On some problems in combinatorial geometry

Nóra Frankl

A thesis submitted for the degree of
Doctor of Philosophy

Department of Mathematics
London School of Economics
and Political Science

May 2020

Declaration

I certify that the thesis I have presented for examination for the PhD degree of the London School of Economics and Political Science is my own work, with the following exceptions.

Chapter 2 is based on the paper 'Almost sharp bounds on the number of discrete chains in the plane' written with Andrey Kupavskii.

Chapter 3 is based on the paper 'Embedding graphs in Euclidean space' written with Andrey Kupavskii and Konrad Swanepoel.

Chapter 4 is based on the paper 'Almost monochromatic sets and the chromatic number of the plane' written with Tamás Hubai and Dömötör Pálvölgyi.

Chapter 5 is based on the paper 'Nearly k-distance sets' written with Andrey Kupavskii.

I warrant that this authorization does not, to the best of my belief, infringe the rights of any third party.

Abstract

Combinatorial geometry is the study of combinatorial properties of geometric objects. In this thesis we consider several problems in this area.

1. We determine the maximum number of paths with k edges in a unit-distance graph in the plane almost exactly. It is only for $k \equiv 1 \pmod 3$ that the answer depends on the maximum number of unit distances in a set of n points which is unknown. This can be seen as a generalisation of the Erdős unit distance problem, and was recently suggested to study by Palsson, Senger and Sheffer (2019). We also obtain almost sharp results for even k in dimension 3.

2. Finding the smallest d for which a given graph can be represented as a unit-distance graph in dimension d is an important problem, that is NP-hard in general. It is closely related to orthonormal representations of graphs, which has many combinatorial and algorithmic applications. Answering questions of Erdős and Simonovits (1980), we show that any graph with less than $\binom{d+2}{2}$ or with maximum degree d can be represented as a unit-distance graph in dimension d.

3. We propose an approach that might lead to a human-verifiable proof of the recent theorem of de Grey that the chromatic number of the plane is at least 5. Our ideas are based on finding so-called almost-monochromatic sets. Motivated by its connections to the chromatic number of the plane, we study questions about finding almost-monochromatic similar copies of a given set in colourings of various base sets under some restrictions on the colouring.

4. We study an approximate version of k-distance sets. We compare its maximum cardinality with the maximum cardinalities of k-distance sets. It turns out that for fixed k and large dimension the two quantities are the same, while for fixed dimension and large k they are very different. We also address a closely related Turán-type problem, studied by Erdős, Makai, Pach, and Spencer: given n points in the d-dimensional space, at most how many pairs of them form a distance that is very close to k given distances, if any two points in the set are at distance at least 1 apart?

5. A set in a normed space is an equilateral set if the distance between any two of its points is the same. It is wide open conjecture that any normed space of dimension d contains an equilateral set of cardinality $d+1$. We find large equilateral sets in a specific family of normed spaces. As a corollary, we confirm the conjecture for those normed spaces whose unit ball is a polytope with at most $\frac{4d}{3}$ opposite pairs of facets.

Acknowledgements

First and foremost I thank my supervisor Konrad Swanepoel for his guidance and support during these four years. I am grateful to him for giving me freedom on choosing problems to work on, while always being available for discussing them. I thank him for having patience with the difficult task of not letting me neglect writing. I am also grateful to my second supervisor Jozef Skokan for his valuable advice on starting an academic path, and for broadening my interest beyond discrete geometry.

I thank LSE for providing me with an LSE PhD Scholarship, and the Department of Mathematics for a motivating working environment. I am grateful for the friendly administrative support of Rebecca Lumb, Enfale Farooq and Kate Barker.

I thank Andrey Kupavskii for working together on several projects, and for inviting me to join his research group at MIPT. I thank Márton Naszódi for recommending applying to LSE, and for discussing interesting problems from time to time. I thank Dömötör Pálvölgyi for hosting me in Budapest for a summer internship in 2018.

I had the pleasure to work with many more collaborators, from whom I have learnt a lot. They are Fabio Botler, Sebastián Bustamante, Jan Corsten, Attila Dankovics, Hiêp Hàn, Tamás Hubai, Sergei Kiselev, János Nagy, Andrea Jiménez, Balázs Patkós, and Alexey Pokrovskiy.

I am grateful to my family and friends for their love and support.

Contents

Chapter 1

Introduction

In this thesis we consider several problems in combinatorial geometry. Some of these problems are related to unit-distance graphs, while others are related to sets that span only few different distances. All the problems we study are in the setting of Euclidean space, with the exception of one which is about other normed spaces.

In the d-dimensional Euclidean space, a graph is a *unit-distance graph* if its vertices are points of the space, and its edges are some pairs of points at distance 1 apart. Unit-distance graphs are in the focus of many important questions in combinatorial geometry. Three of these are the following.

1. What is the maximum number of edges of a unit-distance graph in the d-dimensional space on n vertices?
2. What is the lowest dimension d for which a given graph is a unit-distance graph in the d-dimensional space?
3. How large can be the chromatic number of a unit-distance graph in the plane?

In Chapter 2 we consider a generalisation of the first question about the maximum number of paths in unit distance graphs. In Chapter 3, we investigate the second question in terms of the maximum degree and the number of edges. In Chapter 4 we propose an approach to find a lower bound on the chromatic number of the plane.

It is not hard to see that in the d-dimensional Euclidean space the maximum cardinality of a set having an equal distance between any two of its points is $d+1$. However, the following questions are much more difficult.

4. What is the maximum cardinality of a set in the d-dimensional Euclidean space, whose points span at most k different (positive) distances?

5. What is the maximum cardinality of a set having an equal distance between any two of its points in a given normed space of dimension d?

A set having the property described in the fourth question is called a *k-distance set*. In Chapter 5 we investigate an approximate version of this notion, and consider some related questions. A set having the property described in the fifth question is called an *equilateral set*. In Chapter 6 we find equilateral sets of large cardinality in a specific family of normed spaces.

1.1 Notation

Unless stated otherwise, $\|p - q\|$ denotes the Euclidean distance of $p, q \in \mathbb{R}^d$.

We denote by $[d]$ the set $\{1, \ldots, d\}$, and by $\binom{S}{m}$ the set of all subsets of S of cardinality m. For $j \in \mathbb{R}$ and $S, T \subseteq \mathbb{R}$ let $j + S = \{j + s : s \in S\}$ and $S + T = \{s + t : s \in S, t \in T\}$.

By a subspace of $\mathbb{R}^d$ we mean a linear subspace. For a subspace $X \subseteq \mathbb{R}^d$ we denote by $X^\perp$ the orthogonal complement of X. We write $\text{span}\,(a_1, \ldots, a_k)$ for the subspace spanned by $a_1, \ldots, a_k \in \mathbb{R}^d$.

We use the notation $f(n) = O(g(n))$ if there exists a constant C so that $f(n)/g(n) \leq C$ for every sufficiently large n. We write $f(n) = \Omega(g(n))$ if $g(n) = O(f(n))$, and $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $g(n) = O(f(n))$. We use the notation $f(n) = o(g(n)$ if $f(n)/g(n) \to 0$ as $n \to \infty$. Further we write $f(n) = \tilde{O}(g(n))$ if there exist positive constants C and c such that $f(n)/g(n) \leq C \log^c n$ for every sufficiently large n, $f(n) = \tilde{\Omega}(g(n))$ if $g(n) = \tilde{O}(f(n))$, and $f(n) = \tilde{\Theta}(g(n))$ if $f(n) = \tilde{O}(g(n))$ and $g(n) = \tilde{O}(f(n))$.

1.2 Discrete chains

We prove the results from this section in Chapter 2, which is based on [30].

The first question can also be phrased as asking for the maximum number $u_d(n)$ of unit distance pairs determined by a set of n points in $\mathbb{R}^d$. The planar version, determining $u_2(n)$, is also known as the Erdős unit distance problem. The question dates back to 1946, and despite its long history, the best known upper and lower bounds are still very far apart. For some constants $C, c > 0$, we have

$$n^{1+c/\log\log n} \leq u_2(n) \leq Cn^{4/3},$$

where the lower bound is due to Erdős [14] and the upper bound is due to Spencer, Szemerédi and Trotter [63].

As in the planar case, the best known upper and lower bounds in the 3-dimensional case are also far apart. For every $\varepsilon > 0$ there are some $c, C > 0$, such that

$$cn^{4/3}\log\log n \leq u_3(n) \leq Cn^{295/197+\varepsilon}, \tag{1.1}$$

where the lower bound is due to Erdős [15], and the upper bound is due to Zahl [68]. The latter is a recent improvement upon the upper bound $O(n^{3/2})$ by Kaplan, Matoušek, Safernová, and Sharir [41], and Zahl [69].

A possible way to generalise this problem is to ask for the maximum number of paths of a fixed length in a unit-distance graph on n vertices. Surprisingly, in many cases this question turns out to be more approachable than the original. We consider a slight modification of this problem, that was recently proposed by Palsson, Senger and Sheffer [54]. Let $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_k)$ be a sequence of k positive reals. A $(k+1)$-tuple $(p_1, \ldots, p_{k+1})$ of distinct points in $\mathbb{R}^d$ is called a $(k, \boldsymbol{\delta})$-*chain* if $\|p_i - p_{i+1}\| = \delta_i$ for all $i = 1, \ldots, k$. For every fixed k determine $C_k^d(n)$, the maximum number of $(k, \boldsymbol{\delta})$-chains that can be spanned by a set of n points in $\mathbb{R}^d$, where the maximum is taken over all $\boldsymbol{\delta}$. Note that $C_1^d(n) = u_d(n)$. In the planar case, the following upper bounds were found in [54] in terms of the maximum number of unit distances.

Proposition 1.1 ([54])**.**

$$C_k^2(n) = \begin{cases} O\left(n \cdot u_2(n)^{k/3}\right) & \text{if } k \equiv 0 \pmod 3, \\ O\left(u_2(n)^{(k+2)/3}\right) & \text{if } k \equiv 1 \pmod 3, \\ O\left(n^2 \cdot u_2(n)^{(k-2)/3}\right) & \text{if } k \equiv 2 \pmod 3. \end{cases}$$

If $u_2(n) = O(n^{1+\varepsilon})$ for any $\varepsilon > 0$, which is conjectured to hold, then the upper bounds in the proposition above almost match the lower bounds given in Theorem 1.2. However, as we have already mentioned, determining the order of magnitude of $u_2(n)$ is very far from being done, and in general it proved to be a very hard problem. Thus, it is interesting to obtain "unconditional" bounds, that depend on the value of $u_2(n)$ as little as possible. In [54], the following "unconditional" upper bounds were proved in the planar case.

Theorem 1.2 ([54]). *$C_2^2(n) = \Theta(n^2)$, and for every $k \geq 3$ we have*

$$C_k^2(n) = \Omega\left(n^{\lfloor (k+1)/3 \rfloor + 1}\right)$$

and

$$C_k^2(n) = O\left(n^{2k/5+1+\gamma_k}\right),$$

where $\gamma_k \leq \frac{1}{12}$, and $\gamma_k \to \frac{4}{75}$ as $k \to \infty$.

We determine the value of $C_k^2(n)$ up to a small error term in two thirds of the cases independently of the value of $u_2(n)$, by matching the lower bounds given in Theorem 1.2. Further, we show that in the remaining cases determining $C_k^2(n)$ essentially reduces to determining $u_2(n)$.

Theorem 1.3. *For every integer $k \geq 1$ we have*

$$C_k^2(n) = \tilde{\Theta}\left(n^{\lfloor (k+1)/3 \rfloor + 1}\right) \text{ if } k \equiv 0, 2 \pmod 3,$$

and for any $\varepsilon > 0$ we have

$$C_k^2(n) = \Omega\left(n^{(k-1)/3} u_2(n)\right) \text{ and } C_k^2(n) = O\left(n^{(k-1)/3+\varepsilon} u_2(n)\right) \text{ if } k \equiv 1 \pmod 3.$$

As for the 3-dimensional case, the following was proved in [54].

Theorem 1.4 ([54]). *For any integer $k \geq 2$, we have*

$$C_k^3(n) = \Omega\left(n^{\lfloor k/2 \rfloor + 1}\right),$$

and

$$C_k^3(n) = \begin{cases} O\left(n^{2k/3+1}\right) & \text{if } k \equiv 0 \pmod 3, \\ O\left(n^{2k/3+23/33+\varepsilon}\right) & \text{if } k \equiv 1 \pmod 3, \\ O\left(n^{2k/3+2/3}\right) & \text{if } k \equiv 2 \pmod 3. \end{cases}$$

We improve this upper bound and essentially settle the problem for even k.

Theorem 1.5. *For any integer $k \geq 2$ we have*

$$C_k^3(n) = \tilde{O}\left(n^{k/2+1}\right).$$

Furthermore, for even k we have

$$C_k^3(n) = \tilde{\Theta}\left(n^{k/2+1}\right).$$

We note that for $d \geq 4$ we have $C_k^d(n) = \Theta(n^{k+1})$, thus both the unit distance problem and the generalisation to paths is the most interesting for the $d = 2, 3$ cases. Indeed, we clearly have $C_k^d(n) = O(n^{k+1})$. To see that $C_k^d(n) = \Omega(n^{k+1})$, take two circles of radius $1/\sqrt{2}$ centred at the origin in two orthogonal planes, and place $n/2$ points on each of them. The constant factor for $d > 4$ can be improved by using more than two pairwise orthogonal circles. For even $d \geq 4$ Swanepoel [64] determined the exact value of $C_1^d(n)$ for sufficiently large n.

1.3 Unit-distance embeddings

We prove the results from this section in Chapter 3, which is based on [32].

We say that a graph G is *realizable* in a subset X of $\mathbb{R}^d$, if there exists a unit-distance graph G' in $\mathbb{R}^d$ on a set of vertices $X_0 \subset X$, which is isomorphic to G. We will use this notion for $X = \mathbb{R}^d$ and for $X = \mathbb{S}^{d-1}$, where $\mathbb{S}^{d-1}$ is the sphere of radius $1/\sqrt{2}$ with centre in the origin. Note that the reason for choosing this particular radius is that the distance of two points on a sphere of radius $1\sqrt{2}$ is one if and only if the corresponding vectors are orthogonal.

Erdős, Harary and Tutte [24] introduced the concept of Euclidean dimension $\dim G$ of a graph G. The *Euclidean dimension* $\dim G$ (*spherical dimension* $\dim_S G$) of a graph G is equal to the smallest integer k such that G is realizable in $\mathbb{R}^k$ (on $\mathbb{S}^{k-1} \subset \mathbb{R}^k$).

For example, for the complete graph K_d on d vertices we have $\dim K_d = d-1$. Indeed, it is not hard to see that that K_d is not realizable in $\mathbb{R}^{d-2}$ (we will prove this in Lemma 3.14). At the same time, the vertices of a regular unit simplex provide a realization of K_d in $\mathbb{R}^{d-1}$. Similarly, we have $\dim_S K_d = d$.

It is also well known that $K_{3,3}$ cannot be realized in $\mathbb{R}^3$. However, the dimension of any bipartite graph G is at most 4. Indeed, let S_1 and S_2 be circles of radius $1/\sqrt{2}$ centred at the origin, in orthogonal planes. Then the distance between any two points of S_1 and S_2 is 1. Thus placing every vertex of the first class on S_1 and every vertex of the second class on S_2 is a realization of G. Note that, for this example, it is important that in our definition of a unit distance graph we do not require the edge set to contain all unit-distance pairs. If we had require that, we would arrive to the definition of *faithful realizations* and *faithful dimension*. Some differences between these two types of realizations were investigated by Alon and Kupavskii in [2]. Among other results, they prove that for every d there is a bipartite graph that does not admit a faithful realization in $\mathbb{R}^d$.

In general, it is NP-hard to determine the dimension of a graph [59]. However, there are upper bounds on the dimension, in terms of certain graph parameters. We consider upper bounds in terms of the maximum degree and the number of edges of a graph.

Erdős and Simonovits [25] showed that if G has maximum degree d then $\dim_S G \leq d+2$, which implies $\dim G \leq d+2$. In Theorem 3.1 and Proposition 1.7 we improve these results.

Theorem 1.6. *Let $d \geq 1$ and let $G = (V, E)$ be a graph with maximum degree d. Then G is a unit distance graph in $\mathbb{R}^d$ except if $d = 3$ and G contains $K_{3,3}$.*

Proposition 1.7. *Let $d \geq 2$. Any graph $G = (V, E)$ with maximum degree $d-1$ has spherical dimension at most d.*

Note that, since $\dim K_{d+1} = d$ and and $\dim_S K_d = d$, the two results above are best possible.

Let $f(d)$ denote the least number for which there is a graph with $f(d)$ edges that is not realizable in $\mathbb{R}^d$.

There are some natural upper bounds on $f(d)$. Since K_{d+2} is not realizable in $\mathbb{R}^d$, we have $f(d) \leq \binom{d+2}{2}$. Further, since $K_{3,3}$ cannot be realized in $\mathbb{R}^3$, we have $f(3) \leq 9 < \binom{3+2}{2}$. In [25], Erdős and Simonovits asked if $f(d) = \binom{d+2}{2}$ for $d > 3$. House [39] proved that $f(3) = 9$, and that $K_{3,3}$ is the only graph with 9 edges that can not be realized in $\mathbb{R}^3$. Chaffee and Noble [8] showed that $f(4) = \binom{4+2}{2} = 15$, and there are only two graphs, K_6 and $K_{3,3,1}$, with 15 edges that can not be realized in $\mathbb{R}^4$ as a unit distance graph. Recently, they showed [9] that $f(5) = \binom{5+2}{2} = 21$, and that K_7 is the only graph with 21 edges that cannot be realized in $\mathbb{R}^5$ as a unit distance graph. We answer the above-mentioned question of Erdős and Simonovits as part of the following result.

Theorem 1.8. *Let $d > 3$. Any graph G with less than $\binom{d+2}{2}$ edges can be realized in $\mathbb{R}^d$. If G moreover does not contain $K_{d+2} - K_3$ or K_{d+1}, then it can be realized in $\mathbb{S}^{d-1}$.*

It is necessary to forbid $K_{d+2} - K_3$ and K_{d+1} in the second statement of the above theorem as they cannot be realized in $\mathbb{S}^{d-1}$; see Lemma 3.14.

Ramsey-type questions about unit distance graphs have been studied in [2], and also by Kupavskii, Raigorodskii and Titova [46]. In [2] the first of the following quantities was introduced. Let $f_D(s)$ denote the smallest possible d, such that for any graph G on s vertices, either G or its complement $\overline{G}$ can be realized as a unit distance graph in $\mathbb{R}^d$.

Similarly, we define $f_{SD}(s)$ to be the smallest possible d, such that for any graph G on s vertices, either G or its complement $\overline{G}$ can be realized as a unit distance graph in $\mathbb{S}^{d-1}$.

In [2] it is shown that $f_D(s) = (\frac{1}{2} + o(1))s$. We determine the exact value of $f_{SD}(s)$ and give almost sharp bounds on $f_D(s)$.

Theorem 1.9. *For any* $d, s \geq 1$, $f_{SD}(s) = \lceil (s+1)/2 \rceil$ *and* $\lceil (s-1)/2 \rceil \leq f_D(s) \leq \lceil s/2 \rceil$.

1.4 The chromatic number of the plane

We prove the results and discuss the details of the approach introduced in this section in Chapter 4, which is based on [29].

A *colouring* of a set X is a function $\varphi : X \to A$ for some finite set A. A *k-colouring* of X is a function $\varphi : X \to A$ with $|A| = k$. For a graph $G = (V, E)$ a colouring $\varphi : V \to A$ is a *proper colouring of* G if $(x, y) \in E$ implies $\varphi(x) \neq \varphi(y)$, and it is a proper k-colouring, if in addition $|A| = k$. The *chromatic number* $\chi(G)$ of a graph G is the smallest k for which there exists a proper k-colouring of G.

The *chromatic number of the plane*, denoted by $\chi(\mathbb{R}^2)$, is the chromatic number of the graph $(\mathbb{R}^2, E)$ where $E = \{(x, y) : \|x - y\| = 1\}$. Determining the exact value of $\chi(\mathbb{R}^2)$, also known as the Hadwiger-Nelson problem, is a long standing open problem.

In 2018 de Grey [11] showed that $\chi(\mathbb{R}^2) \geq 5$, improving the long standing previous lower bound $\chi(\mathbb{R}^2) \geq 4$ which was first noted by Nelson (see [60]). The best known upper bound $\chi(\mathbb{R}^2) \leq 7$ was first observed by Isbell (see [60]), and it is conjectured that $\chi(\mathbb{R}^2) = 7$. For history and related results see Soifer's book [60].

It is easy to see that for any unit-distance graph G we have $\chi(\mathbb{R}^2) \geq \chi(G)$. However, a stronger statement is also true. According to the de Bruijn–Erdős theorem [10] the chromatic number of any infinite graph is attained by a finite subgraph. Thus, to determine $\chi(\mathbb{R}^d)$ it is sufficient to determine the maximum chromatic number of a finite unit distance graph.

De Grey constructed a unit-distance graph G on 1581 vertices, and checked that $\chi(G) \geq 5$ by a computer program. Following his breakthrough, a polymath project, Polymath16 [12] was launched with the main goal of finding a human-verifiable proof of $\chi(\mathbb{R}^2) \geq 5$. Following ideas proposed in the Polymath16 project by Pálvölgyi [56], we present an approach that might lead to a human-verifiable proof of $\chi(\mathbb{R}^2) \geq 5$.

We call a collection of unit circles $C = C_1 \cup \cdots \cup C_n$ having a common point O a *bouquet through O*. For a given colouring of $\mathbb{R}^2$, the bouquet C is *smiling* if there is a colour, say blue, such that every circle C_i has a blue point, but O is not blue. We make the following conjecture.

Conjecture 1.10. *For every bouquet C, every colouring of the plane with finitely many but at least two colours contains a smiling congruent copy of C.*

We show that the statement of Conjecture 1.10 would provide a human-verifiable proof of $\chi(\mathbb{R}^2) \geq 5$, and we prove the conjecture for a specific family of bouquets for proper colourings.

Theorem 1.11. *Let $C = C_1 \cup \cdots \cup C_n$ be a bouquet through O and for every i et O_i be the centre of C_i. If O and $O_1, \ldots, O_n$ are contained in $\mathbb{Q}^2$, further O is an extreme point of $\{O, O_1, \ldots, O_n\}$, then Conjecture 4.1 is true for C for proper colourings of $(\mathbb{R}^2, E)$.*

The proof of Theorem 1.11 is through the study of *almost-monochromatic sets*, which we shall introduce below.

Almost monochromatic sets

Let $S \subseteq \mathbb{R}^d$ be a finite set with $|S| \geq 3$, and let $s_0 \in S$. In a colouring of $\mathbb{R}^d$ we call S *monochromatic*, if every point of S has the same colour. A pair (S, s_0) is *almost-monochromatic* if $S \setminus \{s_0\}$ is monochromatic but S is not.

Two sets S and T are *similar*, if there is an isometry f of $\mathbb{R}^d$ and a constant $\lambda \in \mathbb{R}$, $\lambda \neq 0$ with $T = \lambda f(S)$. A *(one way) infinite arithmetic progression* in $\mathbb{R}^d$ is a similar copy of $\mathbb{N}$. A colouring is *arithmetic progression-free* if it does not contain a monochromatic infinite arithmetic progression. Motivated by its connections to the chromatic number of the plane, we propose to study the following problem.

Problem 1.12. *Characterise those pairs (S, s_0) with $S \subseteq \mathbb{R}^d$ and with $s_0 \in S$ for which it is true that every arithmetic progression-free colouring of $\mathbb{R}^d$ contains an almost-monochromatic similar copy of (S, s_0).*

Note that finding an almost-monochromatic *congruent* copy of a given pair (S, s_0) was studied by Erdős, Graham, Montgomery, Rothschild, Spencer, and Strauss [19]. We solve Problem 1.12 in the case when $S \subseteq \mathbb{Z}^d$. A point $s_0 \in S$ is called an *extreme point* of

S if $s_0 \notin \mathrm{conv}(S \setminus \{s_0\})$. From now on we will use the abbreviations AM for almost-monochromatic and AP for arithmetic progression.

Theorem 1.13. *Let $S \subseteq \mathbb{Z}^d$ and $s_0 \in S$. Then there is an AP-free colouring of $\mathbb{R}^d$ without an AM similar copy of (S, s_0) if and only if $|S| > 3$ and s_0 is not an extreme point of S.*

Problem 1.12 is related to and motivated by Euclidean Ramsey theory, a topic introduced by Erdős, Graham, Montgomery, Rothschild, Spencer, and Strauss [18]. Its central question asks for finding those finite sets $S \subseteq \mathbb{R}^d$ for which the following is true. *For every k if d is sufficiently large, then every colouring of $\mathbb{R}^d$ using at most k colours contains a monochromatic congruent copy of S.* Characterising sets having the property described above is a well-studied difficult question, and is in general wide open. For a comprehensive overview see Graham's survey [34].

T is a *homothetic copy* (or *homothet*) of a S, if there is a vector $v \in \mathbb{R}^d$ and a constant $\lambda \in \mathbb{R}$ ($\lambda \neq 0$) with $T = v + \lambda S$. T is a *positive homothetic copy* (or *positive homothet*) of a S, if moreover $\lambda > 0$. The nature of the problem significantly changes if instead of a monochromatic congruent copy we ask for a monochromatic similar copy, or a monochromatic homothetic copy. Gallai proved that if $S \subseteq \mathbb{R}^d$ is a finite set, then every colouring of $\mathbb{R}^d$ using finitely many colours contains a monochromatic positive homothetic copy of S. This statement first appeared in the mentioned form in the book of Graham, Rothschild, and Spencer [35].

A direct analogue of Gallai's theorem for AM sets is not true: there is no AM similar copy of any (S, s_0) if the whole space is coloured with one colour only. However, there are pairs (S, s_0) for which a direct analogue of Gallai's theorem is true for colourings of $\mathbb{Q}$ with more than one colour. In particular, we prove the following result.

Theorem 1.14. *Let $S = \{0, 1, 2\}$ and $s_0 = 0$. Then every finite colouring of $\mathbb{Q}$ with more than one colour contains an AM positive homothet of (S, s_0).*

In general, we could ask whether every non-monochromatic colouring of $\mathbb{R}^d$ contains an AM similar copy of every (S, s_0). This, however, is false, as shown by the following example from [19]. Let $S = \{1, 2, 3\}$ and $s_0 = 2$. If $\mathbb{R}_{>0}$ is coloured red and $\mathbb{R}_{\leq 0}$ is coloured blue, we obtain a colouring of $\mathbb{R}$ without an AM similar copy of (S, s_0). Restricting the colouring to $\mathbb{N}$, using the set of colours $\{0, 1, 2\}$ and colouring every $n \in \mathbb{N}$ with n modulo 3, we obtain a colouring without an AM similar copy of (S, s_0). However, notice that in both examples each colour class contains an infinite monochromatic AP.

$k \diagdown d$	2	3	4	5	6	7	8
2	5	6	10	16	27	29	45
3	7	12	16	≥ 24	≥ 40	≥ 65	≥ 121
4	9	13	≥ 25	≥ 41	≥ 73	≥ 127	≥ 241
5	12	≥ 20	≥ 35	≥ 66	≥ 112	≥ 168	≥ 252
6	13	≥ 21	≥ 40	≥ 96	≥ 141	≥ 281	≥ 505

Table 1.1: Lower bounds on $m_k(d)$.

Therefore, our reason, apart from its connections to the Hadwiger-Nelson problem, for finding AM similar copies of (S, s_0) in AP-free colourings was to impose a meaningful condition to exclude 'trivial' colourings.

1.5 Nearly *k*-distance sets

We prove the results from this section in Chapter 5, which is based on [31].

Let us denote by $m_k(d)$ the cardinality of the largest k-distance set in $\mathbb{R}^d$. We already mentioned that $m_1(d) = d + 1$, and it is also easy to see that $m_k(1) = k + 1$. In 1947, Kelly [42] (answering a question of Erdős) showed that $m_2(2) = 5$. Table 1.1, taken from a paper of Szöllősi and Östergård [26], summarises the best known lower bounds on (and in some cases exact values of) $m_k(d)$ for small values of k and d.

