilon \neq 1$, in this case Q_L has order 2. Therefore Theorem 2.16 (iii) applies to give $O^2(P)$ normal in G, a contradiction. ∎

(7.12.2) *We have $s > p$ unless $d_\epsilon = 2$, $p = 3$ and $n = 6$.*

If $s \leq p$, then $n = d_\epsilon s \leq d_\epsilon p$ by (7.12.1), whereas hypotheses (i) to (v) assert $n > d_\epsilon p$ unless $d_\epsilon = 2$, $p = 3$ and $n = 6$. ∎

(7.12.3) *If $d_\epsilon > 1$, then $L \geq MS$ and $d_\epsilon = 2$. Furthermore, if p is odd, then $p = 3$, $r^a = 2$ and $\epsilon = +$.*

Suppose that $L \geq MS$. If $p = 2$, then, as $d_\epsilon \neq 1$, we have $d_\epsilon = 2$. Suppose that p is odd. Then, as $Q_L \cap X > 1$, $Q_L \cap X \leq F(MS \cap X)$ which has order dividing $(r^a - \epsilon)^s$ unless $p = 3$ and $r^a = 2$. Thus, as $d_\epsilon \neq 1$, $p = 3$, $r^a = 2$ and we have $d_\epsilon = 2$ and $\epsilon = +$.

It remains to show that $L \geq MS$. So for a contradiction assume the contrary. Then $P \leq MS$. If $\mathrm{GL}_{d_\epsilon}^\epsilon(r^a)$ is not soluble, then, as $s \geq p$ by (7.12.2), Theorem 2.15(i) applied to K implies that $L \geq KS \geq E(M)S$. In particular,

$$1 \neq Q_L \cap X \leq Q_K \cap X \leq C_K(E(M)).$$

As $|C_K(E(M))|$ divides $|C_M(E(M))| = (r^a - \epsilon)^s$ and $d_\epsilon > 1$, we have $p = 2$ and $C_K(E(M)) = Z(E(M))$. Since $P \leq MS$, Theorem 2.15(i) implies $O^p(P)$ is normal in MS and so $O^p(P)$ commutes with $Q_L \cap X$, a contradiction as then this group is normal in G. Therefore $\mathrm{GL}_{d_\epsilon}^\epsilon(r^a)$ is soluble and so $r^a \in \{2, 3\}$ and $d_\epsilon = 2$. Notice that if $r^a = 2$, we must have $p = 3$ and if $r^a = 3$, we must have $p = 2$. Furthermore, we have $\epsilon = +$. So now

$$\widehat{M^*} = \begin{cases} \mathrm{GL}_2(2) \wr \mathrm{Sym}(n/2) & \text{when } r^a = 2 \\ \mathrm{GL}_2(3) \wr \mathrm{Sym}(n/2) & \text{when } r^a = 3. \end{cases}$$

In the first case we notice that an element ι^* of $X\iota$ centralizes M. Thus $P \leq M \leq C_X(\iota^*) \cong \mathrm{Sp}_n(2)$. Theorem 6.3 then implies $n = 4$, a contradiction.

Thus $r^a = 3$, $p = 2$ and $\epsilon = +$. In particular, $\widehat{G^*} \cong \mathrm{GL}_n(3)$ and

$$\widehat{M^*}/O_2(\widehat{M^*}) \sim \mathrm{Sym}(3) \wr \mathrm{Sym}(n/2).$$

Furthermore, as n is even, hypothesis (i) means that $n \geq 6$. Hence $F_2^*(MS)/Q_{MS}$ is elementary abelian of order $3^{n/2}$ and MS acts irreducibly on this quotient. Since $O^2(P) \leq$

$F_2^*(MS)$ by Theorem 2.15 (ii), this contradicts Lemma 2.4. ∎

(7.12.4)   *If $d_\epsilon = 1$, then $MS \leq L$.*

Suppose that L does not contain MS. Then $P \leq MS$. Let N be the largest normal soluble subgroup of MS. By hypothesis, $n \geq 5$ and so $MS/N \cong \mathrm{Sym}(n)$. Furthermore, Lemma 2.6 (i) implies that one of the following holds:

(A) $O^p(P) \leq N$; or
(B) PN/N is a rank one isolated p-minimal subgroup of MS/N.

Suppose that p is odd. If (A) holds, then, as $O^p(P) \leq N$ and $G = XS$, we infer that $O^p(P) \leq N \cap X \leq F(M)$ and therefore $O^p(P)$ is abelian. Thus Lemma 2.4 implies that $p = 3$ and $O^3(P)/O_3(O^3(P))$ has order 2^2. In addition, Theorem 2.15 (i) yields $O^3(P)$ is normal in MS. As $O^3(P)Q_{MS}/Q_{MS}$ is not cyclic, Lemma 2.19 now implies that $n \leq 4$, a contradiction.

Suppose that (B) holds. Then, since $n \geq 5$ and p is odd, Theorem 3.3 implies that $MS/N \cong \mathrm{Sym}(5)$ ($\mathrm{Alt}(5) \cong \mathrm{PSL}_2(5)$, $p = 5$) or $\mathrm{Sym}(6)$ ($\mathrm{Alt}(6) \cong \mathrm{PSL}_2(9)$, $p = 3$). If $p = 5$, then $n = 5$ and so $n \leq d_\epsilon p$, against assumption (v). So $p \neq 5$ and we have $p = 3$ with $MS/N \cong \mathrm{Sym}(6)$. Therefore, $X \cong \mathrm{PSL}_6^\epsilon(r^a)$. Put

$$\widehat{M_1^*} = \mathrm{GL}_3^\epsilon(r^a) \wr \mathrm{Sym}(2)$$

and notice that M_1 is normalized by S. By hypothesis (iii), $X \not\cong \mathrm{PSU}_6(2)$ and so $\mathrm{GL}_3^\epsilon(r^a)$ is not soluble. Therefore, as $L \geq E(M_1)S$ by Theorem 2.15(i), we obtain $Q_L \leq Q_{M_1 S} \leq Q_{MS} \leq Q_P$, a contradiction. This shows that $p = 2$.

Suppose that n is odd and consider

$$\widehat{M_2^*} = \mathrm{GL}_{n-1}^\epsilon(r^a) \times \mathrm{GL}_1^\epsilon(r^a) = \mathrm{GL}_{n-1}^\epsilon(r^a) \times (r^a - \epsilon).$$

Then M_2 is S-invariant and $M_2 S$ has no isolated 2-minimal subgroups by the minimal choice of n. Hence $L \geq M_2 S$. But then we again have $Q_L \leq Q_P$, which is impossible. Thus n is even. Set

$$\widehat{M_1^*} = \mathrm{GL}_2^\epsilon(r^a) \wr \mathrm{Sym}(n/2)$$

and

$$\widehat{K_1^*} = \mathrm{SL}_2(r^a)T_1 \wr \mathrm{Sym}(n/2)$$

where $T_1 \in \mathrm{Syl}_2(\mathrm{GL}_2^\epsilon(r^a))$.

Then $\widehat{K_1^*}$ is normalized by $\widehat{S}$. If $\mathrm{SL}_2(r^a)$ is not soluble, then Theorem 2.15 gives $L \geq K_1 S$. As $O_2(\langle K, K_1 \rangle) = 1$, $K \not\leq L$. Therefore $P \leq K$. Since $K/Q_K \cong \mathrm{Sym}(n)$, we have P/Q_M is a rank one isolated 2-minimal subgroup of K/Q_K by Lemma 2.7. By Theorem 3.3 $n \in \{6, 8, 12\}$ as n is even. Furthermore, we have $(\widehat{K_1^*} \cap \widehat{K^*})/Q_{\widehat{K^*}} \cong 2 \wr \mathrm{Sym}(n/2)$. Referring to Example 3.2, we see that $n \neq 12$. If $n = 8$, then $P/Q_K \cong \mathrm{Sym}(4) \wr 2$, but this group is not rank one. Hence $n = 6$ and $P/Q_K \cong \mathrm{Sym}(4) \times 2$. It follows that $P \leq M_2 S$ where $M_2^* \cong \mathrm{GL}_4^\epsilon(r^a) \times \mathrm{GL}_2^\epsilon(r^a)$. Thus P is a rank one isolated 2-minimal subgroup of $\mathrm{SL}_4^\epsilon(r^a)S$ and this contradicts Lemma 7.10 as we assumed $\mathrm{GL}_2^\epsilon(r^a)$ was non-soluble (whereas we have just proved $r^a = 3$).

Thus $\mathrm{GL}_2^\epsilon(r^a)$ is soluble and so $r^a = 3$. Since $p = 2$, we have $\epsilon = -$. So

$$\widehat{M_1^*} = \mathrm{GU}_2(3) \wr \mathrm{Sym}(n/2)$$

and, by hypothesis, $n \geq 8$. Since $N = F(MS)$ is a 2-group, we now have PN/N is a rank one isolated 2-minimal subgroup of $MS/N \cong \mathrm{Sym}(n)$ with $n \geq 8$. Theorem 3.3 states that $n \in \{8, 12\}$.

Suppose that $n = 8$. Then as PN/N is rank one, we have $PN/N \cong 2 \wr \mathrm{Sym}(4)$. Let

$$\widehat{M_2^*} = \mathrm{GU}_4(3) \wr \mathrm{Sym}(2).$$

Then, as we know the structure of P/N in MS/N, we have $P \not\leq M_2S$. Thus $L \geq M_2S$ and $Q_L \leq Q_{M_2S} \leq Q_P$, a contradiction. For $n = 12$, we ponder the S-invariant subgroup

$$\widehat{M_3^*} \cong \mathrm{GU}_8(3) \times \mathrm{GU}_4(3).$$

By Theorem 3.3, $O^2(P)$ is contained in the left hand factor of M_3S. Since this factor has no rank one isolated 2-minimal subgroups by the minimality of n, we have a contradiction. Thus (7.12.4) holds. ∎

(7.12.5) *We have p does not divide s.*

Suppose that p divides s and set

$$\widehat{M_1}^* = \mathrm{GL}_{d_\epsilon p}^\epsilon(r^a) \wr \mathrm{Sym}(s/p)$$

and

$$\widehat{K_1}^* = \mathrm{SL}_{d_\epsilon p}^\epsilon(r^a)T_1 \wr \mathrm{Sym}(s/p)$$

where $T_1 \in \mathrm{Syl}_p(\mathrm{GL}_2^\epsilon(r^a))$. Then M_1 and K_1 are S-invariant and $O_p(\langle MS, E(K_1)S\rangle) = 1$. In particular, as $L \geq MS$ by (7.12.3) and (7.12.4), we have $P \leq E(M_1)S \leq K_1S$. Thus Theorem 2.15(i) applied to K_1S implies that either $p = s$ or $\mathrm{GL}_{d_\epsilon p}^\epsilon(r^a)$ is soluble. If $p = s$, then (7.12.2) gives $d_\epsilon = 2$, $p = 3$ and $n = 6$. Applying (7.12.3) again yields $G = \mathrm{SL}_6(2)$ and $M = \mathrm{SL}_2(2) \wr \mathrm{Sym}(3)$. Since $S \leq M \leq H \cong \mathrm{Sp}_6(2)$ and $O_3(H) = 1$, we have $P \leq H$ contrary to Theorem 6.3. Thus $\mathrm{GL}_{d_\epsilon p}^\epsilon(r^a)$ is soluble and so $d_\epsilon = 1$ and $p \leq 3$ with $r^a \in \{2, 3\}$. If $p = 2$, then $r^a = 3$ and $\epsilon = -$. If $p = 3$, then $r^a = 2$ and $\epsilon = -$. In particular, by the hypothesis (ii) and (iv), we have $n \geq 8$.

Suppose the first scenario occurs. Then, explicitly, we have

$$\widehat{M}^* = \mathrm{GU}_1(3) \wr \mathrm{Sym}(n) = 4 \wr \mathrm{Sym}(n)$$

and

$$\widehat{M_1}^* = \mathrm{GU}_2(3) \wr \mathrm{Sym}(n/2).$$

If n is divisible by 4, then

$$\widehat{M_2}^* = \mathrm{GU}_4(3) \wr \mathrm{Sym}(n/4)$$

is also normalized by $\widehat{S}$. As $O_2(\langle MS, E(M_2)S\rangle) = 1$ and $L \geq MS$, we must have $P \leq E(M_2)S$ and this contradicts Theorem 2.15(i). Therefore, $n = m + 2$ with m divisible by 4 and, as a consequence,

$$\widehat{M_3}^* = \mathrm{GU}_m(3) \times \mathrm{GU}_2(3)$$

is normalized by $\widehat{S}$. Let K_3 be the component of M_3. Then, since $n \geq 10$, $m \geq 8$ and so by minimality $P \not\leq K_3 S$. Hence $L \geq \langle K_3 S, MS \rangle$ which is absurd as $O_2(\langle K_3 S, MS \rangle) = 1$. So this configuration shrivels and therefore $p = 3$, $r^a = 2$ and $\epsilon = -$.

Assume now that $p = 3$, $r^a = 2$ and $\epsilon = -$. In this case, as 3 divides $sd_\epsilon = n$, we have $n \geq 9$ by hypothesis. We first eliminate the smallest case. Thus $n = 9$ and $\widehat{M}^* = \mathrm{GU}_1(2) \wr \mathrm{Sym}(9) = 3 \wr \mathrm{Sym}(9)$ and $\widehat{M_1}^* = \mathrm{GU}_3(2) \wr \mathrm{Sym}(3)$. As $P \leq M_1$ is soluble, M_1/Q_{M_1} must normalize $O^3(P)Q_{M_1}/Q_{M_1}$ which has order either 4 or 8. We calculate that $O^3(P)Q_{M_1}/Q_{M_1} \leq Z(F_3^*(M_1)/Q_{M_1})$ which is elementary abelian of order 8. However, the preimage of $Z(F_3^*(M_1)/Q_{M_1})$ is contained in $M \leq L$ (as $\widehat{M}^* \cap \widehat{M_1}^* = 3 \wr \mathrm{Sym}(3) \wr \mathrm{Sym}(3)$) and this is impossible. Thus $n \neq 9$. We now argue as in the first case. If n is divisible by 9, we set $\widehat{M_2}^* = \mathrm{GU}_9(2) \wr \mathrm{Sym}(n/9)$. Then $E(M_2)S \leq L$ whereas $O_3(\langle MS, E(M_2)S\rangle) = 1$, a contradiction. So we have $n = 9k + 3\ell$ where $\ell \in \{1, 2\}$. Set

$$\widehat{M_3}^* = \begin{cases} \mathrm{GU}_{n-3}(2) \times \mathrm{GU}_3(2) & \ell = 1 \\ \mathrm{GU}_{n-6}(2) \times \mathrm{GU}_3(2) \times \mathrm{GU}_3(2) & \ell = 2 \end{cases}$$

and let $K_3 = E(M_3)$. Then by minimality $P \not\leq K_3 S$ and so $K_3 S \leq L$. But then $Q_L \leq O_3(\langle K_3, MS\rangle) = 1$ a contradiction. This completes the proof of (7.12.5). ∎

We can now wrap up the proof of the theorem. By (7.12.5), p does not divide s and as, by (7.12.2) $s \geq p$, we can write

$$s = jp + k$$

with $1 \leq k \leq p - 1$ and $j > 0$. Let

$$\widehat{M_1}^* = \mathrm{GL}_{d_\epsilon p}^\epsilon(r^a) \wr \mathrm{Sym}(j) \times \mathrm{GL}_{d_\epsilon k}^\epsilon(r^a)$$

and let $\widehat{J_1}^*$ be the first factor and $\widehat{J_2}^*$ the second factor of this expression. These subgroups are normalized by $\widehat{S}$. If $k > 1$, then $O_p(\langle J_1 S, MS\rangle) = O_p(\langle J_2 S, MS\rangle) = 1$ and so Lemma 2.11 implies that $P \leq MS \leq L$, which is of course not the case. Hence $k = 1$ and $J_2 \leq MS$. Suppose that $\mathrm{GL}_{d_\epsilon p}^\epsilon(r^a)$ is soluble. Then $p \in \{2, 3\}$ and $d_\epsilon = 1$. Furthermore, as $k = 1$, p divides $n - 1$, and so

$$\widehat{M_2}^* = \mathrm{GL}_{n-1}^\epsilon(r^a) \times (r^a - \epsilon)$$

is normalized by $\widehat{S}$. But then, as $O_p(\langle MS, M_2\rangle) = 1$, P is a rank one isolated p-minimal subgroup of $M_2 S$, contrary to the minimal choice of n. Hence $\mathrm{GL}_{d_\epsilon p}^\epsilon(r^a)$ is not soluble.

Since $O_p(\langle E(J_1), MS\rangle) = 1$, $P \leq E(J_1)S$ and so Theorem 2.15(i) implies $j = 1$. Hence $s = p + 1$ and

$$\widehat{M_1}^* = \mathrm{GL}_{d_\epsilon p}^\epsilon(r^a) \times \mathrm{GL}_{d_\epsilon}^\epsilon(r^a).$$

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In particular, we have P is a rank one isolated p-minimal subgroup of $J_1 S$. If $p \geq 5$, then $\mathrm{GL}^\epsilon_{d_\epsilon p}(r^a)$ has no rank one isolated p-minimal subgroups by Lemma 7.4. Hence $p \leq 3$. Since $n \geq 5$, $d_\epsilon = 2$.

If $p = 3$, then $r^a = 2$ and $\epsilon = +$ by (7.12.3). Hence $\widehat{J_1} \cong \mathrm{GL}_6(2)$ and this has no rank one isolated 3-minimal subgroups by the minimal choice of n. So $p \neq 3$. Finally suppose that $p = 2$. Then $d_\epsilon p = 4$ and $\epsilon = -$. Hence $\widehat{J_1} = \mathrm{GU}_4(3)$ by Lemma 7.10. But then $n = 6$, is not in the permitted range for the theorem. This concludes the proof of the theorem. $\square$

The proof of Theorem 7.1. Lemmas 7.4, 7.5, 7.6, 7.7, 7.8, 7.10 and 7.11 provide the list of examples tabulated in Table 1 and Table 2 as well as setting the basis for the inductive proof showing that there are no further examples as is presented in Theorem 7.12. Together these statements prove Theorem 7.1. $\square$

8. Projective orthogonal groups

We conclude our investigation of the classical groups by investigating the orthogonal groups. So suppose that r is a prime and that V is a vector space over $\mathrm{GF}(r^a)$ of dimension m equipped with a non-degenerate quadratic form $\mathcal{Q}$ with associated non-degenerate symmetric bilinear form f and assume that the type of the quadratic form is $\epsilon \in \{\pm, 0\}$, 0 indicating that $m = 2n + 1$ is odd. If $(V, \mathcal{Q})$ is a quadratic space, we say that V has type ϵ whenever $\mathcal{Q}$ has type ϵ and typically we suppress mention of $\mathcal{Q}$. When m is even we shall write $m = 2n$. Our standard references for facts about the orthogonal groups are [1, Chapter 7] and [30, Chapter 2].

In particular, we remark that the vectors $v \in V$ with $Q(v) = 0$ are called *singular vectors* and the vectors with $Q(v) \neq 0$ are called *non-singular*. If r is odd and $v \in V$ is non-singular, then either $Q(w)$ is a square for all $w \in \langle v \rangle$ or a non-square for all $w \in \langle v \rangle$. In the former case we say that v has $+$-*type* and in the latter that it has $-$-*type*. If r is even, then all non-singular vectors are of $+$-type. A subspace U of V is called *singular* provided it is isotropic with respect to the bilinear form and every vector in U is singular.

Suppose that $(V, \mathcal{Q})$ has type ϵ. In this section we will be concerned with the following groups:

$$\mathrm{CO}^\epsilon_m(r^a) = \{ X \in \mathrm{GL}_m(r^a) \mid \mathcal{Q}^X = \lambda_X \mathcal{Q} \text{ for some } \lambda_X \in \mathrm{GF}(r^a) \},$$

$$\mathrm{O}^\epsilon_m(r^a) = \{ X \in \mathrm{GL}_m(r^a) \mid \mathcal{Q}^X = \mathcal{Q} \},$$

and

$$\mathrm{SO}^\epsilon_m(r^a) = \{ X \in \mathrm{O}^\epsilon_m(r^a) \mid \det X = 1 \}.$$

If r is odd, then we define $\Omega^\epsilon_m(r^a)$ to be the kernel in $\mathrm{SO}^\epsilon_m(r^a)$ of the spinor norm and if $r = 2$ we define $\Omega^\epsilon_m(r^a)$ to be the kernel of the quasideterminant. For more details about these latter two definitions see [12, page xi] or [30, pages 29, 30 and 31]. The group of all semilinear automorphisms of V that preserve $\mathcal{Q}$ up to semi-similarity is denoted by $\mathrm{C\Gamma O}^\epsilon_m(r^a)$ as in [4, Definition 1.6.4].