If d is large compared to k, then the best general bounds are

$$\binom{d+1}{k} \leq m_k(d) \leq \binom{d+k}{k}. \tag{1.2}$$

The upper bound is by Bannai, Bannai, and Stanton [5], and the lower bound for $k \leq d+1$, follows from the following construction. Take all vectors in $\{0, 1\}^{d+1}$ with exactly k many 1's. Then they lie on a sphere in the hyperplane $\sum x_i = k$ and determine only k distinct scalar products (and thus only k distinct distances).

For fixed d, asking for the maximum cardinality of a k-distance set is the inverse of the Erdős distinct distances problem, which asks for the minimum number of disctinct distances determined by a set of n points in $\mathbb{R}^d$. Erdős [14] conjectured that $m_k(2) = O(k^{1+\varepsilon})$ and showed that $m_k(2) = \Omega(k\sqrt{\log k})$, which is still the best general lower bound. The upper bound $m_k(2) = O(k \log k)$ is a recent break-through of Guth and Katz [37]. The best upper bounds by Solymosi and Vu [62] for $d = 3$ is $m_k(3) = O(k^{5/3+o(1)})$ and for $d \geq 3$ is $m_k(d) = O(k^{(d^2+d-2)/(2d)+o(1)})$. In general, it is conjectured that $m_k(d) = O(k^{d/2+\varepsilon})$.

We study two related quantities. The first is the maximum cardinality of sets where all the distances spanned are very close to k given one. The second is the maximum number of pairs in a set of n points whose distance is very close to k given distances.

A set of points $S \subseteq \mathbb{R}^d$ is called an *ε-nearly k-distance set* if there exist $1 \leq t_1 \leq \ldots \leq t_k$ such that

$$\|p - q\| \in [t_1, t_1 + \varepsilon] \cup \ldots \cup [t_k, t_k + \varepsilon]$$

for all $p \neq q \in S$. We study

$$M_k(d) := \lim_{\varepsilon \to 0} \max\{|S| \ : \ S \text{ is an } \varepsilon\text{-nearly } k\text{-distance set in } \mathbb{R}^d\}.$$

Note that the $t_1 \geq 1$ assumption is important, otherwise we would have $M_k(d) = \infty$. We clearly have $M_k(d) \geq m_k(d)$. The difficulty of relating the maximal cardinalities of k-distance sets and nearly k-distance sets more precisely lies in the fact that, in nearly k-distance sets distances of different order of magnitude may appear. If we additionally assume $\frac{t_{i+1}}{t_i} \leq K$ for some universal constant K in the definition of nearly k-distance sets, a compactness argument would imply that $m_k(d)$ equals this modified $M_k(d)$. An expression equivalent to $M_k(d)$ appears in a paper of Erdős, Makai and Pach [21, page 19], where they speculate that "for k fixed, d sufficiently large probably $M_k(d) = m_k(d)$." We confirm this.

Theorem 1.15. *For a fixed positive integer k we have $M_k(d) = m_k(d)$ if either $d \geq d(k)$ or $k \leq 3$.*

On the other hand, for fixed d and large k the two quantities are very different. We determine the order of magnitude of $M_k(d)$ in this setting. We show that $M_k(d) = \Theta\left(k^d\right)$ holds for any fixed $d \geq 2$. Since by [62] we have $m_k(d) = O\left(k^{\frac{d}{2}+1}\right)$, we obtain that $M_k(d) > m_k(d)$ if k is sufficiently large compared to d. We will also find examples of small k and d for which $M_k(d) > m_k(d)$.

We call a set $S \subseteq \mathbb{R}^d$ *separated* if the distance between any two of its points is at least 1. Let $M_k(d, n)$ denote the maximum M, such that there exist numbers $1 \leq t_1 \leq \ldots \leq t_k$ and a separated set S of n points in $\mathbb{R}^d$ with at least M pairs of points at a distance that falls into $[t_1, t_1 + 1] \cup \cdots \cup [t_k, t_k + 1]$.

This quantity was studied by Erdős, Makai, Pach and Spencer [20, 21, 22, 50]. Their constructions can easily be generalised to show that for any $d \geq 2$, $k \geq 1$ we have

$$M_k(d,n) \geq T(n, m_k(d-1)) = \left(1 - \frac{1}{m_k(d-1)}\right)\frac{n^2}{2} + O(1), \tag{1.3}$$

where $T(n,m)$ denotes the number of edges in a balanced complete m-partite graph on n vertices. Regarding upper bounds, they considered the planar case, and the case of 1 or 2 distances. In these cases, they matched, or closely matched (1.3) from above. In particular, they proved the following results.

- In [22], they showed that

  $$M_1(d,n) \leq T(m_1(d-1), n) \leq \left(1 - \frac{1}{d}\right)\frac{n^2}{2}$$

  holds for sufficiently large n.

- In [20] they proved that for every $\gamma > 0$ if n is sufficiently large, then

  $$M_k(2,n) \leq T(m_k(1), n) + \gamma n^2 \leq \left(1 - \frac{1}{k+1} + 2\gamma\right)\frac{n^2}{2}. \tag{1.4}$$

- Recently, in [21], they proved that for every $\gamma > 0$ if n is sufficiently large, then

  $$M_2(d,n) \leq T(m_2(d-1), n) + \gamma n^2 \leq \left(1 - \frac{1}{m_2(d-1)} + 2\gamma\right)\frac{n^2}{2}. \tag{1.5}$$

  Moreover, for $d \neq 4, 5$ they removed the γn^2 error term.

It is interesting to investigate how $M_k(d,n)$ changes if in its definition we modify the intervals $[t_i, t_i + 1]$ to $[t_i, t_i + f(n)]$ for some function $f = f_d$. It turns out that the threshold for essential changes is $f_d(n) = \Theta(n^{1/d})$. It was shown in [20] that inequality (1.4) remains true with intervals of the form $[t_i, t_i + c\sqrt{n}]$ for some constant $c = c(k,\gamma)$. Similarly, it was shown in [21] that inequality (1.5) remains true for $d \neq 4, 5$ without the γn^2 error term with intervals of the form $[t_i, t_i + cn^{1/d}]$ for some constant $c > 0$.

On the other hand, $M_k(d,n)$ becomes $\binom{n}{2}$ if $f(n) = Cn^{1/d}$ for sufficiently large C. Indeed, a standard volume argument shows that one can find a separated set of n points in $\mathbb{R}^d$ in a ball of radius $Cn^{1/d}$.

Our main result about $M_k(d,n)$ is an extension of (1.5).

Theorem 1.16. *Let $k \geq 1$ be fixed. If either $k \leq 3$ or $d \geq d(k)$, then for sufficiently large n we have*

$$M_k(d,n) = T(m_k(d-1), n).$$

Moreover, the same holds if in the definition of $M_k(d,n)$ we change the intervals to be of the form $[t_i, t_i + cn^{1/d}]$ with some constant $c = c(k,d) > 0$.

It would be interesting to determine $M_k(d,n)$ in terms of $m_k(d-1)$ for all k and d. However, as we will show below, we have

$$T(M_k(d-1), n) \leq M_k(d,n) \leq T(M_k(d), n), \tag{1.6}$$

which gives the impression that the "right quantity" to relate $M_k(d,n)$ to might rather be $M_k(d-1)$.

Proof sketch of (1.6). The lower bound on $M_k(d,n)$ for all $k \geq 1$, $d \geq 2$ is shown by the following construction, which is similar to those that appeared in the work of Erdős, Makai, Pach and Spencer. Embed a $\frac{1}{2}$-nearly k-distance set $S \subseteq \mathbb{R}^{d-1}$ of size $M_k(d-1)$ and with distances $2n^2 \leq t_1 \leq \cdots \leq t_k$ in a hyperplane H in $\mathbb{R}^d$. Replace each point $p \in S$ by an arithmetic progression A_p of length $\lfloor n/M_k(d-1) \rfloor$ or $\lceil n/M_k(d-1) \rceil$ and of difference 1, in the direction orthogonal to H. One can easily check that in $\bigcup_{p \in S} A_p$ there are at least $T(M_k(d-1), n)$ pairs forming a distance in $[t_1, t_1 + 1] \cup \cdots \cup [t_k, t_k + 1]$.

For proving the upper bound on $M_k(d,n)$, by a volume argument we may assume that $t_1 = \Omega(n^{1/d})$. This, together with Turán's theorem and the definition of $M_k(d)$ implies $M_k(d,n) \leq T(M_k(d), n)$. □

It might be that for for every k and d, if n is sufficiently large, then we have

$$M_k(d,n) = T(M_k(d-1), n). \tag{1.7}$$

Motivated by the goal of determining $M_k(d,n)$ in a quantity similar to $M_k(d)$, we will introduce a more technical notion that we call a *flat nearly k-distance set*, and denote its maximum possible cardinality in $\mathbb{R}^d$ by $N_k(d)$. We prove that a relation similar to (1.7) holds with an additive $o(n^2)$ error term if we replace $M_k(d-1)$ by $N_k(d)$.

1.6 Equilateral sets

We prove the results from this section in Chapter 6, which is based on [28].

Let $(X, \|\cdot\|)$ be a normed space. A set $S \subseteq X$ is called *c-equilateral* if $\|x - y\| = c$ for all distinct $x, y \in S$. S is called *equilateral* if it is c-equilateral for some $c > 0$. The *equilateral number* $e(X)$ of X is the cardinality of the largest equilateral set of X. Petty [55] made the following conjecture regarding lower bounds on $e(X)$.

Conjecture 1.17 ([55]). *For all normed spaces X of dimension d, $e(X) \geq d+1$.*

It is easy to see for $d = 2$ (Golab [33] and Kelly [43]). Petty [55] proved Conjecture 1.17 for $d = 3$, and Makeev [51] for $d = 4$. For $d \geq 5$ the conjecture is still open, except for some special classes of norms. The best general lower bound is $e(X) \geq \exp(\Omega(\sqrt{\log d}))$, proved by Swanepoel and Villa [66]. Regarding upper bounds on the equilateral number, a classical result of Petty [55] and Soltan [61] shows that $e(X) \leq 2^d$ for any X of dimension d, with equality if and only if the unit ball of X is an affine image of the d-dimensional cube. For more background on the equilateral number see Section 3 of the survey [65].

The norm $\|\cdot\|_\infty$ of $x \in \mathbb{R}^d$ is defined as $\|x\|_\infty = \max_{1\leq i\leq d} |x_i|$, and ℓ_∞^d denotes the normed space $(\mathbb{R}^d, \|\cdot\|_\infty)$. In [45] Kobos studied subspaces of ℓ_∞^d of codimension 1, and proved the lower bound $e(X) \geq 2^{\lfloor \frac{d}{2} \rfloor}$, which in particular implies Conjecture 1.17 for these spaces for $d \geq 6$.

In the same paper he proposed as a problem to prove Petty's conjecture for subspaces of ℓ_∞^d of codimension 2. In Theorem 1.18 we prove exponential lower bounds on the equilateral number of subspaces of ℓ_∞^d of codimension k. This, in particular, solves Kobos' problem from [45] if $d \geq 9$.

Theorem 1.18. *Let X be a $(d-k)$-dimensional subspace of ℓ_∞^d. Then*

$$e(X) \geq \frac{2^{d-k}}{(d-k)^k}, \tag{1.8}$$

$$e(X) \geq 1 + \frac{1}{2^{k-1}} \sum_{r=1}^{\ell} \binom{d-k\ell}{r} \text{ for every } 1 \leq \ell \leq d/(k+1), \text{ and} \tag{1.9}$$

$$e(X) \geq 1 + \sum_{r=1}^{\ell} \binom{d-2k\ell}{r} \text{ for every } 1 \leq \ell \leq d/(2k+1). \tag{1.10}$$

Note that none of the three bounds follows from the other two in Theorem 1.18, and therefore none of them is redundant. Comparing (1.8) and (1.10), for fixed k we have $\max_\ell \sum_{1\leq r\leq \ell} \binom{d-2k\ell}{r} = O(2^{cd})$ for some $c < 1$, while $\frac{2^{d-k}}{(d-k)^k} = 2^{d-k-k\log(d-k)} = 2^{d-o(d)}$. On the other hand, when we let k vary, it can be as large as $\Omega(d)$ in (1.10) to still give a non-trivial estimate, while k can only be chosen up to $O(d/\log d)$ for (1.8) to be non-trivial. Finally, (1.9) is beaten by (1.8) and (1.10) in most cases, however for $k = 2, 3$ and for small values of d (1.9) gives the best bound.

For two d-dimensional normed spaces X,Y we denote by $d_{BM}(X,Y) = \inf_T\{\|T\|\|T^{-1}\|\}$ their *Banach-Mazur distance*, where the infimum is taken over all linear isomorphisms

$T : X \to Y$. Note that the Banach-Mazur distance is not a metric. However, taking its logarithm, we obtain a metric space, called the *Banach-Mazur compactum*, on the isometry classes of normed spaces. It is not hard to see that $e(X)$ is upper semi-continuous on the Banach-Mazur compactum. This, together with the fact that any convex polytope can be obtained as a section of a cube of sufficiently large dimension (see for example Page 72 of Grünbaum's book [36]) implies that it would be sufficient to prove Conjecture 1.17 for k-codimensional subspaces of ℓ_∞^d for all $1 \le k \le d-4$ and $d \ge 5$. (This was also pointed out in [45].) Unfortunately, our bounds are only non-trivial if d is sufficiently large compared to k. However, we deduce an interesting corollary.

Corollary 1.19. *Let P be an origin-symmetric convex polytope in $\mathbb{R}^d$ with at most $\frac{4d}{3} - \frac{1+\sqrt{8d+9}}{6} = \frac{4d}{3} - o(d)$ opposite pairs of facets. If X is a d-dimensional normed space with P as a unit ball, then $e(X) \ge d+1$.*

There have been some extensions of lower bounds obtained on the equilateral number of certain normed spaces to other norms that are close to them according to the Banach-Mazur distance. These results are based on using the Brouwer Fixed-Point Theorem, first applied in this context by Brass [6] and Dekster [13]. We prove the following.

Theorem 1.20. *Let X be an $(d-k)$-dimensional subspace of ℓ_∞^d, and Y be an $(d-k)$-dimensional normed space such that $d_{BM}(X,Y) \le 1 + \frac{\ell}{2(d-2k-\ell k-1)}$ for some integer $1 \le \ell \le \frac{d-2k}{k}$. Then $e(Y) \ge d - k(2+\ell)$.*

Chapter 2

Discrete chains

2.1 Introduction

Recall that $u_d(n)$ is the maximum possible number of pairs at distance 1 apart in a set of n points in $\mathbb{R}^d$. Let $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_k)$ be a sequence of k positive reals. A $(k+1)$-tuple $(p_1, \ldots, p_{k+1})$ of distinct points in $\mathbb{R}^d$ is a $(k, \boldsymbol{\delta})$-*chain* if $\|p_i - p_{i+1}\| = \delta_i$ for all $i = 1, \ldots, k$. For fixed k we denote by $C_k^d(n)$, the maximum number of $(k, \boldsymbol{\delta})$-chains that can be spanned by a set of n points in $\mathbb{R}^d$, where the maximum is taken over all $\boldsymbol{\delta}$. The main results of this chapter are the following.

Theorem 2.1. *For every integer $k \geq 1$ we have*

$$C_k^2(n) = \tilde{\Theta}\left(n^{\lfloor (k+1)/3 \rfloor + 1}\right) \text{ if } k \equiv 0, 2 \pmod 3,$$

and for any $\varepsilon > 0$ we have

$$C_k^2(n) = \Omega\left(n^{(k-1)/3} u_2(n)\right) \text{ and } C_k^2(n) = O\left(n^{(k-1)/3+\varepsilon} u_2(n)\right) \text{ if } k \equiv 1 \pmod 3.$$

Theorem 2.2. *For any integer $k \geq 2$ we have*

$$C_k^3(n) = \tilde{O}\left(n^{k/2+1}\right).$$

Furthermore, for even k we have

$$C_k^3(n) = \tilde{\Theta}\left(n^{k/2+1}\right).$$

We also improve the lower bound from Theorem 1.4 for odd k. Let $us_3(n)$ be the maximum possible number of pairs at unit distance apart in $X \times Y$, where X is a set of n points in $\mathbb{R}^3$ and Y is a set of n points on a sphere in $\mathbb{R}^3$.

Proposition 2.3. *Let $k \geq 3$ odd. Then we have*

$$C_k^3(n) = \Omega\left(\max\left\{\frac{u_3(n)^k}{n^{k-1}}, us_3(n)n^{(k-1)/2}\right\}\right).$$

By using stereographic projection we obtain that $us_3(n)$ equals the maximum number of incidences between a set of n points and a set of n circles (not necessarily of the same radii) in the plane. Thus we have

$$cn^{4/3} \leq us_3(n) = \tilde{O}\left(n^{15/11}\right).$$

(For the lower bound see [53], and for the upper bound see [1, 3, 52]). Therefore, in general we cannot tell which of the two bounds in Proposition 2.3 is better. However, for large k the second term is larger than the first due to (1.1).

2.2 Preliminaries

We denote by $u_d(m,n)$ the maximum number of incidences between a set of m points and n spheres[1] of fixed radius in $\mathbb{R}^d$. In other words, $u_d(m,n)$ is the maximum number of red-blue pairs spanning a given distance in a set of m red and n blue points in $\mathbb{R}^d$. By the result of Spencer, Szemerédi and Trotter [63], we have

$$u_2(m,n) = O\left(m^{\frac{2}{3}}n^{\frac{2}{3}} + m + n\right). \tag{2.1}$$

For given r and δ we say that a point p is *r-rich* with respect to a set $P \subseteq \mathbb{R}^d$ and to distance δ, if the sphere of radius δ around p contains at least r points of P. If $P \subseteq \mathbb{R}^2$ and $|P| = n^x$, then (2.1) implies that the number of points that are n^α-rich with respect to P and to a given distance δ is

$$O\left(n^{2x-3\alpha} + n^{x-\alpha}\right). \tag{2.2}$$

The bound

$$u_3(m,n) = O\left(m^{\frac{3}{4}}n^{\frac{3}{4}} + m + n\right) \tag{2.3}$$

is due to Zahl [68] and Kaplan, Matoušek, Safernová, and Sharir [41]. It implies that for $P \subseteq \mathbb{R}^3$ with $|P| = n^x$ the number of points that are n^α-rich with respect to P and to a given distance δ is

$$O\left(n^{3x-4\alpha} + n^{x-\alpha}\right). \tag{2.4}$$

[1] circles, if $d = 2$

2.3 Bounds in $\mathbb{R}^2$

For $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_k)$ and $P_1, \ldots, P_{k+1} \subseteq \mathbb{R}^2$ we denote by $\mathcal{C}_k^{\boldsymbol{\delta}}(P_1, \ldots, P_k)$ the family of $(k+1)$-tuples $(p_1, \ldots, p_{k+1})$ with $p_i \in P_i$ for all $i \in [k+1]$, $\|p_i - p_{i+1}\| = \delta_i$ for all $i \in [k]$ and with $p_i \neq p_j$ for $i \neq j$. Let $C_k^{\boldsymbol{\delta}}(P_1, \ldots, P_{k+1}) = |\mathcal{C}_k^{\boldsymbol{\delta}}(P_1, \ldots, P_{k+1})|$ and

$$C_k(n_1, \ldots, n_{k+1}) = \max C_k^{\boldsymbol{\delta}}(P_1, \ldots, P_{k+1}),$$

where the maximum is taken over all choices of $\boldsymbol{\delta}$ and sets $P_1, \ldots, P_{k+1}$ subject to $|P_i| \leq n_i$ for all $i \in [k+1]$.

We have $C_k^2(n) \leq C_k(n, \ldots, n) \leq C_k^2((k+1)n)$. Indeed, for the lower bound choose $P_i = P$ for every $1 \leq i \leq k+1$, and for the upper bound note that $|P_1 \cup \cdots \cup P_{k+1}| \leq (k+1)n$. Since we are only interested in the order of magnitude of $C_k^2(n)$ for fixed k, we are going to bound $C_k(n, \ldots, n)$ instead of $C_k^2(n)$.

In Section 2.3.1, we are going to prove the lower bounds in Theorem 1.18. In Section 2.3.2, we are going to prove an upper bound on $C_k(n, \ldots, n)$, which is almost tight for $k \equiv 0, 2 \pmod 3$. The case $k \equiv 1 \pmod 3$ is significantly more complicated. We will treat the $k = 4$ case separately in Section 2.3.3, and then the general case in Section 2.3.4.

2.3.1 Lower bounds

For completeness, we present constructions for all congruence classes modulo 3. For $k \equiv 0, 2$ they were described in [54].

First, note that $C_0(n) = n$ and $C_1(n, n) = u_2(n, n) = \Theta(u_2(n))$. For $k = 2$, let $P_2 = \{x\}$ for some point x, and let P_1, P_3 be disjoint sets of n points on the unit circle around x. It is easy to see that $C_2^{\boldsymbol{\delta}}(P_1, P_2, P_3) = n^2$ with $\boldsymbol{\delta} = (1, 1)$, implying the lower bound $C_2(n, n, n) = \Omega(n^2)$. To obtain lower bounds in Theorem 1.18, it is thus sufficient to show that

$$C_{k+3}(n, \ldots, n) \geq n C_k(n, \ldots, n).$$

To see this take, a construction with $k+1$ parts $P_1, \ldots, P_{k+1}$ of size n that contains $C_k(n, \ldots, n)$ many $(k, \boldsymbol{\delta})$-chains for some $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_k)$. Next, fix an arbitrary point x on the plane and choose distances $\delta_{k+1}, \delta_{k+2}$ to be sufficiently large so that x can be connected to each of the points in P_{k+1} by a 2-chain with distances δ_{k+2} and δ_{k+1}. Set $P_{k+3} = \{x\}$ and let P_{k+2} be the set of intermediate points of the 2-chains described above. Finally, let $\delta_{k+3} = 1$, and P_{k+4} be a set of n points (disjoint from P_{k+2}) on the unit circle around x. Since every $(k, \boldsymbol{\delta})$-chain from $P_1 \times \cdots \times P_{k+1}$ can be extended to a $(k+3, \boldsymbol{\delta})$-chain in at least n different ways, we obtain that the number of $(k+3, \boldsymbol{\delta})$-chains with $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_{k+3})$ in $P_1 \times \cdots \times P_{k+4}$ is at least $nC_k(n)$.

Note that we can modify this construction in a straightforward way to show that for any given $\boldsymbol{\delta}$ there is a set of n points with $\Omega(n^{k/3+1})$ many $(k, \boldsymbol{\delta})$-chains if $k \equiv 0 \pmod 3$ and with $\Omega(n^{(k+4)/3})$ many $(k, \boldsymbol{\delta})$-chains if $k \equiv 2 \pmod 3$. However, for $k \equiv 1 \pmod 3$, our construction to find sets of n points with $\Omega(n^{(k-1)/3} u_2(n))$ many $(k, \boldsymbol{\delta})$-chains only works if δ_1 is much smaller than δ_2 and δ_3.

2.3.2 Upper bound for the $k \equiv 0, 2 \pmod 3$ cases

We fix $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_k)$ throughout the remainder of Section 2.3 and leave $\boldsymbol{\delta}$ out of the notation. All logs are base 2.

Theorem 2.4. *For any fixed integer $k \geq 0$ and $x, y \in [0,1]$, we have*

$$C_k(n^x, n, \ldots, n, n^y) = \tilde{O}\left(n^{\frac{f(k)+x+y}{3}}\right),$$

where $f(k) = k+2$ if $k \equiv 2 \pmod 3$ and $f(k) = k+1$ otherwise.

Theorem 2.4 implies the upper bounds in Theorem 1.18 for $k \equiv 0, 2 \pmod 3$ by taking $x = y = 1$. It is easier, however, to prove this more general statement than the upper bounds in Theorem 1.18 directly. Having varied sizes of the first and the last groups of points allows for a seamless use of induction.

Proof of Theorem 2.4. The proof is by induction on k. Let us first verify the statement for $k \leq 2$. (Note that, for $k = 0$, we should have $x = y$.) We have

$$C_0(n^x) \leq n^x = O\left(n^{\frac{1+x+y}{3}}\right),$$

$$C_1(n^x, n^y) \leq u_2(n^x, n^y) = O\left(n^{\frac{2}{3}(x+y)} + n^x + n^y\right) = O\left(n^{\frac{2+x+y}{3}}\right), \tag{2.5}$$

$$C_2(n^x, n, n^y) \leq 2n^x n^y = O\left(n^{\frac{4+x+y}{3}}\right), \tag{2.6}$$

where (2.5) follows from (2.1) and (2.6) follows from the fact that each pair (p_1, p_3) can be extended to a 2-chain (p_1, p_2, p_3) in at most 2 different ways.

Next, let $k \geq 3$. Take $P_1, \ldots, P_{k+1} \subseteq \mathbb{R}^2$ with $|P_1| = n^x$, $|P_{k+1}| = n^y$, and $|P_i| = n$ for $2 \leq i \leq k$. Denote by $P_2^\alpha \subseteq P_2$ the set of those points in P_2 that are at least n^α-rich but at most $2n^\alpha$-rich with respect to P_1 and δ_1. Similarly, we denote by $P_k^\beta \subseteq P_k$ the set of those points in P_k that are at least n^β-rich but at most $2n^\beta$-rich with respect to P_{k+1} and δ_k.

Applying a standard dyadic decomposition argument twice implies that

$$\mathcal{C}_k(P_1, P_2 \ldots, P_k, P_{k+1}) = \bigcup_{\alpha,\beta} \mathcal{C}_k(P_1, P_2^\alpha, P_3, \ldots, P_{k-1}, P_k^\beta, P_{k+1}),$$

where the union is taken over all $\alpha, \beta \in \left\{ \frac{i}{\log n} : i = 0, \ldots, \lceil \log n \rceil \right\}$. Since the cardinality of the latter set is at most $\log n + 2$, it is sufficient to prove that for every α and β we have

$$C_k(P_1, P_2^\alpha, P_3, \ldots, P_{k-1}, P_k^\beta, P_{k+1}) = \tilde{O}\left(n^{\frac{f(k)+x+y}{3}}\right). \tag{2.7}$$

To prove this, we consider three cases.

Case 1: $\boldsymbol{\alpha \geq \frac{x}{2}}$. By (2.2) we have $|P_2^\alpha| = O(n^{x-\alpha})$. Therefore the number of pairs $(p_1, p_2) \in P_1 \times P_2^\alpha$ with $\|p_1 - p_2\| = \delta_1$ is at most $O(n^x)$. Since every pair $(p_1, p_2) \in P_1 \times P_2^\alpha$ and every $(k-3)$-chain $(p_4, \ldots, p_{k+1}) \in P_4 \times \cdots \times P_k^\beta \times P_{k+1}$ can be extended to a k-chain $(p_1, \ldots, p_{k+1}) \in P_1 \times \cdots \times P_{k+1}$ in at most two different ways, we obtain

$$C_k(P_1, P_2^\alpha, \ldots, P_k^\beta, P_{k+1}) \leq 4O(n^x) C_{k-3}(P_4, \ldots, P_k^\beta, P_{k+1}).$$

By induction we have

$$C_{k-3}(P_4, \ldots, P_k^\beta, P_{k+1}) = \tilde{O}\left(n^{\frac{f(k-3)+1+y}{3}}\right).$$

These two displayed formulas and the fact that $f(k-3) = f(k) - 3$ imply (2.7).

Case 2: $\boldsymbol{\beta \geq \frac{y}{2}}$. By symmetry, this case can be treated in the same way as Case 1.

Case 3: $\boldsymbol{\alpha \leq \frac{x}{2}}$ and $\boldsymbol{\beta \leq \frac{y}{2}}$. By (2.2) we have $|P_2^\alpha| = O\left(n^{2x-3\alpha}\right)$ and $|P_k^\beta| = O\left(n^{2y-3\beta}\right)$. The number of $(k-2)$-chains in $P_2^\alpha \times P_3 \times \cdots \times P_{k-1} \times P_k^\beta$ is $C_{k-2}(P_2^\alpha, P_3, \ldots, P_{k-1}, P_k^\beta)$, and every $(k-2)$-chain $(p_2, \ldots, p_k) \in P_2^\alpha \times P_3 \times \cdots \times P_{k-1} \times P_k^\beta$ can be extended at most $4n^{\alpha+\beta}$ ways to a k-chain in $P_1 \times P_2^\alpha \times \cdots \times P_k^\beta \times P_{k+1}$. Thus

$$C_k(P_1, P_2^\alpha, \ldots, P_k^\beta, P_{k+1}) \leq 4n^{\alpha+\beta} C_{k-2}(P_2^\alpha, \ldots, P_k^\beta).$$

By induction we have

$$C_{k-2}(P_2^\alpha, \ldots, P_k^\beta) = \tilde{O}\left(n^{\frac{f(k-2)+2x-3\alpha+2y-3\beta}{3}}\right).$$

For $k \equiv 0, 2 \pmod 3$ we have $f(k) \geq f(k-2) + 2$, and thus

$$C_k(P_1, P_2^\alpha, \ldots, P_k^\beta, P_{k+1}) = \tilde{O}\left(n^{\alpha+\beta} n^{\frac{f(k-2)+2x-3\alpha+2y-3\beta}{3}}\right)$$
$$= \tilde{O}\left(n^{\frac{f(k)-2+2x+2y}{3}}\right) = \tilde{O}\left(n^{\frac{f(k)+x+y}{3}}\right).$$

If $k \equiv 1 \pmod 3$ then $f(k) < f(k-2) + 2$, and thus the argument above does not work. However, we then have $f(k) = f(k-1) + 1$, and we can use the bound

$$C_k(P_1, P_2^\alpha, \ldots, P_k^\beta, P_{k+1}) \leq 2n^\alpha C_{k-1}(P_2^\alpha, P_3, \ldots, P_{k+1}),$$

obtained in an analogous way. This gives

$$C_k(P_1, P_2^\alpha, P_3, \ldots, P_{k+1}) = \tilde{O}\left(n^\alpha n^{\frac{f(k-1)+2x-3\alpha+y}{3}}\right) = \tilde{O}\left(n^{\frac{f(k)-1+2x+y}{3}}\right) = \tilde{O}\left(n^{\frac{f(k)+x+y}{3}}\right).$$

□

Remark 2.5. The proof above is not sufficient to obtain an almost sharp bound in the $k \equiv 1 \pmod 3$ case for two reasons. First, for these k any analogue of Theorem 2.4 would involve taking maximums of two expressions, where one contains $u_2(n^x, n)$ and the other contains $u_2(n^y, n)$. However, due to our lack of good understanding of how $u_2(n^x, n)$ changes as x is increasing, this is difficult to work with.

Second, on a more technical side, while Case 1 and Case 2 in the above proof would go through with any reasonable inductive statement, Case 3 would fail. The main reason for this is that C_k as a function of k makes jumps at every third value of k, and remains essentially the same, or changes by $u(n,n)/n$ for the other values of k. Thus one would need to remove three vertices from the path to make the induction work. However, the path has only two ends, and removing vertices other than the endpoints turns out to be intractable.

2.3.3 Upper bound for $k = 4$

In this section we prove the upper bound in Theorem 1.18 for $k = 4$. Let $P_1, \ldots, P_5$ be five sets of n points. We will show that $C_4(P_1, \ldots, P_5) = \tilde{O}(u_2(n)n)$, which is slightly stronger than what is stated in Theorem 1.18.

Instead of (2.2) we need the following more general bound on the number of rich points.

Observation 2.6 (Richness bound)**.** *Let n^y be the maximum possible number of points that are n^α-rich with respect to a set of n^x points and some distance δ. Then we have*

$$n^{y+\alpha} \leq u_2(n^x, n^y), \tag{2.8}$$

or, equivalently

$$n^\alpha \leq \frac{u_2(n^x, n^y)}{n^y}.$$

The proof of (2.8) follows immediately from the definition of n^α-richness and $u_2(n^x, n^y)$.

Let $\Lambda := \left\{\frac{i}{\log n} : i = 0, \dots, \lceil \log n \rceil\right\}^4$. For any $\boldsymbol{\alpha} = (\alpha_2, \alpha_3, \alpha_4, \alpha_5) \in \Lambda$ let $Q_1^{\boldsymbol{\alpha}} = P_1$ and for $i = 2, \dots, 5$ define recursively $Q_i^{\boldsymbol{\alpha}}$ to be the set of those points in P_i that are at least n^{α_i}-rich but at most $2n^{\alpha_i}$-rich with respect to Q_{i-1} and δ_i.

Applying a standard dyadic decomposition argument 4-times implies

$$\mathcal{C}_4(P_1, \dots, P_5) = \bigcup_{\boldsymbol{\alpha} \in \Lambda} \mathcal{C}_4\left(Q_1^{\boldsymbol{\alpha}}, \dots, Q_5^{\boldsymbol{\alpha}}\right).$$

We have $|\Lambda| = \tilde{O}(1)$ and thus, in order to prove the theorem, it is sufficient to show that for every $\boldsymbol{\alpha} \in \Lambda$ we have

$$C_4\left(Q_1^{\boldsymbol{\alpha}}, \dots, Q_5^{\boldsymbol{\alpha}}\right) = O\left(n \cdot u_2(n, n)\right).$$

From now on, fix $\boldsymbol{\alpha} = (\alpha_2, \dots, \alpha_5)$, and denote $Q_i = Q_i^{\boldsymbol{\alpha}}$. Choose $x_i \in [0, 1]$ so that $|Q_i| = n^{x_i}$. Then we have

$$C_4(Q_1, \dots, Q_5) = O\left(n^{x_5+\alpha_5+\alpha_4+\alpha_3+\alpha_2}\right). \tag{2.9}$$

Indeed, each chain $(p_1, \dots, p_5)$ with $p_i \in Q_i$ can be obtained in the following five steps.

- **Step 1:** Pick $p_5 \in Q_5$.
- **Step i ($2 \leq i \leq 5$):** Pick a point $p_{6-i} \in Q_{6-i}$ at distance δ_{6-i} from p_{7-i}.

In the first step we have n^{x_5} choices, and for $i \geq 2$ in the i-th step we have at most $2n^{\alpha_{6-i}}$ choices. Further, by Observation 2.6, for each $i \geq 2$ we have

$$n^{\alpha_i} \leq \frac{u_2(n^{x_{i-1}}, n^{x_i})}{n^{x_i}}. \tag{2.10}$$

Combining (2.9) and (2.10), we obtain

$$C_4(Q_1, \dots, Q_5) = O\left(u_2(n^{x_4}, n^{x_5}) \frac{u_2(n^{x_3}, n^{x_4})}{n^{x_4}} \frac{u_2(n^{x_2}, n^{x_3})}{n^{x_3}} \frac{u_2(n^{x_1}, n^{x_2})}{n^{x_2}}\right). \tag{2.11}$$

By (2.1) we have

$$u_2(n^{x_{i-1}}, n^{x_i}) = O\left(\max\left\{n^{\frac{2}{3}(x_i+x_{i-1})}, n^{x_i}, n^{x_{i-1}}\right\}\right).$$

Note that the maximum is attained on the second (third) term iff $x_{i-1} \le \frac{x_i}{2}$ ($x_i \le \frac{x_{i-1}}{2}$). To bound $C_4(Q_1, \ldots, Q_5)$ we consider several cases depending on which of these three terms the maximum above is attained on for different i.

Case 1: For all $2 \le i \le 5$ we have $u_2(n^{x_{i-1}}, n^{x_i}) = O\left(n^{\frac{2}{3}(x_i+x_{i-1})}\right)$. Then

$$\frac{u_2(n^{x_4}, n^{x_5})u_2(n^{x_3}, n^{x_4})u_2(n^{x_2}, n^{x_3})}{n^{x_2+x_3+x_4}} = O\left(n^{\frac{2}{3}x_5+\frac{1}{3}x_4+\frac{1}{3}x_3-\frac{1}{3}x_2}\right)$$

and

$$\frac{u_2(n^{x_3}, n^{x_4})u_2(n^{x_2}, n^{x_3})u_2(n^{x_1}, n^{x_2})}{n^{x_2+x_3+x_4}} = O\left(n^{-\frac{1}{3}x_4+\frac{1}{3}x_3+\frac{1}{3}x_2+\frac{2}{3}x_1}\right).$$

Substituting each of these two displayed formulas into (2.11) and taking their product, we obtain

$$C_4(Q_1, \ldots, Q_5)^2 = O\left(u_2(n^{x_1}, n^{x_2})u_2(n^{x_4}, n^{x_5}) \cdot n^{\frac{2}{3}x_1+\frac{2}{3}x_3+\frac{2}{3}x_5}\right) = O\left(u_2(n,n)^2 \cdot n^2\right),$$

which concludes the proof in this case.

Case 2: There is an $2 \le i \le 5$ such that

$$\min\{x_{i-1}, x_i\} \le \frac{1}{2}\max\{x_{i-1}, x_i\} \quad \text{and thus} \quad u_2(n^{x_{i-1}}, n^{x_i}) = O\left(\max\{n^{x_{i-1}}, n^{x_i}\}\right). \tag{2.12}$$

We distinguish three cases based on for which i holds.

Case 2.1: (2.12) holds for $i = 2$ or 5. In particular, this implies that $u_2(n^{x_1}, n^{x_2}) = O(n)$ or $u_2(n^{x_4}, n^{x_5}) = O(n)$. The following lemma finishes the proof in this case.

Lemma 2.7. *Let $R_1, \ldots, R_5 \subseteq \mathbb{R}^2$ such that $|R_i| \le n$ for every $i \in [5]$. If $u_2(R_1, R_2) = O(n)$ or $u_2(R_4, R_5) = O(n)$ holds, then $C_4(R_1, \ldots, R_5) = O(n \cdot u_2(n,n))$.*

Proof. We have

$$C_4(R_1, \ldots, R_5) \le 2u_2(R_1, R_2)u_2(R_4, R_5) = O\left(n \cdot u_2(n,n)\right).$$

Indeed, every 4-tuple (r_1, r_2, r_4, r_5) with $r_i \in R_i$ can be extended in at most two different ways to a 4-chain $(r_1, \ldots, r_5) \in R_1 \times \cdots \times R_5$. At the same time, the number of 4-tuples with $\|r_1 - r_2\| = \delta_1$, $\|r_4 - r_5\| = \delta_4$ is at most $u_2(R_1, R_2)u_2(R_4, R_5)$. $\square$

Case 2.2: (2.12) holds for $i = 4$. Note that if $x_4 \leq \frac{x_3}{2} \leq \frac{1}{2}$, then $u_2(n^{x_5}, n^{x_4}) = O(n)$, and we can apply Lemma 2.7 to conclude the proof in this case. Thus we may assume that $x_3 \leq \frac{x_4}{2}$, and hence $u_2(n^{x_4}, n^{x_3}) = O(n^{x_4})$. This means that $n^{\alpha_4} = O(1)$ by Observation 2.6. Thus, to finish the proof of this case, it is sufficient to prove the following claim.

Claim 2.8. *Let $R_1, \dots, R_5 \subseteq \mathbb{R}^2$ such that $|R_i| \leq n$ for all $i \in [5]$ and every point of R_4 is $O(1)$ rich with respect to R_3 and δ_3. Then $C_4(R_1, \dots, R_5) = O\left(n \cdot u_2(n, n)\right)$.*

Proof. Every 4-chain $(r_1, \dots, r_5)$ can be obtained in the following steps.

- Pick a pair $(r_4, r_5) \in R_4 \times R_5$ with $\|r_4 - r_5\| = \delta_4$.
- Choose $r_3 \in R_3$ at distance δ_3 from r_4.
- Pick a point $r_1 \in R_1$.
- Extend (r_1, r_3, r_4, r_5) to a 4-chain.

In the first step, we have at most $u_2(n, n)$ choices, in the third at most n choices, and in the other two steps at most $O(1)$. □

Case 2.3 (2.12) holds for $i = 3$ *only*. Arguing as in Case 2.2, we may assume that $u_2(n^{x_3}, n^{x_2}) = O(n^{x_2})$. Then we have

$$C_4(Q_1, \dots, Q_5) = O\left(u_2(n^{x_4}, n^{x_5}) \frac{u_2(n^{x_3}, n^{x_4})}{n^{x_4}} \frac{u_2(n^{x_2}, n^{x_3})}{n^{x_3}} \frac{u_2(n^{x_1}, n^{x_2})}{n^{x_2}}\right)$$

$$= O\left(u_2(n^{x_1}, n^{x_2}) \cdot n^{\frac{2}{3}(x_4 + x_5) + \frac{2}{3}(x_3 + x_4) - x_4 - x_3}\right) = O\left(u_2(n, n) \cdot n\right),$$

which finishes the proof.

2.3.4 Upper bound for $k \equiv 1 \pmod 3$

We will prove the upper bound in Theorem 1.18 for $k \equiv 1$ by induction. The $k = 1$ case follows from the definition of $u_2(n, n)$, thus we may assume that $k \geq 4$. For the rest of the section fix $\varepsilon' > 0$, and sets $P_1, \dots, P_{k+1} \subseteq \mathbb{R}^2$ of size n, further let $\varepsilon = \frac{\varepsilon'}{4k}$. We are going to show that $C_k(P_1, \dots, P_{k+1}) = O(n^{(k-1)/3+\varepsilon'} u_2(n))$.

The first step of the proof is to find a certain covering of $P_1 \times \cdots \times P_{k+1}$, which resembles the one used for the $k = 4$ case, although is more elaborate.[2] (The goal of this covering is to make the corresponding graph between each of the two consecutive parts 'regular in both directions' in a certain sense.)

Let
$$\Lambda = \left\{ i\varepsilon : i = 0, \ldots, \left\lfloor \frac{1}{\varepsilon} \right\rfloor \right\}^{k+1}.$$
We cover the product $\mathbf{P} = P_1 \times \cdots \times P_{k+1}$ by fine-grained classes $P_1^{\boldsymbol{\gamma}} \times \ldots \times P_{k+1}^{\boldsymbol{\gamma}}$ encoded by the sequence $\boldsymbol{\gamma} = (\boldsymbol{\gamma^1}, \boldsymbol{\gamma^2}, \ldots)$ of length at most $(k+1)\varepsilon^{-1} + 1$ with $\boldsymbol{\gamma^j} \in \Lambda$ for each $j = 1, 2, \ldots$. One property that we shall have is

$$P_1 \times \cdots \times P_{k+1} = \bigcup_{\boldsymbol{\gamma}} P_1^{\boldsymbol{\gamma}} \times \ldots \times P_{k+1}^{\boldsymbol{\gamma}}.$$

To find the covering, first we define a function D that receives a parity digit $j \in \{0, 1\}$, a product set $\mathbf{R} := R_1 \times \ldots \times R_{k+1}$ and a $(k+1)$-tuple $\boldsymbol{\alpha} \in \Lambda$, and outputs a product set $D(j, \boldsymbol{R}, \boldsymbol{\alpha}) = \mathbf{R}(\boldsymbol{\alpha}) = R_1(\boldsymbol{\alpha}) \times \ldots \times R_{k+1}(\boldsymbol{\alpha})$.

Definition of D

- If $j = 1$ then let $R_1(\boldsymbol{\alpha}) := R_1$ and for $i = 2, \ldots, k+1$ define $R_i(\boldsymbol{\alpha})$ iteratively to be the set of points in R_i that are at least n^{α_i}, but at most $n^{\alpha_i + \varepsilon}$-rich with respect to $R_{i-1}(\boldsymbol{\alpha})$ and δ_{i-1}.
- If $j = 0$ then apply the same procedure, but in reverse order. More precisely, let $R_{k+1}(\boldsymbol{\alpha}) = R_{k+1}$ and for $i = k, k-1, \ldots, 1$ define $R_i(\boldsymbol{\alpha})$ iteratively to be the set of points in R_i that are at least n^{α_i} but at most $n^{\alpha_i + \varepsilon}$-rich with respect to $R_{i+1}(\boldsymbol{\alpha})$ and δ_i.

Note that
$$\mathbf{R} = \bigcup_{\boldsymbol{\alpha} \in \Lambda} \mathbf{R}(\boldsymbol{\alpha}). \tag{2.13}$$
For a sequence $\boldsymbol{\gamma} = (\boldsymbol{\gamma^1}, \boldsymbol{\gamma^2}, \ldots)$ with $\boldsymbol{\gamma^j} \in \Lambda$, we define $\mathbf{P}^{\boldsymbol{\gamma}}$ recursively as follows. Let $\mathbf{P}^{\emptyset} := \mathbf{P}$, and for each $j \geq 1$ let

$$\mathbf{P}^{(\gamma^1, \ldots, \gamma^j)} = D(j \pmod 2, \mathbf{P}^{(\gamma^1, \ldots, \gamma^{j-1})}, \boldsymbol{\gamma^j}).$$

[2] This covering brings in the ε-error term in the exponent, that we could avoid in the $k = 4$ case.

We say that a sequence $\boldsymbol{\gamma}$ is *stable at* j if

$$\left|\mathbf{P}^{(\boldsymbol{\gamma}^1,\dots,\boldsymbol{\gamma}^j)}\right| \geq \left|\mathbf{P}^{(\boldsymbol{\gamma}^1,\dots,\boldsymbol{\gamma}^{j-1})}\right| \cdot n^{-\varepsilon}.$$

Otherwise $\boldsymbol{\gamma}$ is *unstable at* j.

Definition 2.9. *Let Υ be the set of those sequences $\boldsymbol{\gamma}$ that are stable at their last coordinate, but are not stable for any previous coordinate, and for which $\mathbf{P}^{\boldsymbol{\gamma}}$ is non-empty.*

The set Υ has several useful properties, some of which are summarised in the following lemma.

Lemma 2.10. *1. Any $\boldsymbol{\gamma} \in \Upsilon$ has length at most $(k+1)\varepsilon^{-1}+1$.*
2. $|\Upsilon| = O_\varepsilon(1)$.
3. $\mathbf{P} = \bigcup_{\boldsymbol{\gamma}\in\Upsilon} \mathbf{P}^{\boldsymbol{\gamma}}$.

Proof. 1. If $\boldsymbol{\gamma}$ is unstable at j then

$$|\mathbf{P}^{(\gamma^1,\dots,\gamma^j)}| \leq |\mathbf{P}^{(\gamma^1,\dots,\gamma^{j-1})}| \cdot n^{-\varepsilon}.$$

Since $|\mathbf{P}| = n^{k+1}$ and $|\mathbf{P}^{\boldsymbol{\gamma}}| \geq 1$, we conclude that $\boldsymbol{\gamma}$ is unstable at at most $(k+1)\varepsilon^{-1}$ indices j.

2. It follows from part 1 by counting all possible sequences of elements from the set Λ of length at most $(k+1)\varepsilon^{-1}+1$. (Note that $|\Lambda| = O_\varepsilon(1)$.)

3. For a nonnegative integer j let $\Lambda^{\leq j}$ be the set of all sequences of length at most j of elements from Λ. Let

$$\Upsilon_j := \left(\Upsilon \cap \Lambda^{\leq j}\right) \cup \Psi_j, \text{ where } \Psi_j := \left\{\boldsymbol{\gamma} \in \Lambda^j : \boldsymbol{\gamma} \text{ is not stable for any } \ell \leq j\right\}.$$

By part 1 of the lemma, $\Upsilon_j = \Upsilon$ for $j > (k+1)\varepsilon^{-1}$. We prove by induction on j that $\mathbf{P} = \bigcup_{\boldsymbol{\gamma}\in\Upsilon_j} \mathbf{P}^{\boldsymbol{\gamma}}$.

Υ_0 consists of an empty sequence, thus the statement is clear for $j = 0$. Next, assume that the statement holds for j. We have

$$\mathbf{P} = \bigcup_{\boldsymbol{\gamma}\in\Upsilon_j} \mathbf{P}^{\boldsymbol{\gamma}} = \bigcup_{\boldsymbol{\gamma}\in\Lambda^{\leq j}} \mathbf{P}^{\boldsymbol{\gamma}} \cup \bigcup_{\boldsymbol{\gamma}\in\Psi_j} \mathbf{P}^{\boldsymbol{\gamma}}.$$

By (2.13) we have that $\mathbf{P}^{\boldsymbol{\gamma}} = \bigcup_{\boldsymbol{\gamma}'} \mathbf{P}^{\boldsymbol{\gamma}'}$ holds for any $\boldsymbol{\gamma} \in \Psi_j$, where the union is taken over the sequences from Λ^{j+1} that coincide with $\boldsymbol{\gamma}$ on the first j entries. This, together with $\boldsymbol{\gamma}' \in \left(\Upsilon \cap \Lambda^{j+1}\right) \cup \Psi_{j+1}$ when $\mathbf{P}^{\boldsymbol{\gamma}'}$ is nonempty finishes the proof.

□

Parts 2 and 3 of Lemma 2.10 imply that in order to complete the proof of the $k \equiv 1 \pmod 3$ case, it is sufficient to show that for any $\boldsymbol{\gamma} \in \Upsilon$ we have

$$C_k(P_1^{\boldsymbol{\gamma}}, \dots, P_{k+1}^{\boldsymbol{\gamma}}) = O\left(u_2(n) \cdot n^{\frac{k-1}{3}+4k\varepsilon}\right). \tag{2.14}$$

From now on fix $\boldsymbol{\gamma} \in \Upsilon$. For each $i = 1, \dots, k+1$ let $R_i := P_i^{\boldsymbol{\gamma}}$ and $Q_i := P_i^{\boldsymbol{\gamma}'}$, where $\boldsymbol{\gamma'}$ is obtained from $\boldsymbol{\gamma}$ by removing the last element of the sequence. Without loss of generality, assume that the length ℓ of $\boldsymbol{\gamma}$ is even. For each $i = 1, \dots, k+1$, choose x_i, y_i such that

$$|Q_i| = n^{x_i}, \qquad |R_i| = n^{y_i}.$$

Let $\alpha_i := \boldsymbol{\gamma}_i^{\ell-1}$ and $\beta_i := \boldsymbol{\gamma}_i^{\ell}$. By the definition of $\mathbf{P}^{\boldsymbol{\gamma}}$ we have that each point in Q_i is at least n^{α_i}-rich but at most $n^{\alpha_i+\varepsilon}$-rich with respect to Q_{i-1} and δ_{i-1}, and each point in R_i is at least n^{β_i}-rich but at most $n^{\beta_i+\varepsilon}$-rich with respect to R_{i+1} and δ_i.

By Observation 2.6, we have

$$n^{\alpha_i} \leq \frac{u_2(n^{x_{i-1}}, n^{x_i})}{n^{x_i}} \quad \text{and} \quad n^{\beta_i} \leq \frac{u_2(n^{y_i}, n^{y_{i+1}})}{n^{y_i}} \leq \frac{u_2(n^{x_i}, n^{x_{i+1}})}{n^{x_i-\varepsilon}}. \tag{2.15}$$

The last inequality follows from two facts: first $u_2(n^{y_i}, n^{y_{i+1}}) \leq u_2(n^{x_i}, n^{x_{i+1}})$ and, second, since $\boldsymbol{\gamma}$ is stable at its last coordinate[3], we have $n^{y_i} = |R_i| \geq |Q_i| \cdot n^{-\varepsilon} = n^{x_i-\varepsilon}$.

In the same fashion as in the beginning of Section 2.3.3, we can show that

$$C_k(R_1, \dots, R_{k+1}) \leq n^{y_1} n^{\beta_1+\dots+\beta_k+k\varepsilon}, \text{ and}$$

$$C_k(R_1, \dots, R_{k+1}) \leq C_k(Q_1, \dots, Q_{k+1}) \leq n^{x_{k+1}} n^{\alpha_{k+1}+\alpha_k+\dots+\alpha_2+k\varepsilon}.$$

Combining the first of these displayed inequalities with (2.15), we have

$$C_k(R_1, \dots, R_{k+1}) \leq u_2(n^{x_1}, n^{x_2}) \prod_{2\leq i\leq k} \frac{u_2\left(n^{x_i}, n^{x_{i+1}}\right)}{n^{x_i}} n^{2k\varepsilon}.$$

Recall that

$$u_2(n^{x_i}, n^{x_{i+1}}) = O\left(\max\{n^{\frac{2}{3}(x_i+x_{i+1})}, n^{x_i}, n^{x_{i+1}}\}\right). \tag{2.16}$$

[3] This is the only place where we use the stability of $\boldsymbol{\gamma}$ directly.

To bound $C_k(R_1, \ldots, R_{k+1})$, we consider several cases based on which of these three terms can be used to bound $u_2(n^{x_i}, n^{x_{i+1}})$ for different values of i.

Case 1: Either $u_2(n^{x_1}, n^{x_2}) = O(n)$ or $u_2(n^{x_k}, n^{x_{k+1}}) = O(n)$ holds. As in the proof of Lemma 2.7, we have

$$C_k(R_1, \ldots, R_{k+1}) \\ \leq \min \left\{2u_2(n^{y_1}, n^{y_2})C_{k-3}(R_4, \ldots, R_{k+1}), 2u_2(n^{y_k}, n^{y_{k+1}})C_{k-3}(R_1, \ldots, R_{k-2})\right\}.$$

By induction we obtain $C_{k-3}(R_4, \ldots, R_{k+1}), C_{k-3}(R_1, \ldots, R_{k-2}) = O\left(n^{\frac{k-4}{3}+\varepsilon} \cdot u_2(n)\right)$. Together with the assumption of Case 1, and the fact that $u_2(n^{y_1}, n^{y_2}) \leq u_2(n^{x_1}, n^{x_2})$ and $u_2(n^{y_k}, n^{y_{k+1}}) \leq u_2(n^{x_k}, n^{x_{k+1}})$, this implies (2.14) and finishes the proof.

Case 2: For some $i = 1, \ldots, (k-1)/3$, one of the following holds:

- $u_2(n^{x_{3i+1}}, n^{x_{3i+2}}) = O(\max\{n^{x_{3i+1}}, n^{x_{3i+2}}\})$;
- $u_2(n^{x_{3i-1}}, n^{x_{3i}}) = O(n^{x_{3i-1}})$;
- $u_2(n^{x_{3i}}, n^{x_{3i+1}}) = O(n^{x_{3i+1}})$.

We will show how to conclude in the first case. The other cases are very similar and we omit the details of their proofs. If $u_2(n^{x_{3i+1}}, n^{x_{3i+2}}) = O(n^{x_{3i+2}})$ then $n^{\alpha_{3i+2}} = O(1)$ by (2.15). Every chain $(r_1, \ldots, r_{k+1}) \in \mathcal{C}_k(Q_1, \ldots, Q_{k+1})$ can be obtained as follows.

1. Pick a $(3i-2)$-chain $(r_1, \ldots, r_{3i-1})$ with $r_j \in Q_j$ for every j.
2. Pick a $(k-3i-1)$-chain $(r_{3i+2}, r_{3i+3}, \ldots, r_{k+1})$ with $r_j \in Q_j$ for every j.
3. Extend $(r_{3i+2}, r_{3i+3}, \ldots, r_{k+1})$ to a $(k-3i-2)$ chain $(r_{3i+1}, r_{3i+2}, \ldots, r_{k+1})$.
4. Connect $(r_1, \ldots, r_{3i-1})$ and $(r_{3i+1}, r_{3i+2}, \ldots, r_{k+1})$ to obtain a k-chain.

In the first step, we have $O\left(n^{\frac{3i-3}{3}+\varepsilon} \cdot u_2(n)\right)$ choices by induction on k. In the second step, we have $\tilde{O}\left(n^{\frac{k-3i+2}{3}}\right)$ choices by the $k \equiv 0 \pmod 3$ case of Theorem 1.18. In the third step, we have at most $n^{\alpha_{3i+2}+\varepsilon} = O(n^{\varepsilon})$ choices. Finally, in the fourth step we have at most 2 choices. Thus the number of k-chains is at most

$$O\left(n^{\frac{3i-3}{3}+\varepsilon} \cdot u_2(n)\right) \cdot \tilde{O}\left(n^{\frac{k-3i+2}{3}}\right) \cdot O\left(n^{\varepsilon}\right) \cdot 2 = O\left(n^{\frac{k-1}{3}+3\varepsilon} \cdot u_2(n)\right),$$

finishing the proof of the first case.