We have the following indices

$$
\begin{aligned}
|\mathrm{C\Gamma O}_m^\epsilon(r^a)\colon \mathrm{CO}_m^\epsilon(r^a)| &= a, \\
|\mathrm{CO}_m^\epsilon(r^a)\colon \mathrm{O}_m^\epsilon(r^a)| &= r^a - 1, \\
|\mathrm{O}_m^\epsilon(r^a)\colon \mathrm{SO}_m^\epsilon(r^a)| &= \begin{cases} 2 & r \text{ odd} \\ 1 & r = 2 \end{cases} \text{ and} \\
|\mathrm{SO}_m^\epsilon(r^a)\colon \Omega_m^\epsilon(r^a)| &= 2.
\end{aligned}
$$

Now we define the projective groups by factoring $Z(\mathrm{CO}_m^\epsilon(r^a))$ which is cyclic of order $r^a - 1$ and we obtain

$$
\mathrm{PC\Gamma O}_m^\epsilon(r^a) \geq \mathrm{PCO}_m^\epsilon(r^a) \geq \mathrm{PO}_m^\epsilon(r^a) \geq \mathrm{PSO}_m^\epsilon(r^a) \geq \mathrm{P\Omega}_m^\epsilon(r^a).
$$

It is easy to see that $|\mathrm{PCO}_m^\epsilon(r^a)/\mathrm{P\Omega}_m^\epsilon(r^a)|$ is a 2-group of order at most 8 and the quotient $\mathrm{PC\Gamma O}_m^\epsilon(r^a)/\mathrm{PCO}_m^\epsilon(r^a)$ is cyclic of order a (see [12, page xi]). Note that if $r = 2$, then $\mathrm{PCO}_m^\epsilon(r^a) = \mathrm{PO}_m^\epsilon(r^a)$. For $m \geq 3$, we have that $\mathrm{PC\Gamma O}_m^\epsilon(r^a)$ is isomorphic to $\mathrm{Aut}(\Omega_m^\epsilon(r^a))$ unless $m = 8$ and $\epsilon = +$.

To fully describe the structure of $\mathrm{PCO}_m^\epsilon(r^a)/\mathrm{P\Omega}_m^\epsilon(r^a)$ when r is odd, we need the notion of the discriminant of $\mathcal{Q}$. We follow [30, page 32]. For $\beta = (v_1, \ldots, v_m)$ a basis of V, let f_β be the matrix $(f(v_i, v_j))$. We say that the *discriminant* $D(\mathcal{Q})$ of $\mathcal{Q}$ is a square if $\det(f_\beta)$ is a square for some, and so any, choice of basis β of V in $\mathrm{GF}(r^a)$ and otherwise it is a non-square. Notice that if r is even, then $D(\mathcal{Q})$ is a square. An elementary calculation shows that the discriminant is an invariant of the form (see [30, (2.5.14)]).

Since our primary interest will be Hypothesis 2.18 with $p \neq r$, we will only be interested in $\mathrm{PCO}_m^\epsilon(r^a)$ when r is odd and $p = 2$. In this case we have

$$
\mathrm{PCO}_{2n}^\epsilon(r^a)/\mathrm{P\Omega}_{2n}^\epsilon(r^a) \cong \begin{cases} \mathrm{Dih}(8) & D(\mathcal{Q}) \text{ a square} \\ 2 \times 2 & D(\mathcal{Q}) \text{ not a square} \end{cases}
$$

by [30, Propositions 2.7.3 and 2.8.2] and $|\mathrm{PCO}_{2n+1}(r^a)/\mathrm{P\Omega}_{2n+1}(r^a)| = 2$ by [30, Proposition 2.6.3].

Lemma 8.1. *Suppose that $m = 2n$ and that r is odd. Then*

(i) If $\epsilon = +$, then $D(\mathcal{Q})$ is a square if and only if $\frac{n(r^a - 1)}{2}$ is even.

(ii) If $\epsilon = -$, then $D(\mathcal{Q})$ is a square if and only if $\frac{n(r^a - 1)}{2}$ is odd.

Proof. This is [30, 2.5.10]. $\qquad\square$

Since it is important for the configurations in Examples 8.9, we remark further that

Lemma 8.2. *For r an odd prime we have $\mathrm{PCO}_4^+(r^a)/\mathrm{P\Omega}_4^+(r^a) \cong \mathrm{PCO}_8^+(r^a)/\mathrm{P\Omega}_8^+(r^a) \cong \mathrm{PCO}_{12}^+(r^a)/\mathrm{P\Omega}_{12}^+(r^a) \cong \mathrm{Dih}(8)$.* $\qquad\square$

We also mention that $\mathrm{CO}_{2n}^{\epsilon}(r^a)$ acts transitively on non-singular vectors of V and this means that it fuses the plus and minus points when r is odd. We also recall that the 2-dimensional orthogonal space of minus type is characterized as being the only orthogonal space which contains no non-zero singular vectors [1]. It is often called *anisotropic* or *definite*. The 2-dimensional orthogonal space of plus type is called *hyperbolic*. We have $\mathrm{O}_2^{\epsilon}(r^a)$ is a dihedral group of order $2(r^a - \epsilon)$.

We also require some explicit information about $\mathrm{CO}_4^+(3)$. So we recount some general facts about $\mathrm{CO}_4^+(r^a)$ which we present as an example.

Example 8.3. *Suppose that r is a prime and let $\{e_1, e_2, f_1, f_2\}$ be a basis for the 4-dimensional space V of $+$-type with quadratic form given by the companion matrix $A = \left(\begin{smallmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{smallmatrix}\right)$ defined over $\mathrm{GF}(r^a)$. We set $\rho = \left(\begin{smallmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right)$ and calculate that*

$$L_1 = \left\langle \left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ \mu & 1 & 0 & 0 \\ 0 & 0 & 1 & -\mu \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & \mu & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\mu & 1 \end{smallmatrix}\right) \mid \mu \in \mathrm{GF}(r^a) \right\rangle \cong \mathrm{SL}_2(r^a),$$

$$L_2 = \left\langle \left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \mu & 1 & 0 \\ -\mu & 0 & 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 & 0 & -\mu \\ 0 & 1 & \mu & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right) \mid \mu \in \mathrm{GF}(r^a) \right\rangle \cong \mathrm{SL}_2(r^a),$$

$$\Omega_4^+(r^a) = L_1 * L_2,$$

and, when $r^a \geq 4$, L_1 and L_2 are components of $\mathrm{CO}_4^+(r^a)$. Now suppose that r is odd. Then

$$\mathrm{SO}_4^+(r^a) = \left\langle \Omega_4^+(r^a), \left(\begin{smallmatrix} \lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right) \right\rangle,$$

$$\mathrm{O}_4^+(r^a) = \left\langle \mathrm{SO}_4^+(r^a), \rho \right\rangle \text{ and}$$

$$\mathrm{CO}_4^+(r^a) = \left\langle \mathrm{O}_4^+(r^a), \left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right) \right\rangle$$

where λ is a primitive element of $\mathrm{GF}(r^a)$. We note that the two normal subgroups isomorphic to $\mathrm{SL}_2(r^a)$ in $\Omega_4^+(r^a)$ are exchanged in $\mathrm{O}_4^+(r^a)$ and are normal in $\mathrm{SO}_4^+(r^a)$. The specific fact that we require is that the largest normal 2-subgroup of $\mathrm{CO}_4^+(3)$ coincides with the largest normal 2-subgroup of $\Omega_4^+(3)$.

Lemma 8.4. *Suppose that V is 8-dimensional orthogonal $\mathrm{GF}(r^a)$-space of $+$-type, $G = \mathrm{O}_8^+(r^a)$, $r^a \geq 4$ and $H = \Omega_8^+(r^a)$. Let $F = \mathrm{O}_4^+(r^a) \wr \mathrm{Sym}(2)$ preserve the decomposition $V = U_1 \perp U_2$ where U_1 and U_2 are 4-dimensional subspaces of V of $+$-type. Then $F \leq G$ and $F \cap H$ acts acts transitively on the components of F.*

Proof. We regard the base group of F as $F_1 \times F_2$ where $F_1 \cong F_2 \cong \mathrm{O}_4^+(r^a)$ considered as 8×8 matrices. Let τ be the antidiagonal matrix element with non-zero entries all equal to 1. Then $\tau \in F$ and $F_1^\tau = F_2$.

We adopt the notation introduced in Example 8.3. Then, when $r^a \geq 4$, the components L_1 and L_2 become components of F. In particular, we see that F has exactly 4 components and we denote them by $L_1, L_2, L_1^\tau, L_2^\tau$ where L_1 and L_2 are in F_1. Notice that $\det \tau = 1$ and $\dim C_V(\tau)$ is even. Hence $\tau \in \Omega_8^+(r^a)$. Let $\theta = (\rho, \rho)$. Then $L_1^\theta = L_2$ and $L_3^\theta = L_4$.

Moreover, $\det\theta = \det\rho^2 = 1$ and the quasideterminant is $-1^{\dim C_V(\theta)} = 1$. Hence $\theta \in \Omega_8^+(r^a)$. Thus $\langle\tau, \theta\rangle \leq F \cap H$ and acts transitively on the components of F. $\qquad\square$

One final general point is illustrated in the following lemma.

Lemma 8.5. *Suppose that r is odd. Then $O_1(r^a) \wr \mathrm{Sym}(m)$ is a subgroup of $O_m^\epsilon(r^a)$ provided either m is odd or $\epsilon = \begin{cases} - & m \equiv 2 \pmod 4 \text{ and } r^a \equiv 3 \pmod 4 \\ + & \text{otherwise} \end{cases}$.*

Proof. See [30, Proposition 4.2.15]. $\qquad\square$

We now start our investigation of Hypothesis 2.18 when $X \cong \mathrm{P}\Omega_m^\epsilon(r^a)$. Recall that $\Omega_5(r^a) \cong \mathrm{PSp}_4(r^a)$ and $\mathrm{P}\Omega_6^\epsilon(r^a) \cong \mathrm{PSL}_4^\epsilon(r^a)$ (see [30, Proposition 2.9.1]). So in light of Theorems 6.3 and 7.1 we shall assume that $m \geq 7$. In addition, when m is odd, we do not consider $r = 2$ as in this case the orthogonal groups are isomorphic to the symplectic groups $\mathrm{PSp}_{m-1}(2^a)$. Thus we have

$$X = F^*(G) = O^p(G) \cong \mathrm{P}\Omega_m^\epsilon(r^a)$$

with $m \geq 7$, $\epsilon \in \{\pm, 0\}$. Importantly, when $m = 2n = 8$ and $p = 3$, we also assume that G/X does not contain an element of order 3 inducing the triality automorphism or the triality automorphism multiplied by a field automorphism on X (so the triality automorphism is not involved in any way). The possibilities when the triality automorphism has an influence is the subject of Theorem 8.13. Thus G can be identified with a subgroup of $\mathrm{P}\Gamma\mathrm{CO}_m^\epsilon(r^a)$ which contains $X \cong \mathrm{P}\Omega_m^\epsilon(r^a)$. So that we can exploit the action of G on V, we let $\widehat{G}$ be a subgroup of $\Gamma\mathrm{CO}_m^\epsilon(r^a)$ which contains $Z(\mathrm{CO}_m^\epsilon(r^a))$ and is such that $G = \widehat{G}/Z(\mathrm{CO}_m^\epsilon(r^a))$. We extend this notation to subgroups of G. The most convenient place to indicate the structure of certain subgroups of $\mathrm{C}\Gamma\mathrm{O}_m^\epsilon(r^a)$ is in $O_m^\epsilon(r^a)$ and so we adopt a convention which is similar to that in Section 7. This means that we shall specify certain subgroups $\widehat{M}^*$ of $O_m^\epsilon(r^a)$ and then

$$\widehat{M} = \widehat{G} \cap \widehat{M}^*$$

and M is the image of $\widehat{M}$ in G.

We first focus our investigations on the even dimensional case. So we have

$$m = 2n \text{ and } \epsilon = \pm$$

until the proof of Theorem 8.11 is complete. Set

$$d_0 = \frac{1}{2}\mathrm{lcm}(2, d)$$

where as usual $d = \operatorname{ord}_p(r^a)$ when p is odd and, when $p = 2$, $d = \begin{cases} 1 & r^a \equiv 1 \pmod 4 \\ 2 & r^a \equiv 3 \pmod 4 \end{cases}$. In particular, this means $d_0 = 1$ when $p = 2$. Define

$$\eta = \begin{cases} + & \text{when } p \text{ odd and } d \text{ odd} \\ + & \text{when } p = 2 \text{ and } r^a \equiv 1 \pmod 4 \\ - & \text{when } p \text{ odd and } d \text{ even} \\ - & \text{when } p = 2 \text{ and } r^a \equiv 3 \pmod 4 \end{cases}$$

and

$$s = \begin{cases} \left\lfloor \frac{n}{d_0} \right\rfloor & \text{when } d_0 \text{ does not divide } n \\ \left\lfloor \frac{n}{d_0} \right\rfloor & \text{when } d_0 \text{ divides } n \text{ and } \eta^{\left\lfloor \frac{n}{d_0} \right\rfloor} = \epsilon \\ \left\lfloor \frac{n}{d_0} \right\rfloor - 1 & \text{when } d_0 \text{ divides } n \text{ and } \eta^{\left\lfloor \frac{n}{d_0} \right\rfloor} \neq \epsilon. \end{cases}$$

Additionally, set

$$\theta = \epsilon \eta^s$$

and put

$$\widehat{M}^* = \mathrm{O}_{2d_0}^{\eta}(r^a) \wr \operatorname{Sym}(s) \times \mathrm{O}_{2(n-d_0 s)}^{\theta}(r^a).$$

Then

Lemma 8.6. *The subgroup $\widehat{M}^*$ contains a Sylow p-subgroup of $\mathrm{O}_{2n}^{\epsilon}(r^a)$.*

Proof. See [18, page 114] or [56] for p odd and [11] for $p = 2$. $\qquad\square$

The subgroup $\widehat{M}^*$ corresponds to a decomposition of V as

$$V = V_1 \perp \cdots \perp V_s \perp V_0$$

where, when $V_0 \neq 0$, V_0 is a $2(n-d_0 s)$-dimensional orthogonal space of θ-type and $V_1, \ldots, V_s$ are $2d_0$-dimensional orthogonal spaces of η-type. In particular, we note that, if p is odd, $\widehat{S} \cap \widehat{X}$ centralizes V_0 and, if $p = 2$ and $\dim V_0 = 2$, then $\widehat{S} \cap \widehat{X}$ acts on V_0 as a Sylow 2-subgroup of $\mathrm{O}_2^{\theta}(r^a)$.

Lemma 8.7. *Suppose that p is odd. If $V_0 \neq 0$, then $\dim V_0 = 2$ and V_0 has $-$-type. That is $\theta = -$ and $n - d_0 s = 1$.*

Proof. Suppose that $V_0 \neq 0$. Then V_0 is centralized by $\widehat{S} \cap \widehat{X}$. Assume that V_0 contains a non-zero singular vector and so $S \cap X$ is contained in a parabolic subgroup of X. This contradicts Lemma 4.6 (as the triality automorphism is not present by assumption). So V_0 only contains non-singular vectors. Hence $\dim V_0 = 2$ and $\theta = -$. This proves the lemma. $\square$

The next lemma allows us to take advantage of our results on rank one isolated p-minimal subgroups in projective linear and unitary groups.

Lemma 8.8. *Suppose that $\widehat{X} \cong \Omega_{2n}^{\epsilon}(r^a)$, $d_0 = 1$, $s = n$ and $\eta^s = \epsilon$. Assume that p is odd. Then a Sylow p-subgroup of $\widehat{X}$ is contained in the subgroup $\widehat{H}^* = \mathrm{GL}_n^{\epsilon}(r^a)$ of $\Omega_{2n}^{\epsilon}(r^a)$.*

Proof. To see that $\Omega_{2n}^{\epsilon}(r^a)$ contains such subgroups when $\eta^n = \eta^s = \epsilon$, we cite [30, Tables 4.2.A,4.3.A]. Since $d_0 = 1$, recalling the definition of d_- at the beginning of Section 7 and using the definition of d_0 and η yields

$$(d_-, \eta) = \begin{cases} (2, +) & d = 1 \\ (1, -) & d = 2 \end{cases}.$$

If $\epsilon = +$, we then see that $\mathrm{GL}_n^{\eta}(r^a)$ has a monomial subgroup $(r^a - \eta) \wr \mathrm{Sym}(n)$ and this plainly contains a Sylow p-subgroup of $\mathrm{O}_2^{\eta}(r^a) \wr \mathrm{Sym}(n)$ which, by Lemma 8.6, contains a Sylow p-subgroup of $\Omega_{2n}^{\epsilon}(r^a)$. If $\epsilon = -$, then $\eta = -$, $d_- = 1$ and n is odd as $\epsilon = \eta^n$. This time we see $\mathrm{GL}_n^{-}(r^a)$ has a monomial subgroup $(r^a + 1) \wr \mathrm{Sym}(n)$ which contains a Sylow p-subgroup of $\mathrm{O}_2^{-}(r^a) \wr \mathrm{Sym}(n)$ and the result follows. $\qquad\square$

The influence of the examples of rank one isolated p-minimal subgroups in symmetric groups given by Theorem 3.3 is evident in Example 8.9. Remember that $\mathrm{O}_2^{-}(3) \cong \mathrm{Dih}(8)$.

Example 8.9. *The following examples with $X \cong \mathrm{P}\Omega_{2n}^{\epsilon}(r^a)$ and P a rank one isolated p-minimal subgroup in G can be easily verified using Lemma 8.2 to obtain $\mathrm{Out}(X)$.*

(i) $p = 2$, $X \cong \mathrm{P}\Omega_8^{+}(3)$, G/X is isomorphic to a subgroup of $\mathrm{Out}(X) \cong \mathrm{Dih}(8)$,

$$\widehat{L}^* = \mathrm{O}_4^{+}(3) \wr \mathrm{Sym}(2)$$

and

$$\widehat{P}^* = \widehat{M}^* = \mathrm{O}_2^{-}(3) \wr \mathrm{Sym}(4);$$

(ii) $p = 2$, $X \cong \mathrm{P}\Omega_{12}^{+}(3)$, G/X is isomorphic to a subgroup of $\mathrm{Out}(X) \cong \mathrm{Dih}(8)$,

$$\widehat{L}^* = \mathrm{O}_4^{+}(3) \wr \mathrm{Sym}(3)$$

and

$$\widehat{P}^* = \mathrm{O}_2^{-}(3) \wr \mathrm{Sym}(4) \times \mathrm{O}_2^{-}(3) \wr 2 < \widehat{M}^*;$$

(iii) $p = 3$, $d = 2$, $G \cong \mathrm{P}\Omega_8^{+}(r^a)$ with $r^a \equiv 2, 5 \pmod 9$,

$$\widehat{L}^* = \mathrm{O}_2^{-}(r^a) \wr \mathrm{Sym}(4)$$

and

$$P \sim 3_+^{1+2}.\mathrm{SL}_2(3) \times 3.$$

We note that Example 8.9 (iii) has the example in Lemma 7.7(i) in its genetic make up owing to the isomorphism $\mathrm{P}\Omega_6^{-}(r^a) \cong \mathrm{PSU}_4(r^a)$. Observe that in Example 8.9 (iii), a is coprime to 3 and so field automorphism cannot be added to the configurations to produce further examples. Indeed, as in the previous sections, field automorphisms do not cause any difficulties as the over-groups of $S \cap X$ in X that we select are always invariant under the standard field automorphism which raises matrix entries to some power of r.

Remark 8.10. *We single out Example 8.9 (i) for further scrutiny, looking at the possible extensions of* $P\Omega_8^+(3)$. *We follow* [25]. *Let* $X \cong P\Omega_8^+(3)$, *set* $S_0 = S \cap X$ *and* $Z = Z(S_0)$. *Then* $\mathcal{P}_X(S_0) = \{R_1, R_2, R_3, R_4, R_5\}$, $L = C_G(Z) = \langle R_1, R_2, R_3, R_4 \rangle$ *and* P *is uniquely determined as* $R_5 S$. *For* $1 \le i < j \le 4$, *we have* $\langle R_5, R_i, R_j \rangle \cong 2^6{:}\Omega_6^+(2)$. *Setting, for* $1 \le i \le 4$, $R_{5i} = \langle R_5, R_i \rangle$, *we have* $R_{5i}/Q_{R_{5i}} \cong \mathrm{SL}_3(2)$. *Now, by the Frattini Argument,* $\mathrm{Out}(X) \cong \mathrm{Sym}(4)$ *can be made to act on* $\mathcal{P}_X(S)$. *When it does, it fixes* R_5 *and permutes the other members as a natural* $\mathrm{Sym}(4)$. *We are only interested in a Sylow* 2-*subgroup of* $\mathrm{Out}(X)$ *and so we are interested in the action of a* $\mathrm{Dih}(8)$ *subgroup. It acts transitively on* $\{R_1, \ldots, R_4\}$. *Thus, we observe that every subgroup* G *of* $\mathrm{Aut}(X)$ *which contains* X *has a rank one isolated* 2-*minimal subgroup. Now the elements of* $\mathrm{Dih}(8)$, *which normalize* R_i *for some* $1 \le i \le 4$, *act as transpositions on* $\{R_1, \ldots, R_4\}$. *Thus* $P = R_5 S$ *and there exists* $P_1 \in P_G(S)$, *such that* $\langle P, P_1 \rangle / O_2(\langle P, P_1 \rangle) \cong \mathrm{SL}_3(2)$ *if and only if* G/X *is cyclic of order at most* 2 *and if* $G \ne X$, *then* S *acts as a transposition on* $\{R_1, \ldots, R_4\}$. *This means that* $G \le \langle X, r \rangle$ *where* r *is a reflection. Using* ATLAS *notation, this corresponds to* G/X *acting as* 2_2 *on* X. *Finally, we mention that this configuration of subgroups forms a geometry investigated by Kantor and which he calls a GAB* [25].