If $u_2(n^{x_{3i+1}}, n^{x_{3i+2}}) = O(n^{x_{3i+1}})$ then $n^{\beta_{3i+1}} = O(n^{\varepsilon})$ by (2.15).[4] We proceed similarly in this case, but we count the k-chains now in $R_1 \times \ldots \times R_{k+1}$ instead in $Q_1 \times \ldots \times Q_{k+1}$ (and get an extra factor of n^{ε} in the bound). In all cases, we obtain (2.14).

Case 3: Neither the assumptions of Case 1 nor that of Case 2 hold. We define four sets S', S'_+, S'_{++}, and S'_- of indices in $\{2, \ldots, k\}$ as follows. Let

$$S' := \left\{ i \,:\, u_2(n^{x_i}, n^{x_{i-1}}) = O(n^{\frac{2}{3}(x_i + x_{i-1})}) \text{ and } u_2(n^{x_{i+1}}, n^{x_i}) = O(n^{\frac{2}{3}(x_{i+1}+x_i)}) \right\},$$

$$S'_+ := \Big\{ i : u_2(n^{x_i}, n^{x_{i-1}}) = O(n^{\frac{2}{3}(x_i + x_{i-1})}) \text{ and } u_2(n^{x_{i+1}}, n^{x_i}) = O(n^{x_i}), \text{ or}$$
$$u_2(n^{x_i}, n^{x_{i-1}}) = O(n^{x_i}) \text{ and } u_2(n^{x_{i+1}}, n^{x_i}) = O(n^{\frac{2}{3}(x_{i+1}+x_i)}) \Big\},$$

$$S'_{++} := \left\{ i : u_2(n^{x_i}, n^{x_{i-1}}) = O(n^{x_i}) \text{ and } u_2(n^{x_{i+1}}, n^{x_i}) = O(n^{x_i}) \right\}, \text{ and}$$

$$S'_- := \Big\{ i : u_2(n^{x_i}, n^{x_{i-1}}) = O(n^{\frac{2}{3}(x_i + x_{i-1})}) \text{ and } u_2(n^{x_{i+1}}, n^{x_i}) = O(n^{x_{i+1}}), \text{ or}$$
$$u_2(n^{x_i}, n^{x_{i-1}}) = O(n^{x_{i-1}}) \text{ and } u_2(n^{x_{i+1}}, n^{x_i}) = O(n^{\frac{2}{3}(x_{i+1}+x_i)}) \Big\}.$$

Since the conditions of Case 2 are not satisfied, we have

$$\{2, \ldots, k\} \subseteq S' \cup S'_+ \cup S'_{++} \cup S'_-.$$

Indeed, for each $i \in \{2, \ldots, k\}$, there are 9 possible pairs of maxima in (2.16) with $i, i+1$. The four sets above encompass 6 possibilities. In total, there are 4 possible pairs of maxima with only the two last terms from (2.16) used. For $i \equiv 1, 2 \pmod 3$, any of those 4 are excluded due to the first condition in Case 2 (in fact, then $i \in S' \cup S'_-$). If $i \equiv 0 \pmod 3$, then the second and the third condition in Case 2 rule out all possibilities but the one defining S'_{++}.

From these it directly follows that if $i \in S'_{++}$, then $i-1, i+1 \in S'_-$, while if $i \in S'_+$ then one of $i-1, i+1$ is in S'_-. (Recall that $i \in S'_+ \cup S'_{++}$ only if $i \equiv 0 \pmod 3$.) These together imply

$$|S'_+| + 2|S'_{++}| \leq |S'_-|. \tag{2.17}$$

We partition $\{2, \ldots, k\}$ as follows: let $S_- = S'_-, S = S' \setminus S'_-, S_+ = S'_+ \setminus (S'_- \cup S')$ and $S_{++} = \{2, \ldots, k\} \setminus S'_- \cup S' \cup S'_+$. Note that the analogue of (2.17) holds for the new sets.

[4] This is the key application of (2.15), and the reason why we needed a decomposition with regularity in both directions between the consecutive parts.

That is, we have

$$|S_+| + 2|S_{++}| \leq |S_-|.$$

Recall that

$$C_k(R_1, \ldots, R_{k+1}) \leq u_2(n^{x_1}, n^{x_2}) \prod_{2 \leq i \leq k} \frac{u_2\left(n^{x_i}, n^{x_{i+1}}\right)}{n^{x_i}} n^{2k\varepsilon}. \tag{2.18}$$

Since the assumptions of Case 1 and 2 do not hold, we have $2, k \in S$. Indeed, $2, k \neq 0 \pmod 3$ and thus $2, k \notin S_+, S_{++}$. Further, if say $k \in S_- = S'_-$ then by the definition of S'_- we either have $u_2(n^{x_{k+1}}, n^{x_k}) = O(n)$, or $u_2(n^{x_k}, n^{x_{k-1}}) = O(n^{x_{k-1}})$. The first case cannot hold since the assumption of Case 1 does not hold. Further, the second case cannot hold either, since it would imply $x_k \leq \frac{x_{k-1}}{2} \leq \frac{1}{2}$, meaning $u_2(n^{x_{k+1}}, n^{x_k}) = O(n)$. Using $2, k \in S$ and expanding (2.18), we obtain

$$\begin{aligned} &C_k(R_1, \ldots, R_{k+1}) \\ &\quad \leq n^{2k\varepsilon} u_2(n^{x_1}, n^{x_2}) n^{-\frac{1}{3}x_2} n^{\frac{2}{3}x_{k+1}} \prod_{\substack{i \in S, \\ i \neq 2}} n^{\frac{1}{3}x_i} \prod_{i \in S_+} n^{\frac{2}{3}x_i} \prod_{i \in S_{++}} n^{x_i} \prod_{i \in S_-} n^{-\frac{1}{3}x_i}, \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} &C_k(R_1, \ldots, R_{k+1}) \\ &\quad \leq n^{2k\varepsilon} u_2(n^{x_k}, n^{x_{k+1}}) n^{-\frac{1}{3}x_k} n^{\frac{2}{3}x_1} \prod_{\substack{i \in S, \\ i \neq k}} n^{\frac{1}{3}x_i} \prod_{i \in S_+} n^{\frac{2}{3}x_i} \prod_{i \in S_{++}} n^{x_i} \prod_{i \in S_-} n^{-\frac{1}{3}x_i}. \end{aligned} \tag{2.20}$$

Taking the product of (2.19) and (2.20) we obtain

$$\begin{aligned} &C_k(R_1, \ldots, R_{k+1})^2 \leq \\ &n^{4k\varepsilon} \cdot u_2(n^{x_1}, n^{x_2}) u_2(n^{x_k}, n^{x_{k+1}}) n^{\frac{2}{3}(x_1 + x_{k+1})} \left(\prod_{\substack{i \in S, \\ i \neq 2,k}} n^{\frac{1}{3}x_i} \prod_{i \in S_+} n^{\frac{2}{3}x_i} \prod_{i \in S_{++}} n^{x_i} \prod_{i \in S_-} n^{-\frac{1}{3}x_i} \right)^2 \\ &\qquad \leq n^{4k\varepsilon} \cdot u_2(n,n)^2 \cdot n^{2\left(\frac{2}{3} + \frac{1}{3}|S \setminus \{2,k\}| + \frac{2}{3}|S_+| + |S_{++}|\right)} = u_2(n,n)^2 \cdot n^{\frac{2(k-1)}{3} + 4k\varepsilon}. \end{aligned}$$

The last equality follows from $|S_+| + 2|S_{++}| \leq |S_-|$, which is equivalent to $\frac{2}{3}|S_+| + |S_{++}| \leq \frac{1}{3}(|S_+| + |S_{++}| + |S_-|)$, and from the fact that S, S_+,S_{++}, and S_- partition $\{2, \ldots, k\}$. This finishes the proof.

2.4 Bounds in $\mathbb{R}^3$

Similarly as in the planar case, for $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_k)$ and $P_1 \ldots, P_{k+1} \subseteq \mathbb{R}^3$ we denote by $\mathcal{C}_k^{3,\boldsymbol{\delta}}(P_1, \ldots, P_k)$ the family of $(k+1)$-tuples $(p_1, \ldots, p_{k+1})$ with $p_i \in P_i$ for all $i \in [k+1]$ and with $\|p_i - p_{i+1}\| = \delta_i$ for all $i \in [k]$. Let $C_k^{3,\boldsymbol{\delta}}(P_1, \ldots, P_{k+1}) = |\mathcal{C}_k^{3,\boldsymbol{\delta}}(P_1, \ldots, P_{k+1})|$ and

$$C_k^3(n_1, \ldots, n_{k+1}) = \max C_k^{3,\boldsymbol{\delta}}(P_1, \ldots, P_{k+1}),$$

where the maximum is taken over all choices of $\boldsymbol{\delta}$ and sets $P_1, \ldots, P_{k+1}$ subject to $|P_i| \leq n_i$ for all $i \in [k+1]$.

Similarly to the planar case, we have $C_k^3(n) \leq C_k^3(n, \ldots, n) \leq C_k^3\left((k+1)n\right)$. Since we are only interested in the order of magnitude of $C_k^3(n)$ for fixed k, sometimes we are going to work with $C_k^3(n, \ldots, n)$ instead of $C_k^3(n)$.

2.4.1 Lower bounds

For completeness, we recall the constructions from [54] for even $k \geq 2$. For every even $2 \leq i \leq k$, let $P_i = \{p_i\}$ be a single point such that the unit spheres centred at p_i and p_{i+2} intersect in a circle. Further, let P_1 and P_{k+1} be a set of n points contained in the unit sphere centred at p_2 and p_k respectively. Finally, for every odd $3 \leq i \leq k-1$, let P_i be a set of n points contained in the intersection of the unit spheres centred at p_{i-1} and p_{i+1}. Then $P_1 \times \cdots \times P_{k+1}$ contains $n^{\frac{k}{2}+1}$ many $(k, \boldsymbol{\delta})$-chains for $\boldsymbol{\delta} = (1, \ldots, 1)$, since every element of $P_1 \times \cdots \times P_{k+1}$ is a $(k, \boldsymbol{\delta})$-chain, and $|P_1 \times \cdots \times P_{k+1}| = n^{\frac{k}{2}+1}$.

Next, we prove the lower bounds for odd $k \geq 3$ given in Proposition 2.3.

Proof of Proposition 2.3. First we show that $C_k^3(n) = \Omega\left(\frac{u_3(n)^k}{n^{k-1}}\right)$. Take a set $P' \subset \mathbb{R}^3$ of size n that contains $u_3(n)$ point pairs at unit distance apart. It is a standard exercise in graph theory to show that since $u_3(n)$ is superlinear, there is $P \subset P'$ such that $\frac{n}{2} \leq |P| \leq n$ and for every $p \in P$ there are at least $\frac{u_3(n)}{4n}$ points $p' \in P$ at distance 1 from p. Then P contains $\Omega\left(\frac{u_3(n)^k}{n^{k-1}}\right)$ many $(k, \boldsymbol{\delta})$-chains with $\boldsymbol{\delta} = (1, \ldots, 1)$.

To prove $C_k^3(n) = \Omega\left(us_3(n) n^{(k-1)/2}\right)$, we modify and extend the construction used for $k-1$ as follows. Let $P_1, \ldots, P_{k-1}$ be as in the construction for $(k-1)$-chains (from the even case). Further, let P_k be a set of n points on the unit sphere around p_{k-1}, and P_{k+1} be a set of n points such that $u_3(P_k, P_{k+1}) = us_3(n)$. Since every $(p_1, \ldots, p_{k+1}) \in P_1 \times \cdots \times P_{k+1}$ with $\|p_k - p_{k+1}\| = 1$ is a $(k, \boldsymbol{\delta})$-chain, we obtain that $P_1 \times \cdots \times P_{k+1}$ contains $\Omega\left(us_3(n) n^{(k-1)/2}\right)$ many $(k, \boldsymbol{\delta})$-chains with $\boldsymbol{\delta} = (1, \ldots, 1)$. □

2.4.2 Upper bound

We again fix $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_k)$ throughout the section and, omit it from the notation. The following result with $x = 1$ implies the upper bound in Theorem 1.5.

Theorem 2.11. *For any fixed integer $k \geq 0$ and $x \in [0,1]$, we have*

$$C_k^3(n^x, n, \ldots, n) = \tilde{O}\left(n^{\frac{k+1+x}{2}}\right).$$

Proof. The proof is by induction on k. For $k = 0$ the bound is trivial, and for $k = 1$ it follows from (2.3).

For $k \geq 2$ let $P_1, \ldots, P_{k+1} \subseteq \mathbb{R}^3$ be sets of points satisfying $|P_1| = n^x$, and $|P_i| = n$ for $2 \leq n \leq k+1$. Denote by $P_2^\alpha \subseteq P_2$ the set of those points in P_2 that are at least n^α-rich but at most $2n^\alpha$-rich with respect to P_1 and δ_1.

A standard dyadic decomposition argument implies

$$\mathcal{C}_k^3(P_1, P_2 \ldots, P_{k+1}) = \bigcup_{\alpha \in \Lambda} \mathcal{C}_k^3(P_1, P_2^\alpha, P_3, \ldots, P_{k+1}),$$

where $\Lambda := \{\frac{i}{\log n} : i = 0, 1, \ldots, \lfloor \log n \rfloor\}$. Since $|\Lambda| = \tilde{O}(1)$, it is sufficient to prove that, for every $\alpha \in \Lambda$, we have

$$C_k^3(P_1, P_2^\alpha, P_3, \ldots, P_{k+1}) = \tilde{O}\left(n^{\frac{k+1+x}{2}}\right).$$

Assume that $|P_2^\alpha| = n^y$. The number of $(k-1)$-chains in $P_2^\alpha \times P_3 \times \cdots \times P_{k+1}$ is at most $C_{k-1}^3(n^y, n, \ldots, n)$, and each of them may be extended in $2n^\alpha$ ways. By induction, we get

$$C_k^3(P_1, P_2^\alpha, P_3, \ldots, P_{k+1}) = \tilde{O}\left(n^\alpha \cdot n^{\frac{k+y}{2}}\right),$$

and we are done as long as

$$2\alpha + k + y \leq k + 1 + x. \tag{2.21}$$

To show this, we consider several cases depending on the value of α. Note that $\alpha \leq x$.

- If $\alpha \geq \frac{2x}{3}$, then by (2.4) we have $y \leq x - \alpha$, and the LHS of (2.21) is at most $\alpha + k + x \leq 1 + k + x$.
- If $\frac{x}{2} \leq \alpha \leq \frac{2x}{3}$ then by (2.4) we have $y \leq 3x - 4\alpha$. The LHS of (2.21) is at most $k + 3x - 2\alpha \leq k + 2x \leq k + 1 + x$.
- If $\alpha \leq \frac{x}{2}$ then we use a trivial bound $y \leq 1$. The LHS of (2.21) is at most $2\alpha + k + 1 \leq x + k + 1$.

□

2.5 Further problems

We can generalise the problem of determining the maximum number of chains to determine the maximum number of copies of a fixed tree as follows. Let $T = (V, E)$ be a tree on $k+1$ vertices $V = \{v_1, \ldots, v_{k+1}\}$, and with edges $E = \{e_1, \ldots, e_k\}$. For a sequence $\boldsymbol{\delta} = \{\delta_1, \ldots, \delta_k\}$ a $(k+1)$-tuple of disjoint points $(p_1, \ldots, p_{k+1})$ in $\mathbb{R}^2$ is a $(T, \boldsymbol{\delta})$-tree, if for every edge $e_\ell = (v_i, v_j)$ we have $\|p_i - p_j\| = \delta_\ell$. What is the maximum possible number $C_T^d(n)$ of $(T, \boldsymbol{\delta})$-trees in a set of n points, where the maximum is taken over all $\boldsymbol{\delta}$? We make the following conjecture.

Conjecture 2.12. *For every tree T there are integers m, ℓ such that $C_T(n) = \Theta(n^m u_2(n)^\ell)$.*

One of the simplest trees to consider are subdivisions of stars with one vertex of degree 3. Let $T_{\ell,3}$ be tree on $3\ell + 1$ vertices, with one (central) vertex of degree 3, and 3 paths on ℓ vertices joined to the central vertex. The problem even for these trees turns out to be more difficult than the problem about chains, and for tackling it new ideas are needed. It is easy to see that for $T = T_{2,3}$ we have $C_T(n) = \Omega(n^3)$ (by fixing the central vertex), however finding matching upper bounds seems challenging.

Problem 2.13. *Is it true that $C_T(n) = \Theta(n^3)$ for $T = T_{2,3}$?*

For $T = T_{3,3}$ by generalising the constructions in from Section 2.4.1 in two different ways, we obtain that $C_T(n) = \Omega(u_2(n)^\ell)$ (by fixing the central vertex) and $C_T(n) = \Omega(n^{\ell+1})$ (by fixing the vertices that are the neighbours of the leaves). This example shows that even if Conjecture 2.12 is true, we might not be able to determine the value of m and ℓ in some cases, as they can depend on the value of $u_2(n)$.

Problem 2.14. *Is it true that $C_T(n) = \Theta(\max\{n^{\ell+1}, u_2(n)^\ell\})$ for $T = T_{3,3}$?*

Chapter 3

Unit-distance embeddings

3.1 Introduction

A graph $G = (V, E)$ is a *unit-distance graph* in Euclidean space $\mathbb{R}^d$, if $V \subset \mathbb{R}^d$ and

$$E \subseteq \{(x, y) : x, y \in V, \|x - y\| = 1\}.$$

(Remember that we do not require the edge set of a unit distance graph to contain all unit-distance pairs.) A graph G is *realizable* in a subset X of $\mathbb{R}^d$, if there exists a unit distance graph G' in $\mathbb{R}^d$ on a set of vertices $X_0 \subseteq X$, which is isomorphic to G.

We denote by $\mathbb{S}^{d-1}$ the sphere of radius $1/\sqrt{2}$ in $\mathbb{R}^d$ with centre in the origin. The *Euclidean dimension* $\dim G$ (*spherical dimension* $\dim_S G$) of a graph G is equal to the smallest integer k such that G is realizable in $\mathbb{R}^k$ (on $\mathbb{S}^{k-1} \subset \mathbb{R}^k$).

We prove the following results.

Theorem 3.1. *Let $d \geq 1$ and let $G = (V, E)$ be a graph with maximum degree d. Then G is a unit distance graph in $\mathbb{R}^d$ except if $d = 3$ and G contains $K_{3,3}$.*

Proposition 3.2. *Let $d \geq 2$. Any graph $G = (V, E)$ with maximum degree $d - 1$ has spherical dimension at most d.*

Theorem 3.3. *Let $d > 3$. Any graph G with less than $\binom{d+2}{2}$ edges can be realized in $\mathbb{R}^d$. If G moreover does not contain $K_{d+2} - K_3$ or K_{d+1}, then it can be realized in $\mathbb{S}^{d-1}$.*

We also consider the following Ramsey-type notion. Let $f_D(s)$ be the smallest possible d such that for any graph G on s vertices, either G or its complement $\overline{G}$ can be realized as a unit distance graph in $\mathbb{R}^d$. Similarly, $f_{SD}(s)$ is be the smallest possible d, such that for any graph G on s vertices, either G or its complement $\overline{G}$ can be realized as a unit distance graph in $\mathbb{S}^{d-1}$.

Theorem 3.4. *For any $d, s \geq 1$, $f_{SD}(s) = \lceil (s+1)/2 \rceil$ and $\lceil (s-1)/2 \rceil \leq f_D(s) \leq \lceil s/2 \rceil$.*

3.2 Maximum degree

We use the following lemma of Lovász in the proofs of the results on bounded maximum degrees.

Lemma 3.5 ([47])**.** *Let $G = (V, E)$ be a graph with maximum degree k and let $k_1, \ldots, k_\alpha$ be non-negative integers such that $k_1 + \cdots + k_\alpha = k - \alpha + 1$. Then there is a partition $V = V_1 \cup \cdots \cup V_\alpha$ of the vertex set into α parts such that the maximum degree in $G[V_i]$ is at most k_i, $i = 1, \ldots, \alpha$.*

The proof of Proposition 3.2 is a simple induction.

Proof of Proposition 3.2. The proof is by induction on d. For $d = 2$ and $d = 3$ the theorem is easy to verify. (Note that for $d = 3$ it also follows from Proposition 3.7 below.) Let $V = V_1 \cup V_2$ be a partition as in Lemma 3.5 for $\alpha = 2$, $k_1 = \lfloor \frac{d-2}{2} \rfloor$, and $k_2 = \lceil \frac{d-2}{2} \rceil$. Then by the induction hypothesis, $G[V_i]$ can be represented on a $\mathbb{S}^{k_i}$ in $\mathbb{R}^{k_i+1}$. Represent $G[V_1]$ and $G[V_2]$ on $\mathbb{S}^{k_1}$ and $\mathbb{S}^{k_2}$ in orthogonal subspaces of dimension $k_1 + 1$ and $k_2 + 1$, respectively. Since the distance between any point in $\mathbb{S}^{k_1}$ and any point in $\mathbb{S}^{k_2}$ is 1, and both spheres are subspheres of $\mathbb{S}^{d-1}$, we obtain a representation of G in $\mathbb{S}^{d-1}$. □

In the proof of Theorem 3.1 we use Lemma 3.6, which is a strengthening of a special case of Lemma 3.5, and Proposition 3.7, which gives an embedding of cycles in sufficiently general position on the 2-sphere.

Lemma 3.6. *Let $d \geq 4$ and let $G = (V, E)$ be a graph with maximum degree at most d.*

If d is even, then there is a partition $V = V_1 \cup \cdots \cup V_{d/2}$ such that the maximum degree of $G[V_i]$ is at most 1 for $1 \leq i < d/2$, the maximum degree of $G[V_{d/2}]$ is at most 2, and any $v \in V_{d/2}$ of degree 2 in $G[V_{d/2}]$ has exactly 2 neighbours in each V_i.

If d is odd, then there is a partition $V = V_1 \cup \cdots \cup V_{(d-1)/2}$ such that the maximum degree of $G[V_i]$ is at most 1 for $1 \leq i < (d-3)/2$, the maximum degree of $G[V_{(d-3)/2}]$ and $G[V_{(d-1)/2}]$

is at most 2*, any degree* 2 *vertex in* $G[V_{(d-3)/2}]$ *has exactly* 2 *neighbours in each* V_i *for* $i \leq (d-5)/2$ *and exactly* 3 *neighbours in* $V_{(d-1)/2}$*, and any degree* 2 *vertex of* $G[V_{(d-1)/2}]$ *has at least* 2 *neighbours in each* V_i *for* $i \leq (d-3)/2$ *and at most* 3 *neighbours in* $V_{(d-3)/2}$*.*

Proof. d is even: Let $V = V_1 \cup \cdots \cup V_{d/2}$ be a partition for which $\sum_{i=1}^{d/2} e(G[V_i])$ is minimal, where $e(G[V_i])$ denotes the number of edges in $G[V_i]$. For such a partition, each $v \in V_i$ is joined to at most 2 vertices in V_i, otherwise we could move v into some other part V_j to decrease the sum of the $e(G[V_i])$. Similarly, any $v \in V_i$ joined to exactly 2 other vertices in V_i has exactly 2 neighbours in each V_j. Hence we can move each degree 2 vertex of $G[V_i]$ one by one to $V_{d/2}$ without changing $\sum_{i=1}^{d/2} e(G[V_i])$, thus preserving the above two properties.

d is odd: Let $V = V_1 \cup \cdots \cup V_{(d-1)/2}$ be a partition for which $\sum_{i=1}^{(d-1)/2} e(G[V_i])$ is minimal. Again, for such a partition each $v \in V_i$ is joined to at most 2 vertices in V_i. If $v \in V_i$ is joined to exactly 2 other vertices in V_i, then it has at most 3 neighbours in one of the V_j's and exactly 2 neighbours in all the others. So we can move each degree 2 vertex of $G[V_i]$ one by one to $V_{(d-3)/2}$ or to $V_{(d-1)/2}$, keeping $\sum_{i=1}^{d/2} e(G[V_i])$ unchanged. To obtain the final partition, we move the degree 2 vertices of $G[V_{(d-3)/2}]$ to $V_{(d-1)/2}$, except for those with 3 neighbours in $V_{(d-1)/2}$. Finally, note that a vertex of degree 2 in $G[V_{(d-1)/2}]$ is joined to at least 2 vertices in each V_i ($i \leq (d-3)/2$), hence is joined to at most 3 vertices in $V_{(d-3)/2}$. □

The following proposition states that paths and cycles can be realized on $\mathbb{S}^2$ in sufficiently general position. Note that when a 4-cycle is realized on $\mathbb{S}^2$, there is always a pair of non-adjacent points that are diametrically opposite on the sphere.

Proposition 3.7. *Any graph with maximum degree* 2 *can be realized on* $\mathbb{S}^2$ *such that the following two properties hold:*

1. *For no* 3 *distinct vertices* a*,* b*, and* c*, does there exist a vertex at distance* 1 *from all three.*

2. *No* 4 *vertices are on a circle, unless the* 4 *vertices consist of two pairs of diametrically opposite points coming from two distinct* 4*-cycles.*

In the proof of the proposition we use ideas from the correction [49] to the paper [48] of Lovász, Saks and Schrijver. A graph $G = (V, E)$ is called *k-degenerate* if any subgraph of G has a vertex of degree at most k.

Let $G = (V, E)$ be a $(d-1)$-degenerate graph, and label its vertices as $V = \{v_1, \dots, v_n\}$ such that $|\{v_j : j < i \text{ and } v_i v_j \in E\}| \leq d-1$ for all i. We realize G in $\mathbb{S}^{d-1}$ using a random process. For any linear subspace A of $\mathbb{R}^d$ of dimension at least 1, there is a unique probability measure on the subsphere $A \cap \mathbb{S}^{d-1}$ that is invariant under orthogonal transformations of A, namely the Haar measure μ_A. Given the Haar measure μ on $\mathbb{S}^{d-1}$, μ_A on $A \cap \mathbb{S}^{d-1}$ can be obtained as the pushforward of μ by the normalized projection $\tilde{\pi}_A \colon \mathbb{S}^{d-1} \setminus A^\perp \to A \cap \mathbb{S}^{d-1}$ given by $\tilde{\pi}_A(x) = (\sqrt{2}|\pi_A(x)|)^{-1}\pi_A(x)$, where $\pi_A \colon \mathbb{R}^d \to A$ is the orthogonal projection onto A.

We now embed G as follows. We first choose u_1 distributed uniformly from $\mathbb{S}^{d-1}$ (that is, according to μ). Then for each $i = 2, \dots, n$, we do the following sequentially. Let $L_i = \operatorname{span}\{u_j : j < i \text{ and } v_i v_j \in E\}$, and choose u_i uniformly from $L_i^\perp \cap \mathbb{S}^{d-1}$ (according to $\mu_{L_i^\perp}$) and independently of $\{u_j : j < i\}$.

Since each L_i has dimension at most $d-1$, this process is well defined. If G has maximum degree at most $d-1$, then for any permutation σ of $[n]$, the ordering $(v_{\sigma(1)}, \dots, v_{\sigma(n)})$ has the property that $|\left\{v_{\sigma(j)} : j < i \text{ and } v_{\sigma(i)} v_{\sigma(j)} \in E\right\}| \leq d-1$ for all i, and we can follow the above random process to embed G, thus obtaining a probability distribution ν_σ on the collection of realizations of G in $\mathbb{S}^{d-1}$. As pointed out in [49], for different σ we may obtain different probability distributions ν_σ. Nevertheless, as shown in [49], if G does not contain a big complete bipartite graph, then any two such measures are *equivalent*, that is, they have the same sets of measure 0, or equivalently, the same sets of measure 1. We say that an event A holds *almost surely* (*a.s.*) with respect to some probability distribution if it holds with probability 1.

Lemma 3.8 ([49]). *For any graph $G = (V, E)$ that does not contain a complete bipartite graph on $d+1$ vertices, for any two permutations σ and τ of $\{1, \dots, n\}$, the distributions ν_σ and ν_τ are equivalent.*

This lemma is used in [49] to show that under the same condition, the above random process gives a realization of the graph such that the points are in general position almost surely.

Theorem 3.9 ([49, 48]). *For any graph $G = (V, E)$ that does not contain a complete bipartite graph on $d+1$ vertices, the above random process gives a realization of G such that for any set of at most d vertices of G, the embedded points are linearly independent.*

We now apply Lemma 3.8 and Theorem 3.9 to prove Proposition 3.7.

Proof of Proposition 3.7. Note that G is a disjoint union of paths and cycles. If we remove a vertex from each 4-cycle, we obtain a graph $G' = (V', E')$ with $V' = \{v_1, \ldots, v_n\} \subseteq V$ that does not contain a complete bipartite graph on 4 vertices (that is, a 4-cycle or $K_{1,3}$). Take a random realization of G' as described above, and then add back the removed vertices as follows. If a was removed from the cycle $av_iv_jv_k$ with this cyclic order, then embed a as the point $-v_j$ opposite v_j. We also denote a by $-v_j$. We claim that this realization satisfies the conditions of the proposition almost surely.

We want to avoid certain configurations on some small number of vertices. By Lemma 3.8 it is always enough to show that if we start with these few vertices then almost surely they do not form a prohibited configuration.

First we have to see that after adding back the removed vertices, we have a unit distance realization of G almost surely. By Theorem 3.9, we have a realization of G' almost surely, and for any c with neighbours b and d, we have that $b \neq \pm d$ a.s. and that no point is diametrically opposite c. By adding back $a = -c$, we then also have b and d at distance 1 from a.

Suppose next that some vertex v is at distance 1 to a, b, and c. If any of these vertices are in $V \setminus V'$, we may replace them by their diametrically opposite point which is in V, and we still have that v is at distance 1 to a, b, and c, and $v, a, b, c \in V'$. Since v is not adjacent to all three in G', we may assume without loss of generality that $va \notin E'$. If we then randomly embed G' using an ordering that starts with v and a, we obtain a.s. that $\|v - a\| \neq 1$, which is a contradiction (by Lemma 3.8). Therefore, no vertex of G is at distance 1 to three distinct vertices of G.

We next show that no 4 distinct vertices $w_1, w_2, w_3, w_4 \in V$ of G will be realized on a circle a.s., where $w_i = \varepsilon_i v_i$ for some $\varepsilon_i \in \{\pm 1\}$ and $v_i \in V'$, $i = 1, 2, 3, 4$, unless we have $w_1 = -w_2$ and $w_3 = -w_4$ after relabelling. Suppose first that v_1, v_2, v_3, v_4 are distinct, and let $H := G[v_1, v_2, v_3, v_4]$ and $H' = G[w_1, w_2, w_3, w_4]$. Note that $v_i \mapsto w_i$ is an isomorphism from H to H'. Since G does not contain a 4-cycle or $K_{1,3}$, $d_H(v_i) \leq 1$ for some $i = 1, 2, 3, 4$. Without loss of generality, $d_H(v_4) = d_{H'}(w_4) \leq 1$, and if $d_H(v_4) = 1$, then $v_3v_4 \in E'$. Then $\dim L_4^{\perp} \geq 2$, and it follows that after choosing u_3, the fourth point u_4 and $-u_4$ will a.s. not be on the circle through εu_1, $\varepsilon_2 u_2$, $\varepsilon_3 u_3$, since the great circle of $\mathbb{S}^2$ orthogonal to u_3 intersects each of the 8 circles through any of $\pm u_1$, $\pm u_2$, $\pm u_3$ in at most 2 points.

Next suppose that v_1, v_2, v_3, v_4 consist of exactly 3 distinct vertices, say with $w_3 = v_3 = v_4$ and $w_4 = -v_3 = -v_4$. Since u_1, u_2, u_3 are linearly independent a.s., none of the 8 triples

$\{\varepsilon_1 u_1, \varepsilon_2 u_2, \varepsilon_3 u_3\}$ where $(\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{\pm 1\}^3$, lie on a great circle a.s., hence w_4 is not on the circle through w_1, w_2, w_3 a.s.

The only remaining case is where v_1, v_2, v_3, v_4 consist of exactly 2 distinct vertices, say with $w_1 = -w_2 = v_1$ and $w_3 = -w_4 = v_2$. It follows that w_1 and w_2 are embedded as opposite points on $\mathbb{S}^2$, and w_3 and w_4 are too. □

Proof of Theorem 3.1. For $d = 1$ and $d = 2$, the theorem is trivial. For $d = 3$, we use Proposition 3.7 as follows. First we remove vertices of degree 3 in G from V one by one. Let $W \subset V$ be the set of removed vertices. Each $w \in W$ has exactly 3 neighbours in V, W is an independent set of G, and the maximum degree in $G[V \setminus W]$ is at most 2. Now we represent $G[V \setminus W]$ on $\mathbb{S}^2$ as in Proposition 3.7. Finally, we embed the removed vertices in W one by one as follows. For any circle on $\mathbb{S}^2$, there are exactly 2 points at distance 1 from the circle. (They are not necessarily on the sphere.) For any $w \in W$, we choose one of these two points determined by the circle through the 3 neighbours of w. It remains to show that there are at most 2 vertices in W that determine the same circle. First note that at most 2 vertices in W can have the same set of neighbours, because G does not have $K_{3,3}$ as a component. Also, if $w_1 \in W$ and $w_2 \in W$ have different sets of neighbours, then their neighbours span different circles on $\mathbb{S}^2$. Otherwise, if the neighbours of w_1 and w_2 lie on the same circle C, then by Proposition 3.7, w_1 and w_2 have a common neighbour v on C that lies on a 4-cycle in $G[V \setminus W]$, so v will have degree 4 in G, a contradiction.

For $d > 3$ we consider two cases depending on the parity of d.

Case 1: d is even. Let $V = V_1 \cup \cdots \cup V_{d/2}$ be a partition as in Lemma 3.6. Remove vertices of degree 2 in $G[V_{d/2}]$ from $V_{d/2}$ until the maximum degree of each remaining vertex in $V_{d/2}$ is at most 1 in $G[V_{d/2}]$. Let $W \subset V_{d/2}$ be the set of removed vertices. Then W is an independent set of G, any $w \in W$ has exactly 2 neighbours in $V_{d/2}$, and the maximum degree of a vertex in $G[V_{d/2} \setminus W]$ is at most 1. Hence $G[V \setminus W] = G[V_1 \cup \cdots \cup V_{(d/2)-1} \cup (V_{d/2} \setminus W)]$ can be represented on $\mathbb{S}^{d-1}$ as follows. As $G[V_i]$ for $1 \leq i < d/2$ and $G[V_{d/2} \setminus W]$ have maximum degree 1, they can be realized on circles of radius $1/\sqrt{2}$ and centre the origin o in pairwise orthogonal 2-dimensional subspaces of $\mathbb{R}^d$. We can also ensure that no two vertices are diametrically opposite on a circle.

Then we add the vertices of W one by one to this embedding. Each vertex $w \in W$ has exactly 2 neighbours on each circle, so the set $N(w)$ of d neighbours of w span an affine hyperplane H not passing through o, hence they lie on a subsphere of $\mathbb{S}^{d-1}$ of radius less than $1/\sqrt{2}$. It follows that there are exactly 2 points in $\mathbb{R}^d \setminus \mathbb{S}^{d-1}$ at distance 1 from $N(w)$, both on the line through o orthogonal to H. We choose one of these points to embed w.

It remains to show that there are at most two $w \in W$ that determine the same subsphere, and that two different subspheres determine disjoint pairs of points at distance 1. There are no 3 vertices in W with the same set of neighbours, since the maximum degree in $V_{d/2}$ is at most 2. If some two vertices w_1 and w_2 from W have different sets of neighbours $N(w_1) \neq N(w_2)$, then they have different pairs of neighbours on at least one of the orthogonal circles, so the affine hyperplanes H_1 and H_2 spanned by $N(w_1)$ and $N(w_2)$ are different. If H_1 and H_2 are parallel, then the two subspheres $H_1 \cap \mathbb{S}^{d-1}$ and $H_2 \cap \mathbb{S}^{d-1}$ have different radii, and the pair of points at distance 1 from $H_1 \cap \mathbb{S}^{d-1}$ are disjoint from the pair of points at distance 1 from $H_2 \cap \mathbb{S}^{d-1}$. If H_1 and H_2 are not parallel, the pairs of points at distance 1 from $H_1 \cap \mathbb{S}^{d-1}$ and from $H_2 \cap \mathbb{S}^{d-1}$ lie on different lines through o (and none can equal o), and so are also disjoint. Therefore, all points from W can be placed.

Case 2: d is odd. Let $V = V_1 \cup \cdots \cup V_{(d-3)/2} \cup V_{(d-1)/2}$ be a partition as in Lemma 3.6. First we embed $V \setminus V_{(d-3)/2} = V_1 \cup \cdots \cup V_{(d-5)/2} \cup V_{(d-1)/2}$ on $\mathbb{S}^{d-3}$ as follows. As each $G[V_i]$ for $1 \leq i \leq (d-5)/2$ has maximum degree 1, the $G[V_i]$ can be realized on circles of radius $1/\sqrt{2}$ and with centre in the origin o in pairwise orthogonal 2-dimensional subspaces of $\mathbb{R}^d$. We can also ensure that from $V_1 \cup \cdots \cup V_{(d-5)/2}$ no two vertices are diametrically opposite on a circle. Since the maximum degree of $G[V_{(d-1)/2}]$ is at most 2, $V_{(d-1)/2}$ can be embedded on a 2-sphere S of radius $1/\sqrt{2}$ and centre o in a subspace orthogonal to the subspace spanned by $= V_1 \cup \cdots \cup V_{(d-5)/2}$, as described in Proposition 3.7. We will denote by C the circle of radius $1/\sqrt{2}$ and with centre o in the plane orthogonal to the subspace spanned by $\mathbb{S}^{d-3}$.

Before treating the general case, we show that we can add $V_{(d-3)/2}$ to the embedding, assuming that $V_{(d-1)/2}$ is embedded in S in general position in the sense that no four points of $V_{(d-1)/2}$ lie on the same circle and no three points of $V_{(d-1)/2}$ lie on a great circle of S. With this assumption, embedding $V_{(d-3)/2}$ is very similar to the embedding of $V_{d/2}$ in the even case. First we find an independent set $W \subseteq V_{(d-3)/2}$ such that the maximum degree of $G[V_{(d-3)/2} \setminus W]$ is at most 1, and each $w \in W$ has exactly two neighbours in $V_{(d-3)/2}$. Then we embed $V_{(d-3)/2} \setminus W$ on C such that no two vertices are in opposite positions. Note that $V \setminus W$ is embedded in $\mathbb{S}^{d-1}$. Finally, we embed the vertices of W one by one. Each vertex $w \in W$ has exactly two neighbours in V_i for $1 \leq i \leq (d-3)/2$ and three neighbours in $V_{(d-1)/2}$. By the general position assumption the affine hyperplane spanned by the set of neighbours $N(w)$ of w does not contain the origin. Thus there are exactly 2 points in $\mathbb{R}^d \setminus \mathbb{S}^{d-1}$ at distance 1 from $N(w)$. We choose one of these points to embed w. An argument similar to the one that was used in the even case shows that there are at most two $w \in W$ that determine the same hyperplane, and two different hyperplanes determine disjoint pairs of points.

We now turn to the general case. As before, we would like to choose an independent set $W \subseteq V_{(d-3)/2}$ such that the maximum degree of $G[V_{(d-3)/2} \setminus W]$ is at most 1 and each $w \in W$ has exactly two neighbours in $V_{(d-3)/2}$. However, this is not enough: Note that if $V_{(d-1)/2}$ is not in general position, then it is possible that there is a vertex $w \in V_{(d-3)/2}$ for which $N_1(w) := N(w) \cap V_{(d-1)/2}$ spans a great circle on S. Hence the points that are at distance 1 from $N(w)$ are the poles of the circle spanned by $N_1(w)$ on S. In addition, in this case the points that are at distance 1 from $N(w)$ are determined by $N_1(w)$. Thus if for $w_1, w_2 \in W$ we have $N(w_1) \neq N(w_2)$ but $N_1(w_1)$ and $N_1(w_2)$ span the same great circle on S, then the pair of points where w_1 and w_2 can be embedded, are the same. Thus, we have to impose some more properties on the independent subset W.

Recall that $V_{(d-1)/2}$ is embedded on the 2-sphere S as in Proposition 3.7. Therefore, three vertices $a, b, c \in V_{(d-1)/2}$ can only span a great circle if two of them are opposite vertices of a 4-cycle that are embedded in antipodal points. We assign an ordered triple (a, b, c) to a, b, c if they span a great circle with a and b being antipodal. By the properties of the embedding of $V_{(d-1)/2}$ on S, we have that (a, b, c) and (e, f, g) span the same great circle if and only if one of the following two statements hold.

1. $\{a, b\} = \{e, f\}$, $c = g$, and no vertex from $V_{(d-1)/2}$ is embedded in the point antipodal to $c = g$. (That is, $c = g$ is not part of a pair of opposite vertices of a 4-cycle that was embedded in an antipodal pair.)

2. $\{a, b, c, e, f, g\} = \{h, i, j, k\}$ consist of two pairs of points $\{h, i\}$ and $\{j, k\}$ that are opposite vertices of two 4-cycles.

If for $w_1, w_2 \in V_{(d-3)/2}$, $N_1(w_1)$ and $N_1(w_2)$ are as in the first statement, they span the same great circle if and only if $N_1(w_1) = N_1(w_2) = \{a, b, c\}$. Since a and b have degree 2 in $G[V_{(d-1)/2}]$, by Lemma 3.6 they are each joined to at most 3 vertices in $V_{(d-3)/2}$, hence there are at most three vertices $w_1, w_2, w_3 \in V_{(d-3)/2}$ for which $N_1(w_1) = N_1(w_2) = N_1(w_3) = \{a, b, c\}$. We will call such a triple $\{w_1, w_2, w_3\}$ a *conflicting triple*.

If for $w_1, w_2 \in V_{(d-3)/2}$, $N_1(w_1)$ and $N_1(w_2)$ are as in the second statement, they span the same great circle in S if and only if $N_1(w_1), N_1(w_2) \subseteq \{h, i, j, k\}$. By Lemma 3.6, any vertex from $\{h, i, j, k\}$ has at most three neighbours in $V_{(d-3)/2}$, and so there are at most four vertices $w_1, w_2, w_3, w_4 \in V_{(d-3)/2}$ for which $N_1(w_1), N_1(w_2), N_1(w_3), N_1(w_4) \subseteq \{h, i, j, k\}$. If there are 4 such vertices we will call them a *conflicting* 4*-tuple*, while if there are 3, we will also call them a *conflicting triple.*

We will also call both a conflicting triple and a conflicting 4-tuple a *conflicting set.* Note that any two different conflicting sets are disjoint. Recall that by the properties of the embedding of $V_{(d-1)/2}$ given by Proposition 3.7, if three vertices on S span a great circle, no vertex from $V_{(d-1)/2}$ is embedded in the poles of this circle. It follows that it is sufficient for an embedding to find $W \subseteq V_{(d-3)/2}$ with the following properties.

1. W is an independent set.
2. If $w \in W$, then w has exactly two neighbours in $V_{(d-3)/2}$ *(in order for w to have exactly 3 neighbours in $V_{(d-1)/2}$).*
3. $V_{(d-3)/2} \setminus W$ can be embedded on C, such that if $a, b \in V_{(d-3)/2} \setminus W$ are neighbours of some $w \in W$, then a and b are not in opposite positions *(in order to guarantee that if for $w_1, w_2 \in W$ the neighbour sets $N_1(w_1)$ and $N_1(w_2)$ span different circles, then $N(w_1)$ and $N(w_2)$ define different hyperplanes.)*
4. W contains at most two points of any conflicting set *(in order to guarantee that the neighbours of at most two vertices from W can define the same hyperplane).*

Once we find such a W, we can proceed as in the particular case considered above. In the remaining part of the proof we construct such a W.

Note that the connected components of $G[V_{(d-3)/2}]$ are paths and cycles. We embed paths of length at most 3 and cycles of length 4 on C. Let $\mathcal{H}$ be the set of the remaining connected components of $G[V_{(d-3)/2}]$. We need the following simple claim.

Claim 3.10. *Let $H \in \mathcal{H}$ be a cycle of length not equal to 4 or a path of length at least 4. Then $V(H)$ can be partitioned into sets A_H and B_H, so that:*

1. *$H[B_H]$ is a matching containing only vertices of degree 2 in H (that is, not containing endpoints of P).*
2. *For any maximal independent set $W' \subset B_H$ the graph $H[A_H \cup W']$ has connected components of size ≤ 4.*

Proof. Such a partition is very easy to achieve — simply choose the edges in B_H "greedily", in the path case starting from a vertex next to the endpoint of a path. □

For each $H \in \mathcal{H}$ we denote the partition given by Claim 3.10 as $V(H) = A_H \cup B_H$, and select a maximal independent W_H from each B_H, $H \in \mathcal{H}$, in a specific way to be explained below, and put $W := \bigcup_{H \in \mathcal{H}} W_H$. First, let us verify that for any choice of W, we can make sure that the properties 1–3 are satisfied. First, clearly, W is an independent set. Second, by the choice of B in each component $H \in \mathcal{H}$, each vertex in W has degree 2. Third, since each connected component of $H \setminus W$, $H \in \mathcal{H}'$, has size at most 4, it can be realized on the circle C such that vertices from different connected components are not in opposite position. Thus, if $w \in W$ has neighbours in different connected components of $H \setminus W$, then the property 3 is satisfied for w. If both neighbours of w are in the same component of $H \setminus W$, then H is a cycle of length 3 or 5, and $H \setminus W = H \setminus \{w\}$ is a path of length 1 or 3. In both cases the neighbours of w form an angle of $\pi/2$ and thus are not in opposite positions.

To conclude the proof, it remains to choose W in such a way that property 4 is also satisfied. Recall that $G[M]$ is a matching, where $M := \bigcup_{H \in \mathcal{H}} B_H$, and $W \subset M$ has exactly 1 vertex from each edge of $G[M]$. The vertices from M may belong to several conflicting sets, but, since different conflicting sets are disjoint, each vertex belongs to at most one of them.

We add some new edges to $G[M]$ to obtain G' as follows. For each conflicting triple, we add an edge between two of its vertices that were not connected before, and for each conflicting 4-tuple we add two vertex disjoint edges that connect two-two of its vertices that were not connected before. It is clear that finding such edges is possible. Moreover, the added set of edges forms a matching. Thus, the graph G' is a union of two matchings, and therefore does not have odd cycles. Hence, G' is bipartite, and it has an independent set W which contains exactly one vertex from each edge in G'. This is the desired independent set, since no independent set in G' intersects a conflicting group in more than two vertices. □

3.3 Number of edges

In this section we prove Theorem 3.3 after some preparation.

Lemma 3.11. *Let $d \geq 2$ and let x be a vertex of degree at most $d-2$ in a graph G. If $G - x$ can be realized on $\mathbb{S}^{d-1}$ as a unit distance graph, then G can also be represented on*

$\mathbb{S}^{d-1}$.

Proof. The neighbours of x span a linear subspace of dimension at most $d-2$, so there is a great circle from which to choose x. □

Corollary 3.12. *Any $(d-2)$-degenerate graph has spherical dimension at most d.*

The above corollary also follows from the proof of [25, Proposition 2].

In the proof of Theorem 3.3 we need the following simple lemma.

Lemma 3.13. *If the complement of a graph H on $d+k$ vertices has a matching of size k, then H can be realized on $\mathbb{S}^{d-1}$. In particular, the graph of the d-dimensional cross-polytope can be realized on $\mathbb{S}^{d-1}$.*

Proof. Let $v_1, \dots, v_{d+k}$ be the vertices of H, labelled so that v_i is not joined to v_{d+i} $(i = 1, \dots, k)$. Let vectors $e_1, e_2, \dots, e_d \in \mathbb{S}^{d-1}$ form an orthogonal basis. Map v_i to e_i and v_{d+i} to $-e_i$ $(i = 1, \dots, k)$. This is the desired realization: e_i is at distance 1 from $\pm e_j$ whenever $j \neq i$. □

Proof of Theorem 3.3. Define $g(2) = 3$, $g(3) = 8$ and $g(d) = \binom{d+2}{2} - 1$ for $d \geq 4$. We show by induction on $d \geq 2$ that if $G = (V, E)$ has at most $g(d)$ edges, then G can be embedded in $\mathbb{R}^d$, and if G furthermore does not contain K_{d+1} or $K_{d+2} - K_3$, then G can be embedded in the sphere $\mathbb{S}^{d-1}$ of radius $1/\sqrt{2}$. This is easy to verify for $d = 2$. From now on, assume that $d \geq 3$, and that the statement is true for dimension $d-1$.

Remove vertices of degree at most $d-2$ one by one from G until this is not possible anymore. If nothing remains, Corollary 3.12 gives that G can be embedded in $\mathbb{S}^{d-1}$. Thus, without loss of generality, a subgraph H of minimum degree at least $d-1$ remains. We first show that if H contains K_{d+1} or $K_{d+2} - K_3$, then G can be embedded in $\mathbb{R}^d$.

Suppose that H contains $K_{d+2} - K_3$. Then H cannot have more than $d+2$ vertices, otherwise, since each vertex of H has degree at least $d-1$, $|E(H)| \geq \binom{d+2}{2} - 3 + d - 1 > g(d)$, a contradiction. Therefore, H is contained in $K_{d+2} - e$, which can be embedded in $\mathbb{R}^d$ as two regular d-simplices with a common facet. Note that this embedding has diameter $\sqrt{2 + 2/d} < 2$. There are at most two edges of G that are not in H. Then the degrees of the vertices in $V(G) \setminus V(H)$ are at most 2, so they can easily be embedded in $\mathbb{R}^d$.

Suppose next that H contains K_{d+1} but not $K_{d+2} - K_3$. If H has more than one vertex outside K_{d+1}, then $|E(H)| \geq \binom{d+1}{2} + d - 1 + d - 2 > g(d)$, a contradiction. If H has a vertex outside K_{d+1}, then this vertex is joined to at least $d-1$ vertices of K_{d+1}, and it follows that H contains $K_{d+2}-K_3$, a contradiction. Therefore, $H = K_{d+1}$. There are at most $g(d) - \binom{d+1}{2} \le d$ edges between $V(H)$ and $V(G) \setminus V(H)$. Therefore, some $v \in H$ is not joined to any vertex outside H. Then $H - v = K_d$ can be embedded in $\mathbb{S}^{d-1}$, hence by Lemma 3.11, $G - v$ can be embedded in $\mathbb{S}^{d-1}$. Since v is only joined to the d vertices in $V(H-v)$, we can embed it in $\mathbb{R}^d \setminus \mathbb{S}^{d-1}$ so that it has distance 1 to all its neighbours.

We may now assume that H does not contain K_{d+1} or $K_{d+2}-K_3$. It will be sufficient to show in this case that H can be embedded in $\mathbb{S}^{d-1}$, as it then follows by Lemma 3.11 that G can also be embedded in $\mathbb{S}^{d-1}$.

If H has at most $d+1$ vertices, then H is a proper subgraph of K_{d+1}, and we are done by Lemma 3.13.

Suppose next that H has $d+2$ vertices. Then the complement $\overline{H}$ has maximum degree at most 2. If $\overline{H}$ does not have two independent edges, then its edges are contained in a K_3, and H contains $K_{d+2}-K_3$, a contradiction. Therefore, $\overline{H}$ has two independent edges, and we are done by Lemma 3.13.

Thus without loss of generality, H has at least $d+3$ vertices. Let v be a vertex of maximum degree in H. If v is adjacent to all other vertices of H, then v has degree at least $d+2$, hence $|E(H-v)| \le g(d) - (d+2) \le g(d-1)$, and, since H does not contain K_{d+1} or $K_{d+2}-K_3$, the graph $H-v$ does not contain K_d or $K_{d+1}-K_3$. Therefore, by induction, $H-v$ is embeddable in a subsphere $\mathbb{S}^{d-2}$. We then embed v as a point on $\mathbb{S}^{d-1}$ orthogonal to this $\mathbb{S}^{d-2}$.

Thus without loss of generality, each vertex v of maximum degree Δ has a non-neighbour w. We may also assume that $\Delta \ge d$, otherwise Proposition 3.2 gives that H is embeddable in $\mathbb{S}^{d-1}$. Then $|E(H-v-w)| \le g(d) - \Delta - (d-1) \le g(d) - d - (d-1) \le g(d-1)$. By induction, either $H-v-w$ is embeddable in $\mathbb{S}^{d-1} \cap H$, where H is a hyperplane passing through the origin, and then v and w can be embedded as the two points on $\mathbb{S}^{d-1}$ orthogonal to H, or $H-v-w$ contains a K_d or a $K_{d+1}-K_3$.

Case 1: *For any v of maximum degree and any w that is non-adjacent to v, $H-v-w$ contains a d-clique K.* Since H does not contain K_{d+1}, v has a non-neighbour x in K. Then $H-v-x$ contains another d-clique K'. If K and K' intersect in at most $d-2$ vertices, then $K \cup K'$ has at least $\binom{d+2}{2} - 4$ edges, hence $|E(H)| \ge d + \binom{d+2}{2} - 4 > g(d)$, a contradiction. Therefore, K and K' intersect in exactly $d-1$ vertices, and $K \cup K'$ has at least $\binom{d+1}{2} - 1$ edges. Since H has at least $d+3$ vertices, there exists a vertex $y \ne v$ not in

$K \cup K'$. Then $|E(H)| \geq \deg(v) + \deg(y) - 1 + \binom{d+1}{2} - 1 \geq d + (d-1) - 1 + \binom{d+1}{2} - 1 > g(d)$, a contradiction.

Case 2: *Some vertex $v \in H$ of maximum degree $\Delta \geq d$ has a non-neighbour w such that $H - v - w$ contains a $K_{d+1} - K_3$.* Then $|E(H)| \geq \Delta + \deg(w) + \binom{d+1}{2} - 3 \geq g(d)$. Since also $|E(H) \leq g(d)$, it follows that $H - v - w = K_{d+1} - K_3$, v has degree $\Delta = d$, and w has degree $d-1$. Let v_1, v_2, v_3 be the pairwise non-adjacent vertices in $H - v - w$. If v is joined to at most 2 of the v_i and w is joined to at most 1 of the v_i, $i = 1, 2, 3$, then the components of $H[v, w, v_1, v_2, v_3]$ are paths of length at most 3, hence can be realized on a great circle C of $\mathbb{S}^{d-1}$, and the remaining K_{d-2} can be realized on the subsphere orthogonal to C. Otherwise, either v is joined to all of v_1, v_2, v_3, or w is joined to at least two of them. Note that v has a non-neighbour other than w in H, and w has at least 2 non-neighbours other than v in H. It follows that there are two different vertices $w_1, w_2 \in V(H) \setminus \{v, w\}$ such that vw_1 and ww_2 are non-adjacent pairs and $|\{w_1, w_2\} \cap \{v_1, v_2, v_3\}| \leq 1$. Thus, we can find three disjoint pairs of non-adjacent vertices in H and apply Lemma 3.13. □

3.4 Ramsey results

Lemma 3.14. *The graphs K_{d+2} and $K_{d+3} - K_3$ cannot be embedded in $\mathbb{R}^d$. The graphs K_{d+1} and $K_{d+2} - K_3$ cannot be embedded in $\mathbb{S}^{d-1}$.*

Proof. Embeddability 1n $\mathbb{S}^{d-1}$ reduces to statements about orthonormal vectors, since $\mathbb{S}^{d-1}$ has radius $1/\sqrt{2}$, hence the endpoints of an edge of a unit-distance graph on $\mathbb{S}^{d-1}$ are orthogonal when viewed as unit vectors. It is then immediate that K_{d+1} cannot be realized in $\mathbb{S}^{d-1}$.

We next show by induction on d that $G = K_{d+2} - K_3$ cannot be realized on a sphere of any radius in $\mathbb{R}^d$. This is easy to see for $d = 1$ and 2. For $d \geq 3$, choose a $v \in G$ that is joined to all other vertices. Then $G - v$ is contained in the intersection of the sphere with the unit sphere centred at v. This gives an embedding of $K_{d+1} - K_3$ in a subsphere on a hyperplane of $\mathbb{R}^d$, which contradicts the induction hypothesis.