The examples above, together with those presented in Section 7 for the projective linear and unitary groups in dimension 4, form the examples which give the base of our main inductive proof.

Theorem 8.11. *Assume that Hypothesis 2.18 holds and that* $X \cong P\Omega_{2n}^\epsilon(r^a)$ *with* $n \ge 4$. *If* $2n = 8$ *and* $\epsilon = +$, *further assume that* G *embeds into* $\mathrm{PC\Gamma O}_8^+(r^a)$. *Then either* $p = r$ *or the possibilities for* G, P *and* L *are listed in Example 8.9 and displayed in lines 22 and 23 of Table 1 and line 11 of Table 2.*

Proof. We assume that $p \ne r$, $n \ge 4$ and that (n, ϵ, r^a, p, P) is not as in one of the examples given in Example 8.9 and seek a contradiction. Furthermore, assume that n is chosen minimally subject to this condition and G having a rank one isolated p-minimal subgroup P. We refer to the set

$$\{(4, +, 3, 2), (6, +, 3, 2), (4, +, r^a, 3) \mid r^a \equiv 2, 5 \pmod 9\}$$

as the enchanted set but we have to understand that for tuples in the set we really need to consider the relevant p-minimal subgroup as well.

Thus, the minimality of n means that $P\Omega_{2k}^\mu(r^a)$ with $4 \le k < n$ and (k, μ, r^a, p) not in the enchanted set has no rank one isolated p-minimal subgroup.

(8.11.1) *We have* $n = sd_0$.

We suppose that $sd_0 < n$ and work for a contradiction. Then $\dim V_0 = 2$, V_0 has $-$–type by Lemma 8.7. So $\theta = -$, $n - d_0 s = 1$.

First assume that p is odd. Let

$$\widehat{M_1^*} = \mathrm{O}_{2(n-1)}^{-\epsilon}(r^a)$$

be the subgroup of $\widehat{G}$ which fixes all the vectors in V_0 and acts on $V_1 \perp \cdots \perp V_s$. Then M_1 is S-invariant and, as p is odd, $\mathrm{O}_2^-(r^a)$ has p'-order, $Q_{M_1 S} = 1$. Hence $P \le M_1 S$ by

Theorem 2.16 (ii). In particular, P is a rank one isolated p-minimal subgroup in $M_1 S$. Thus either $(n-1, -\epsilon, r^a, p)$ is in the enchanted set or $2(n-1) = 6$ which falls outside the inductive setting of the lemma. If the latter case holds, we note that $P\Omega_6^\epsilon(r^a) \cong \mathrm{PSL}_4^\epsilon(r^a)$ and apply Theorem 7.1 to obtain $\widehat{M_1} = \mathrm{O}_6^-(r^a)$, $p = 3$, $r^a \equiv 2, 5 \pmod 9$. But then $n = 4$, $\epsilon = +$, $d_0 = 1$ and so $s = 4$, whereas we know $s = 3$. So this possibility bows out. Thus $(n-1, -\epsilon, r^a, p)$ is in the enchanted set. Since p is odd, we obtain $(n-1, -\epsilon, r^a, p) = (4, +, r^a, 3)$ with $r^a \equiv 2, 5 \pmod 9$ and $d = 2$ is as in Example 8.9(iii).This yields $d_0 = 1$ and, by the definition of η, $\eta = -$. As V_0 has minus type, this time we obtain $s = 5$ rather than 4. Hence p is not odd.

Assume that $p = 2$. Then, as $d \le 2$, $d_0 = 1$. Now d_0 divides n and so, as $s = n - 1$, we must have $\eta^n = \eta^{n/d_0} \ne \epsilon$ from the definition of s. Since $\eta^{n-1}\theta = \eta^s\theta = \epsilon$, we have $\theta \ne \eta$. Therefore

$$\widehat{M}^* = \mathrm{O}_2^\eta(r^a) \wr \mathrm{Sym}(n-1) \times \mathrm{O}_2^\theta(r^a)$$

is contained in the group

$$\mathrm{O}_{2(n-1)}^{\eta^{n-1}}(r^a) \times \mathrm{O}_2^\theta(r^a)$$

Let

$$\widehat{M_1}^* = \mathrm{O}_{2(n-1)}^{\eta^{n-1}}(r^a)$$

be the left hand factor of the displayed over-group of $\widehat{M}$. We claim $P \le M_1 S$. If not then $\widehat{M_1} \le \widehat{L}$ and so $Q_L \le Q_{M_1 S}$. Notice that $[Q_{M_1 S}, O^2(M_1 S)] = 1$. Thus $[\widehat{Q_L}, O^2(\widehat{M_1})] = 1$. Furthermore, $\widehat{Q_L \cap X} \le C_{\widehat{X}}(O^2(\widehat{M_1}))$ where $C_X(O^2(M_1))$ is isomorphic to a subgroup of $\mathrm{SO}_2^\theta(r^a)$. In particular, $Q_L \cap X$ has order 2 and $\widehat{Q_L \cap X}$ is elementary abelian of order 4 and is centralized by $O^2(\widehat{L})$. Since $V = C_V(x) \perp C_V(y)$ for x and y the elements of $\widehat{Q_L \cap X}$ which are not in $Z(\widehat{X})$ and since $C_V(x)$ and $C_V(y)$ are $\widehat{M_1}$-invariant, we deduce that V_0 and $V_0^\perp$ are invariant under the action of $O^2(\widehat{L})$. Hence we have $M_1 S \le L = O^2(L)S \le MS$ and the maximality of L implies that $L = MS$.

If $r^a \ne 3$, then, as $r^a \equiv \eta \pmod 4$ and $\theta \ne \eta$, $r^a - \theta \equiv 2 \pmod 4$ and so $O_2(\mathrm{O}_2^\theta(r^a)) \cong O_2(\mathrm{Dih}(2(r^a - \theta)))$ has order two. This means that Q_L also has order 2, contrary to Theorem 2.16 (iii). Hence $r^a = 3$, $\eta = -$, $\theta = +$ and $\widehat{M_1}\widehat{S} = \widehat{L}$ as $\mathrm{O}_2^\theta(3) = \mathrm{O}_2^+(3)$ is elementary abelian of order 4. Since Q_L is not contained in $Z(L)$ by Theorem 2.16 (iii), Q_L has order 4 and is acted upon non-trivially by S. Further, S contains an element exchanging the two reflections in $\mathrm{O}_2^+(3)$ which are contained in $Q_{\widehat{L}}$. In particular, $|Q_L \cap X| = 2$ and so $Q_L \cap X = Z(L)$. By Theorem 2.16 (i), we now have $Q_L \cap X \le Q_P$ as $O^2(P) \ne X$ and we additionally know that $[Q_{\widehat{L}}, \widehat{S}]$ negates V_0 and fixes every vector in $V_0^\perp$.

Notice that $\widehat{L} = N_{\widehat{G}}([Q_{\widehat{L}}, \widehat{S}])$ by the maximality of L. In particular, $\widehat{P}$ does not normalize $[Q_{\widehat{L}}, \widehat{S}]$. Set $Z_{\widehat{P}} = \langle [Q_{\widehat{L}}, \widehat{S}]^{\widehat{P}} \rangle \le Z(Q_{\widehat{P}})$. Then $Z_{\widehat{P}}$ is an elementary abelian group which can be considered as a $\mathrm{GF}(2)P/Q_P$-module. Notice that, as Q_L has order 4, $|Q_L Q_P / Q_P| = 2$ and is normalized by the unique maximal subgroup of P which contain S. Since $P/Q_P \in \mathcal{L}_1(2)$, Lemma 2.4 implies that $O^2(P/Q_P) \cong 3, 3^2, 3_+^{1+2}$ or has order 5. Because $[Z_{\widehat{P}}, Q_{\widehat{L}}] = [\widehat{S}, Q_{\widehat{L}}]$ has order 2, and $O^2(P/Q_P)Q_L Q_P / Q_P$ is generated by three involutions, we have

$|[Z_{\widehat{P}}, O^2(\widehat{P})Q_{\widehat{L}}]| \leq 2^3$. Since $O^2(P)$ does not centralize Z_P and $C_{O^2(P)}(Z_P)$ is normalized by S, we deduce from the fact that 3^2 and 5 do not divide $|\operatorname{GL}_3(2)|$ that $P/Q_P \cong \operatorname{Sym}(3)$. As $[Q_{\widehat{P}}, Q_{\widehat{L}}] \leq Z_{\widehat{P}}$, we have $O^2(\widehat{P}) \cong \operatorname{Alt}(4)$ and $Q_{\widehat{L}} \cap \widehat{X} = [O_2(O^2(\widehat{P})), Q_{\widehat{L}}] \neq 1$. Hence $Z_{\widehat{P}} = O^2(\widehat{P})$. Let $[Q_{\widehat{L}}, \widehat{S}] = \langle x \rangle$ where x negates V_0 and centralizes $V_0^{\perp}$. Then we may write $Z_{\widehat{P}}^{\#} = \{x, y, xy\}$ with x, y and xy all conjugate in $\widehat{P}$. Suppose that $V_0 = [V, y]$, then xy centralizes V_0, a contradiction as xy then centralizes V. If $V_0 \cap [V, y] = 0$, then V_0 is centralized by y, which means that xy negates V_0 and we are in the previous case, which is impossible. Hence $[V, Z_{\widehat{P}}] = [V, x] + [V, y]$ is a 3-space. Since $[V, Z_{\widehat{P}}] > [V, x]$ and $\widehat{S}$ leaves $[V, Z_{\widehat{P}}]$ invariant, this contradicts the fact that the subspaces of V/V_0 which are $\widehat{S}$-invariant all have dimension at least 2. This contradiction demonstrates that $P \leq M_1 S$.

Since $P \leq M_1 S$, Lemma 2.2 implies that P is a rank one isolated 2-minimal subgroup of $\widehat{M_1 S}$. The minimality of n yields that $n = 4$ and $O^2(M_1 S/Q_{M_1 S}) \cong \operatorname{P\Omega}_6^-(3)$ by Lemma 7.10 or $(n-1, \eta^{n-1}, r^a, 2)$ is in the enchanted set. In the latter case, we have $n \in \{5, 7\}$ and $O^2(M_1 S/Q_{M_1 S})$ is described in Example 8.9 (i) and (ii). In any case $r^a = 3$.

Assume first that $2n = 8$. Then V_1, V_2 and V_3 all have $-$-type, V_0 is of $+$-type and they all have dimension 2. Furthermore, we can arrange notation so that V_0 and V_3 are $\widehat{S}$-invariant and V_1 and V_2 are permuted non-trivially by $\widehat{S}$. We have that $X \cong \operatorname{P\Omega}_8^-(3)$. Let

$$\widehat{M_2^*} = \operatorname{O}_6^+(3) \times \operatorname{O}_2^-(3)$$

stabilize the decomposition $(V_1 + V_2 + V_0) \perp V_3$. Then M_2 is S-invariant. By Lemma 7.10, $M_2 S \leq L$. Now consider

$$\widehat{M_3^*} = \operatorname{O}_4^+(3) \times \operatorname{O}_4^-(3)$$

which stabilizes the decomposition $(V_1 + V_2) \perp (V_0 + V_3)$. Again M_3 is S-invariant. Since $O_2(\langle M_2 S, M_3 S \rangle) = 1$ and $L \geq M_2 S$, we have $P \leq M_3 S$. Thus $\widehat{P} \leq \widehat{M_3 \widehat{S}} \cap \widehat{M \widehat{S}}$ and this group stabilizes $V_3 + V_0$ and V_0. Hence P stabilizes $V_0^{\perp} \cap (V_0 + V_3) = V_3$ and consequently the decomposition $(V_1 + V_2 + V_0) \perp V_3$. Therefore $P \leq L$, a contradiction.

Now suppose that $(n-1, \eta^{n-1}, r^a, 2) = (4, +, 3, 2)$. Then $2n = 10$ and $\eta = -$ as $r^a = 3$ and so $\theta = +$. Therefore $X \cong \operatorname{P\Omega}_{10}^+(3)$. Thus

$$\widehat{M_2^*} = \operatorname{O}_8^+(3) \times \operatorname{O}_2^+(3),$$

is the stabilizer of the decomposition

$$V = (V_1 + V_2 + V_3 + V_4) \perp V_0$$

with $\dim V_i = 2$ and, if i is positive, V_i has $-$-type. From Example 8.9 (i), we have $\widehat{L}$ contains $\widehat{J}$ where

$$\widehat{J^*} = \operatorname{O}_4^+(3) \wr \operatorname{Sym}(2) \times \operatorname{SO}_2^+(3) \leq \widehat{M_2^*}.$$

By considering the information in Example 8.3 about $\operatorname{CO}_4^+(3)$, we see that $Q_{\widehat{L}} \leq Q_{\widehat{L} \cap \widehat{J}}$ is isomorphic to a subgroup of

$$\widehat{D^*} = (\operatorname{Q}_8 * \operatorname{Q}_8) \times (\operatorname{Q}_8 * \operatorname{Q}_8) \times 2^2.$$

Therefore $\Phi(Q_{\widehat{L}}) \leq \Phi(\widehat{D}^*)$ and, as $[V, \Phi(\widehat{D}^*)] = V_1 + V_2 + V_3 + V_4$, if Q_L is not elementary abelian, then, as Q_L is S-invariant, $\widehat{L} \leq \widehat{M_1^*}$ which is a contradiction. Thus Q_L is elementary abelian and, as Q_L is normalized by J, $Q_{\widehat{L}} \leq Z(\widehat{D})$. Returning to the description of P given in Example 8.9(i), we see that

$$\widehat{P} \leq \widehat{M} = O_2^-(3) \wr \mathrm{Sym}(4) \times O_2^+(3).$$

We know

$$Q_{\widehat{L}} \leq Z(\widehat{D}) \leq Z(\widehat{D}^*) = \langle d_1, d_2, d_3, d_4 \rangle,$$

where $d_1 = \mathrm{diag}((-1)^4, 1^6), d_2 = \mathrm{diag}(1^4, (-1)^4, 1^2), d_3 = \mathrm{diag}(1^8, -1, 1)$ and $d_4 = \mathrm{diag}(1^9, -1)$. However, as $\widehat{P} \leq \widehat{M}$, $Q_{\widehat{P}} \geq Z(\widehat{D})$ and hence we have a contradiction. Thus $(n - 1, \eta^{n-1}, r^a, 2) \neq (4, +, 3, 2)$.

It remains to deal with the case $(n - 1, \eta^{n-1}, r^a, 2) = (6, +, 3, 2)$. So $2n = 14$ and again $\theta = +$. Then

$$\widehat{M} = O_2^-(3) \wr \mathrm{Sym}(6) \times O_2^+(3)$$

and the Sylow 2-subgroup of $\widehat{G}$ may be assumed to leave the subspaces $W_1 = V_1 + V_2 + V_3 + V_4$, $W_2 = V_5 + V_6$ and V_0 invariant. Thus $\widehat{S}$ is contained in the subgroups

$$\widehat{M_2^*} = \mathrm{Stab}_{\widehat{G}}(W_1) \cong O_8^+(3) \times O_6^+(3)$$

and

$$\widehat{M_3^*} = \mathrm{Stab}_{\widehat{G}}(W_2) \cong O_{10}^+(3) \times O_4^+(3).$$

The description of P given in Example 8.9(ii) shows that $P \leq K_1 S$ where $\widehat{K_1} = \Omega_8^+(3)$ is in the first factor of $\widehat{M_2^*}$. Considering $\widehat{M_3}$ and writing it as $\widehat{J_1}\widehat{J_2}\widehat{S}$ with $\widehat{J_1} = \Omega_{10}^+(3)$ and $\widehat{J_2} = \Omega_4^+(3)$, we have $J_1 S \geq K_1 S \geq P$. But then P is isolated in $J_1 S$ and this contradicts the minimality of n. We have proved that $n = sd_0$. $\blacksquare$

(8.11.2) $s \geq p$.

For $p = 2$ or 3, this follows because of the requirements on the size of n. So $p \geq 5$ and $s \geq p$ follows from Lemma 4.5 as $G \neq P$. $\blacksquare$

Because of (8.11.1) we now have

$$\widehat{M^*} = O_{2d_0}^\eta(r^a) \wr \mathrm{Sym}(s).$$

(8.11.3) $L \geq MS$.

Suppose on the contrary that $P \leq MS$. Let $\widehat{N^*}$ be the base group of $\widehat{M^*}$. Suppose that N is not soluble. Then, from $O_p(Z(N)E(N)S) = Q_{MS} \leq Q_P$, we deduce that $P \leq Z(N)E(N)S$. Since the individual components of N are not normalized by MS, Theorem 2.15 (i) combined with (8.11.2) implies that $P \leq Z(N)Z(E(N))S$ and then that $O^p(P) \leq Z(N)Z(E(N))$ which is a 2-group. Now $L \cap Z(N)E(N)S$ is a normal subgroup of

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$Z(N)E(N)S = O^p(P)(L \cap Z(N)E(N)S)$. But then $O_p(L \cap Z(N)E(N)S)$ is also a 2-group, a contradiction. Hence N is soluble. Therefore $d_0 = 1$, $s = n$ and

$$\widehat{M}^* \cong \mathrm{O}_2^\eta(r^a) \wr \mathrm{Sym}(n).$$

(8.11.3.1) N is a 2-group and $p = 2$.

Assume that N is not a 2-group. Then, as $P \leq MS$, $Q_P \geq Q_{MS} = Q_{NS}$. It follows that $P \leq NS$ and $O^p(P) \leq N$. In particular, P is soluble and so by Theorem 2.15 (i), $O^p(P)Q_{NS}$ is normalized by MS. The structure of $O^p(P)Q_{MS}/Q_{MS}$ is now given by Lemma 2.4 and, in particular, we have $p \in \{2, 3\}$.

Suppose that $p = 3$. Consider $Q_P \geq Q_{MS} = O_3(F(N)S)$ and so $P \leq F(N)S$ and $O^3(P) \leq F(N)$ which is abelian. It follows from Lemma 2.4 that $O^3(P)$ is elementary abelian of order 4. Since $\Omega_1(O_2(F(\widehat{N})))$ is isomorphic to the natural $\mathrm{GF}(2)M/N$-permutation module, we may apply Lemma 2.19 to obtain a contradiction as $n \geq 4$. Therefore $p = 2$ and $O^p(P)Q_{MS}/Q_{MS}$ has order 3, 3^2, 3^3 or 5 by Lemma 2.4. Since $NS/F(N)$ is a 2-group, we have $O^2(P) \leq F(N)$ which is abelian. In particular, $O^2(P)$ is normal in MS.

Since N is not a 2-group, $r^a > 3$. Write $s = n = 2j + k$ with $0 \leq k \leq 1$. If $k = 0$, then set

$$\widehat{M}_2^* = \mathrm{O}_4^+(r^a) \wr \mathrm{Sym}(j)$$

and notice that this group contains $\widehat{N}$. In particular, $P \leq M_2 S$ and $O^2(P)Q_{M_2 S}$ is not normal in $M_2 S$ as $O_{2,3}(M_2 S) = O_{2,5}(M_2 S) = 1$. Therefore, as $M_2 S/Q_{M_2 S}$ does not have normal components, we have a contradiction to Theorem 2.15 (i). Hence $k = 1$ and $n \geq 5$. This time set

$$\widehat{M}_2^* = \mathrm{O}_{2(n-1)}^{\eta^{n-1}}(r^a) \times \mathrm{O}_2^\eta(r^a).$$

Then $M_2 S \geq N$ and so $P \leq M_2 S$. Let $\widehat{K}_1$ be the first factor of $\widehat{M}_2$. If $P \leq K_1 S$, then, as $p = 2$, we have $r^a = 3$ by the minimality of n, a contradiction as $r^a > 3$. Hence $P \leq K_2 S$ where $\widehat{K}_2 \cong \mathrm{O}_2^\eta(r^a)$ is the second factor of $\widehat{M}_2$. But then $O^2(P)$ is normal in M_2 as well as in MS and so $O^2(P) \leq O_t(\langle M_2, M \rangle) = 1$ for $t \in \{3, 5\}$, a contradiction. We have proved that N is a 2-group. Finally, as p divides $|N|$, $p = 2$. This proves (8.11.3.1).

By (8.11.3.1), N is a 2-group and $p = 2$. Hence $N \leq Q_{MS}$ and P/Q_{MS} is a rank one isolated 2-minimal subgroup of $MS/Q_{MS} \cong \mathrm{Sym}(n)$ if $n > 4$ and $MS/Q_{MS} \cong \mathrm{Sym}(3)$ if $n = 4$. Obviously MS is 2-minimal when $n = 4$. If $n \geq 5$, we may call upon Theorem 3.3 to obtain that $n \in \{5, 6, 8, 12\}$.