This also implies that $K_{d+3} - K_3$ cannot be embedded in $\mathbb{R}^d$.

Suppose that K_{n+1} can be embedded in $\mathbb{R}^d$. Without loss of generality, we then have unit vectors $v_1, \dots, v_n$ such that the distance between any two v_i is 1. It then follows from the identity $\sum_{i=1}^n \sum_{j=1}^n \lambda_i \lambda_j \|v_i - v_j\|^2 = 2(\sum_{i=1}^n \lambda_i) \sum_{i=1}^n \lambda_i \|v_i\|^2 - 2\| \sum_{i=1}^n \lambda_i v_i\|^2$ that $v_1, \dots, v_n$ are linearly independent, hence $n \leq d$. □

Proof of Theorem 3.4. Consider the graph G on $2d$ vertices which is a union of a K_{d+1} and $d-1$ isolated vertices. Then G contains K_{d+1} and $\overline{G}$ contains $K_{d+2}-K_3$. By Lemma 3.14, neither of these graphs can be embedded in $\mathbb{S}^{d-1}$. It follows that $f_{SD}(s) \geq \lceil (s+1)/2 \rceil$.

To prove $f_{SD}(s) \leq \lceil (s+1)/2 \rceil$, we show that if the edges of the complete graph on $2d-1$ vertices are coloured with red and blue, then either the graph spanned by the red (denoted by G_r) or the graph spanned by the blue edges (denoted by G_b) can be embedded on $\mathbb{S}^{d-1}$.

The proof is by induction on d. It is easy for $d = 1, 2$. For $d > 2$: If the maximum degree of G_r or G_b is at most $d-1$, we are done by Proposition 3.2. So we may assume that there are two vertices, v_r and v_b, of degree at least d in G_r and G_b respectively. By the induction we may assume that $G_r[V - v_r - v_b]$ is realizable on $\mathbb{S}^{d-2}$. If the edge (v_r, v_b) is blue, we put v_r and v_b in the poles of the $(d-2)$-sphere on which $G_r[V - v_r - v_b]$ is embedded. Otherwise v_b has at most $d-3$ neighbours in $G_r[V - v_r]$. In this case we first add v_b on the $(d-2)$-sphere (on which $G_r[V - v_r - v_b]$ is embedded), and then we can put v_r in one of the poles of the $(d-2)$-sphere.

To obtain the lower bound on $f_D(s)\lceil (s-1)/2 \rceil$, consider the graph G which is the union of K_{d+2} and d isolated vertices. Then G contains K_{d+2} and $\overline{G}$ contains $K_{d+3} - K_3$. Neither of these can be embedded in $\mathbb{R}^d$ by Lemma 3.14.

To prove $f_D(s) \leq \lceil s/2 \rceil$, we show that if the edges of the graph on $2d$ vertices are coloured with red and blue, then either G_r or G_b can be embedded in $\mathbb{R}^d$. For any vertex $v \in V$ we have $d_{G_r}(v) + d_{G_b}(v) = 2d-1$, so either $d_{G_r} \leq d-1$ or $d_{G_b} \leq d-1$. Hence we may assume that there are at most d vertices that have degree larger than $d-1$ in G_r. Let W be the set of vertices $v \in V$ with $d_{G_r}(v) \leq d-1$. $|V \setminus W| \leq d$, so we can embed $G_r(V)$ on $\mathbb{S}^{d-1}$. Then we add the vertices of W to this embedding one by one as follows. If $w \in W$ has a neighbour in W, then it has at most $d-2$ neighbours in $V \setminus W$, thus we remove it from W and embed it on $\mathbb{S}^{d-1}$. We repeat this until W is an independent set. Now for each vertex $w \in W$ there is at least a circle (which is not necessarily contained in $\mathbb{S}^{d-1}$) in which we can embed w, so we embed them one by one. □

3.5 Additional questions

In Theorem 3.1 we proved that any graph with maximum degree d can be embedded in $\mathbb{R}^d$ unless $d = 3$ and G has $K_{3,3}$ as a component. We suspect that a slightly stronger statement holds.

Problem 3.15. *Is it true that for $d > 3$ any graph with maximum degree d, except K_{d+1}, has spherical dimension at most d?*

This is false for $d = 3$: the 3-cube (even the 3-cube with a vertex removed) cannot be embedded on $\mathbb{S}^2$; neither can the graphs on the vertices $a_1, \ldots, a_n, b_1, \ldots, b_n$ with edge set $\{(a_i, b_j) : j = i-1, i, i+1 \bmod n\}$ where $n \geq 3$ odd.

The lower and upper bound on $f_D(s)$ in Theorem 3.4 are very close, but it still does not give the exact value of $f_D(s)$. We conjecture that the lower bound is sharp.

Problem 3.16. *Is it true that for any graph G on $2d+1$ vertices, either G or $\overline{G}$ has dimension at most d?*

Chapter 4

The chromatic number of the plane

4.1 Introduction

A *colouring* of a set X is a function $\varphi : X \to A$ for some finite set A. A *k-colouring* of X is a function $\varphi : X \to A$ with $|S| = k$. For any graph $G = (V, E)$ a colouring $\varphi : V \to A$ is a *proper colouring of* G if $(x, y) \in E$ implies $\varphi(x) \neq \varphi(y)$, and it is a proper k-colouring, if in addition $|A| = k$. The *chromatic number* $\chi(G)$ of a graph G is the smallest k for which there exists a proper k-colouring of G. We present an approach that might lead to a human-verifiable proof of $\chi(\mathbb{R}^2) \geq 5$, following ideas proposed by Pálvölgyi [56]. A collection of unit circles $C = C_1 \cup \cdots \cup C_n$ having a common point O a *bouquet through* O. For a given colouring of $\mathbb{R}^2$, the bouquet C is *smiling* if there is a colour, say blue, such that every circle C_i has a blue point, but O is not blue. We make the following conjecture.

Conjecture 4.1. *For every bouquet C, every colouring of the plane with finitely many but at least two colours contains a smiling congruent copy of C.*

In Section 4.5 we show that the statement of Conjecture 4.1 would provide a human-verifiable proof of $\chi(\mathbb{R}^2) \geq 5$. We prove the conjecture for a specific family of bouquets for proper colourings of the plane.

Theorem 4.2. *Let $C = C_1 \cup \cdots \cup C_n$ be a bouquet through O and for every i et O_i be the centre of C_i. If O and $O_1, \ldots, O_n$ are contained in $\mathbb{Q}^2$, further O is an extreme point of $\{O, O_1, \ldots, O_n\}$, then Conjecture 4.1 is true for C for proper colourings.*

In Section 4.5.2 we prove a more general statement which implies Theorem 4.2. We also prove a statement similar to that of Conjecture 4.1 for concurrent lines. We call a collection of lines $L = L_1 \cup \cdots \cup L_n$ with a common point O a *pencil through* O. The pencil L is *smiling* if there is a colour, say blue, such that every line L_i has a blue point, but O is not blue.

Theorem 4.3. *For every pencil L, every colouring of the plane with finitely many but at least two colours contains a smiling congruent copy of L.*

4.1.1 Almost-monochromatic sets

Let $S \subseteq \mathbb{R}^d$ be a finite set with $|S| \geq 3$, and let $s_0 \in S$. In a colouring of $\mathbb{R}^d$ we call S *monochromatic*, if every point of S has the same colour. A pair (S, s_0) is *almost-monochromatic* if $S \setminus \{s_0\}$ is monochromatic but S is not.

Two sets S and T are similar, if there is an isometry f and a constant $\lambda \in \mathbb{R}$ $\lambda \neq 0$ with $T = \lambda f(S)$. An *infinite arithmetic progression* in $\mathbb{R}^d$ is a similar copy of $\mathbb{N}$. A colouring is *arithmetic progression-free* if it does not contain a monochromatic infinite arithmetic progression. We use abbreviations AM for almost-monochromatic and AP for arithmetic progression.

Motivated by its connections to the chromatic number of the plane,[1] we propose to study the following problem.

Problem 4.4. *Characterise those pairs (S, s_0) with $S \subseteq \mathbb{R}^d$ and with $s_0 \in S$ for which it is true that every arithmetic progression-free finite colouring of $\mathbb{R}^d$ contains an almost-monochromatic similar copy of (S, s_0).*

We solve Problem 4.4 in the case when $S \subseteq \mathbb{Z}^d$. A point $s_0 \in S$ is called an *extreme point* of S if $s_0 \notin \text{conv}(S \setminus \{s_0\})$.

Theorem 4.5. *Let $S \subseteq \mathbb{Z}^d$ and $s_0 \in S$. Then there is an AP-free colouring of $\mathbb{R}^d$ without an AM similar copy of (S, s_0) if and only if $|S| > 3$ and s_0 is not an extreme point of S.*

We prove Theorem 4.5 in full generality in Section 4.3.1. The 'only if' direction follows from a stronger statement, Theorem 4.16. In Section 2 we consider the 1-dimensional case. We prove some statements similar to Theorem 4.5 for $d = 1$, and illustrate the ideas that are used to prove the theorem in general.

[1]The connection is described in details in the the proof of Theorem 4.26.

T is a *homothetic copy* (or *homothet*) of a S, if there is a vector $v \in \mathbb{R}^d$ and a constant $\lambda \in \mathbb{R}$ ($\lambda \neq 0$) with $T = v + \lambda S$. T is a*positive homothetic copy* (or *positive homothet*) of a S, if moreover $\lambda > 0$. We prove the following result in Section 4.4.

Theorem 4.6. *Let $S = \{0, 1, 2\}$ and $s_0 = 0$. Then every finite colouring of $\mathbb{Q}$ with more than one colour contains an AM positive homothet of (S, s_0).*

4.2 The line

In this section we prove a statement slightly weaker than Theorem 4.5 for $d = 1$. The main goal of this section to illustrate some of the ideas that we use to prove Theorem 4.5, but in a simpler case. Note that in $\mathbb{R}$ the notion of similar copy and homothetic copy is the same.

Theorem 4.7. *Let $S \subseteq \mathbb{Z}$ and $s_0 \in S$. Then there is an AP-free colouring of $\mathbb{N}$ and of $\mathbb{R}$ without an AM positive homothetic copy of (S, s_0) if and only if $|S| > 3$ and s_0 is not an extreme point of S.*

To prove Theorem 4.7 it is sufficient to prove the 'if' direction only for $\mathbb{R}$ and the 'only if' direction only for $\mathbb{N}$. Thus it follows from the three lemmas below, that consider cases of Theorem 4.7 depending on the cardinality of S and on the position of s_0.

Lemma 4.8. *If s_0 is an extreme point of S, then every finite AP-free colouring of $\mathbb{N}$ contains an AM positive homothetic copy of (S, s_0).*

Lemma 4.9. *If $|S| = 3$, then every AP-free finite colouring of $\mathbb{N}$ contains an AM positive homothetic copy of (S, s_0).*

Lemma 4.10. *If $S \subseteq \mathbb{R}$, $|S| > 3$ and s_0 is not an extreme point of S, then there is an AP-free finite colouring of $\mathbb{R}$ without an AM positive homothetic copy of (S, s_0).*

Before turning to the proofs, recall Van der Waerden's theorem [67] and a corollary of it. A colouring is a *k-colouring* if it uses at most k colours.

Theorem 4.11 (Van der Waerden [67]). *For every $k, \ell \in \mathbb{N}$ there is an $N(k, \ell) \in \mathbb{N}$ such that every k-colouring of $\{1, \ldots, N(k, \ell)\}$ contains an ℓ-term monochromatic AP.*

Corollary 4.12 (Van der Waerden [67]). *For every $k, \ell \in \mathbb{N}$ and for every k-colouring of $\mathbb{N}$ there is a $t \leq N(k, \ell)$ such that there are infinitely many monochromatic ℓ-term AP of the same colour with difference t.*

Proof of Lemma 4.8. Let $S = \{p_1, \ldots, p_n\}$ with $1 < p_1 < \cdots < p_n$ and φ be an AP-free colouring of $\mathbb{N}$. If s_0 is an extreme point of S, then either $s_0 = p_1$ or $s_0 = p_n$.

Case 1: $s_0 = p_n$. By Theorem 4.11 φ contains a monochromatic positive homothet $M + \lambda([1, p_n) \cap \mathbb{N})$ of $[1, p_n) \cap \mathbb{N}$ of colour, say, blue. Observe that since φ is AP-free there is a $q \in M + \lambda([p_n, \infty) \cap \mathbb{N})$ which is not blue. Let $M + q\lambda$ be the smallest non-blue element in $M + \lambda([p_n, \infty) \cap \mathbb{N})$. Then $(\lambda(q - p_n) + M + \lambda S, M + \lambda q)$ is an AM homothet of (S, s_0).

Case 2: $s_0 = p_1$. By Corollary 4.12 there is a $\lambda \in \mathbb{N}$ such that φ contains infinitely many monochromatic congruent copies of $\lambda((1, p_n] \cap \mathbb{N})$, say of colour blue. Without loss of generality, we may assume that infinitely many of these monochromatic copies are contained in $\lambda\mathbb{N}$. Since φ is AP-free, $\lambda\mathbb{N}$ is not monochromatic, and thus there is an i such that $i\lambda$ and $(i+1)\lambda$ are of different colours. Consider a blue interval $M + \lambda((1, p_n] \cap \mathbb{N})$ such that $M + \lambda > i\lambda$, and let q be the largest non-blue element of $[1, M + \lambda) \cap \lambda\mathbb{N}$. This largest element exists since λi and $\lambda(i+1)$ are of different colour. Then $(q - \lambda p_1 + \lambda S, q)$ is an AM homothet of (S, s_0). □

Proof of Lemma 4.9. Let $S = \{p_1, p_2, p_3\}$ with $1 < p_1 < p_2 < p_3$ and φ be an AP-free colouring of $\mathbb{N}$. We may assume that $s_0 = p_2$, otherwise we are done by Lemma 4.8.

There is an $r \in \mathbb{Q}_{>0}$ such that $\{q_1, q_2, q_3\}$ is a positive homothet of S if and only if $q_2 = rq_1 + (1 - r)q_3$. Fix an $M \in \mathbb{N}$ for which $Mr \in \mathbb{N}$. We say that I is an interval of $c + \lambda\mathbb{N}$ of length ℓ if there is an interval $J \subseteq \mathbb{R}$ such that $I = J \cap (c + \lambda\mathbb{N})$ and $|I| = \ell$.

Proposition 4.13. *Let I_1 and I_3 be intervals of $\lambda\mathbb{N}$ of length $2M$ and M respectively such that $\max I_1 < \min I_3$. Then there is an interval $I_2 \subseteq \lambda\mathbb{N}$ of length M such that $\max I_1 < \max I_2 < \max I_3$, and for every $q_2 \in I_2$ there are $q_1 \in I_1$ and $q_3 \in I_3$ such that $\{q_1, q_2, q_3\}$ is a positive homothetic copy of S.*

Proof. Without loss of generality we may assume that $\lambda = 1$. Let I_1^L be the set of the M smallest elements of I_1. By the choice of M for any $q_3 \in \mathbb{N}$ the interval $rI_1^L + (1 - r)q_3$ contains at least one natural number. Let q_3 be the smallest element of I_3 and $q_1 \in I_1^L$ such that $rq_1 + (1 - r)q_3 \in \mathbb{N}$. Then $I_2 = \{r(q_1 + i) + (1 - r)(q_3 + i) : 0 \leq i < M\}$ is an interval of $\mathbb{N}$ of length M satisfying the requirements, since $q_1 + i \in I_1$ and $q_3 + i \in I_3$. □

We now return to the proof of Lemma 4.9. Let I be an interval of $\mathbb{N}$ of length $2M$. By Theorem 4.12 there is a $\lambda \in \mathbb{N}$ such that φ contains infinitely many monochromatic copies of λI of the same colour, say of blue. Moreover, by the pigeonhole principle there is a $c \in \mathbb{N}$ such that infinitely many of these blue copies are contained in $c + \lambda\mathbb{N}$, and without loss of generality we may assume that $c = 0$.

Consider a blue interval $[a\lambda, a\lambda + 2M\lambda - \lambda]$ of $\lambda\mathbb{N}$ of length $2M$. Since φ is AP-free, $[a\lambda + 2M\lambda, \infty) \cap \lambda\mathbb{N}$ is not completely blue. Let $q\lambda$ be its smallest element which is not blue and let $I_1 = [q\lambda - 2M\lambda, (q-1)\lambda] \cap \lambda\mathbb{N}$. Let I_3 be the blue interval of length M in $\lambda\mathbb{N}$ with the smallest possible $\min I_3$ for which $\max I_1 < \min I_3$. Then Proposition 4.13 provides an AM positive homothet of (S, s_0).

Indeed, consider the interval I_2 given by the proposition. There exists a $q_2 \in I_2$ which is not blue, otherwise every point of I_2 is blue, contradicting the minimality of $\min I_3$. But then there are $q_1 \in I_1$, $q_3 \in I_3$ such that $(\{q_1, q_2, q_3\}, q_2)$ is an AM homothet copy of (S, s_0). □

Proof of Lemma 4.10. S contains a set S' of 4 points with $s_0 \in S'$ such that s_0 is not an extreme point of S'. Thus we may assume that $S = \{p_1, p_2, p_3, p_4\}$ with $p_1 < p_2 < p_3 < p_4$ and that $s_0 = p_2$ or $s_0 = p_3$. We construct the colouring for these two cases separately. First we construct a colouring φ_1 of $\mathbb{R}_{>0}$ for the case of $s_0 = p_3$, and a colouring φ_2 of $\mathbb{R}_{\geq 0}$ for the case of $s_0 = p_2$. Then we extend the colouring in both cases to $\mathbb{R}$.

Construction of φ_1 ($s_0 = p_3$) : Fix K such that $K > \frac{p_4 - p_2}{p_2 - p_1} + 1$ and let $\{0, 1, 2\}$ be the set of colours. We define φ_1 as follows. Colour $(0, 1)$ with colour 2, and for every $i \in \mathbb{N} \cup \{0\}$ colour $[K^i, K^{i+1})$ with i modulo 2. The colouring φ_1 defined this way is AP-free, since it contains arbitrarily long monochromatic intervals of colours 1 and 2. Thus we only have to show that it does not contain an AM positive homothet of (S, s_0).

Consider a positive homothet $c + \lambda S = \{r_1, r_2, r_3, r_4\}$ of S with $r_1 < r_2 < r_3 < r_3$. If $\{r_1, r_2, r_3, r_4\} \cap [0, 1) \neq \emptyset$, then $\{r_1, r_2, r_3, r_4\}$ cannot be AM. Thus we may assume that $\{r_1, r_2, r_3, r_4\} \cap [0, 1) = \emptyset$.

Note that by the choice of K we have

$$Kr_2 > r_2 + \frac{p_4 - p_2}{p_2 - p_1} r_2 = r_2 + \frac{p_4 - p_2}{p_2 - p_1} \left(\lambda(p_2 - p_1) + r_1\right) \geq r_2 + \lambda(p_4 - p_2) = r_4.$$

Hence $\{r_2, r_3, r_4\}$ is contained in the union of two consecutive intervals of the form $[K^i, K^{i+1})$. This means that $(\{r_1, \ldots, r_4\}, r_3)$ cannot be AM since either $\{r_2, r_3, r_4\}$ is monochromatic, or r_2 and r_4 have different colours.

Construction of φ_2 ($s_0 = p_2$): Fix K such that $K > \frac{p_4 - p_2}{p_2 - p_1} + 1$, let $L = K \cdot \left\lceil \frac{p_3 - p_1}{p_4 - p_3} \right\rceil$ and let $\{0,\dots,2L\}$ be the set of colours. We define φ_2 as follows. For each odd $i \in \mathbb{N} \cup \{0\}$, divide the interval $[L \cdot K^i, L \cdot K^{i+1})$ into L equal half-closed intervals, and colour the j-th of them with colour j. For even $i \in \mathbb{N} \cup \{0\}$ divide the interval $[L \cdot K^i, L \cdot K^{i+1})$ into L equal half-closed intervals, and colour the j-th of them with colour $L + j$. That is, for $j = 1,\dots,L$ we colour $[L \cdot K^i + (j-1)(K^{i+1} - K^i), L \cdot K^i + j(K^{i+1} - K^i))$ with colour j if i is odd, and with colour $j + L$ if i is even. Finally, colour the points in $[0, L)$ with colour 0.

φ_2 defined this way is AP-free, since it contains arbitrarily long monochromatic intervals of colours $1,\dots,2L$. Thus we only have to show it does not contain an AM positive homothetic copy of (S, s_0).

Consider a positive homothet $c + \lambda S = \{r_1, r_2, r_3, r_4\}$ of S with $r_1 < r_2 < r_3 < r_4$. If $\{r_1, r_2, r_3, r_4\} \cap [0, L) \neq \emptyset$, then $(\{r_1, r_2, r_3, r_4\}, r_2)$ cannot be AM, thus we may assume that $\{r_1, r_2, r_3, r_4\} \cap [0, L) = \emptyset$. Note that by the choice of K we again have

$$Kr_2 > r_2 + \frac{p_4 - p_2}{p_2 - p_1} r_2 = r_2 + \frac{p_4 - p_2}{p_2 - p_1}(\lambda(p_2 - p_1) + r_1) \geq r_2 + \lambda(p_4 - p_2) = r_4.$$

This means that $\{r_2, r_3, r_4\}$ is contained in the union of two consecutive intervals of the form $[L \cdot K^i, L \cdot K^{i+1})$, which implies that if $(\{r_1, r_2, r_3, r_4\}, r_2)$ is AM, then $\{r_3, r_4\}$ is contained in an interval $[L \cdot K^i + (j-1)(K^{i+1} - K^i), L \cdot K^i + j(K^{i+1} - K^i))$ for some $1 \leq j \leq L$. However, then by the choice of L we have that r_1 is either contained in the interval $[L \cdot K^i, L \cdot K^{i+1})$ or in the interval $[L \cdot K^{i-1}, L \cdot K^i)$. Indeed,

$$\begin{aligned} r_3 - r_1 \leq \left\lceil \frac{r_3 - r_1}{r_4 - r_3} \right\rceil (r_4 - r_3) &\leq \left\lceil \frac{r_3 - r_1}{r_4 - r_3} \right\rceil (K^{i+1} - K^i) \\ &= \left\lceil \frac{p_3 - p_1}{p_4 - p_3} \right\rceil (K^{i+1} - K^i) = L(K^i - K^{i-1}). \end{aligned}$$

Thus, if r_1 has the same colour as r_3 and r_4, then r_1 is also contained in the interval $[L \cdot K^i + (j-1)(K^{i+1} - K^i), L \cdot K^i + j(K^{i+1} - K^i))$, implying that $(\{r_1, r_2, r_3, r_4\}, r_2)$ is monochromatic.

We now extend the colouring to $\mathbb{R}$ in the case of $s_0 = p_3$. Let φ_2' be a colouring of $\mathbb{R}_{\geq 0}$ isometric to the reflection of φ_2 over 0. Then φ_2' contains no AM positive homothet of (S, s_0). If further we assume that φ_1 and φ_2' use disjoint sets of colours then the union of φ_1 and φ_2' is an AP-free colouring of $\mathbb{R}$ containing no AM positive homothet of (S, s_0). We can extend the colouring similarly in the case of $s_0 = p_2$. $\square$

4.3 Higher dimensions

In this section we prove Theorem 4.5.

4.3.1 Proof of the 'if' direction of Theorem 4.5

Let $S \subseteq \mathbb{R}^d$ such that $|S| > 3$ and s_0 is not an extreme point of S. To prove the 'if' direction of Theorem 4.5, we prove that there is an AP-free colouring of $\mathbb{R}^d$ without an AM similar copy of (S, s_0). (Note that for the proof of Theorem 4.5, it would be sufficient to prove this for $S \subseteq \mathbb{Z}^d$.)

Recall that $C \subseteq \mathbb{R}^d$ is a *convex cone* if for every $x, y \in C$ and $\alpha, \beta \geq 0$, the vector $\alpha x + \beta y$ is also in C. The *angle* of C is $\sup_{x,y \in C \setminus \{o\}} \angle(x, y)$.

We partition $\mathbb{R}^d$ into finitely many convex cones $C_1 \cup \cdots \cup C_m$, each of angle at most $\alpha = \alpha(d, S)$, where $\alpha(d, S)$ will be set later. We colour the cones with pairwise disjoint sets of colours as follows. First, we describe a colouring φ of the closed circular cone $C = C(\alpha)$ of angle α around the line $x^1 = \cdots = x^d$. Then for each i we define a colouring φ_i of C_i using pairwise disjoint sets of colours in a similar way. More precisely, let f_i be an isometry with $f_i(C_i) \subseteq C$, and define φ_i such that it is isometric to φ on $f_i(C_i)$.

It is not hard to see that it is sufficient to find an AP-free colouring φ of C without an AM similar copy of (S, s_0). Indeed, since the cones C_i are coloured with pairwise disjoint sets of colours, any AP or AM similar copy of (S, s_0) is contained in one single C_i.

We now turn to describing the colouring φ of C. Note that by choosing α sufficiently small we may assume that $C \subseteq \mathbb{R}^d_{\geq 0}$. For $x \in \mathbb{R}^d$ let $\|x\|_1 = |x^1| + \cdots + |x^d|$. Then for any $x \in \mathbb{R}^d$ we have

$$\|x\| \leq \|x\|_1 \leq \sqrt{d}\|x\|. \tag{4.1}$$

Let $S = \{p_1, \ldots, p_n\}$ and fix K such that

$$K > 1 + 2\sqrt{d} \max_{p_i, p_j, p_l, p_\ell \in S, p_i \neq p_j} \frac{\|p_k - p_\ell\|}{\|p_i - p_j\|}.$$

For a sufficiently large L, to be specified later, we define $\varphi : C \to \{0, 1, \ldots, 2L\}$ as

$$\varphi(x) = \begin{cases} 0 & \text{if } \|x\|_1 < L \\ j & \text{if for some even } i \in \mathbb{N} \text{ and } j \in [L] \text{ we have} \\ & \|x\|_1 \in [L \cdot K^i + (j-1)(K^{i+1} - K^i), L \cdot K^i + j(K^{i+1} - K^i)) \\ L + j & \text{if for some odd } i \in \mathbb{N} \text{ and } j \in [L] \text{ we have} \\ & \|x\|_1 \in [L \cdot K^i + (j-1)(K^{i+1} - K^i), L \cdot K^i + j(K^{i+1} - K^i)). \end{cases}$$

φ is AP-free since any half-line in C contains arbitrarily long monochromatic sections of colours $1, \dots, 2L$. Thus we only have to show that it does not contain an AM similar copy of (S, s_0). Let $(\{r_1, \dots, r_n\}, q_0)$ be a similar copy of (S, s_0), with

$$\|r_1\|_1 \le \|r_2\|_1 \le \cdots \le \|r_n\|_1. \tag{4.2}$$

Claim 4.14. *$\{\|r_2\|_1, \dots, \|r_n\|_1\}$ is contained in the union of two consecutive intervals of the form $[L \cdot K^j, L \cdot K^{j+1})$.*

Proof. For any r_i with $i \ge 2$ we have

$$\begin{aligned}
\|r_i\|_1 &\le \|r_2\|_1 + \|r_i - r_2\|_1 \\
&\le \|r_2\|_1 + \sqrt{d}\|r_i - r_2\| && \text{by (4.1)} \\
&= \|r_2\|_1 + \sqrt{d}\frac{\|r_i - r_2\|}{\|r_2 - r_1\|}\|r_2 - r_1\| \\
&< \|r_2\|_1 + \frac{K-1}{2} \cdot \|r_2 - r_1\| && \text{by the definition of } K \\
&\le \|r_2\|_1 + \frac{K-1}{2}(\|r_2\| + \|r_1\|) && \text{by the triangle inequality} \\
&\le \|r_2\|_1 + \frac{K-1}{2}(\|r_2\|_1 + \|r_1\|_1) && \text{by (4.1)} \\
&\le K\|r_2\|_1 && \text{by (4.2).}
\end{aligned}$$

□

Assume now that $(\{r_1, \dots, r_n\}, q_0)$ is AM. Note that $\|x\|_1$ is $\sqrt{d}$ times the length of the projection of x on the $x^1 = \cdots = x^d$ line for $x \in \mathbb{R}^d_{\ge 0}$. Thus for any similar copy $\psi(S)$ of S we have $\|\psi(s_0)\|_1 \in \operatorname{conv}\{\|p\|_1 : p \in \psi(S \setminus \{s_0\})\}$, and we may assume that $q_0 \ne r_1, r_n$. This means that $\varphi(r_1) = \varphi(r_n)$, and there is exactly one $i \in \{2, \dots, n-1\}$ with $\varphi(r_i) \ne \varphi(r_1)$. This, by Claim 4.14 and by the definition of φ, is only possible if $i = 2$ and there are $i \in \mathbb{N}$ and $j \in [L]$ such that

$$\|r_3\|_1, \dots, \|r_{n-1}\|_1 \in [L \cdot K^i + (j-1)(K^{i+1} - K^i), L \cdot K^i + j(K^{i+1} - K^i)).$$

The following claim finishes the proof.