Suppose that $n = 12$ and consider the subgroup

$$\widehat{M}_2^* = \mathrm{O}_{16}^+(r^a) \times \mathrm{O}_8^+(r^a)$$

which is normalized by $\widehat{S}$. Let K be the component of the left hand factor of $\widehat{M}_2^*$. Using Example 3.2 and Theorem 3.3 reveals that

$$\widehat{P} \leq \widehat{J}^* = \mathrm{O}_2^\eta(r^a) \wr \mathrm{Sym}(8) \times \mathrm{O}_2^\eta(r^a) \wr (\mathrm{Sym}(2) \wr \mathrm{Sym}(2)).$$

Therefore $P \leq KS$ which contradicts the minimal choice of n.

If $n = 8$, then set
$$\widehat{M_2^*} = \mathrm{O}_8^+(r^a) \wr \mathrm{Sym}(2).$$
Then M_2 is S-invariant and $|Q_{M_2}| = 2$. Hence $P \leq M_2 S$ by Theorem 2.16 (ii) and this contradicts Theorem 2.15(i). Therefore $n \neq 8$.

For $n = 6$, we let
$$\widehat{M_2^*} = \mathrm{O}_8^+(r^a) \times \mathrm{O}_4^+(r^a)$$
and
$$\widehat{M_3^*} = \mathrm{O}_4^+(r^a) \wr \mathrm{Sym}(3)$$
be such that M_2 and M_3 are S-invariant. Then, as the 2-minimal subgroups in $\mathrm{Sym}(6)$ are either $\mathrm{Sym}(4) \times \mathrm{Sym}(2)$ or $\mathrm{Sym}(2) \wr \mathrm{Sym}(3)$, we either have $P \leq MS \cap M_2 S$ and $L \geq M_3 S$ or $P \leq MS \cap M_3 S$ and $L \geq M_2 S$. If $r^a > 3$, then we get $Q_L \leq Q_P$ which is absurd. Thus $r^a = 3$. Suppose that $P \leq M_3 S$. Then $L \geq M_2 S$ and so $Q_L \leq Q_{M_2 S} \leq Q_{M_3 S} \leq Q_P$, whence this case fails. Finally, if $P \leq M_2 S$, then $L \geq M_3 S$ and we have the configuration described in Example 8.9 (ii), a contradiction.

Consider the possibility that $n = 5$. Then
$$\widehat{M} = \mathrm{O}_2^\eta(r^a) \wr \mathrm{Sym}(5)$$
which is a 2-minimal group as N is a 2-group by (8.11.3.1). Thus $P = MS$. Let
$$\widehat{M_2^*} = \mathrm{O}_8^+(r^a) \times \mathrm{O}_2^\eta(r^a).$$
Then $P \nleq M_2 S$ and so $M_2 S \leq L$. In particular, $Q_L \leq N = Q_P$, a contradiction.

Finally we have $n = 4$. Then
$$\widehat{M^*} = \mathrm{O}_2^\eta(r^a) \wr \mathrm{Sym}(4)$$
and $P = MS$. Let
$$\widehat{M_2^*} = \mathrm{O}_4^+(r^a) \wr \mathrm{Sym}(2).$$
Then $P \nleq M_2 S$. Hence $L \geq M_2 S$ and $r^a = 3$ for otherwise $|Q_L| = 2$. This contradicts the choice of n as this configuration is on our list.

Having considered all the possibilities we conclude that $P \nleq MS$ and so (8.11.3) holds. ∎

Since $L \geq MS$ by (8.11.3), we must have that MS is a p-local subgroup of G. If $d_0 > 1$, then $F(MS)$ is a 2-group. Thus $p = 2$ and in fact $d_0 = 1$, which is a contradiction. Therefore $d_0 = 1$. Combining this comment with (8.11.1) we have the following result.

(8.11.4) *We have $\widehat{M^*} \cong \mathrm{O}_2^\eta(r^a) \wr \mathrm{Sym}(n)$.* ∎

Our next claim is

(8.11.5) $n = s > p$.

By (8.11.2), $s \geq p$ and by (8.11.4) $s = n$. Suppose that $n = p$. Then $p \geq 5$ since $n \geq 4$. Lemma 8.8 implies that $\widehat{S}$ normalizes $\widehat{H} \cong \mathrm{GL}_n^\epsilon(r^a)$. If $P \leq HS$ and $O^p(P) \nleq F(H)$, then Lemma 2.7 and Theorem 7.1 together imply that $p \leq 3$, a contradiction. If $O^p(P) \leq F(H)$,

then $O^p(P)$ centralizes $S \cap X$ and consequently leaves invariant each V_i, $1 \leq i \leq p$. But then, using (8.11.3), $O^p(P) \leq MS \leq L$, a contradiction. Therefore, $P \not\leq HS$. But then $L \geq \langle MS, H \rangle$ which is not a p-local subgroup of G. Thus $n \neq p$ and so, by (8.11.2), $n = s > p$. ∎

Write $n = \ell p + k$ where $0 \leq k \leq p - 1$. Then put

$$\widehat{M_1^*} = O_{2\ell p}^{\eta^{\ell p}}(r^a)$$

such that this group centralizes

$$V_{\ell p + 1} + \cdots + V_n.$$

Then S normalizes M_1.

Assume that $\ell p \neq n$. Since $O_p(\langle M_1 S, MS \rangle) = 1$, $P \leq M_1 S$ and P is a rank one isolated p-minimal subgroup in $M_1 S$. If $p \geq 5$, this contradicts the minimal choice of n. Therefore, $p \leq 3$.

Suppose that $p = 3$. Then the minimality of n implies that $\ell p = 3$ and that $n = 4$ or $n = 5$. In particular, we have that

$$\widehat{M_1^*} = O_6^-(r^a)$$

with $r^a \equiv 2, 5 \pmod 9$ by the minimality of n. In addition, $G = X$ as X has no field automorphisms of order 3. Therefore $P \cap M_1 \cong 3_+^{1+2}{:}\mathrm{SL}_2(3)$. This then means that if $n = 4$, $X \cong \mathrm{P}\Omega_8^+(r^a)$ with $r^a \equiv 2, 5 \pmod 9$ and this case is listed in Example 8.9 (iii). Hence $n = 5$ and we have $\widehat{G} = \widehat{X} \cong \Omega_{10}^-(r^a)$. By Lemma 8.8, we may select

$$\widehat{M_2^*} = \mathrm{GU}_5(r^a)$$

so that M_2 is S-invariant. If $P \leq M_2 S$, then by Theorem 7.1 $O^p(P) \leq F(M_2 S)$. But then P centralizes S and we have $P \leq MS$ whereas $MS \leq L$ by (8.11.3), a contradiction. Therefore $M_2 S \leq L$ and, since $O_3(\langle M_2 S, MS \rangle) = 1$, we have a contradiction.

Finally consider the case $p = 2$. Then, as P is a rank one isolated 2-minimal subgroup of $M_1 S$ where $\widehat{M_1^*} \cong O_{4\ell}^+(r^a)$, the minimality of n and the fact that $n = 2\ell + 1$ implies that either $2n = 10$ or $2n = 14$ and in any case $r^a = 3$. But then we read the descriptions of the rank one isolated 2-minimal subgroups of $O_8^+(3)$ and $O_{12}^+(3)$ from Example 8.9 (i) and (ii) and see that $P \leq MS$, and this contradicts (8.11.3). Hence we conclude that $n = \ell p$ and, as $n \neq p$ by (8.11.5), we have $\ell \geq 2$.

Let

$$\widehat{M_1^*} = O_{2p}^{\eta^p}(r^a) \wr \mathrm{Sym}(\ell).$$

Then M_1 is normalized by S. If $L \geq M_1$, then $L \geq \langle M, M_1 \rangle$ which means that $Q_L = 1$. Thus $P \leq M_1 S$. Assume that $(p, r^a) \neq (2, 3)$. Then Theorem 2.15 (i) implies that $O^p(P)$ is a normal soluble subgroup of $M_1 S$. Hence $O^p(P) \leq O_2(M_1 S) \leq MS$, which is a contradiction. Thus $(p, r^a) = (2, 3)$. If ℓ is even we consider the $\widehat{S}$-invariant subgroup

$$\widehat{M_2^*} = O_8^+(3) \wr \mathrm{Sym}(\ell/2).$$

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Then again $P \leq M_2 S$ and we obtain a contradiction from Theorem 2.15 (i). Thus $\ell = 2j+1$ for some $j \geq 1$. We now set

$$\widehat{M_3^*} = \mathrm{O}_{8j}^+(3) \times \mathrm{O}_4^+(3).$$

By considering a component $\widehat{K}_3$ of the first factor and noting that L cannot contain $\langle K_3, MS \rangle$, we deduce the $P \leq K_3 S$ and thus $8j = 8$ by the minimality of n. Therefore $2n = 12$ and $P \cap K_3$ is as in Example 8.9(ii). But then $P \leq L$ and this contradiction completes the proof of the theorem. $\qquad\square$

We next grapple with orthogonal groups defined in odd dimension $m = 2n + 1$.

Theorem 8.12. *Assume that Hypothesis 2.18 holds and that $X \cong \mathrm{P\Omega}_{2n+1}(r^a)$ with r odd and $n \geq 3$. Then either $p = r$ or $p = 2$, $X \cong \mathrm{P\Omega}_7(3)$ and P and L are as described in line 21 of Table 1.*

Proof. Supposing that $p \neq r$ we show that $p = 2$ and $X \cong \mathrm{P\Omega}_7(3)$. The stabilizer of a non-singular point in V is isomorphic to $\mathrm{O}_{2n}^\epsilon(r^a) \times 2$ for some $\epsilon = \pm$. Since $|\mathrm{O}_{2n+1}(r^a){:}\mathrm{O}_{2n}^\epsilon(r^a)| = (r^{na} + \epsilon)r^b$ for some b, $\widehat{S}$ fixes either a plus point or a minus point when acting on V. Let ϵ be the type of this point and set

$$\widehat{M^*} = \mathrm{O}_{2n}^\epsilon(r^a) \times 2$$

where as usual we may assume that M is S-invariant. Notice that $2n \geq 6$. Therefore Q_M has order dividing 2. Hence P is a rank one isolated p-minimal subgroup of MS by Theorem 2.16 (iii). Thus Theorem 8.11 implies that $n \in \{3, 4, 6\}$ and $p \in \{2, 3\}$.

Suppose first that $p = 2$. Then $r^a = 3$ by Theorems 7.1 and 8.11.

If $n = 3$, then $X \cong \mathrm{P\Omega}_7(3)$ and

$$\widehat{M^*} = \mathrm{O}_6^-(3) \times 2.$$

Since $\mathrm{O}_7(3)$ contains the subgroup $\widehat{M}_1 = \mathrm{O}_1(3) \wr \mathrm{Sym}(7)$ by Lemma 8.5 and since $\mathrm{Sym}(7)$ and $\mathrm{Alt}(7)$ have no rank one isolated 2-minimal subgroups by Theorem 3.3, we have $L \geq M_1 S$. Because $M_1 S$ is a maximal subgroup of G, we have $L = M_1 S$. Hence $(L \cap M)/Q_{L \cap M}$ contains a subgroup isomorphic to $\mathrm{Alt}(6)$. The examples in $\mathrm{P\Omega}_6^-(3) \cong \mathrm{PSU}_4(3)$ tabulated in Table 1 show that P/Q_{MS} centralizes an involution and $P/Q_{MS} \cong R_1 S/Q_{MS}$ from Example 7.9 (or up to changing notation $R_3 S$). This is the specified configuration in $\mathrm{P\Omega}_7(3)$ and so we may assume that $n \neq 3$.

Assume now that $n = 4$ (and $p = 2$). Then $X \cong \mathrm{P\Omega}_9(3)$ and, by Theorem 8.11, as $P \leq MS$ we have

$$\widehat{P} \leq \widehat{M^*} = \mathrm{O}_2^-(3) \wr \mathrm{Sym}(4) \times 2 \leq \widehat{J^*}$$

where $\widehat{J^*} \cong \mathrm{O}_1(3) \wr \mathrm{Sym}(8) \times 2$. Let

$$\widehat{M_1^*} = \mathrm{O}_1(3) \wr \mathrm{Sym}(9).$$

Then $\widehat{M}_1$ is $\widehat{S}$-invariant by Lemma 8.5. Since M_1 has no rank one isolated 2-minimal subgroups by Theorem 3.3, we have that $L \geq M_1 S$ But then $\widehat{L} \cap \widehat{M}_1 \geq \widehat{J} \geq \widehat{P}$. Therefore $P \leq L$ and this configuration is eliminated.

Finally suppose that $n = 6$ and $p = 2$. Then define

$$\widehat{M}_1^* = \mathrm{O}_1(3) \wr \mathrm{Sym}(13).$$

By Theorem 3.3, $L \geq M_1 S$ which means that $\widehat{L} \cap \widehat{M} \geq \widehat{J}$ where $\widehat{J}^* = 2 \wr \mathrm{Sym}(12) \times 2$ and this contradicts the information given about $\widehat{L}$ in Example 8.9 and Theorem 8.11. We conclude that $p = 3$.

Assume now that $p = 3$. Then we have $n = 3$ or 4 and in both cases $r^a \equiv 2, 5 \pmod{9}$. In particular, a is not divisible by 3 and so $G = X$.

If $n = 3$, then $X = \mathrm{P}\Omega_7(r^a)$, $\widehat{M}^* \cong \mathrm{O}_6^-(r^a) \times 2$ with $r^a \equiv 2, 5 \pmod{9}$. We have that $P \sim 3_+^{1+2}.\mathrm{SL}_2(3)$ and so $|Z(\widehat{S})|$ is cyclic of order 3 and $\dim[V, Z(\widehat{S})] = \dim[V, \widehat{S}] = 6$. As $Q_{\widehat{L}} \geq Z(\widehat{S})$, we have $[V, Q_{\widehat{L}}] = [V, \widehat{S}]$ is normalized by $\widehat{L}$ and $\widehat{M}$. Thus $L \leq M \geq P$, a contradiction.

Assume that $n = 4$. Then Theorem 8.11 gives

$$\widehat{M}^* \cong \mathrm{O}_8^+(r^a) \times 2$$

and again we have that $r^a \equiv 2, 5 \pmod{9}$. Set

$$\widehat{M}_1^* = \mathrm{O}_6^-(r^a) \times \mathrm{O}_3(r^a).$$

Write $\widehat{M}_1 = \widehat{K}_1 \times \widehat{K}_2$ where $\widehat{K}_1$ is the left hand component and $\widehat{K}_2$ is the right hand factor of $\widehat{M}_1$. Since $Q_{M_1} = 1$, $P \leq M_1 S$. Because $P \leq MS$, we conclude that $P \leq K_1 S$. Thus $K_2 \leq L$ and, as $\widehat{L} \cap \widehat{M} \leq \widehat{J}^* = \mathrm{O}_2^-(r^a) \wr \mathrm{Sym}(4) \times 2$, we conclude that $L \geq \langle L \cap M, K_2 \rangle$ and so $Q_L = 1$, a contradiction. $\square$

Finally in this section we come to the situation when $p = 3$, $X \cong \mathrm{P}\Omega_8^+(r^a)$ and G is not contained in $\mathrm{PC\Gamma O}_8^+(r^a)$. So the triality automorphism of X is exerting an influence.

Theorem 8.13. *Assume Hypothesis 2.18 holds with $p = 3$, $X \cong \mathrm{P}\Omega_8^+(r^a)$ and G is not a subgroup of $\mathrm{PC\Gamma O}_8^+(r^a)$. Then, letting $\eta \equiv r^a \pmod 3$, we have $r^a - \eta \equiv 3, 6 \pmod 9$ with $r^a \neq 4$, $G = \mathrm{P}\Omega_8^+(r^a){:}3$, $P \sim 3_+^{1+4}{:}\mathrm{SL}_2(3)$, and $L = (\Omega_2^\eta(r^a)^4).(2.(r^a-1,2))^3.\mathrm{Sym}(4).3$. (These example being displayed on lines 12 and 13 of Table 2.)*

Proof. We call on the maximal subgroups of groups G with $F^*(G) \cong \mathrm{P}\Omega_8^+(r^a)$ as listed in [27] and succinctly tabulated in [4, Table 8.50]. By Theorem 2.16 (ii), we know $Q_L \cap X \neq 1$ hence $L \cap X$ is a 3-local subgroup of X. Let $r^a \equiv \eta \pmod 3$. Then by [27] there are exactly two 3-local maximal subgroups M and M_1 of X that contain $S \cap X$ and are such that MS and $M_1 S$ are maximal 3-local subgroups of $G = XS$. These are

$$M = (\Omega_2^\eta(r^a)^4).(2.\mathbf{d})^3.\mathrm{Sym}(4)$$

and

$$M_1 = (\Omega_2^\eta(r^a) \times \frac{1}{\mathbf{d}}\mathrm{GL}_3^\eta(r^a)).[2\mathbf{d}].$$

where $\mathbf{d} = (2, r^a - 1)$ (the notation $\mathbf{d}$ is used to distinguish this parameter from our standard parameter d).

Let $A = O_3(M)$. Then A is homocyclic of order 3^{4b} for some integer $b \geq 1$. If $B \leq S \cap X$ is abelian of the same order as A with $A \neq B$, then $S \cap X = AB$ and $Z(S \cap X) = A \cap B$. This contradicts the action of S on A as described in Lemma 2.19. Hence A is the unique abelian subgroup of its order in $S \cap X$. In particular, A is a characteristic subgroup of $S \cap X$ and so is normalized by MS. Now $M/C_M(A)$ is a subgroup of $2 \wr \mathrm{Sym}(4)$ and $Z(M/C_M(A))$ has order 2. Let F be the preimage of the latter group in M. Then F is normal in MS and M/F is an elementary abelian group of order 2^4 extended by $\mathrm{Sym}(3)$. Now consider $O^2(M/F)$ of order 48 with $O_2(M/F)$ elementary abelian of order 2^4. Since $S \cap X$ acts fixed-point-freely on $O^2(M/F)$ as a cyclic group of order 3, there are exactly $5 = 4+1$ subgroups of $O^2(M/F)$ which are $(S \cap X)$-invariant. As S is a 3-group, there are at least two of these groups which are S-invariant. Thus we find distinct subgroups of D_1 and D_2 say which contain F, are normalized by S and have order 4 mod F. Therefore SD_1 and SD_2 are distinct over-groups of S in MS. Now $O^2(M \cap M_1) \leq (S \cap X)C_M(A)$ whereas $O^2((S \cap X)D_iC_M(A)/C_M(A))$ has order at least 12. Hence neither of $(S \cap X)D_1$ nor $(S \cap X)D_2$ is contained in M_1. Suppose that $L = M_1$. Then as D_1S and D_2S are not contained in M_1, $O^3(P) \leq D_1 \cap D_2 = F$ and $(M_1 \cap M_2)F = M$. Since $O^2(M \cap M_1) \leq (S \cap X)C_M(A)$, this is impossible. Therefore $P \leq M_1$ and $L = M$.

Let $K_1 \leq M_1$ be such that $K_1 \cong \frac{1}{\mathbf{d}}\mathrm{GL}_3^\eta(r^a)$. Assume that $r^a \neq 2$ when $\epsilon = -$. In this case $\mathrm{GL}_3^\epsilon(r^a)$ is not soluble. Since $K_1 \not\leq M$, $P \leq K_1S$ and so Lemma 7.5 yields $r - \eta \equiv 3, 6$ (mod 9), $r^a \neq 4$, and $P \cap K_1 \sim 3_+^{1+2}{:}\mathrm{SL}_2(3)$. By [4, Table 8.50], G contains a subgroup $H \cong \Omega_8^+(2){:}3$ which contains P and so we read the structure of P from the description given in [27] (see also ATLAS [12]). This completes the proof of the theorem. $\qquad\square$

9. EXCEPTIONAL GROUPS OF LIE TYPE

We continue our investigations of groups G satisfying Hypothesis 2.18 by considering the possibilities which arise when X is an exceptional group of Lie type defined over a field of order r^a. During this section we also make the blanket assumption that $p \neq r$. We let $\widehat{X}$ be the universal version of X and $\widehat{G}$ be such that $\widehat{X}$ is a universal exceptional group of Lie type and assume that $G = \widehat{G}/F(\widehat{G})$ satisfies Hypothesis 2.18. Throughout this section, we have $X \notin \mathcal{L}_1(p)$ and so $X \neq O^p(P)$ and Lemma 4.5 implies that if $p \geq 5$, then X does not have abelian Sylow p-subgroups. We begin this confrontation by showing that d is very small.

Lemma 9.1. *We have* $d = \mathrm{ord}_p(r^a) \leq 2$.