Claim 4.15. *If L is sufficiently large and α is sufficiently small, then $\|r_1\|_1$ is contained in $[L \cdot K^{i-1}, L \cdot K^i) \cup [L \cdot K^i, L \cdot K^{i+1})$.*

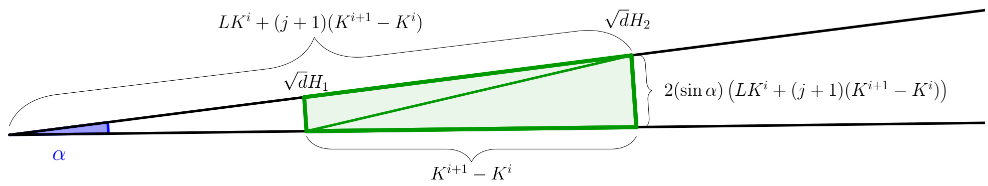

Figure 4.1: $T \cap C(\alpha)$

The claim indeed finishes the proof. By the definition of φ then $\varphi(r_1) = \varphi(r_n)$ implies

$$\|r_1\|_1 \in [L \cdot K^i + (j-1)(K^{i+1} - K^i), L \cdot K^i + j(K^{i+1} - K^i)).$$

But then we have

$$\|r_2\|_1 \in [L \cdot K^i + (j-1)(K^{i+1} - K^i), L \cdot K^i + j(K^{i+1} - K^i))$$

as well, contradicting $\varphi(r_2) \neq \varphi(r_1)$.

Proof of Claim 4.15. It is sufficient to show that $\|r_{n-1}\|_1 - \|r_1\|_1 < LK^i - LK^{i-1}$. We have

$$\|r_{n-1}\|_1 - \|r_1\|_1 \leq \sqrt{d}\|r_{n-1} - r_1\| = \sqrt{d}\|r_n - r_{n-1}\| \frac{\|r_{n-1} - r_n\|}{\|r_n - r_{n-1}\|} < \|r_{n-1} - r_n\| \frac{K-1}{2},$$

by (4.1) and by the definition of K. Let H_1 and H_2 be the hyperplanes orthogonal to the line $x^1 = \cdots = x^d$ at distance $\frac{1}{\sqrt{d}}(L \cdot K^i + (j-1)(K^{i+1} - K^i))$ and $\frac{1}{\sqrt{d}}(L \cdot K^i + j(K^{i+1} - K^i))$ from the origin respectively. Since

$$\|r_n\|_1, \|r_{n-1}\|_1 \in [L \cdot K^i + (j-1)(K^{i+1} - K^i), L \cdot K^i + j(K^{i+1} - K^i)),$$

r_n and r_{n-1} are contained in the intersection T of $C(\alpha)$ and the slab bounded by the hyperplanes H_1 and H_2.

Thus $\|r_{n-1} - r_n\|$ is bounded by the length of the diagonal of the trapezoid which is obtained as the intersection of T and the 2-plane through r_n, r_{n-1} and the origin. Scaled by $\sqrt{d}$, this is shown in Figure 4.1.

From this, by the triangle inequality we obtain

$$\|r_{n-1} - r_n\| \leq \frac{1}{\sqrt{d}} \left(K^{i+1} - K^i + 2(\sin\alpha)\left(LK^i + (j+1)(K^{i+1} - K^i)\right)\right)$$
$$\leq \frac{1}{\sqrt{d}} \left(K^{i+1} - K^i + 2\sin\alpha \cdot LK^{i+1}\right) \leq \frac{2}{\sqrt{d}} \left(K^{i+1} - K^i\right),$$

where the last inequality holds if α is sufficiently small. Combining these inequalities and choosing $L = \frac{K^2}{\sqrt{d}}$ we obtain the desired bound $\|r_{n-1}\|_1 - \|r_n\|_1 < LK^i - LK^{i-1}$, finishing the proof of the claim. □

4.3.2 Proof of the 'only if' direction of Theorem 4.5

The 'only if' direction follows from Theorem 4.7 in the case of $d = 1$, and from the following stronger statement for $d \geq 2$ (since in this case s_0 is an extreme point of S).

Theorem 4.16. *Let $S \subseteq \mathbb{Z}^d$ and $s_0 \in S$ be an extreme point of S. Then for every k there is a constant $\Lambda = \Lambda(d, S, k)$ such that the following is true. Every k-colouring of $\mathbb{Z}^d$ contains either an AM similar copy of (S, s_0) or a monochromatic similar copy of $\mathbb{Z}^d$ with an integer scaling ratio $1 \leq \lambda \leq \Lambda$.*

Before the proof we need some preparation.

Lemma 4.17. *There is an $R > 0$ such that for any ball D of radius at least R the following is true. For every $p \in \mathbb{Z}^d$ outside D and is at distance at most 1 from D there is a similar copy (S', s'_0) of (S, s_0) in $\mathbb{Z}^d$ such that $s'_0 = p$ and $S' \setminus \{s'_0\} \subset D$.*

Let $\mathbb{Q}_N = \left\{\frac{a}{b} : a, b \in \mathbb{Z}, b \leq N\right\} \subseteq \mathbb{Q}$.

Proof. Since s_0 is an extreme point of S, there is a hyperplane that separates s_0 from $S \setminus \{s_0\}$. Thus if R is sufficiently large, there is an $\varepsilon > 0$ with the following property. If p is outside D and is at distance at most 1 from D, then there is a congruent copy (S'', s''_0) of (S, s_0) with $s''_0 = p$ and such that every point of $S'' \setminus \{s''_0\}$ is contained in D at distance at least 2ε from the boundary of D.

Let $\mathbb{Q}_N = \left\{\frac{a}{b} : a, b \in \mathbb{Z}, b \leq N\right\} \subseteq \mathbb{Q}$. We use the fact that $O(\mathbb{R}^d) \cap \mathbb{Q}^{d\times d}$, the set of rational rotations, is dense in $O(\mathbb{R}^d)$ (see for example [58]). This, together with the compactness of balls implies that we can find an $N = N(\varepsilon) \in \mathbb{N}$ and (S^*, s'_0*) in $\mathbb{Q}_N^d$ which is a rotation of (S'', s''_0) around p, ε-close to (S'', s''_0). With this $S^* \setminus \{s^*_0\}$ is contained in D. Moreover, if R is sufficiently large, then enlarging (S^*, s'_0*) from s'_0* by $N!$, $S' \setminus \{s'_0\}$ is contained in $D \cap \mathbb{Z}^d$. □

The proof of the following variant of Gallai's theorem can be found in the Appendix.

Theorem 4.18 (Gallai)**.** *Let $S \subseteq \mathbb{Z}^d$ be finite. Then there is a $\lambda(d, S, k) \in \mathbb{Z}$ such that every k-colouring of $\mathbb{Z}^d$ contains a monochromatic positive homothet of S with an integer scaling ratio bounded by $\lambda(d, S, k)$.*

Proof of Theorem 4.16. Let R be as in Lemma 4.17 and let H be the set of points of $\mathbb{Z}^d$ contained in a ball of radius R. By Theorem 4.18 there is a monochromatic, say blue, homothetic copy $H_0 = c + \lambda H$ of H for some integer $\lambda \leq \lambda(d, H, k)$. Without loss of generality we may assume that $H_0 = B(O, \lambda R) \cap \lambda\mathbb{Z}^d$ for some $O \in \mathbb{Z}^d$, where $B(O, \lambda R)$ is the ball of radius λR centred at O.

Consider a point $p \in \lambda\mathbb{Z}^d \setminus H$ being at distance at most λ from H_0. If p is not blue then using Lemma 4.17 we can find an AM similar copy of (S, s_0). Thus we may assume that any point $p \in \lambda\mathbb{Z}^d \setminus H_0$ which is λ close to H_0 is blue as well.

By repeating a similar procedure, we obtain that there is either an AM similar copy of (S, s_0), or every point of $H_i = B(O, \lambda R + i\lambda) \cap \lambda\mathbb{Z}^d$ is blue for every $i \in \mathbb{N}$. But the latter means $\lambda\mathbb{Z}^d$ is monochromatic, which finishes the proof. □

4.3.3 Finding an almost-monochromatic positive homothet

The following statement shows that it is not possible to replace an AM similar copy of (S, s_0) with a positive homothet of (S, s_0) in the 'only if' direction of Theorem 4.5.

Proposition 4.19. *Let $S \subseteq \mathbb{Z}^d$ such that S is not contained in a line and $s_0 \in S$. Then there is an AP-free colouring of $\mathbb{R}^d$ without an AM positive homothet of (S, s_0).*

Proof. We may assume that $|S| = 3$ and thus $S \subseteq \mathbb{R}^2$. Since the problem is affine invariant, we may further assume that $S = \{(0, 1), (1, 1), (1, 0)\}$ with $s_0 = (1, 1)$, $s_1 = (0, 1)$ and $s_2 = (1, 0)$. First we describe a colouring of $\mathbb{R}^2$ and then we extend it to $\mathbb{R}^d$.

For every $i \in \mathbb{N}$ let Q_i be the square $[-4^i, 4^{i-1}] \times [-4^i, 4^{i-1}]$, and $Q_0 = \emptyset$. Further let H_+ be the open half plane $x < y$ and H_- be the closed half plane $x \geq y$. We colour $\mathbb{R}^2$ using four colours, green, blue, red and yellow as follows (see also Figure 4.2).

- **Green:** For every odd $i \in \mathbb{N}$ colour $(Q_i \setminus Q_{i-1}) \cap H_+$ with red.
- **Blue:** For every even $i \in N$ colour $(Q_i \setminus Q_{i-1}) \cap H_+$ with yellow.

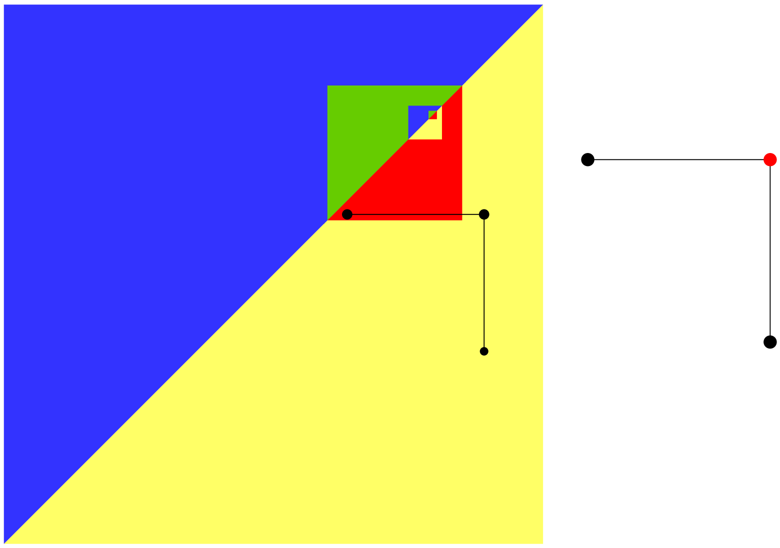

Figure 4.2: A 4-colouring avoiding AM homothets of (S, s_0).

- **Red:** For every odd $i \in \mathbb{N}$ colour $(Q_i \setminus Q_{i-1}) \cap H_-$ with green.
- **Yellow:** For every even $i \in \mathbb{N}$ colour $(Q_i \setminus Q_{i-1}) \cap H_-$ with blue.

A similar argument that we used before shows that this colouring φ_1 is AP-free. Thus we only have to check that it contains no AM positive homothet of (S, s_0). Let S' be a positive homothet of S. First note that we may assume that S' is contained in one of the half planes bounded by the $x = y$ line, otherwise it is easy to see that it cannot be AM. Thus by symmetry we may assume that $s'_0 \in Q_i \setminus Q_{i-1} \cap H_+$ for some $i \in \mathbb{N}$.

If the y-coordinate of s'_0 is smaller than -4^{i-1}, then $s'_1 \in Q_i \setminus Q_{i-1}$, and hence S' cannot be AM. On the other hand, if the y-coordinate of s'_0 is at least -4^{i-1}, then $\|s'_0 - s'_1\| \leq 2 \cdot 4^{i-1}$. This means that the y-coordinate of s'_2 is at least $-(4^{i-1} + 2 \cdot 4^{i-1}) > -4^i$. Thus, in this case s'_2 is contained in $s'_1 \in Q_i \setminus Q_{i-1}$, and hence S' cannot be monochromatic.

To finish the proof, we extend the colouring to $\mathbb{R}^d$. Let $T \cong \mathbb{R}^{d-2}$ be the orthogonal complement of $\mathbb{R}^2$. Fix an AP-free colouring φ of T using the colour set $\{1, 2\}$. Further let φ_2 be a colouring of $\mathbb{R}^2$ isometric to φ_1, but using a disjoint set of colours. For every $t \in T$ colour $\mathbb{R}^2 + t$ by translating φ_i if $\varphi(t) = i$. This colouring is AP-free and does not contain any AM positive homothet of (S, s_0). □

4.4 Colouring $\mathbb{Q}$

Before proving Theorem 4.6, note that we could not replace $S = \{0, 1, 2\}, s_0 = 0$ with any arbitrary pair (S, s_0) where s_0 is an extreme point of S. For example, let $S = \{0, 1, 2, 3, 4\}$, $s_0 = 0$, and colour $\mathbb{Q}$ as follows. Write each non-zero rational as $2^t \frac{p}{q}$ where p and q are odd, and colour it red if t is even and blue if t is odd. This colouring is non-monochromatic, but does not contain any AM homothet of (S, s_0) (in fact not even similar copies).

Indeed, let $c+\lambda S = \{r_1, r_2, r_3, r_4, r_5\} = \{r_1, r_1+\lambda, r_1+2\lambda, r_1+3\lambda, r_1+4\lambda\}$ be a homothet of S. Note that for any $x, y, \alpha \in \mathbb{Q}$ we have that x and y are of the same colour if and only if αx and αy have the same colour. That is, multiplying each r_i with the same α does not change the colour pattern. Thus we may assume that $r_1 = 2^a b$ and $\lambda = 2^t p$ for some $a, t \in \mathbb{N}$ and odd integers b, p.

If $a < t$, then $\{r_1, r_2, r_3, r_4, r_5\}$ is monochromatic, thus we may assume that $t \leq a$ and divide by 2^t, to obtain $\{2^{a-t}b, 2^{a-t}b+p, 2^{a-t}b+2p, 2^{a-t}b+3p, 2^{a-t}b+4p\} \subseteq \mathbb{Z}$. But then two of $2^{a-t}b+p, 2^{a-t}b+2p, 2^{a-t}b+3p, 2^{a-t}b+4p$ are odd, and one of them is 2 mod 4, implying that $\{2^{a-t}b+p, 2^{a-t}b+2p, 2^{a-t}b+3p, 2^{a-t}b+4p\}$ cannot be monochromatic.

To prove Theorem 4.6 we need the following lemma.

Lemma 4.20. *If a k-colouring φ of $\mathbb{Z}$ does not contain an AM positive homothet of (S, s_0) then every colour class is a two-way infinite AP. Moreover, there is an $F = F(k)$ such that φ is periodic with F.*

Proof of Theorem 4.6. Let $\varphi_{\mathbb{Q}}$ be a colouring of $\mathbb{Q}$ without an AM positive homothet of (S, s_0). For any $x \in \mathbb{Q}$, we define a colouring φ of $\mathbb{Z}$ as $\varphi(n) = \varphi_{\mathbb{Q}}(\frac{xn}{F(k)})$. Applying Lemma 4.20 for φ implies that $\varphi_{\mathbb{Q}}(0) = \varphi(0) = \varphi(F(k)) = \varphi_{\mathbb{Q}}(x)$. this means 0 and x have the same colour in $\varphi_{\mathbb{Q}}$. □

Proof of Lemma 4.20. We first prove that if a and $a+d$ for some $d > 0$ have the same colour, say red, then every point in the two-way infinite AP $\{a+id \mid i \in \mathbb{Z}\}$ is also red.

For this first we show that $\{a-id \mid i \in \mathbb{N}\}$ is red. Indeed, if it is not true, let $j \in \mathbb{N}$ be the smallest such that $a-jd$ is not red. But then $\{a-jd, a-(j-1)d, a-(j-2)d\}$ is AM, a contradiction.

Second, we show that $\{a+id \mid i \in \mathbb{N}\}$ contains at most one element of any other colour. Assume to the contrary that there are at least two blue elements in $\{a+id \mid i \in \mathbb{N}\}$, and let $a + jd$, $a + kd$ be the two smallest with $j < k$. But then since $a + jd - (k - j)d$ is red, we have that $\{a + jd - (k - j)d, a + jd, a + kd\}$ is AM, a contradiction.

Finally we show that every element $\{a + id \mid i \in \mathbb{N}\}$ is red. Assume that $(a + jd)$ is the largest non-red element. (By the previous paragraph this largest j exists.) But then, since $a + (j + 1)d$ and $a + (j + 2)d$ are red, we obtain that $\{a + jd, a + (j + 1)d, a + (j + 2)d\}$ is AM, a contradiction.

If d is the smallest difference between any two red numbers, this shows that the set of red numbers is a two-way infinite AP. Thus to finish the first half of the claim, we only have to show that if a colour is used once, then it is used at least twice. But this follows from the fact that the complement of the union of finitely many AP is either empty or contains an infinite AP.

In order to prove the second part, it is sufficient to show that there is an N_k depending on k such that the following is true. If $\mathbb{Z}$ is covered by k disjoint AP, then the difference of any of these AP's is at most N_k. Thus, by considering densities, the following claim finishes the proof of Lemma 4.20.

Claim 4.21. *There is an N_k such that if for $x_1, \ldots, x_k \in \mathbb{N}$ we have $\sum_{i=1}^{k} \frac{1}{x_i} = 1$, then $x_i \leq N_k$ for all $1 \leq i \leq k$.*

Proof. We prove by induction on k that for every $c \in \mathbb{R}^+$ there is a number $N_k(c)$ such that if we have $\sum_{i=1}^{k} \frac{1}{x_i} = c$, then $x_i \leq N_k(c)$ for all $1 \leq i \leq k$. For $k = 1$ setting $N_1(c) = \lfloor \frac{1}{c} \rfloor$ is a good choice.

For $k > 1$ notice that the *smallest* number whose reciprocal is in the sum is at most $\lfloor \frac{k}{c} \rfloor$. Thus we obtain $N_k(c) \leq \max_{i=1}^{\lfloor \frac{k}{c} \rfloor} \left(\max(i; N_{k-1}(c - \frac{1}{i})\right)$. □

This finishes the proof of Lemma 4.20, and thus also of Theorem 4.6. □

It would be interesting to find a complete characterisation of those pairs (S, s_0) for which Theorem 4.6 holds.

Question 4.22. *For which (S, s_0) does the statement similar to that of Theorem 4.6 holds?*

4.5 Smiling bouquets and the chromatic number of the plane

For a graph $G = (V, E)$ with a given origin (distinguished vertex) $v_0 \in V$ a colouring φ with $\varphi : V \setminus \{v_0\} \to \binom{[k]}{1}$ and $\varphi(v_0) \in \binom{[k]}{2}$ is a proper k-colouring with *bichromatic origin*

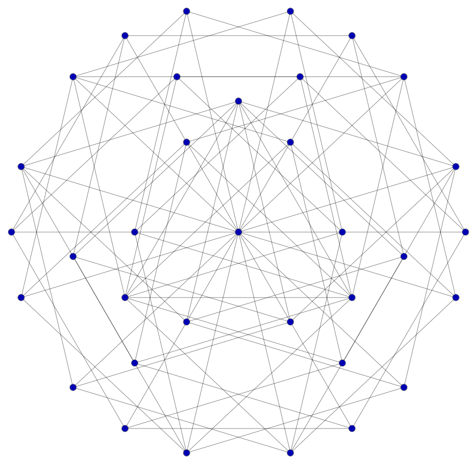

Figure 4.3: A 34 vertex graph without a 4-colouring if the origin is bichromatic.

v_0, if $(v, w) \in E$ implies $\varphi(v) \cap \varphi(w) = \emptyset$. There are unit-distance graphs with not too many vertices that do not have a 4-colouring with a certain bichromatic origin. Figure 4.3 shows such an example, the 34-vertex graph G_{34}, posted by Hubai [40] in Polymath16. Finding such graphs has been motivated by an approach to find a human-verifiable proof of $\chi(\mathbb{R}^5) \geq 5$, proposed by the Pálvölgyi [56] in Polymath16.

$G_3 4$ is the first example found whose chromatic number can be verified quickly without relying on a computer. To see this, note that the vertices connected to the central vertex have to be coloured with two colours, and they can be decomposed into three 6-cycles. Using this observation and the symmetries of the graph, we obtain that there are only two essentially different ways to colour the neighbourhood of the central vertex. In both cases, for the rest of the vertices a systematic back-tracking strategy shows in a few steps that there is no proper colouring with four colours.

Theorem 4.23 with $G = G_{34}$ shows that a human-verifiable proof of Conjecture 4.1 for colourings of the plane with 4 colours would provide a human-verifiable proof of $\chi(\mathbb{R}^2) \geq 5$. Note that G_{34} was found by a computer search, and for finding other similar graphs one might rely on a computer program. Thus, the approach we propose is human-verifiable, however it might be computer-assisted.

For a graph G with origin v_0 let $\{C_1, \ldots, C_n\}$ be the set of unit circles whose centres are the neighbours of v_0, and let $C(G, v_0) = C_1 \cup \cdots \cup C_n$ be the bouquet through v_0.

Theorem 4.23. *If there is a unit-distance graph $G = (V, E)$ with $v_0 \in V$ which does not have a proper k-colouring with bichromatic origin v_0, and Conjecture 4.1 is true for*

$C(G, v_0)$, then $\chi(\mathbb{R}^2) \geq k+1$.

Proof of Theorem 4.23. Assume for a contradiction that there is a proper k-colouring φ of the plane. Using φ we construct a proper k-colouring of G with bichromatic origin $v_0 \in V$.

Let $v_1, \dots, v_n$ be the neighbours of the origin v_0, and C_j be the unit circle centred at v_j. Then $C = C_1 \cup \dots \cup C_n$ is a bouquet through v_0. If Conjecture 4.1 is true for φ, then there is a smiling congruent copy $C' = C'_1 \cup \dots C'_n$ of C through v'_0. That is, there are points $p_1 \in C'_1, \dots, p_n \in C'_n$ with $\ell = \varphi(p_1) = \dots = \varphi(p_n) \neq \varphi(v'_0)$.

For $i \in [n]$ let v'_i be the centre of C'_i. We define a colouring φ' of G as $\varphi'(v_0) = \{\varphi(v'_0), \ell\}$ and $\varphi'(v_i) = \varphi(v'_i)$ for $v \in V \setminus \{v_0\}$. We claim that φ' is a proper k-colouring of G with a bichromatic origin v_0, contradicting our assumption.

Indeed, if $v_i \neq v_0 \neq v_j$ then for $(v_i, v_j) \in E$ we have $\varphi'(v_i) \neq \varphi'(v_j)$ because $\varphi(v'_i) \neq \varphi(v'_j)$. For $(v_0, v_i) \in E$, we have $\varphi'(v_i) \neq \varphi(v_0)$ because $\varphi(v'_i) \neq \varphi(v'_0)$, and $\varphi'(v_i) \neq \ell$ because $\varphi(v'_i) \neq \ell$ since $\|v'_i - p_i\| = 1$. This finishes the proof of Theorem 4.23. □

4.5.1 Smiling pencils

In this section we prove Theorem 4.3. We start with the following simple claim.

Claim 4.24. *For every pencil L through O there is an $\varepsilon > 0$ for which the following is true. For any circle C of radius R if a point p is at distance at most εR from C, then there is a congruent copy L' of L through p such that every line of L' intersects C.*

Proof. It is sufficient to prove the following. If C is a unit circle and p is sufficiently close to C, then there is a congruent copy L' of L through p such that every line of L' intersects C.

Note that if p is contained in the disc bounded by C, clearly every line of every congruent copy L' of L through p intersects C. Thus we may assume that p is outside the disc.

Let $0 < \alpha < \pi$ be the largest angle spanned by lines in L. If p is sufficiently close to C, then the angle spanned by the tangent lines of C through p is larger than α. Thus, any congruent copy L' of L through p can be rotated around p so that every line of the pencil intersects C. □

Proof of Theorem 4.3. Assume for contradiction that φ is a colouring using at least two colours, but there is a pencil L such that there is no congruent smiling copy of L.

First we obtain a contradiction assuming that there is a monochromatic, say red, circle C of radius r. We claim that then every point p inside the disc bounded by C is red. Indeed, translating L to a copy L' through p, each line L'_i will intersect C, and so have a red point. Thus p must be red.

A similar argument together with Claim 4.24 shows that if there is a non-red point at distance at most εr from C, we would find a congruent smiling copy of L through p. Thus there is a circle C' of radius $(1+\varepsilon)r$ concentric with C, such that every point of the disc bounded by C' is red. Repeating this argument, we obtain that every point of $\mathbb{R}^2$ is red contradicting the assumption that φ uses at least 2 colours.

To obtain a contradiction, we prove that there exists a monochromatic circle. For $1 \le i \le n$ let α_i be the angle of L_i and L_{i+1}. Fix a circle C, and let $a_1, \dots, a_n \in C$ be points such that if $c \in C \setminus \{a_1, \dots, a_n\}$, then the angle of the lines connecting c with a_i and c with a_{i+1} is α_i. By Gallai's theorem there is a monochromatic (say red) set $\{a'_1, \dots, a'_n\}$ similar to $\{a_1, \dots, a_n\}$. Let C' be the circle that contains $\{a'_1, \dots, a'_n\}$. Then C' is monochromatic. Indeed, if there is a point p on C' for which $\varphi(p)$ is not red, then by choosing L'_j to be the line connecting p with a'_j we obtain $L' = L'_1 \cup \dots \cup L'_n$, a smiling congruent copy of L. □

4.5.2 Conjecture 4.1 for lattice-like bouquets

Using the ideas from the proof of Theorem 4.16, we prove Conjecture 4.1 for a broader family of bouquets.

Lattices

A *lattice* $\mathcal{L}$ in the plane generated by two linearly independent vectors v_1 and v_2 is the set $\mathcal{L} = \mathcal{L}(v_1, v_2) = \{n_1 v_1 + n_2 v_2 : n_1, n_2 \in \mathbb{Z}\}$. We call a lattice $\mathcal{L}$ *rotatable* if for every $0 \le \alpha_1 < \alpha_2 \le \pi$ there is an angle $\alpha_1 < \alpha < \alpha_2$ and scaling factor $\lambda = \lambda(\alpha_2, \alpha_1)$ such that $\lambda\alpha(\mathcal{L}) \subset \mathcal{L}$, where $\alpha(L)$ is the rotated image of $\mathcal{L}$ by angle α around the origin. For example, $\mathbb{Z}^2$, the triangular grid, and $\{n_1(1,0) + n_2(0,\sqrt{2}) : n_1, n_2 \in \mathbb{Z}\}$ are rotatable, but $\mathcal{L} = \{n_1(1,0) + n_2(0,\pi) : n_1, n_2 \in \mathbb{Z}\}$ is not.[2]

The rotatability of $\mathcal{L}$ allows us to extend Lemma 4.17 from $\mathbb{Z}^2$ to $\mathcal{L}$. This leads to an extension of Theorem 4.16 to rotatable lattices.