Proof. Assume that $d > 2$. Then, by the definition of d, $p \geq 5$ and so $\widehat{X}$ does not have abelian Sylow p-subgroups. Hence the parameter b defined in Lemma 4.3 is non-zero. Now, if $d = 3$, then $p \geq 7$, and for $p = 5$, $d = 4$ (as $d > 2$). So $pd \geq 20$. In particular, for $b = \sum_{c \geq 0} n_{dp^c} > 0$, we must have $n_{dp^c} \neq 0$ for some c. Since $pd \geq 20$, Table 3 and Lemma 4.3 yield $X \cong \mathrm{E}_8(r^a)$ and as d divides $p - 1$, $p = 5$ with $pd = 20$. So 5 divides $\Phi_4(r^a) = r^{2a} + 1$. It follows that $\mathrm{E}_8(r^a)$ has Sylow 5-subgroups of order $5(\Phi_4(r^a)_5)^4$. Using [35, Theorem and Table 5.1], we see that X contains a subgroup $K \cong \mathrm{PGU}_5(r^{2a}).4$ and

that there is a unique such subgroup up to conjugacy in X. By comparing the orders of X and K, we infer that K contains a Sylow 5-subgroup of X. Using that K is unique up to conjugacy, the Frattini argument implies that we may assume that KS is a subgroup of G. Since K is not a 5-local subgroup of G, we get $P \leq KS$ from Theorem 2.16 (ii). By Lemma 2.2 P is a rank one isolated 5-minimal subgroup of KS. However, Theorem 7.1 shows that KS has no such subgroups. Hence $d \leq 2$ and the lemma holds. $\qquad\square$

Lemma 9.2. *We have* $X \not\cong {}^2\mathrm{B}_2(2^a)$ *and* $X \not\cong {}^2\mathrm{G}_2(3^a)$ *with* $a > 1$.

Proof. If $X \cong {}^2\mathrm{B}_2(2^a)$, then, as 3 does not divide $|X|$, $p > 3$ and the Sylow p-subgroups of $S \cap X$ are abelian by Lemma 4.3. Thus $X \not\cong {}^2\mathrm{B}_2(2^a)$. Suppose that $X \cong {}^2\mathrm{G}_2(3^a)$ with $a > 1$. Then, $X \notin \mathcal{L}_1(p)$ and again by Lemma 4.3, the Sylow p-subgroups of X are abelian when p is odd and we know they are elementary abelian of order 8 when $p = 2$. Hence, by Lemma 4.5, $p = 2$. Since $|\mathrm{Out}(X)| = a$ and a is odd, we have $G = X$ and this contradicts Lemma 2.14. $\qquad\square$

Lemma 9.3. *Suppose that* $X \cong \mathrm{G}_2(r^a)'$. *Then one of the following holds.*
    (i) $p = 2$, $G \cong \mathrm{G}_2(3)$, $P \sim 4^2.\mathrm{Dih}(12)$ *and* $L \sim 2^{1+4}_+{:}3^2.2$.
    (ii) $p = 2$, $G \cong \mathrm{G}_2(3).2$, $P \sim 4^2.\mathrm{Dih}(12).2$ *and* $L \sim 2^{1+4}_+{:}3^2.2^2$.
    (iii) $p = 3$, $G = X \cong \mathrm{G}_2(r^a)$ *with* $r^a \equiv 4, 7 \pmod 9$, $P \cong \mathrm{SU}_3(3)$ *and* $L \sim \mathrm{SL}_3(r^a).2$.
    (iv) $p = 3$, $G = X \cong \mathrm{G}_2(r^a)$ *with* $r^a \equiv 2, 5 \pmod 9$, $P \cong \mathrm{SU}_3(3)$ *and* $L \sim \mathrm{SU}_3(r^a).2$.
    (v) $p = 3$, $r^a = 2$ *and* $P = X$.

Proof. If $p > 3$, then Lemma 4.3 shows that $S \cap X$ is abelian, a contradiction. Therefore $p \leq 3$. If $r^a = 3$ and $p = 2$, we refer the reader to [29] (or [12]) to verify the details needed to confirm that (i) and (ii) hold. So assume that $r^a \neq 3$. Let $\epsilon = \pm$ be defined so that, if $d = 1$, $\epsilon = +$ and if $d = 2$, $\epsilon = -$.

Suppose that $p = 2$ and G does not involve the graph automorphism of X. By [29, Theorem A], there exist two maximal subgroups

$$
\begin{aligned}
K_1 &\sim (q - \epsilon, 3).\mathrm{PSL}_3^\epsilon(r^a).2; \text{ and} \\
K_2 &\sim 2.(\mathrm{PSL}_2(r^a) \times \mathrm{PSL}_2(r^a)).2
\end{aligned}
$$

containing $S \cap X$ and normalized by S. Since K_1 is not a 2-local subgroup, L does not contain K_1 and, as $r^a > 3$, Q_{K_2} has order 2 and so, as $X \notin \mathcal{L}_1(2)$, Theorem 2.16(iii) implies that L does not contain K_2. Thus $P \leq K_1 \cap K_2$. Applying Theorem 7.1 for $P \leq K_1 S$ and noting that P centralizes an involution, we have $r^a \equiv 3, 5 \pmod 8$, $r \neq 5$ (appearing in lines 10 and 16 of Table 1). In particular, a is odd and so $G = X$ and $|S| = 2^6$. Thus, again using [29], G contains two further over-groups of S, namely, $M_1 = \mathrm{G}_2(2)$ and $M_2 = 2^3{:}\mathrm{SL}_3(2)$. Since L is a 2-local subgroup of G we must have $L = M_2$. But, as $r \neq 3$, the structure of $L \cap K_1$ given by Theorem 7.1 is inconsistent with $L = M_2$. We conclude that if $p = 2$, then G contains the graph automorphism of X. Thus $p = 2$ and $r = 3$. Again suppose that $r^a \neq 3$. By [29, Theorem B], K_2 is invariant under the graph automorphism. Furthermore, the two components of K_2 are conjugate in $K_2 S$. Thus

$P \not\leq K_2S$ by Theorem 2.15(i). Hence $L \geq K_2$ and we obtain $|Q_L| = 2$ which is against Theorem 2.16(iii). This proves that $p \neq 2$.

Now assume that $p = 3$. Because of the isomorphism $\mathrm{G}_2(2)' \cong \mathrm{SU}_3(3)$ we see that (v) holds in this special case. So suppose that $r^a \neq 2$. This time as well as using [29] we need to employ [13]. We see that X contains a unique maximal 3-local subgroup K_1 which is therefore S-invariant:

$$K_1 \sim \mathrm{SL}_3^\epsilon(r^a).2.$$

Since $L \cap X$ is a maximal 3-local subgroup of X, we have $K_1 = L \cap X$. Thus, as SK_1/K_1 acts faithfully on $K_1/F(K_1)$, $|Q_L| = 3$ and 3 divides $r^a - \epsilon$ (and is inverted in K_1).

Since $Z(S) \leq K_1' \cong \mathrm{SL}_3^\epsilon(r^a)$, we have that $Z(S)$ is cyclic. Application of Lemma 2.5 shows that $|S| = 3^3$. Thus $S \cong 3_+^{1+2}$, $r^a - \epsilon \equiv 3, 6 \pmod 9$ and $G = X$. In this exceptional case, again using [13, 29], there is a further subgroup $K_2 \leq X$ with $K_2 \cong \mathrm{G}_2(2)$. Since $\mathrm{G}_2(2) \cong \mathrm{SU}_3(3){:}2$, we may take $P = K_2'$ to obtain the configurations in parts (iii) and (iv). $\square$

Lemma 9.4. *Suppose that $X \cong {}^3\mathrm{D}_4(r^a)$. Then $p = 2$, $X \cong {}^3\mathrm{D}_4(3)$, $L \sim (\mathrm{SL}_2(3) * \mathrm{SL}_2(27)).2$ and $P \sim 4^2.\mathrm{Dih}(12)$.*

Proof. As in the last lemma we only need to consider $p = 2$ and $p = 3$.

Suppose that $p = 2$. By [28], X has a subgroup $K_1 \cong \mathrm{G}_2(r^a)$ and, as $\Phi_3(r^a)$, $\Phi_6(r^a)$ and $\Phi_{12}(r^a)$ are all odd, Table 3 can be used to see that K_1 has odd index in X. Hence we may assume that K_1 is S-invariant. Since K_1 is not a 2-local subgroup of X, $P \leq K_1S$ by Theorem 2.16 (ii). Thus Lemma 9.3 implies that $r^a = 3$ from which we deduce $G = X$ and then $P \sim 4^2.\mathrm{Dih}(12)$. Now setting $K_2 = C_G(Z(S \cap X))$, we have $P \not\leq K_2$ and so $L = K_2 \sim (\mathrm{SL}_2(3) * \mathrm{SL}_2(27)).2$. This is the described configuration.

Suppose that $p = 3$. Then, using [28, Theorem] there is a subgroup K of X such that

$$K \cap X \sim ((q^2 + \epsilon q + 1) * \mathrm{SL}_3^\epsilon(r^a)).3.2$$

where $\epsilon \equiv r^a \pmod 3$. By Lemma 4.3, the subgroup $K \cap X$ is normalized by S and [28, Theorem] implies that $K \cap X$ is the unique maximal 3-local subgroup of X. Hence $L = KS$. If $a = 3^t$ for some $t \geq 0$, then [28, Theorem] implies that KS is the unique maximal subgroup containing S and we have that G is 3-minimal, a contradiction. Now write $a_3 = 3^t$ and assume that $a > 3^t$. Then X contains a subgroup $K_1 \cong {}^3\mathrm{D}_4(r^{3^t})$ and this group contains a Sylow 3-subgroup of X and can be chosen to be normalized by S. Hence $P \leq K_1S$ and is a rank one isolated 3-minimal subgroup in K_1S. However we have already noted that such groups are themselves 3-minimal and so we have a contradiction. $\square$

Lemma 9.5. *We have $X \not\cong {}^2\mathrm{F}_4(2^a)'$.*

Proof. Suppose that $X \cong {}^2\mathrm{F}_4(2^a)'$. Then $p \geq 3$ while, by Lemma 4.3 we have $p \leq 3$ as otherwise $S \cap X$ is abelian. Therefore $p = 3$ and, since a is odd, $d = 2$. Thus by Lemmas 4.3, $|S \cap X| = 3(\Phi_2(r^a)_3)^2$. If $r^a = 2$, we have $G = X \cong {}^2\mathrm{F}_4(2)'$ and we turn to [58] (or the

ATLAS[12]) to see that there are no 3-local subgroups which are maximal subgroups of G. Thus there are no candidates for L in this case. Hence we have $a \geq 3$. From [35, Table 5.1], X contains a unique conjugacy class of maximal subgroups with representative $K \sim \mathrm{PGU}_3(2^a).2$. As $|K|_3 = 3(\Phi_2(r^a)_3)^2$, we may assume that S normalizes K. Since $Q_L \cap X > 1$ and $O_3(K) = 1$, we have that $KS \geq P$ by Lemma 2.2 (iii). In particular, by Lemma 7.5, $2^a \equiv 2,5 \pmod 9$ and therefore $G = X$ and $S \cong 3_+^{1+2}$. But then there is a subgroup $Y \cong {}^2\mathrm{F}_4(2)$ contained in X with $S \leq Y$. Since Y is not a 3-local subgroup of X, P is a rank one isolated 3-minimal subgroup of Y. However we have already shown that this is impossible and thus the lemma holds. $\qquad \square$

For the final lemmas in this section we rely heavily on the existence of certain large subgroups of the exceptional groups. These are the so-called subgroups of maximal rank. The notation $W(\Phi)$ represents the Weyl group of the root system of type Φ.

Lemma 9.6. *We have $X \ncong \mathrm{F}_4(r^a)$.*

Proof. Suppose that $X \cong \mathrm{F}_4(r^a)$. If $p > 3$, then Lemma 4.3 shows that $S \cap X$ is abelian which is impossible. Therefore $p \in \{2,3\}$. Notice that, as $p \neq r$, the graph automorphism of $\mathrm{F}_4(2^a)$ makes no appearance in this discussion and so $|G/X|$ is a divisor of a and consists of images of field automorphisms.

Suppose that $p = 2$. Then, by [35, Table 5.1], as $\Phi_3(r^a)$, $\Phi_6(r^a)$ and $\Phi_{12}(r^a)$ are all odd, there is a maximal subgroup $K \leq X$ of odd index with $K \sim 2.\Omega_9(r^a) = C_X(t)$ where t is an involution and because G/X consists of field automorphisms, we can assume that K is normalized by S. By Lemma 8.12, KS has no isolated p-minimal subgroups and so $L = KS$ and $|Q_L| = 2$. But then Theorem 2.16 (i) provides a contradiction. Hence $p \neq 2$.

Assume that $p = 3$. Then from [35, Table 5.1] X contains a subgroup K_1 with

$$K_1 \sim (2, r^a - 1)^2.\mathrm{P\Omega}_8^+(r^a).\mathrm{Sym}(3)$$

From Table 3 (in Section 4) and Lemma 4.3, we calculate that K_1 contains a Sylow 3-subgroup of X. Since K_1 is not a 3-local subgroup of X and L normalizes $Q_L \cap X \neq 1$ by Theorem 2.16 (ii), P is a rank one isolated 3-minimal subgroup of K_1 by Lemma 2.2 (iii). Exploiting Theorem 8.13 yields $\eta \equiv r^a \pmod 3$, $r^a - \eta \equiv 3,6 \pmod 9$ with $r^a \neq 4$ and that

$$P \sim 3_+^{1+4}.\mathrm{SL}_2(3).$$

In particular, $P \leq C_G(Z(S))$.

Again using [35, Table 5.1] and the fact $r^a - \eta \equiv 3,6 \pmod 9$, we see that $G = X$ has a maximal over-group of S with

$$K_2 \sim 3.(\mathrm{PSL}_3^\eta(r^a) \times \mathrm{PSL}_3^\eta(r^a)).3.2$$

(with each of the components normal in K_2 as they originate from a subsystem subgroup of type $\mathrm{A}_2 \times \mathrm{A}_2$ with roots of different lengths). From the maximality of K_2, we deduce that $K_2 = N_G(Z(S))$ and so we have $P \leq K_2$. Additionally,

$$K_3 = (r^a - \eta)^4.W(\mathrm{F}_4)$$

can be assumed to contain S. Since $O_3(P)$ is extraspecial of order 3^5, $P \not\leq K_3$. Hence $L \geq K_3$.

Assume that $r^a \neq 2$. Let R_1 and R_2 be the components of K_2. Then, as $P \leq R_1 S$ or $P \leq R_2 S$ by Lemma 2.3, without loss of generality we may suppose that $R_1 S \leq L$. But then, as $r^a \neq 2$, $Q_L \leq Q_{R_1 S} \cong 3^{1+2}_+$. Since $P \not\leq L$, L does not normalize $Z(S)$ so we deduce Q_L has order 3^2. On the other hand $L \geq K_3$ and so Q_L has order 3^4. Clearly this is impossible. It remains to contemplate $r^a = 2$. For this case we consult [43] or the ATLAS [12] to see that there is a unique maximal 3-local subgroup in X and so $L = N_G(Z(S))$. But then we have $P \leq L$, a contradiction. $\square$

Lemma 9.7. *Assume that* $X \cong \mathrm{E}_6(r^a)$ *or* $^2\mathrm{E}_6(r^a)$. *Then* $p = 3$ *and either*

(i) $G = X \cong {}^2\mathrm{E}_6(2)$, $P \sim 3^{2+1+1+2+2}.\mathrm{SL}_2(3)$ *and*

$$L \sim 3.(\mathrm{PSU}_3(2) \times \mathrm{PSU}_3(2) \times \mathrm{PSU}_3(2)).3.\mathrm{Sym}(3) \sim 3^{1+6}_+.(\mathrm{Q}_8)^3.3^2.2;$$

or

(ii) $G \cong {}^2\mathrm{E}_6(2).3$, $P \sim 3^{2+1+1+2+2+1}.\mathrm{SL}_2(3)$ *and*

$$L \sim 3.(\mathrm{PSU}_3(2) \times \mathrm{PSU}_3(2) \times \mathrm{PSU}_3(2)).3^2.\mathrm{Sym}(3) \sim 3^{1+6}_+.(\mathrm{Q}_8)^3.3^3.2.$$

Proof. Write $\mathrm{E}_6^-(r^a) = {}^2\mathrm{E}_6(r^a)$ and $\mathrm{E}_6^+(r^a) = \mathrm{E}_6(r^a)$. By Lemma 9.1, $d \in \{1, 2\}$ and by Lemma 4.3, $p \in \{2, 3, 5\}$ as $X \notin \mathcal{L}_1(p)$.

Suppose that $p = 5$. Then $G = X$ and Lemma 4.3 further yields $(d, \epsilon) = (1, +)$ or $(2, -)$. Consider the subgroup

$$K = (2, r^a - 1).(\mathrm{PSL}_2(r^a) \times \mathrm{PSL}_6^\epsilon(r^a)).(2, r^a - 1)$$

of X given in [35, Table 5.1] (remember in [35] they include the outer diagonal automorphism which we do not have). Then K is S-invariant and contains a Sylow 5-subgroup of X (which has order $5(\Phi_d(r^a)_5)^6$). This group is not a 5-local subgroup of X and so $P \leq KS$ by Theorem 2.16 (ii). Since $p = 5$, Theorems 5.1 and 7.1 provide a contradiction unless $r^a = 4$ and $O^5(P) \cong \mathrm{PSL}_2(5)$ coincides with the first factor of K. In particular, we now have $X \cong {}^2\mathrm{E}_6(4)$ and that L contains the right hand component J of K. In particular, Q_L has order 5. Now let

$$K_1 = \mathrm{P\Omega}^-_{10}(4) \times 5.$$

Then, by [35, Table 5.1], K_1 is a subgroup of X and so can be selected to contain S. Theorem 8.11 implies that $L \geq K_1$ and, as K_1 is maximal by [35, Theorem], we have $J \leq L = K_1$ But $|J|$ does not divide $|K_1|$, a contradiction. Hence $p \neq 5$.

Suppose that $p = 3$. We first consider the possibilities $(d, \epsilon) = (2, +)$ or $(1, -)$. In these cases, as $(r^a - \epsilon, 3) = 1$, $|S \cap X| = 3^2(\Phi_d(r^a)_3)^4$ and by [36, Corollary 5], there is a unique conjugacy class of subgroups of X containing $S \cap X$ such that $K_1 \cong \mathrm{F}_4(r^a)$. Using Theorem 2.16 (ii) yields $P \leq K_1 S$, and so Lemma 9.6 delivers a contradiction. So we have $(d, \epsilon) = (1, +)$ or $(2, -)$. In this instance, $|S \cap X| = \frac{3^4(\Phi_d(r^a)_3)^6}{3}$ as X is the simple group, rather than the universal group. We have to adjust subgroups presented in [35] by the same factor of 3 as they consider the adjoint group and so their subgroups intersect X in

a subgroup of index 3. Thus, employing [35, Table 5.1], there is a maximal subgroup K of G containing S with

$$K \cap X \sim 3.(\mathrm{PSL}_3^\epsilon(r^a) \times \mathrm{PSL}_3^\epsilon(r^a) \times \mathrm{PSL}_3^\epsilon(r^a)).3.\mathrm{Sym}(3).$$

Assume that $r^a \neq 2$. Then K is not soluble. The components of K are permuted transitively by K (see [35, Table A]), and so $P \not\leq K$ by Theorem 2.15 (i). Hence $K = L$ and Q_L has order 3. Now we have K_1 containing S such that $K_1 \cap X$ is isomorphic to a subgroup of index 3 in $(r^a - \epsilon)^6.W(\mathrm{E}_6)$, where $W(\mathrm{E}_6)$ is the Weyl group of type E_6 (see [35, Table 5.2]). Furthermore, $O_3(K_1 \cap X)$ has order $\frac{((r^a - \epsilon)_3)^6}{3}$ and is contained in K. Thus $Q_L \leq C_X(O_3(K_1 \cap X))$ and so $Q_L \leq O_3(K_1)$. Thus $L \geq K_1$. Since $W(\mathrm{E}_6) \cong \Omega_6^-(2)$, we see that K_1 cannot be a subgroup of K. So $r^a = 2$ and $X \cong \mathrm{E}_6^-(2)$ with $G \cong \mathrm{E}_6^-(2)$ or $\mathrm{E}_6^-(2).3$.

Suppose that $G = X$. Then there exists $K_2 \geq S$, with $K_2 \cong \mathrm{Fi}_{22}$. Since $Q_{K_2} = 1$, we have $P \leq K_2$. Applying (10.2.17) (which is independent of the results here) we find

$$P \sim 3^{2+1+1+2+2}.\mathrm{SL}_2(3).$$

As $Z(S)$ is not normalized by P, we have $L = K$. This yields statement (i) of the theorem.

Assume that $G > X$. Then $G \sim \mathrm{E}_6^-(2).3$ and the subgroup $K_3 \sim 3^6.W(\mathrm{E}_6)$ is a maximal subgroup of G. Now

$$K/Q_K \sim (\mathrm{Q}_8 \times \mathrm{Q}_8 \times \mathrm{Q}_8).3_+^{1+2}.2$$

(this structure can be gleaned from $\mathrm{E}_8(2)$), Hence S acts irreducibly on $O_2(K/Q_K)/\Phi(O_2(K/Q_K))$ and so $O^2(K)$ is 3-minimal. As $K/Q_K \notin \mathcal{L}_1(3)$, we infer that $L = K$ and $P \leq K_3$. Since K_3 has exactly two 3-minimal subgroups, we obtain the configuration in part (ii).

So suppose that $p = 2$. Let

$$K = (4, r^a - \epsilon).(\mathrm{P}\Omega_{10}^\epsilon(r^a) \times (r^a - \epsilon)/(4, r^a - \epsilon)).(4, r^a - \epsilon).$$

Then, by [34, Table 1], K is a maximal subgroup of X which is fixed by the graph automorphism of X and has odd index in X. Hence K can be chosen to be normalized by S. Since groups Y with $F^*(Y) \cong \mathrm{P}\Omega_{10}^\epsilon(r^a)$ have no rank one isolated 2-minimal subgroups by Theorem 8.11, the left hand component J of K is contained in L. Hence Q_L centralizes J and so from the structure of K, Q_L is normal in KS as the graph and field automorphisms do not centralize J. The maximality of KS in G means that $L = KS$ and also $Q_L = Q_{KS} \leq X$. If $r^a - \epsilon \equiv 2 \pmod 4$, then $|Q_L| = 2$ and, as $O^2(P)$ is not normal in G, we have a contradiction to Theorem 2.16 (iii). Hence 4 divides $r^a - \epsilon$. Now the subgroup denoted by N_ϵ in [34, Table 1] can be chosen to be normalized by S. This subgroup is just the normalizer of a split maximal torus. Thus

$$\widehat{N}_\epsilon^* \sim (r^a - \epsilon)^6.W(\mathrm{E}_6)$$

and $N_\epsilon \not\leq L = KS$. Therefore $N_\epsilon S$ must contain P. Since KS has maximal rank $F(N_\epsilon) \leq K$ and indeed $Q_{KS} \leq F(N_\epsilon)$. But then $Q_L = Q_{KS} \leq O_2(F(N_\epsilon)) \leq Q_P$, a contradiction. This completes the proof of Lemma 9.7. $\square$

Lemma 9.8. *Suppose that* $X \cong E_7(r^a)$. *Then* $p = 2$, $X \cong E_7(3)$, $|G/X| \leq 2$,

$$P \cap X \sim 4^7.2^5.2^3 SL_2(2)$$

and

$$L \cap X \sim 2^3.(PSL_2(3)^7).2^3.SL_3(2).$$

Furthermore, $P \cap X$ is a rank one isolated 2-minimal subgroup of $J \cap X \sim (SL_2(3) * \Omega_{12}(3)).2$

Proof. As $O^p(P) \neq X$, Lemmas 4.5 and 4.3 show that we have to consider $p \in \{2, 3, 5, 7\}$. In addition, by Lemma 9.1, $d \in \{1, 2\}$. Set $\epsilon = +$, if $d = 1$ and $\epsilon = -$, if $d = 2$. By Lemma 4.3 we have

$$|S \cap X| = |W(E_7)|_p (r^a - \epsilon)_p / (r^a - \epsilon, 2)_p.$$

We freely use the lists of maximal subgroups as in [35, Table 5.1]. Let $\widetilde{X}$ be the adjoint version of X. So X has index $(r^a - 1, 2)$ in $\widetilde{X}$. It will be convenient to present certain subgroups H of G by describing an S-invariant over-group of $H \cap \widetilde{X}$ which we denote by $\widetilde{H}$ in $\widetilde{X}$. Thus $H = \widetilde{H}$ when p is odd and otherwise $|\widetilde{H}/H| \leq 2$.

Choose $K_1 \geq S$ so that K_1 normalizes a maximal torus of type $(r^a - \epsilon)^7$. Then

$$\widetilde{K}_1 = (r^a - \epsilon)^7 : W(E_7).$$

Suppose that p is odd. Then we read that

$$W(E_7) \cong 2 \times Sp_6(2)$$

has no rank one isolated p-minimal subgroups from Theorem 6.3. Therefore, in these cases, if $P \leq K_1$, $O^p(P) \leq X$ and so $O^p(P) \leq F(K_1)$. By Theorem 2.15 (i), $O^p(P)$ is a normal subgroup of $K_1 S$. Since $F(K_1)$ is abelian, Lemma 2.4 implies that $p = 3$ and $O^3(P)$ is elementary abelian of order 4. As K_1 has composition factors of order 2^6 and 2 in $\Omega_1(O_2(F(K_1)))$, this is impossible. Hence $L \geq K_1$ and consequently $Q_L \leq F(K_1)$, $\Omega_1(Q_L) = \Omega_1(Q_{K_1})$ has order p^7 and is an irreducible $K_1/F(K_1)$-module.

For $p \in \{3, 5\}$, the subgroup

$$\widetilde{K}_2 = (3, r^a - \epsilon).(E_6^\epsilon(r^a) \times (r^a - \epsilon)/(3, r^a - \epsilon)).(3, r^a - \epsilon)$$

can be chosen to contain $S \cap X$ and be S-invariant. From Lemma 9.7 we get $L \geq K_2$ and so Q_L is cyclic contrary to $|\Omega_1(Q_L)| = p^7$.

For $p = 7$, we set

$$\widetilde{K}_3 = f.PSL_8(r^a).g.(2 \times (2/f))$$

where f and g are powers of 2 as described in [35, Table 5.1]. Since K_3 is not a 7-local subgroup, we have $P \leq K_3 S$ and this contradicts Theorem 7.1.

Next suppose that $p = 2$. Then r^a is odd and we select subgroups K_4 and K_5 containing S (see [34, Table 1]) such that

$$\widetilde{K}_4 \sim 2.(PSL_2(r^a) \times P\Omega_{12}^+(r^a)).2$$

and

$$\widetilde{K}_5 \sim 2^3.(PSL_2(r^a)^7).2^4.PSL_3(2)$$

with K_5 transitive on the components of K_5 when $r^a > 3$ [35, Table A and Table 5.1].

Assume that $r^a \neq 3$. Then, invoking Theorem 2.15 (i), we see $P \not\leq K_5$ and so $L \geq K_5$. In particular, Q_L is elementary abelian of order 8. On the other hand, the right hand factor of K_4 is also contained in L by Theorem 8.11 and this means Q_L is contained in a quaternion group, a contradiction.

Thus $r^a = 3$. Suppose that $L = K_4$. Then Q_L is quaternion of order 8 and $Q_L \leq Q_{K_5} \leq Q_P$ for a contradiction. Therefore, $L = K_5$ and $P \leq K_4$. The structure and location of P now follows from Theorem 8.11. However, the description that we present comes from the fact that $P \leq \widetilde{K}_6 \sim 4^7.(2 \times \mathrm{Sp}_6(2))$ and in this group it corresponds to the 2-minimal subgroup which does not normalize an isotropic 3-space in the natural $\mathrm{Sp}_6(2)$-module. This completes the proof of Lemma 9.8. $\qquad\square$

Lemma 9.9. *Assume that* $X \cong \mathrm{E}_8(r^a)$. *Then* $p = 3$, $G = X \cong \mathrm{E}_8(2)$, $P \sim 3^{[12]}.\mathrm{SL}_2(3)$ *and* $L \sim 3^2.(\mathrm{PSU}_3(2) \times \mathrm{PSU}_3(2) \times \mathrm{PSU}_3(2) \times \mathrm{PSU}_3(2)).3^2.\mathrm{GL}_2(3)$.

Proof. We have $d \in \{1, 2\}$ and $p = 2, 3, 5$ or 7. As usual we set $\epsilon = +$ if $d = 1$ and $\epsilon = -$ if $d = 2$. We have

$$|S \cap X| = |W(\mathrm{E}_8)|_p (r^a - \epsilon)_p.$$

Let K_1 be a subgroup of G which contains S and has

$$K_1 \cap X \sim (r^a - \epsilon)^8.W(\mathrm{E}_8).$$

Suppose that $p \in \{2, 7\}$. Then, by [35, Table 5.1] there exists a maximal subgroup K_2 of G with

$$K_2 \cap X \sim (r^a - 1, 2).\mathrm{P\Omega}_{16}^+(r^a).(r^a - 1, 2).$$

Notice that $K_1 \cap K_2$ contains the subgroup of shape

$$(r^a - \epsilon)^8.W(\mathrm{D}_8)$$

and so as $W(\mathrm{D}_8)$ contains a Sylow p-subgroup of $W(\mathrm{E}_8)$, we may suppose that K_2 contains S. By Theorem 8.11, we must have $L = K_2$ and then $|Q_L| \leq 2$. Hence Theorem 2.16 (i) delivers the knockout punch.

Suppose that $p = 5$. Using [35, Table 5.1] we see that G has a maximal subgroup K_3 with

$$K_3 \cap X \sim 5.(\mathrm{PSL}_5^\epsilon(r^a) \times \mathrm{PSL}_5^\epsilon(r^a)).5.4.$$

This time we can organize $K_1 \cap K_3$ to contains $(r^a - \epsilon)^8.(W(\mathrm{A}_4) \times W(\mathrm{A}_4))$ and this contains a Sylow 5-subgroup of X. Thus we may suppose that K_3 contains S. Using Theorem 7.1 we obtain that $P \not\leq K_3$. Hence $L = K_3$ but, as P is not soluble by Lemma 2.4, Theorem 8.11 implies $L \geq K_1$ and of course $K_1 \not\leq K_3$, a contradiction. Thus $p \neq 5$.

Finally assume that $p = 3$. Then by [35, Table 5.1] there are maximal subgroups K_4 and K_5 of G containing S such that

$$K_4 \cap X \sim 3.(\mathrm{PSL}_3^\epsilon(r^a) \times \mathrm{E}_6^\epsilon(r^a)).3.2$$

and

$$K_5 \cap X \sim 3^2.(\mathrm{PSL}_3^\epsilon(r^a) \times \mathrm{PSL}_3^\epsilon(r^a) \times \mathrm{PSL}_3^\epsilon(r^a) \times \mathrm{PSL}_3^\epsilon(r^a)).3^2.\mathrm{GL}_2(3).$$

Furthermore, from the construction of K_4 and K_5 we see that

$$K_4 \cap K_5 \cap X \sim 3^2.(\mathrm{PSL}_3^\epsilon(r^a) \times \mathrm{PSL}_3^\epsilon(r^a) \times \mathrm{PSL}_3^\epsilon(r^a) \times \mathrm{PSL}_3^\epsilon(r^a)).3^2.\mathrm{Sym}(3).$$

Assume that K_5 is not soluble. Then, as the components of K_5 are permuted transitively by K_5 (see [35, Table A]), $L \geq K_5$ by Theorem 2.15 (i). It follows that $P \leq K_4$ and, from the structure of $K_4 \cap K_5$, we infer that $P \leq RS$ where R is the component of K_4 with $R \sim 3{\cdot}\mathrm{E}_6^\epsilon(r_a)$. But then we have a contradiction from Lemma 9.7 (as then K_5 is soluble). Thus we infer that K_5 is indeed soluble and hence $r^a = 2$, $d = 2$ and $\epsilon = -$. Now suppose that $P \leq K_5$. Then by Lemma 9.7, $P \not\leq RS$ and so $L \geq RS \sim (3_+^{1+2} * 3{\cdot}^2\mathrm{E}_6(2).3.2)$. But then $Q_L \leq Q_{K_5} \leq Q_P$, a contradiction. Therefore, $L = K_5$. It follows that $P \leq RS$ and we have the example displayed in the lemma. $\qquad\square$

10. Sporadic simple groups

The main battle is now over, we are left with the task of mopping up the remaining insurgents. Throughout this section Hypothesis 2.18 holds and X will be a sporadic simple group. An observation that we invoke frequently in this section is contained in our first lemma. For the smaller sporadic simple groups, where the data we give can be meaningful, we present descriptions of P as well as of L. For the larger groups, as the structure of P is less well defined, we just give the structure of the maximal subgroup L. In addition, throughout this section, we freely use the lists of maximal subgroups of the sporadic simple groups as listed in the appropriate sections of [12, 60].

Lemma 10.1. $S \cap X$ is non-abelian. In particular $|S \cap X| \geq p^3$.

Proof. If p is odd, then, as $|\mathrm{Out}(X)|$ divides 2 this is Lemma 2.14. If $p = 2$, then we have $G = X \cong \mathrm{J}_1$ and again Lemma 2.14 applies. $\qquad\square$

Theorem 10.2. *For X a sporadic simple group, the conclusions in Theorem 1.6 (i)(c), (ii)(b), (iii) and (iv) hold.*

Proof. We deal with each of the 26 possibilities for X in increasing order. Before we begin we note that by Lemma 10.1 we can assume that X does not have abelian Sylow p-subgroups, by Lemma 2.12, $N_G(S)$ is not a maximal subgroup of G and by Lemma 2.2 L is a maximal subgroup of G. We make these three preliminary checks for each of the possibilities for X to limit the primes p that need to be investigated.

(10.2.1) *We have $X \not\cong \mathrm{M}_{11}$.*

By Lemma 10.1, $p = 2$. Let $H \leq G$ with $H \cong \mathrm{M}_{10}$. Then H is a 2-minimal subgroup of G and $Q_H = 1$. Hence by Lemma 2.2, $H = P$, and so X is not isomorphic to M_{11} as $H \notin \mathcal{L}_1(2)$. $\qquad\blacksquare$

(10.2.2) *Suppose that $X \cong \mathrm{M}_{12}$. Then one of the following holds.*
(i) $p = 2$, $G = X$, $P \sim 2_+^{1+4}.\mathrm{Sym}(3)$ and $L \sim 4^2.2.\mathrm{Dih}(12)$.

(ii) $p = 2$, $|G/X| = 2$, $P \sim 2^{1+4}_+.\mathrm{Sym}(3).2$ *and* $L \sim 4^2.2.\mathrm{Dih}(12).2$.

(iii) $p = 2$, $G = X$, $P \sim 4^2.2.\mathrm{Dih}(12)$ *and* $L \sim 2^{1+4}_+.\mathrm{Sym}(3)$.

(iv) $p = 2$, $|G/X| = 2$, $P \sim 4^2.2.\mathrm{Dih}(12).2$ *and* $L \sim 2^{1+4}_+.\mathrm{Sym}(3).2$.

(v) $p = 3$, $G = X$, $P \sim 3^2{:}\mathrm{SL}_2(3)$ *and* $L \sim 3^2{:}\mathrm{GL}_2(3)$.

We have $p \in \{2, 3\}$ by Lemma 10.1 and consulting [60, Table 5.1] yields the five listed possibilities. ∎

(10.2.3) *We have $X \not\cong \mathrm{J}_1$.*

Since J_1 has abelian Sylow p-subgroups for all prime p, (10.2.3) holds. ∎

(10.2.4) *Suppose that $X \cong \mathrm{M}_{22}$. Then $p = 2$ and one of the following holds.*

(i) $G = X$, $P \sim 2^4{:}\mathrm{Sym}(5)$ *and* $L \sim 2^4{:}\mathrm{Alt}(6)$.

(ii) $|G/X| = 2$, $P \sim 2^5{:}\mathrm{Sym}(5)$ *and* $L \sim 2^4{:}\mathrm{Sym}(6)$.

(iii) $G = X$, $P \sim 2^{1+2+1+2}{:}\mathrm{Sym}(3)$ *and* $L \sim 2^4{:}\mathrm{Sym}(5)$.

(iv) $|G/X| = 2$, $P \sim 2^{1+2+1+2+1}{:}\mathrm{Sym}(3)$ *and* $L \sim 2^{4+1}{:}\mathrm{Sym}(5)$.

The claim follows from Lemma 10.1, using [60, Table 5.1] and the maximality of L. ∎

(10.2.5) *Suppose that $X \cong \mathrm{J}_2$. Then one of the following holds.*

(i) $p = 2$, $G = X$, $P \sim 2^{2+4}.\mathrm{Sym}(3)$ *and* $L \sim 2^{1+4}_-.\mathrm{Alt}(5)$.

(ii) $p = 2$, $|G/X| = 2$, $P \sim 2^{2+4+1}.\mathrm{Sym}(3)$ *and* $L \sim 2^{1+4}_-.\mathrm{Sym}(5)$.

(iii) $p = 2$, $G = X$, $P \sim 2^{1+4}_-.\mathrm{Alt}(5)$ *and* $L \sim 2^{2+4}.(3 \times \mathrm{Sym}(3))$.

(iv) $p = 2$, $|G/X| = 2$, $P \sim 2^{1+4}_-.\mathrm{Sym}(5)$ *and* $L \sim 2^{2+4}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3))$.

(v) $p = 3$, $G = X$, $P \cong \mathrm{PSU}_3(3)$ *and* $L \sim 3.\mathrm{PGL}_2(9)$.

We have $p \leq 3$. For $p = 2$ we use [60, Table 5.4] to get (i), (ii), (iii) and (iv). For $p = 3$, $G = X$ has a unique maximal 3-local subgroup containing S and so this must be L. Since $\mathrm{PSU}_3(3) \in \mathcal{L}_1(3)$, we have (iv). ∎

(10.2.6) *Suppose that $X \cong \mathrm{M}_{23}$. Then $p = 2$, $G = X$, $P \sim 2^4.\mathrm{Sym}(5)$ and $L \sim 2^4.\mathrm{Alt}(7)$.*

In this case $p = 2$. Since G has a maximal 2-local subgroup $H \sim 2^4.\mathrm{Alt}(7)$ and $\mathrm{Alt}(7)$ has no isolated 2-minimal subgroups for $p = 2$ by Theorem 3.3, we infer that $L = H$. Therefore P is contained in the 2-local subgroup K of shape $2^4.\mathrm{Sym}(5)$ and, as this group is 2-minimal, $P = K$ and we are done. ∎

(10.2.7) *Suppose that $X \cong \mathrm{HS}$. Then $p = 2$ and one of the following holds.*

(i) $G = X$, $P \sim 4 * 2^{1+4}_+.\mathrm{Sym}(5)$ *and* $L \sim 4^3.\mathrm{PSL}_3(2)$.

(ii) $|G/X| = 2$, $P \sim 2^{1+6}_+.\mathrm{Sym}(5)$ *and* $L \sim 4^3.(2 \times \mathrm{PSL}_3(2))$.

(iii) $G = X$, $P \sim 2^{2+1+2+1+2}.\mathrm{Sym}(3)$ *and* $L \sim 4 * 2_+^{1+4}.\mathrm{Sym}(5)$.
(iv) $|G/X| = 2$, $P \sim 2^{2+1+2+1+2+1}.\mathrm{Sym}(3)$ *and* $L \sim 2_+^{1+6}.\mathrm{Sym}(5)$.

We have $p \in \{2, 5\}$ as $S \cap X$ is non-abelian. Suppose that $p = 5$. Then $G = X$ and, as there are no maximal 5-local subgroups containing S, there are no candidates for L in this case. For $p = 2$, L is either of the maximal 2-local subgroups containing S and P is the unique 2-minimal subgroup not contained in L. ∎

(10.2.8) *Suppose that* $X \cong \mathrm{J}_3$. *Then* $p = 2$ *and one of the following holds.*
 (i) $G = X$, $P \sim 2_-^{1+4}.\mathrm{Alt}(5)$ *and* $L \sim 2^{2+4}.(3 \times \mathrm{Sym}(3))$.
 (ii) $|G/X| = 2$, $P \sim 2_-^{1+4}.\mathrm{Sym}(5)$ *and* $L \sim 2^{2+4}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3))$.
 (iii) $G = X$, $P \sim 2^{2+4}.\mathrm{Sym}(3)$ *and* $L \sim 2_-^{1+4}.\mathrm{Alt}(5)$.
 (iv) $|G/X| = 2$, $P \sim 2^{2+4+1}.\mathrm{Sym}(3)$ *and* $L \sim 2_-^{1+4}.\mathrm{Sym}(5)$.

We have $p \leq 3$. Also, for $p = 3$, $N_G(S)$ is a maximal subgroup of G and so this case is impossible. Thus $p = 2$ and we have the result. ∎

(10.2.9) *Suppose that* $X \cong \mathrm{M}_{24}$. *Then* $p = 2$, $G = X$ *and either* $L \sim 2^4.\mathrm{Alt}(8)$, $L \sim 2^6.3.\mathrm{Sym}(6)$ *or* $L \sim 2^6.(\mathrm{PSL}_3(2) \times \mathrm{Sym}(3))$. *Furthermore, in each case* P *is uniquely determined and* $P/Q_P \cong \mathrm{Sym}(3)$.

We have $p \leq 3$. Furthermore, for $p = 3$, G has no maximal subgroup which is also a 3-local subgroup. Hence $p = 2$. Now L is one of the three maximal 2-local subgroups containing S and P the unique 2-minimal subgroup not contained in L. ∎

(10.2.10) *Suppose* $X \cong \mathrm{McL}$. *Then* $p = 3$, $G = X$, $P \sim 3^4.\mathrm{Alt}(6) \sim 3^4.\mathrm{PSL}_2(9)$ *and* $L \sim 3_+^{1+4}.2.\mathrm{Sym}(5)$.

This time $p \leq 5$ and, for $p = 5$, $N_G(S)$ is a maximal subgroup of G, so we infer that $p \leq 3$. For $p = 3$, P is contained in a group of shape $3^4.\mathrm{M}_{10}$ or $3_+^{1+4}.\mathrm{Sym}(5)$. The former case gives our example. In the latter case we use Theorem 3.3 to see that $\mathrm{Sym}(5)$ does not contain any rank one isolated 3-minimal subgroups for $p = 3$. Thus (10.2.10) holds in this case.
Suppose $p = 2$. If $G \neq X$, then G has a 2-minimal subgroup $\mathrm{PSL}_3(4).2^2$ which must be P by Lemma 2.2 (iii). But then $O^2(P/Q_P) \notin \mathcal{L}_1(2)$ and we have a contradiction. Therefore $G = X$. Let $S \leq H \leq G$ with $H \sim 2^4.\mathrm{Alt}(7)$. Then H/Q_H has no rank one isolated 2-minimal subgroups by Theorem 3.3 and so $H = L$. But then there are two different choices for L, a contradiction. ∎

(10.2.11) *Suppose* $X \cong \mathrm{He}$. *Then* $G = X$, $p = 2$, $P \sim 2^{1+1+2+2+1+2}.\mathrm{Sym}(3)$ *and* $L \sim 2^6.3.\mathrm{Sym}(6)$. *There are two possibilities for the pair* (P, L) *and they are exchanged by an outer automorphism of* G.

We have $p \leq 3$ as the normalizer of a Sylow 7-subgroup is maximal in G. For $p = 3$, there is a member R of $\mathcal{P}_G(S)$ of shape $2^6.3_+^{1+2}$ ($\leq 2^6.3.\mathrm{Sym}(6)$) which has $Q_R = 1$. Thus $R \not\leq L$ and so $P = R$ and this contradicts $P \in \mathcal{L}_1(3)$.

Therefore $p = 2$. Suppose $G > X$ and let $H \in \mathcal{P}_G(S)$. Then $H/Q_H \cong \mathrm{Sym}(3) \wr \mathrm{Sym}(2)$ or $\mathrm{PSL}_3(2).2$. As in both cases $H/Q_H \notin \mathcal{L}_1(2)$ there are no candidates for P. Therefore $G = X$. In this case there are three maximal 2-local subgroups, two, K_1 and K_2, of shape $2^6{:}3 \cdot \mathrm{Sym}(6)$ and K_3 of shape $2_+^{1+6}.\mathrm{PSL}_3(2)$. The group $K_1 \cap K_2$ has shape $2^{4+4}{:}(\mathrm{Sym}(3) \times \mathrm{Sym}(3))$ and this subgroup accounts for two of the 2-minimal subgroups in each of K_1 and K_2. The remaining 2-minimal subgroup of K_1 and K_2 each centralize a 2-central involution and so are contained in K_3 and these are the two separate candidates for P. ∎

(10.2.12) *Suppose $X \cong \mathrm{Ru}$. Then $p = 2$, $G = X$ and either*
   (i) $P \sim 2^{1+4+6}.\mathrm{Sym}(5)$ and $L \sim 2^{3+8}.\mathrm{PSL}_3(2)$; or
   (ii) $P \sim 2^{2+1+2+2+1+1+2+2}.\mathrm{Sym}(3)$ and $L \sim 2^{1+4+6}.\mathrm{Sym}(5)$.

We have $p \leq 5$ and, for $p = 5$, $N_G(S)$ is maximal in G so $p \leq 3$. For $p = 3$, G contains a 3-minimal subgroup $K \sim 2^6.3_+^{1+2}$ with $Q_K = 1$. Therefore, this group must be P and we have a contradiction as $O^3(P/Q_P) \notin \mathcal{L}_1(3)$. Hence $p = 2$ and, after consulting [60, Table 5.11], we have the two listed possibilities. ∎

(10.2.13) *Suppose that $X \cong \mathrm{Suz}$. Then $|G/X| \leq 2$ and one of the following holds.*
   (i) $p = 2$, $P/Q_P \cong \mathrm{Sym}(3)$ and $L \cap X \sim 2_-^{1+6}.\mathrm{PSU}_4(2)$, $2^{4+6}.3.\mathrm{Alt}(6)$, or $2^{2+8}.(\mathrm{Alt}(5) \times \mathrm{Sym}(3))$.
   (ii) $p = 3$, $G = X$, $P \sim 3^{1+1+2+2}.\mathrm{SL}_2(3)$ and $L \sim 3^5.\mathrm{M}_{11}$.

We have $p \leq 3$. For $p = 3$, G has a maximal subgroup $H \sim 3^5.\mathrm{M}_{11}$ containing S. Since, by (10.2.1), H has no rank one isolated 3-minimal subgroups, for $p = 3$, we have $L = H$ and $P \leq K \sim 3.\mathrm{PSU}_4(3){:}2$. So (ii) holds. For $p = 2$, we obtain (i) from the maximal subgroups of X and G containing $S \cap X$. Each of these subgroups determines a unique 2-minimal rank one subgroup which it does not contain and so we see that $P/Q_P \cong \mathrm{Sym}(3)$. ∎

(10.2.14) *Suppose that $X \cong \mathrm{O'N}$. Then $p = 2$, $|G/X| \leq 2$, $P/Q_P \cong \mathrm{Sym}(3)$ and $L \cap X \sim 4.\mathrm{PSL}_3(4).2$.*

As usual, we have $p \in \{2, 7\}$ and since G contains no maximal 7-local subgroups we infer that $p = 2$. Let $H \leq G$ be the maximal subgroup with $H \cap X \sim 4.\mathrm{PSL}_3(4).2_1$. Then H is 2-minimal but $O^2(H/Q_H) \notin \mathcal{L}_1(2)$ and so we conclude that $L = H$. This forces the given description of P. ∎

(10.2.15) *Suppose that $X \cong \mathrm{Co}_3$. Then one of the following holds.*

  (i) $p = 2$, $P/Q_P \cong \mathrm{Sym}(3)$ and $L \sim 2^{2+6}.3_+^{1+2}.2^2$.
  (ii) $p = 3$ and $P \sim 3_+^{1+4}.\mathrm{SL}_2(9)$ and $L \sim 3^5.2.\,\mathrm{M}_{11}$.

We have $p \leq 5$. For $p = 5$, G contains no maximal 5-local subgroups and so $p \leq 3$. Assume that $p = 3$. Then G contains a maximal subgroup $H \sim 3^5.2.\,\mathrm{M}_{11}$ containing S. Since, by (10.2.1), H has no rank one isolated 3-minimal subgroups for $p = 3$, we conclude that $L = H$. So (ii) holds.

For $p = 2$, let $H_1 \sim 2\cdot\mathrm{Sp}_6(2)$, $H_2 \sim 2^{2+6}.3_+^{1+2}.2^2$ and $H_3 \sim 2^4.\,\mathrm{Alt}(8)$ be the maximal 2-local subgroups of G containing S. By Theorem 2.16(iii), $L \neq H_1$ and so $P \leq H_1$. Define $Q_i = Q_{H_i}$, $i = 1, 2, 3$. Assume that $L = H_3$. Then

$$H_{13} = L \cap H_1 \sim 2^{1+3+3}.\mathrm{PSL}_3(2)$$

and, as $O_2(H_{13}/Q_1)$ is a uniserial module for $H_{13}/O_2(H_{13}) \cong \mathrm{PSL}_3(2)$, we infer that H_{13} has a unique normal subgroup of order 2^4 which must therefore be Q_3.

Now choose K with $S \leq K \leq H_1$ and $K \sim 2^{1+1+4}.\mathrm{Sp}_4(2)$. Then $P \leq K$. Assume for a moment that $|Q_3 Q_K / Q_K| \geq 4$. Then $|[Q_K, Q_3]Q_1/Q_1| \geq 4$ from the structure of the natural $\mathrm{GF}(2)\mathrm{Sp}_4(2)$-modules and this means that $|Q_K \cap Q_3| \geq 2^3$ which in turn gives $2^4 \geq |Q_3| \geq 2^5$. This being impossible we infer that $|Q_3 Q_K / Q_K| \leq 2$. But then

$$Q_3 Q_K / Q_K \leq Z(S/Q_K) \leq O_2(P/Q_K)$$

and we have a contradiction. Therefore $L \neq H_3$. The only remaining possibility is that described in (i). ∎

(10.2.16) *Suppose that $X \cong \mathrm{Co}_2$. Then $G = X$ and one of the following holds.*

  (i) $p = 2$, $P/Q_P \cong \mathrm{Sym}(3)$ and $L \sim 2^{10}.(\mathrm{M}_{22}{:}2)$.
  (ii) $p = 2$, $P/Q_P \cong \mathrm{Sym}(5)$ and $L \sim 2_+^{1+8}.\mathrm{Sp}_6(2)$.
  (iii) $p = 2$, $P/Q_P \cong \mathrm{Sym}(3)$ and $L \sim 2^{4+10}.(\mathrm{Sym}(3) \times \mathrm{Sym}(5))$.
  (iv) $p = 3$, $P \sim 3^4.\mathrm{PSL}_2(9)$ and $L \sim 3_+^{1+4}.2_-^{1+4}.\,\mathrm{Sym}(5)$.

We have $p \leq 3$. Suppose $p = 3$. Since $G = X$ contains a maximal subgroup K which is isomorphic to McL, we infer that $P \leq K$ and the structure of P follows from (10.2.10). Let $S \leq H \leq G$ with $H \sim 3_+^{1+4}.2_-^{1+4}.\,\mathrm{Sym}(5)$. Then $L = H$ and this proves (iv).

For $p = 2$, L can be any of the maximal 2-local subgroups and so we get (i), (ii) or (iii). ∎

(10.2.17) *Suppose that $X \cong \mathrm{Fi}_{22}$. Then one of the following holds.*

  (i) $p = 2$, $P/Q_P \cong \mathrm{Sym}(3)$ and $L \cap X \sim 2^{10}.\,\mathrm{M}_{22}$.
  (ii) $p = 2$, $P/Q_P \cong \mathrm{Sym}(3)$ and $L \cap X \sim 2 \times 2_+^{1+8}.\mathrm{PSU}_4(2).2$.
  (iii) $p = 2$, $P/Q_P \cong \mathrm{Sym}(5)$ and $L \cap X \sim 2^{5+8}.(\mathrm{Sym}(3) \times \mathrm{Alt}(6))$.
  (iv) $p = 3$, $G = X$, $P \sim 3^{2+1+2}.3_+^{1+2}.\mathrm{SL}_2(3)$ and $L \sim 3_+^{1+6}.2^{1+2+2+2}.3^{1+1}.2$.

Examples (i), (ii) and (iii) all extend uniquely to $\mathrm{Aut}(X)$.

This time $p \leq 3$. Suppose $p = 3$. There is a unique maximal 3-local subgroup containing S and so
$$L \sim 3^{1+6}_+.2^{1+2+2+2}.3^{1+1}.2$$
Furthermore, $P \leq K \cong \mathrm{P}\Omega_7(3)$ is the 3-minimal subgroup which corresponds to the middle node of the B_3 Dynkin diagram of K. Therefore
$$P \sim 3^{2+1+2}.3^{1+2}_+.\mathrm{SL}_2(3)$$
and this is (iv).

If $p = 2$, then L can be any of the three maximal 2-local subgroups (see [48, page 72]). So (i), (ii) and (iii) hold. ∎

(10.2.18) *Suppose $X \cong \mathrm{HN}$. Then one of the following holds.*
  (i) $p = 2$, $|G{:}X| \leq 2$, $P/Q_P \cong \mathrm{Sym}(3)$, $L \cap X \sim 2^{1+8}_+.(\mathrm{Alt}(5) \wr \mathrm{Sym}(2))$.
  (ii) $p = 5$, $G = X$, $P \sim 5^{2+1+2}.\mathrm{SL}_2(5)$ *and* $L \sim 5^{1+4}_+.2^{1+4}_-.5.4$.

We have $p \leq 5$. Suppose that $p = 5$ and let H be the maximal 5-local subgroup with $H \sim 5^{1+4}_+.2^{1+4}_-.5.4$. Then, as $p = 5$, P is not soluble and so $L = H$. Thus (ii) holds.

If $p = 3$, then $G = X$ contains a subgroup $H \sim 3^{1+4}_+.4.\mathrm{Alt}(5)$. Since H contains no rank one isolated 3-minimal subgroups, we have $L = H$. Let K be the 3-local subgroup with $K \sim 3^4.2.(\mathrm{PSL}_2(3) \times \mathrm{PSL}_2(3)).4$. Since $N_K(S)$ is a maximal subgroup of K, Lemma 2.12 implies that P is normal in K. Since $S \leq P$, this means that $P = K$ contrary to $P/Q_P \in \mathcal{L}_1(3)$. So $p \neq 3$.

Suppose $p = 2$. Then X has maximal 2-local subgroup H with
$$H \sim 2^{1+8}_+.(\mathrm{Alt}(5) \wr \mathrm{Sym}(2)).$$
Theorem 2.15 (i) applied to H/Q_H implies that that $L = H$. There is only one other maximal over-group of $S \cap X$ and it has shape $2^{3+2+6}.(3 \times \mathrm{PSL}_3(2))$. Thus we observe the example in (i) and its extension to $\mathrm{Aut}(X)$. ∎

(10.2.19) *Suppose $X = \mathrm{Ly}$. Then $p = 5$, $P \sim 5^{2+1+2}.\mathrm{SL}_2(5)$ and $L \sim 5^{1+4}_+.4.\mathrm{Sym}(6)$.*

For $p \geq 7$, S is cyclic. So $p \leq 5$. Suppose that $p = 5$. Let $S \leq H \leq G$ with
$$H \sim 5^{1+4}_+.4.\mathrm{Sym}(6).$$
Since $4.\mathrm{Sym}(6)$ has no rank one isolated 5-minimal subgroup, $P \not\leq H$ and so $L = H$. The only other maximal over-group of S in X has shape $5^3.\mathrm{SL}_3(5)$ and this group has just two 5-minimal subgroups, one of which is in L and the other is $P \sim 5^{2+1+2}.\mathrm{SL}_2(5)$. This gives the example in (10.2.19).

Suppose $p = 3$. Then G has maximal subgroups
$$H \sim 3^5.(2 \times \mathrm{M}_{11})$$

and
$$K \sim 3^{2+4}.2.\,\mathrm{Alt}(5).\mathrm{Dih}(8).$$
By (10.2.1), H has no rank one isolated 3-minimal subgroups and so $L = H$ and $P \leq K$. As $N_K(S)$ is a maximal subgroup of K, Lemma 2.12 implies P is normal in K and so P involves $\mathrm{Alt}(5)$ contrary to $P/Q_P \in \mathcal{L}_1(3)$. Thus $p \neq 3$.

Suppose that $p = 2$. Then G has a subgroup $H \sim 3.\,\mathrm{McL}\,.2$ of odd index. Using (10.2.10) $P \nleq H$ and so $L = H$. Since $Q_H = 1$, we have a contradiction. Hence $p \neq 2$. ∎

(10.2.20) *Suppose $X = \mathrm{Th}$. Then one of the following holds.*
   (i) $p = 2$, $P/Q_P \cong \mathrm{Sym}(3)$ and $L \sim 2_+^{1+8}.\,\mathrm{Alt}(9)$.
   (ii) $p = 3$, $\mathcal{P}_G(S) = \{P_1, P_2\}$, *for* $i = 1, 2$, $P = P_i$ *and* $L = P_{3-i}N_G(S)$. *Furthermore,*
   $P/Q_P \cong \mathrm{SL}_2(3)$.

We use the maximal subgroups as given in [37, 38]. By Lemma 10.1, we have $p \leq 5$ and $p = 5$ is impossible as $N_G(S)$ is a maximal subgroup of G. Suppose that $p = 3$. There are two maximal subgroups $H \sim [3^9].\,\mathrm{GL}_2(3)$ and $K \sim [3^9].\,\mathrm{GL}_2(3)$ containing S. This leads to exactly two rank one 3-minimal subgroups. This proves (ii).

For $p = 2$, we have $H \sim 2_+^{1+8}.\,\mathrm{Alt}(9)$ and $K \sim 2^5.\mathrm{SL}_5(2)$ are the only maximal subgroups containing S. Using Theorem 3.3, $L = H$ and so $P/Q_P \cong \mathrm{Sym}(3)$. ∎

(10.2.21) *Suppose that $X \cong \mathrm{Fi}_{23}$. Then one of the following holds.*
   (i) $p = 2$, $P/Q_P \cong \mathrm{Sym}(3)$ and $L \sim 2^{11}.\,\mathrm{M}_{23}$.
   (ii) $p = 2$, $P/Q_P \cong \mathrm{Sym}(5)$ and $L \sim 2^{6+2\cdot4}.(\mathrm{Alt}(7) \times \mathrm{Sym}(3))$.
   (iii) $p = 3$, $P/Q_P \cong \mathrm{SL}_2(3)$ and $L \sim 3_+^{1+8}.2_-^{1+6}.3_+^{1+2}.2.\,\mathrm{Sym}(4)$.

We use the maximal subgroups as presented in [31]. We have $p \leq 3$ as otherwise S is abelian.

Suppose that $p = 3$ and let $S \leq H \leq G$ with
$$H \sim 3_+^{1+8}.2_-^{1+6}.3_+^{1+2}.2.\,\mathrm{Sym}(4).$$
Since H is soluble and $O^3(P)$ is not normal in H, Theorem 2.15 (ii) implies that $L = H$. Thus
$$P \leq K \sim 3^{3+1+3+3}.(2 \times \mathrm{SL}_3(3))$$
and this gives the structure of P.

Suppose $p = 2$. Let $H_1 \sim 2^{6+8}.(\mathrm{Sym}(3) \times \mathrm{Alt}(7))$ and $H_2 \sim 2^{11}.\,\mathrm{M}_{23}$ be maximal subgroups of G containing S. If $L = H_1$, $P \leq H_2$ and looking at (10.2.6) we obtain $P/Q_{H_2} \sim 2^4.\,\mathrm{Sym}(5)$. Thus (ii) holds in this case. If $L = H_2$, then, as $\mathrm{Alt}(7)$ has no rank one isolated 2-minimal subgroups, we obtain $P \sim 2^{10+4}.(\mathrm{Sym}(3) \times \mathrm{Dih}(8))$. This gives (i). ∎

From here on, our strategy for cornering the rank one isolated 2-minimal subgroups in the larger sporadic simple groups takes advantage of the work of Ronan and Stroth on minimal parabolic systems in the sporadic simple groups [48]. In all of the coming

cases, the Sylow 2-subgroup is self-normalizing and so their notion of a minimal parabolic subgroup coincides with our notion of a 2-minimal subgroup. The possibilities for P and L in each case can be read from the data which they supply. The up-shot is that L can be any maximal 2-local subgroup and that this determines P. So here we see that the large sporadic simple groups behave like groups of Lie type in defining characteristic 2.

(10.2.22) *Suppose $X \cong \mathrm{Co}_1$. Then one of the following holds.*

(i) *$p = 2$, L can be any of the four maximal 2-local subgroups containing S, and $P/Q_P \cong \mathrm{Sym}(3)$.*

(ii) *$p = 3$ and L can be any of the three maximal 3-local subgroups containing S and $P/Q_P \cong \mathrm{SL}_2(3)$. (Note here that the two maximal subgroups $N(3C^2)$ in [60, Table 5.4] (see also [12]) need to be deleted, see the Modular Atlas [24].)*

(iii) *$p = 5$, $P/Q_P \cong \mathrm{SL}_2(5)$ or $\mathrm{PSL}_2(5)$ and L is one of the two maximal 5-local subgroups of G which contain S.*

When $p \geq 7$, S is abelian so $p \leq 5$. For $p = 5$, there are just two over-groups of the Sylow 5-subgroup and both of them are candidates for both L and $PN_G(S)$ and P is uniquely determined in the latter subgroup. For $p = 3$, there are three maximal 3-local subgroups each of which is eligible to be L. For $p = 2$, we refer to [48, page 74]. ∎

(10.2.23) *Suppose $X \cong \mathrm{J}_4$. Then $p = 2$, $P/Q_P \cong \mathrm{Sym}(3)$ and $L \sim 2_+^{1+12}.3.\mathrm{M}_{22}.2$ or $2^{3+12}.(\mathrm{SL}_3(2) \times \mathrm{Sym}(5))$ or $P/Q_P \cong \mathrm{Sym}(5)$ and $L \sim 2^{11}.\mathrm{M}_{24}$.*

If $p \leq 5$ and $p \neq 11$, then S is cyclic and we are done. If $p = 11$, then $N_G(S)$ is a maximal subgroup of G and so we cannot have $p = 11$. For $p = 3$, S is contained in $H \sim 2^{11}.\mathrm{M}_{24}$. Since $Q_H = 1$, $P \leq H$. As $P \in \mathcal{L}_1(3)$, $O^3(P)$ cannot be normal in H so (10.2.9) provides a contradiction. For $p = 2$, see [48, page 75]. ∎

(10.2.24) *Suppose $X \cong \mathrm{Fi}'_{24}$. Then one of the following holds.*

(i) *$p = 2$, $|G/X| \leq 2$, $P/Q_P \cong \mathrm{Sym}(3)$ and L is any of the four maximal 2-local subgroups of G containing S.*

(ii) *$p = 3$, $P/Q_P \cong \mathrm{SL}_2(3)$ and $L \sim 3_+^{1+10}.\mathrm{SU}_5(2).2$ or $L \sim 3^{2+4+8}.(\mathrm{Alt}(5) \times 2.\mathrm{Alt}(4)).2$.*

For $p > 7$ and $p = 5$, S is abelian and so $p \in \{2, 3, 7\}$. For $p = 7$, S is contained in He.2 and we obtain a contradiction via (10.2.11).

Suppose $p = 3$. Let $K \sim 3^7.\Omega_7(3)$, $J \sim 3_+^{1+10}.\mathrm{PSU}_5(2){:}2$ and $M \sim 3^{2+4+8}.(\mathrm{Alt}(5) \times 2.\mathrm{Alt}(4)).2$ be subgroups of X normalized by S. If $L \geq K$, then $P \leq JS$. By Theorem 7.1, we have $(L \cap J)/Q_J \sim 3^4.\mathrm{Sym}(5)$ and this is impossible as K has no such 3-local subgroups. Thus $L \neq K$. The remaining candidates for L are included as possibilities in part (ii).

If $p = 2$, then (i) holds. ∎

(10.2.25) *Suppose $X \cong \mathbb{B}$. Then one of the following holds.*

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(i) $p = 2$, $P/Q_P \cong \mathrm{Sym}(3)$ *and* $L \sim 2^{3+32}.(\mathrm{SL}_3(2) \times \mathrm{Sym}(5))$, $2^{2+10+20}.(\mathrm{Sym}(3) \times \mathrm{M}_{22})$, *or* $2^{1+22}.\mathrm{Co}_2$ *or* $P/Q_P \cong \mathrm{Sym}(5)$ *and* $L \sim 2^{1+8+16}.\mathrm{Sp}_8(2)$.

(ii) $p = 3$, $P/Q_P \cong \mathrm{SL}_2(3)$, *and* $L \sim 3_+^{1+8}.2_-^{1+6}.\mathrm{O}_6^-(2)$.

(iii) $p = 5$, $P/Q_P \cong \mathrm{SL}_2(5)$ *and* $L \sim 5_+^{1+4}.2_-^{1+4}.\mathrm{Alt}(5).4$.

If $p \geq 7$, then S is abelian. So we have $p \leq 5$.

Suppose $p = 5$ and let $S \leq H$ with $H \sim 5_+^{1+4}.2_-^{1+4}.\mathrm{Alt}(5).4$. Then P cannot be contained in H as otherwise $O^5(P) \leq F_5^*(H)$ which is soluble whereas P is not. Therefore $L = H$ and $P \leq J \sim 5^3.\mathrm{SL}_3(5)$. Therefore (iii) holds.

Suppose that $p = 3$ and let $S \leq H$ with $H \sim 3_+^{1+8}.2_-^{1+6}.\mathrm{O}_6^-(2)$. Then again $P \leq H$ leads to P being soluble and, as $O^3(P)Q_H/Q_H$ is normal in H, this forces $|O^3(P)Q_H/Q_H| \geq 2^7$ contrary to $P/Q_P \in \mathcal{L}_1(3)$. Thus (ii) holds.

If $p = 2$, then (i) holds. $\blacksquare$

(10.2.26) *Suppose that $X \cong \mathbb{M}$. Then one of the following holds.*

(i) $p = 2$, $P/Q_P \cong \mathrm{Sym}(3)$ *and L is any of the maximal 2-local subgroups of G containing S.*

(ii) $p = 3$, $P/Q_P \cong \mathrm{SL}_2(3)$ *and* $L \sim 3_+^{1+12}.2.\mathrm{Suz}.2$ *or* $L \sim 3^{2+5+5\cdot2}.(\mathrm{M}_{11} \times \mathrm{GL}_2(3))$.

(iii) $p = 5$, $P/Q_P \cong \mathrm{SL}_2(5)$ *and* $L \sim 5_+^{1+6}.4.\mathrm{J}_2.2$.

(iv) $p = 7$, $P/Q_P \cong \mathrm{SL}_2(7)$ *and* $L \sim 7_+^{1+4}.6.\mathrm{Sym}(7)$.

We have $p \leq 7$ as the other Sylow subgroups are abelian.

For $p = 7$ select $S \leq H \leq G$ with $H \sim 7_+^{1+4}.6.\mathrm{Sym}(7)$. Then $P \not\leq H$ and so $L = H$. Hence (iv) holds. For $p = 5$ choose $S \leq H \leq G$ with $H \sim 5_+^{1+6}.4.\mathrm{J}_2.2$. By (10.2.5) $P \not\leq H$ and so $L = H$. Therefore (iii) holds. For $p = 3$ choose $S \leq H \leq G$ with $H \sim 3_+^{1+12}.2.\mathrm{Suz}.2$. If $H = L$, then part (ii) holds. Suppose that $H \neq L$. Then (10.2.13) implies that $(H \cap L)/Q_{H\cap L} \sim 2.\mathrm{M}_{11}.2$. Thus $L \sim 3^{2+5+5\cdot2}.(\mathrm{M}_{11} \times \mathrm{GL}_2(3))$ and the second option in (ii) holds. If $p = 2$, then (i) holds using [48]. $\blacksquare$

Having looked at all 26 sporadic simple groups, we have completed the proof of Theorem 10.2.

$\square$

11. THE PROOF OF THEOREMS 1.5 AND 1.6 AND THEIR COROLLARIES

This section contains the proofs of the results stated in Section 1. We start with the proof of Theorem 1.6.

Proof of Theorem 1.6. Suppose that G is a finite group, $X = F^*(G)$ is a non-abelian simple group, $S \in \mathrm{Syl}_p(G)$ and P is a rank one isolated p-minimal subgroup of G. By Lemma 2.17, P is a rank one isolated p-minimal subgroup of XS. This means that Hypothesis 2.18 holds. We assume that X is not a group of Lie type defined in characteristic p. Using [58], we

see that $^2\mathrm{F}_4(2)'$ has isolated 2-minimal subgroups and so this is included as case (i)(b) of Theorem 1.6.

Now set $L = L_G(P, S)$. We consider the various possibilities for X.

If X is an alternating group, then Theorem 3.3 is applicable. In case (i) of Theorem 3.3, we have $p = 2$ and $X \cong \mathrm{Alt}(5) \cong \mathrm{SL}_2(4)$ contrary to our assumption that X is not a group of Lie type in characteristic p. Similarly, if Theorem 3.3 (ii) and (iii) hold, then we have $X \not\cong \mathrm{Alt}(8)$. The remaining possibilities are that $X \cong \mathrm{Alt}(6)$ or $\mathrm{Alt}(12)$ and these are listed in part (i)(a) of the theorem.

Suppose that X is a group of Lie type defined in characteristic r with $r \neq p$. Our objective here is to show that $p \in \{2, 3\}$ and verify the details presented in Tables 1 and 2.

If $X \cong \mathrm{PSL}_2(r^a)$, then Theorem 5.1 applies with $r \neq p$ to give $p \in \{2, 3\}$. The cases listed in Theorem 5.1 (ii) occupy lines 1 and 2 of Table 1. The examples itemized in Theorem 5.1 (iii) and (iv) are presented in lines 3, 4 and 5 of Table 1. From case (v) of Theorem 5.1, we eliminate $\mathrm{PSL}_2(7)$ as it is isomorphic to $\mathrm{SL}_3(2)$, but we keep $\mathrm{PSL}_2(9)$ even though it is isomorphic to $\mathrm{Sp}_4(2)'$ just as in the alternating case. This is lines 6, 7 and 8 of Table 1. We omit (vi) of Theorem 5.1 because $\mathrm{Alt}(5) \cong \mathrm{SL}_2(4)$. The final case of Theorem 5.1 is included as lines 1 and 2 of Table 2. We next consider $X \cong \mathrm{PSp}_{2n}(r^a)$ with $n \geq 2$. The possibilities for X are presented in Theorem 6.3 from where we read $p \in \{2, 3\}$. We do not include $\mathrm{PSp}_4(3) \cong \mathrm{PSU}_4(2)$ and so for $p = 2$, we just have $X \cong \mathrm{PSp}_6(3)$ and the possibilities for L and P are listed in line 8 of Table 1. Theorem 7.1 shows that if X a projective linear or unitary group, then it is listed in lines 9 to 20 of Table 1 or lines 3 to 10 of Table 2. For $X \cong \mathrm{P}\Omega_m^\epsilon(r^a)$ with $\epsilon \in \{\pm, 0\}$ and G a subgroup of $\mathrm{P}\mathrm{C}\Gamma\mathrm{O}_m^\epsilon(r^a)$, we may suppose that $m \geq 7$ and apply Theorems 8.11 and 8.12 to obtain the groups listed in lines 21, 22 and 23 of Table 1 and line 11 of Table 2. In the special case that $X \cong \mathrm{P}\Omega_m^\epsilon(r^a)$ and G is not contained in $\mathrm{P}\mathrm{C}\Gamma\mathrm{O}_m^\epsilon(r^a)$, we apply Theorem 8.13 and present the results on lines 12 and 13 of Table 2. Moving on to the exceptional groups of Lie type defined in characteristic $r \neq p$, we have the results from Section 9. Lemma 9.2, shows that X cannot be a Suzuki or a small Ree group and Lemma 9.5 shows that X cannot be a large Ree group. The case of $X \cong \mathrm{G}_2(r^a)$ is the subject of Lemma 9.3. This states that $p \leq 3$. If $p = 2$ the results are recorded in line 24 of Table 1 while in the case of $p = 3$ the result occupies lines 14 and 15 of Table 2. The triality twisted groups are skinned in Lemma 9.4 and the only interesting morsel appears with $p = 2$ and $r^a = 3$. This example is listed on line 25 of Table 1. By Lemma 9.7, the groups $X \cong \mathrm{E}_6(r^a)$ and $^2\mathrm{E}_6(r^a)$ only have rank one isolated p-minimal subgroups for $p = 3$ and $X \cong {}^2\mathrm{E}_6(2)$ as presented in line 16 of Table 2. For $X \cong \mathrm{E}_7(r^a)$, Lemma 9.8 gives $p = 3$, $r^a = 2$ and the examples are on the final line 26 of Table 1. Finally for the groups of Lie type, Lemma 9.9 addresses the case $X \cong \mathrm{E}_8(r^a)$ and states that $p = 3$ and $r^a = 2$ and this completes Table 2 with line 17. Thus Theorem 1.6(i)(b) and (ii)(a) hold.

For X a sporadic simple groups, we simply refer to Theorem 10.2 to obtain Theorem 1.6(i)(c), (ii)(b), (iii) and (iv). This completes the proof of Theorem 1.6. $\qquad\square$

Proof of Corollary 1.7. By Lemma 2.9, we have $PZ(X)/Z(X)$ is a rank one isolated p-minimal subgroup of $G/Z(X)$. Thus Theorem 1.6 applies as claimed. $\qquad\square$

Proof of Corollary 1.8. This is a simple observation following from Theorem 1.6. $\qquad\square$

Proof of Theorem 1.5. Suppose that G is a finite group, p is a prime, P is a rank one isolated p-minimal subgroup of G, $O_p(G) = 1$ and $O^p(P)$ is not normal in G. As LY contains all the members $\mathcal{P}_G(S)$ and $N_G(S)$, we have $G = LY$ by Lemma 2.1. Furthermore, $Y \leq F_p^*(G)$ and Y/Q_Y is quasisimple by Theorem 2.15 (i) and (ii). Since $P \leq YS$, we have that P is a rank one isolated p-minimal subgroup of YS by Lemma 2.2 (iii). Hence, as $O_p(G) = 1$, $F_p^*(YS)/Q_{YS} \cong Y/Q_Y \cong Y$ is described by Corollary 1.7 and this proves the result. $\qquad\square$

Proof of Corollary 1.9. Suppose that G is a finite simple group, p is a prime, $S \in \mathrm{Syl}_p(G)$ and assume that G is completely isolated. We begin with some useful observations on over-groups of S. Let H be a proper subgroup of G containing S. Then $H \leq L_G(P, S)$ for some $P \in \mathcal{P}_G(S)$ and $H/O_p(H)$ is also completely isolated. In particular, $O_p(H) \neq 1$ and $\{L_G(P,S) \mid P \in \mathcal{P}_G(S)\}$ is the set of maximal subgroups of G containing S. Using Lemma 2.1 we have $H = \langle \mathcal{P}_H(S) \rangle N_H(S)$. By Lemma 2.2(ii) $N_G(S)$ normalizes every p-minimal subgroup of G containing S. Hence $K = \langle \mathcal{P}_H(S) \rangle N_G(S) \leq G$ and $H \leq K$. Now $\mathcal{P}_H(S) = \mathcal{P}_G(S)$ would force, by Lemma 2.1, $K = G$. Since G is simple, we then get $G = \langle \mathcal{P}_H(S) \rangle \leq H$, a contradiction. Therefore $\mathcal{P}_H(S) \neq \mathcal{P}_G(S)$ and consequently there exists $P \in \mathcal{P}_G(S) \setminus \mathcal{P}_H(S)$. Thus $H \leq L_G(P, S)$ whence $1 \neq O_p(L_G(P,S)) \leq O_p(H)$. That $H/O_p(H)$ is completely isolated follows from Lemma 2.7 and the assertion on maximal over-groups from Lemma 2.2(v).

We now assume that G is not a group of Lie type in characteristic p. Hence G is isomorphic to one of the simple groups itemized in parts (i), (ii), (iii) and (iv) of Theorem 1.6. We check through the groups listed there, starting with part (i). So $p = 2$. Part (i)(a) gives $G \cong \mathrm{Alt}(6)$ or $\mathrm{Alt}(12)$. The former is easily seen to be completely isolated but for the latter, using the notation of Example 3.2, for P a rank one isolated 2-minimal subgroup of G $L_G(P)$ is the stabilizer of the partition $\{\{1,2\},\{3,4\},\{5,6\},\{7,8\},\{9,10\},\{11,12\}\}$ in $\Omega = \{1, \ldots, 12\}$. Since the stabilizer of $\{1,\ldots,8\}$ in G is a maximal subgroup of G, we conclude that $\mathrm{Alt}(12)$ is not completely isolated. Picking up cases (i)(b), we have that $G = {}^2\mathrm{F}_4(2)'$ and [58] implies that G is completely isolated.

Next we look at Table 1. Since G is simple, we may ignore lines 1 – 5 and lines 10 and 12 – 16 as well as lines 6 and 7 (as $\mathrm{Alt}(6)$ already done). Lines 17, 18 and 19 indicate that $\mathrm{PSU}_4(3)$ is completely isolated (see Example 7.9). From lines 9 (with $r^a = 3$) and 11 and 12 we have that $\mathrm{PSU}(3, 3)$ is also completely isolated. Line 8, with $G \cong \mathrm{PSp}_6(3)$, gives for P a rank one isolated 2-minimal subgroup is not soluble whereas $L_G(P, S)$ is soluble. Therefore $\mathrm{PSp}_6(3)$ is not completely isolated. Suppose $G \cong \mathrm{PSU}_3(r^a)$, with $r^a > 3$. Let t be an involution in $Z(S)$. Since $C_G(t)$ involves $\mathrm{PSL}_2(r^a)$, $C_G(t)$ cannot be a subgroup of $L_G(P^\star, S)$ for some $P^\star$. So $\mathrm{PSU}_3(r^a)$ with $r^a > 3$ is not completely isolated. Similarly

for $PSU_6(3)$, the stabilizer of an appropriate 4-space (in $SU_6(3)$, then projected) shows it cannot be completely isolated. And, $P\Omega_8^+(3)$ and $P\Omega_{12}^+(3)$ may be dealt with in the same manner (with, respectively stabilizers of a decomposition into four 2-spaces and stabilizer of an 8-space). While for $G \cong P\Omega_7^+(3)$, $H \leq G$ with $H \sim 2^6 . \text{Alt}(7)$ has $H/O_p(H)$ is not completely isolated, whence neither is $P\Omega_7^+(3)$. For $G \cong G_2(3)$, we have, from [29], $H \leq G$ with $H \cong PSU_3(3) : 2$ and $O_2(H) = 1$, which rules out $G_2(3)$. For lines 25 and 26 we have, for $L = L_G(P, S)$, $L/O_2(L)$ is not completely isolated (for $G \cong E_7(3)$, using Theorem 2.15).

We now move onto (i)(d). Consulting [60] (or [12]) and [48] we see that M_{12}, J_2, J_3 and Suz are completely isolated, while the remaining listed sporadic groups are not, as we see shortly. Noting that $\text{Sym}(5)$ is not completely isolated and that $G \cong M_{22}, \text{HS}, \text{Ru}$ or M_{23} has an over-group of S with $H/O_2(H) \cong \text{Sym}(5)$, we conclude that none of these groups is completely isolated. Likewise $M_{22}:2$ is not completely isolated which in turn means $\text{Co}_2, \text{Fi}_{22}, \text{Fi}_{23}, \mathbb{B}$ (with, respectively, $H/O_2(H) \cong M_{22}:2, M_{22}, M_{23}, \text{Co}_2$) are not. Now M_{24} too fails to be completely isolated by (10.2.9) and [48] (P_2 in their notation is not isolated). This cascades to eliminate $\text{Co}_1, \text{Fi}_{24}', J_4, \mathbb{M}$. For $G \cong \text{He}$, the fact that $|\mathcal{P}_G(S)| = 3$ (see [48]) and (10.2.11) deals with this case. When $G \cong O'N, \text{Co}_3, \text{HN}$ the maximal subgroups, respectively, $4^3.PSL_3(2), 2 \cdot \text{Sp}_6(2)$ and $[2^{11}](PSL_3(2) \times 3)$ are not of the form $L_G(P^\star, S)$ for some $P^\star$ and for $G \cong \text{Th}$, $\text{Alt}(9)$ not being completely isolated means that we have proved part (i) of the corollary.

We now examine Table 2 with $p = 3$. Line 1 gives $SL_2(8)$ which is one of our examples. Again G being simple means we only need consider lines $7 - 11$ and $14 - 17$. For $G \cong PSU_4(r^a)$ ($r^a \equiv 2, 5 \pmod 9, r^a \neq 2$), the remarks following Lemma 7.7 show that G is not completely isolated. For lines $8 - 10$, the given isolated rank one 3-minimal subgroup P cannot be isomorphic to a subgroup of $L_G(P, S)$. And line 11 has $\Omega_6^-(r^a)S$ (in $\Omega_8^+(r^a)$, projected to G – see Theorem 8.11) which is not contained $L_G(P^\star, S)$ for any $P^\star$. Lines $14, 15, 16$ are eliminated as $O_3(SU_3(3)) = 1 = O_3(\text{Fi}_{22})$ (Fi_{22} is an over-group of S in ${}^2E_6(2)$). Finally, using [35, Table 5.2] we see that $E_8(2)$ has an over-group of S of shape $3^8.(2.P\Omega_8^+(2).2)$ which is not a subgroup of any $L_G(P^\star, S)$. Turning to (ii)(b) of Theorem 1.6, we first consider the groups $\text{McL}, \text{Co}_3, \text{Fi}_{23}, \text{Fi}_{24}', \mathbb{B}, \mathbb{M}$. In these cases, the possibilities for P and $L_G(P, S)$ are given in Section 10 and from there we read that L is not $3^4.\text{Alt}(6)$ when $G = \text{McL}$, $3^{1+4}.4.\text{Sym}(6)$ when $G = \text{Co}_3$, $3^3.[3^7].(2 \times SL_3(3))$ when $G = \text{Fi}_{23}$, $3^7.P\Omega_7(3)$ when $G = \text{Fi}_{24}'$, $3^2.3^3.3^6.(\text{Sym}(4) \times 2.\text{Sym}(4))$ when $G = \mathbb{B}$ or $3^{1+12}.2.\text{Suz}.2$ when $G = \mathbb{M}$. For $G \cong J_2, \text{Co}_2, \text{Fi}_{22}$, using [60] we see that S has an over-group $H \cong PSU_3(3), \text{McL}, P\Omega_7(3)$ in the respective cases and for each of these groups $O_3(H) = 1$. Hence these groups are not completely isolated. If $G \cong \text{Suz}$, then, by [60], G has a subgroup of shape $3^5.M_{11}$ with M_{11} not completely isolated and so Suz is not completely isolated. The only sporadic groups remaining in Theorem 1.6 (ii)(b) are M_{12} and Co_1 and these can be easily checked to be completely isolated, using [60].

Finally we look at parts (iii) and (iv). The only possibility is Co_1, as $\text{HN}, \mathbb{B}, \mathbb{M}$ have subgroups of shape, respectively $5^2.5.5^2.4.\text{Alt}(5), 5^3.PSL_3(5), 5^2.5^2.5^4.(\text{Sym}(3) \times GL_5(2))$ and, for $p = 7$, $7^2.7.7^2.GL_2(7)$ which are not subgroups of $L_G(P^\star, S)$ for any $P^\star$. This completes the proof of the corollary.

□

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