Theorem 4.25. *Let $\mathcal{L}$ be a rotatable lattice, $S \subseteq \mathcal{L}$ be finite and s_0 be an extreme point*

[2] For another characterization of rotatable lattices, see `https://mathoverflow.net/a/319030/955`.

of S. Then for every $k \in N$ there exists a constant $\Lambda = \Lambda(\mathcal{L}, S, k)$ such that the following is true. In every k-colouring of $\mathcal{L}$ there is either an AM similar copy of (S, s_0) with a positive scaling factor bounded by Λ, or a monochromatic positive homothetic copy of $\mathcal{L}$ with an integer scaling factor $1 \leq \lambda \leq \Lambda$.

The proof of extending Lemma 4.17 to rotatable lattices is analogous to the original one, so is the proof of Theorem 4.25 to the proof of Theorem 4.16. Thus, we omit the details.

Lattice-like bouquets

Let $C = C_1 \cup \cdots \cup C_n$ be a bouquet through O, and for $i \in [n]$ let O_i be the centre of C_i. We call C *lattice-like* if O is an extreme point of $\{O, O_1, \ldots, O_n\}$ and there is a rotatable lattice $\mathcal{L}$ such that $\{O, O_1, \ldots, O_n\} \subseteq \mathcal{L}$. Similarly, we call a unit-distance graph $G = (V, E)$ with an origin $v_0 \in V$ *lattice-like* if there is a rotatable lattice $\mathcal{L}$ such that v_0 and its neighbours are contained in $\mathcal{L}$, and v_0 is not in the convex hull of its neighbours.

Since $\mathbb{Z}^2$ is a rotatable lattice, Theorem 4.2 is a direct corollary of the result below.

Theorem 4.26. *If C is a lattice-like bouquet, then every proper k-colouring of $\mathbb{R}^2$ contains a smiling congruent copy of C.*

This implies the following, similarly as Conjecture 4.1 implied Theorem 4.23.

Theorem 4.27. *If there exists a lattice-like unit-distance graph $G = (V, E)$ with an origin v_0 that does not admit a proper k-colouring with bichromatic origin v_0, then $\chi(\mathbb{R}^2) \geq k+1$.*

In the proof of Theorem 4.26, we need a simple geometric statement.

Proposition 4.28. *Let $C = C_1 \cup \cdots \cup C_n$ be a bouquet through O, and let $\mathcal{O} = \{O_1, \ldots, O_n\}$, where O_j is the centre of C_j. Then for every $0 < \lambda \leq 2$ there are n points $P_1, \ldots, P_n$ such that $P_j \in C_j$ and $\{P_1, \ldots, P_n\}$ is congruent to $\lambda\mathcal{O}$.*

Proof. For $\lambda = 2$ let P_j be the image of O reflected in O_j. Then $P_j \in C_j$, and $\{P_1, \ldots, P_n\}$ can be obtained by enlarging $\mathcal{O}$ from O with a factor of 2. For $\lambda < 2$, scale $\{P_1, \ldots, P_n\}$ by $\frac{\lambda}{2}$ from O obtaining $\{P'_1, \ldots, P'_n\}$. Then there is an angle α such that rotating $\{P'_1, \ldots, P'_n\}$ around O by α, the rotated image of each P'_j is on C_j. □

Proof of Theorem 4.26. Let $C = C_1 \cup \cdots \cup C_n$ be the lattice-like bouquet through O, O_i be the centre of C_i for $i \in [k]$, and $\mathcal{L}$ be the rotatable lattice containing $S = \{O, O_1, \ldots, O_n\}$. Consider a proper k-colouring φ of $\mathbb{R}^2$ and let $\delta \in \mathbb{Q}$ to be chosen later.

By Theorem 4.25, the colouring φ either contains an AM similar copy of $\delta(S, s_0)$ with a positive scaling factor bounded by $\lambda(\mathcal{L}, S, k)$, or a monochromatic similar copy of $\delta\mathcal{L}$ with an integer scaling factor bounded by $\lambda(\mathcal{L}, S, k)$.

If the first case holds and δ is chosen so that $\delta\lambda(\mathcal{L}, S, k) \leq 2$, Proposition 4.28 provides a smiling congruent copy of C. Now assume for contradiction that the first case does not hold. Then there is a monochromatic similar copy $\mathcal{L}'$ of $\delta\mathcal{L}$ with an integer scaling factor λ bounded by $\lambda(\mathcal{L}, S, k)$. However, if we choose $\delta = \frac{1}{\lambda(\mathcal{L},S,k)!}$, then for any $1 \leq \lambda \leq \lambda(\mathcal{L}, S, k)$ we have $\delta\lambda = \frac{1}{N_\lambda}$ for some $N_\lambda \in \mathbb{N}$. But this would imply that there are two points in the infinite lattice $\lambda\delta\mathcal{L}$ at distance 1, contradicting that φ is a proper colouring $\mathbb{R}^2$. □

4.6 Further problems and concluding remarks

We mainly focused on finding AM sets *similar* to a given one. However, it is also interesting to find AM sets *congruent* to a given one. In this direction, Erdős, Graham, Montgomery, Rothschild, Spencer and Straus made the following conjecture.

Conjecture 4.29 (Erdős et al. [19])**.** *Let $s_0 \in S \subset \mathbb{R}^2$, $|S| = 3$. There is a non-monochromatic colouring of $\mathbb{R}^2$ that contains no AM congruent copy of (S, s_0) if and only if S is collinear and s_0 is not an extreme point of S.*

As noted in [19], the 'if' part is easy; colour $(x, y) \in \mathbb{R}^2$ red if $y > 0$ and blue if $y \leq 0$. In fact, this colouring also avoids AM similar copies of such S. Conjecture 4.29 was proved in [19] for the vertex set S of a triangle with angles $120°, 30°$, and $30°$ with any $s_0 \in S$. It was also proved for any isosceles triangle in the case when s_0 is one of the vertices on the base, and for an infinite family of right-angled triangles.

Much later, the same question was asked independently in a more general form by the Pálvölgyi [57]. In a comment to this question on the MathOverflow site, a counterexample (to both the MathOverflow question and Conjecture 4.29) was pointed out by user 'fedja' [27], which we sketch below.

Let $S = \{0, 1, s_0\}$ where $s_0 \notin [0, 1]$ is a transcendental number. Then there is a field automorphism τ of $\mathbb{R}$ over $\mathbb{Q}$ such that $\tau(s_0) \in (0, 1)$. Colour x red if $\tau(x) > 0$ and blue if $\tau(x) \leq 0$. Suppose that there is an AM similar copy $\{a, a + b, a + bs_0\}$ of (S, s_0). Then these points are mapped by τ to $\tau(a)$, $\tau(a)+\tau(b)$ and $\tau(a)+\tau(b)\tau(s_0)$. Since $\tau(s_0) \in (0, 1)$, we have $\tau(a) < \tau(a) + \tau(b)\tau(s_0) < \tau(a) + \tau(b)$, so $\{a, a + b, a + bs_0\}$ cannot be an AM copy.

Straightforward generalisations of our arguments from Section 4.5 would also imply lower bounds for the chromatic number of other spaces. For example, if C is a lattice-like bouquet of spheres, then every proper k-colouring of $\mathbb{R}^d$ contains a smiling congruent copy of C. This implies that if one can find a lattice-like unit-distance graph with an origin v_0 that does not admit a proper k-colouring with bichromatic origin v_0, then $\chi(\mathbb{R}^d) \geq k+1$. Possibly one can even strengthen this further; in $\mathbb{R}^d$ it could be even true that there is a d-*smiling* congruent copy of any bouquet C, meaning that there are d colours that appear on each sphere of C. This would imply $\chi(\mathbb{R}^d) \geq k+d-1$ if we could find a lattice-like unit-distance graph with an origin v_0 that does not admit a proper k-colouring with d-chromatic origin v_0.

In our AP-free colourings that avoid AM similar copies of certain sets, we often use many colours. It would be interesting to know if constructions with fewer colours exist, particularly regarding applications to the Hadwiger-Nelson problem. In Lemma 4.10 (and also in the colouring used for proving the 'if' direction of Theorem 4.5) the number of colours we use is not even uniformly bounded. Are there examples with uniformly bounded number of colours?

On of our main questions is about characterising those pairs (S, s_0) for which in every colouring of $\mathbb{R}^d$ we either find an AM similar copy of (S, s_0) or an *infinite* monochromatic AP. However, regarding applications to the Hadwiger-Nelson problem the following, weaker version would also be interesting to consider: Determine those (S, s_0) with $S \subseteq \mathbb{R}^d$ and $s_0 \in S$ for which there is a $D = D(k, S)$ such that the following is true. For every n in every k-colouring of $\mathbb{R}^D$ there is an AM similar copy of (S, s_0) or an n-term monochromatic AP with difference $t \in \mathbb{N}$ bounded by D. Note that there are pairs for which the property above does not hold when colouring $\mathbb{Z}$. For example let $S = \{-2, -1, 0, 1, 2\}$, $s_0 = 0$, and colour $i \in \mathbb{Z}$ red if $\lfloor i/D \rfloor \equiv 0 \bmod 2$ and blue if $\lfloor i/D \rfloor \equiv 1 \bmod 2$.

4.A Appendix

For completeness, we prove the stronger version of Gallai's theorem, Theorem 4.18. We use the Hales-Jewett theorem, following the proof from [35].

A *combinatorial line* in $[n]^N$ is a a collection of n points, $p_1, \dots, p_n$, such that for some fix $I \subset [N]$ for every $i \in I$ the coordinate $(p_j)_i$ is the same for every j, while for $i \notin I$ the coordinate $(p_j)_i = j$ for every j.

Theorem 4.A.1 ([38])**.** *For every n and k there is an N such that every k-colouring of $[n]^N$ contains a monochromatic combinatorial line.*

Proof of Theorem 4.18. Suppose that we want to find a monochromatic positive homothet of some finite set $S = \{s_1, \dots, s_n\}$ from $\mathbb{Z}^d$ in a k-colouring of $\mathbb{Z}^d$. Choose an N that satisfies the conditions of the Hales-Jewett theorem for n and k. Choose an injective embedding of $[n]^N$ into $\mathbb{Z}^d$ given by $\Psi(x_1, \dots, x_N) = \sum_{i=1}^N \lambda_i s_{x_i}$, where the λ_i's are to be specified later. Then every combinatorial line is mapped into a positive homothet of S, with scaling $\sum_{i \in I} \lambda_i$ for some non-empty $I \subset [N]$. Therefore, applying the Hales-Jewett theorem for the pullback of the k-colouring of our space gives a monochromatic homothet of S.

We still have to specify how we choose the numbers λ_i. For Ψ to be injective, we need that $\sum_{i=1}^N \lambda_i (s_{x_i} - s_{x'_i}) \neq 0$ if $\underline{x} \neq \underline{x}'$. These can be satisfied for some $1 \leq \lambda_i \leq n^N$ by choosing them sequentially. This means that the scaling of the obtained monochromatic homothet is at most $\sum_{i=1}^N \lambda_i \leq N n^N$. □

Note that the proof above works in every abelian group of sufficiently large cardinality, thus $\mathbb{Z}^d$ in Theorem 4.18 can be replaced with $\mathbb{R}^d$ or any lattice $\mathcal{L}$.

Chapter 5

Nearly k-distance sets

5.1 Introduction

For $k \geq 1, d \geq 0$ a set $S \subseteq \mathbb{R}^d$ is a k-distance set in $\mathbb{R}^d$ if $|\{\|p-q\| : p, q \in S, p \neq q\}| \leq k$. We denote by $m_k(d)$ the cardinality of the largest k-distance set in $\mathbb{R}^d$. For $k \geq 1, d \geq 0$ a set $S \subseteq \mathbb{R}^d$ is an *ε-nearly k-distance set* if there exist $1 \leq t_1 \leq \ldots \leq t_k$ such that

$$\|p-q\| \in [t_1, t_1+\varepsilon] \cup \ldots \cup [t_k, t_k+\varepsilon]$$

for all $p \neq q \in S$. Let

$$M_k(d) := \lim_{\varepsilon \to 0} \max\{|S| \ : \ S \text{ is an } \varepsilon\text{-nearly } k\text{-distance set in } \mathbb{R}^d\}.^{1}$$

Note that the $t_1 \geq 1$ assumption is important, otherwise we would have $M_k(d) = \infty$.

A set S is *separated* if the distance between any two of its points is at least 1. Let $M_k(d, n)$ denote the maximum M, such that there exist numbers $1 \leq t_1 \leq \ldots \leq t_k$ and a separated set S of n points in $\mathbb{R}^d$ with at least M pairs of points at a distance that falls into $[t_1, t_1+1] \cup \cdots \cup [t_k, t_k+1]$.

In this chapter we will prove the following results.

Theorem 5.1. *For every integer $k \geq 1$ there exist $d(k)$ such that $M_k(d) = m_k(d)$ if $d \geq d(k)$. Moreover $d(k) = 1$ if $k \leq 3$.*

We denote by $T(m, n)$ the number of edges in a complete balanced m-partite graph on n vertices. Note that $T(m, n) = \left(1 - \frac{1}{m}\right)\frac{n^2}{2} + O(1)$ for fixed m.

[1] Note that the case of $d = 0$ is trivial, as we have $m_k(0) = M_k(0) = 1$ for any k. However it will be convenient to use it later in the proofs.

Theorem 5.2. *Let $k \geq 1$ be fixed. If either $k \leq 3$ or $d \geq d(k)$, then for sufficiently large n we have*

$$M_k(d, n) = T(m_k(d-1), n).$$

Moreover, the same holds if in the definition of $M_k(d, n)$ we change the intervals to be of the form $[t_i, t_i + cn^{1/d}]$ with some constant $c = c(k, d) > 0$.

Overview

In Section 5.1.1 we compare $m_k(d)$ and $M_k(d)$. We give a lower bound on $M_k(d)$ in terms of some $m_{k_i}(d_i)$ for some $k_i \leq k$ and $d_i \leq d$. We also describe concrete examples that show $m_k(d) = M_k(d)$ does not always hold.

In Section 5.1.2 we introduce the more technical notion of *flat sets* and *flat nearly k-distance sets.*

In Section 5.1.3 we state the main results involving the maximum cardinalities of flat nearly k-distance sets. We explain how do they imply the results stated in the Introduction.

In section 5.2 we prove the results stated in Section 5.1.3.

5.1.1 Comparing k-distance sets and nearly k-distance sets

In this subsection, we relate the quantities $M_k(d)$ and $m_k(d)$. The difficulty of relating them lies in the fact that in nearly k-distance sets, distances of different orders of magnitude may appear. In Proposition 5.4 we exploit this fact. However, if we additionally assume that $\frac{t_{i+1}}{t_i} \leq K$ for some universal constant K in the definition of nearly k-distance sets, a compactness argument would imply that $m_k(d)$ equals this modified $M_k(d)$ (see Lemma 5.10 later).

For $d \geq 0$ and $k \geq 1$ let us define

$$M'_k(d) := \max\left\{\prod_{i=1}^{s} m_{k_i}(d_i) \,:\, \sum_{i=1}^{s} k_i = k, \sum_{i=1}^{s} d_i = d\right\}. \tag{5.1}$$

Conjecture 5.3. *$M_k(d) = M'_k(d)$ holds for all but finitely many pairs $k, d \geq 1$.*

We do not have have any examples with $M_k(d) > M'_k(d)$. However, there are constructions, that we will describe later, that suggest there could be some. In Theorem 5.5 we show that the conjecture holds for every k and sufficiently large d.

$k \diagdown d$	2	3	4	5	6	7	8
2	5	6	10	16	27	29	45
3	7	12	16	≥ 24	≥ 40	≥ 65	≥ 121
4	9	13	≥ 25	≥ 41	≥ 73	≥ 127	≥ 241
5	12	≥ 20	≥ 35	≥ 66	≥ 112	≥ 168	≥ 252
6	13	≥ 21	≥ 40	≥ 96	≥ 141	≥ 281	≥ 505

Table 5.1: Bounds on $m_k(d)$ from [26].

Proposition 5.4. *$M_k(d) \geq M'_k(d)$ holds for every $1 \leq k, d$.*

Proof. Let $\sum_{i=1}^{s} k_i = k$ and $\sum_{i=1}^{s} d_i = d$. Then there is an ε-nearly k-distance set in $\mathbb{R}^d$ of cardinality $\prod_{i=1}^{s} m_{k_i}(d_i)$ given by the following construction. For each i let S_i be a k_i-distance set in $\mathbb{R}^{d_i}$ of cardinality $m_{k_i}(d_i)$ and such that the distances in S_i are much larger (in terms of ε) than the distances in S_{i-1}. Then $S_1 \times \cdots \times S_s$ is a ε-nearly k-distance set in $\mathbb{R}^d$ of cardinality $\prod_{i=1}^{s} m_{k_i}(d_i)$. $\square$

Examples with fixed k or d. It is not true that $M_k(d) = m_k(d)$ holds for every k and d. There are several examples of k and d for which we need more than one multiplicative term to maximise (5.1), and hence $M_k(d) \geq M'_k(d) > m_k(d)$. Some of these examples we list below. When needed, we rely on the information from Table 5.1.

- In $\mathbb{R}^2$ the largest cardinality of a 6-distance set is 13, while the product of two arithmetic progressions of length 4 ($d_1 = d_2 = 1$, $k_1 = k_2 = 3$ in (5.1)) gives an ε-nearly 6-distance sets of cardinality 16. Thus $M_6(2) \geq M'_6(2) \geq 16 > m_2(6)$.
- In $\mathbb{R}^3$ the cardinality of the largest 4-distance set is 13, while we can construct an ε-nearly 4-distance set of cardinality $15 = 3 \cdot 5$ as a product of arithmetic progression of length 3 and a 2-distance set on the plane of cardinality 5 ($d_1 = 1$, $d_2 = 2$, $k_1 = 2$, $k_2 = 2$ in (5.1)) . Thus $M_4(3) \geq M'_4(3) \geq 15 > m_4(3)$.
- In $\mathbb{R}^2$ the cardinality of a k-distance set is $O_{\varepsilon'}\left(k^{\frac{1}{1-\varepsilon'}}\right)$ by [37], while the product of two arithmetic progressions of length $(\lfloor k/2 \rfloor + 1)$ and of length $(\lceil k/2 \rceil + 1)$ gives an ε-nearly k-distance set of cardinality $(\lfloor k/2 \rfloor + 1)(\lceil k/2 \rceil + 1) \geq k^2/4$.

- In $\mathbb{R}^d$ for any $\varepsilon_0 >$ the cardinality of a k-distance set is $O\left(k^{\frac{d}{2}+1}\right)$ by [62] and [37]. On the other hand, the product of d arithmetic progressions of size $\lfloor k/d \rfloor + 1$ gives an ε-nearly k-distance set of cardinality $(\lfloor k/d \rfloor + 1)^d \geq (k/d)^d$.

The largest 5-distance set in $\mathbb{R}^2$ is of cardinality 12. We may construct an ε-nearly 5-distance set using product-type constructions as described in the list above, also of cardinality 12. In addition, we can construct an ε-nearly 5-distance set of size 12 that is not of this product construction, and neither does it have the structure of a 5-distance set. Take a large equilateral triangle, and in each of its vertices put a rhombus of a much smaller size with angles 30° and 60° such that the angle of the corresponding sides of the rhombus and the triangle is 90° as shown on Figure 5.1. This example makes us suspect that there could be some exceptions to Conjecture 5.3. Though we also believe that there are only finitely many examples with $M_k(d)$ points that are not products of k_i-distance sets.

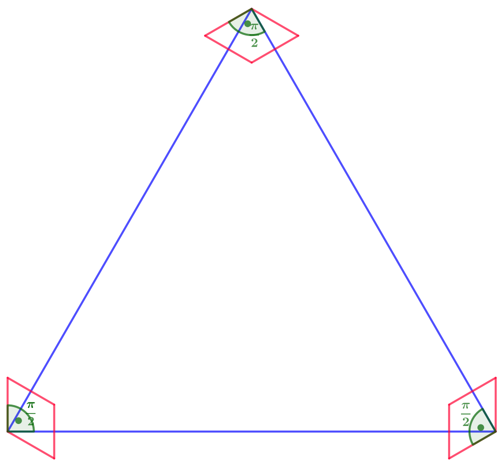

Figure 5.1: An ε-nearly 5-distance set in $\mathbb{R}^2$ that is not product-type

5.1.2 Flat sets

The angle between a vector v and subspace Λ is the minimum of the angles between v and the vectors $w \in \Lambda$.

For $1 \leq d \leq d'$ we say that a set of vectors $V \subseteq \mathbb{R}^{d'}$ is (d, α)-*flat* if there exists a subspace

Λ dimension d such that the angle between any $v \in V$ and Λ is at most α. If Λ is such a subspace, we say that V is *(d, α)-flat* with respect to Λ. A set $P \subseteq \mathbb{R}^{d'}$ is *(p, d, α)-flat* (for some $p \in P$) if $\{p - q : q \in P\}$ is (d, α)-flat. We call a set P *(d, α)-flat* if P is (p, d, α)-flat for every $p \in P$. We say P is *globally (d, α)-flat* if $\{p - q : p, q \in P\}$ is (d, α)-flat. We also say that P is $(p, 0, \alpha)$-flat (for any $p \in P$), and $(0, \alpha)$-flat if $|P| \leq 1$.

Note that, for any $d \geq 2$ and $\beta < \arcsin d^{-1/2}$, (d, α)-flatness for any α does not in general imply global (d, β)-flatness. This is shown for example for $d = 2$ by the set $\{(0, 0, 1), (0, 0, 0), (K, 0, 0), (K, 1, 0)\}$, where $K = K(\alpha, \beta)$ is sufficiently large. However, if for some universal constant K a set S is (p, d, α)-flat for some $p \in S$ and $\frac{\|p_1 - p_2\|}{\|q_1 - q_2\|} \leq K$ for each $p_1, p_2, q_1 \neq q_2 \in S$, then S is globally $(d, \gamma\alpha)$-flat, where γ is a constant depending on K and d. (See Lemma 5.16 later.)

For $0 \leq d \leq d'$ let $N_k(d', d)$ be the largest number N such that for every $\varepsilon, \alpha > 0$ there exists a (d, α)-flat ε-nearly k-distance set in $\mathbb{R}^{d'}$ of cardinality N. Note that $N_k(d', 0) = 1$. For $d \geq 1$ we denote $N_k(d) := N_k(d, d - 1)$. Then we have $M_k(d - 1) \leq N_k(d) \leq M_k(d)$. Indeed, any ε-nearly k-distance set in $\mathbb{R}^{d-1}$ is a $(d - 1, 0)$-flat ε-nearly k-distance set in $\mathbb{R}^d$.

The main reason for introducing the notion of flatness is that the behaviour of $M_k(d, n)$ is asymptotically determined by the value of $N_k(d)$. We suspect though that $N_k(d) = M_k(d)$ for every k and d.

Note the following essential difference between "flat" and "globally flat" constructions. It is not true in general that for any β if $\alpha, \varepsilon > 0$ are sufficiently small then any $(d-1, \alpha)$-flat ε-nearly k-distance S of cardinality $N_k(d)$ is globally $(d - 1, \beta)$-flat.

Indeed, for example, for any α we can construct $(3, \alpha)$-flat ε-nearly 2-distance sets of cardinality $N_2(4) = 6$ in $\mathbb{R}^4$ as follows. Consider an equilateral triangle $\{p_1, p_2, p_3\}$ in $\mathbb{R}^4$ of side length K spanning an affine subspace H of dimension 2. For each $i \in [3]$ let $p_i - q_i$ be any vector of length 1 orthogonal to $p_j - p_k$, $\{i, j, k\} = [3]$. It is not hard to check that $P = \{p_1, p_2, p_3, q_1, q_2, q_3\}$ is a $(3, \alpha)$-flat ε-nearly 2-distance set if $K = K(\alpha, \varepsilon)$ is sufficiently large. However, if $p_1 - q_1$ and $p_2 - q_2$ are orthogonal, then P is not globally $(3, \pi/6)$-flat.

Almost flat sets

We need the following more technical variant of α-flatness, which is however crucial for proving Theorem 5.2. For $d \geq 0$ we say that P is *almost (d, α)-flat* if P is (p, d, α)-flat for all but at most two $p \in P$. Note that this means if P is almost $(0,\alpha)$-flat then $|P| \leq 2$. Let $A_k(d',d)$ denote the largest number N such that for any $\varepsilon, \alpha > 0$ there exists an almost (d,α)-flat ε-nearly k-distance set in $\mathbb{R}^{d'}$ of cardinality N. For $d \geq 1$ we denote $A_k(d) := A_k(d, d-1)$. Note that $A_k(d', 0) = 2$.

Let us summarise the trivial inequalities between the different parameters we introduced:

$$m_k(d) \leq M'_k(d) \leq M_k(d) \leq N_k(d',d) \leq A_k(d',d) \leq M_k(d'), \tag{5.2}$$

for any $d' \geq d \geq 0$.

5.1.3 Main results

Let us stress that all the sets that we consider are separated, which we assume tacitly for the rest of the chapter. Theorem 5.5 below implies Theorem 5.1.

Theorem 5.5. *For any $d' \geq d \geq 1$ we have $A_k(d',d) = N_k(d+1) = M_k(d) = m_k(d)$ if one of the following holds.*

(i) If $d \geq d(k)$, where $d(k)$ is some constant depending on k.

(ii) If $k \leq 3$.

For fixed d and large k we prove the following simple estimate.

Theorem 5.6. *We have $M_k(d) = \Theta\left(k^d\right)$ and $N_k(d) = \Theta\left(k^{d-1}\right)$ for any fixed $d \geq 2$.*

We conjecture that if $k = rd + q$ for $0 \leq q < d$ and k is sufficiently large compared to d, then $M_k(d) = (\lceil k/d \rceil + 1)^{d-q}(\lfloor k/d \rfloor + 1)^q = (k/d)^d + o(k^d)$.

Another main result of the chapter is the following theorem, that concerns the relation of $N_k(d)$ and $M_k(d,n)$.

Theorem 5.7. *For any $d \geq 2$, $k \geq 1$, $\gamma > 0$ there exists n_0, such that for any $n \geq n_0$ we have*

$$T(n, N_k(d)) \leq M_k(d,n) \leq T(n, N_k(d)) + \gamma n^2 \tag{5.3}$$

Moreover, (5.3) *remains valid if in the definition of $M_k(d,n)$ we change the intervals of the form $[t_i, t_i+1]$ to intervals of the form $[t_i, t_i + cn^{1/d}]$ for some constant $c = c(k,d,\gamma) > 0$.*

Theorem 5.7 combined with Theorem 5.5 already gives the value of $M_2(d,n)$, $M_3(d,n)$ and $M_k(d,n)$ for $d \geq d_0(k)$ asymptotically in terms of $m_2(d)$, $m_3(d)$ and $m_k(d)$.

The sharp values in Theorem 5.2 follow from Theorem 5.8 and the lower bound in Theorem 5.7, combined with the fact that $A_k(d) = m_k(d-1)$ in the cases covered in Theorem 5.2.

Theorem 5.8. *For any $d \geq 2$ and $k \geq 1$ there exists n_0, such that for any $n \geq n_0$ we have*

$$M_k(d,n) \leq T(n, A_k(d)) \leq \left(1 - \frac{1}{A_k(d)}\right)\frac{n^2}{2}. \tag{5.4}$$

Moreover, (5.4) *remains valid if in the definition of $M_k(d,n)$ we change the intervals of the form $[t_i, t_i+1]$ to intervals of the form $[t_i, t_i + cn^{1/d}]$ for some constant $c = c(k,d)$.*

As we mentioned in the Introduction of the thesis, $M_k(d,n) \leq T(n, M_k(d))$ follows easily from Turán's theorem. Hence the difficulty in proving Theorem 5.7 lies in bounding $M_k(d,n)$ by the maximal cardinality of $(d-1,\alpha)$-flat nearly k-distance sets. Similarly, the difficulty in proving Theorem 5.2 is bounding $M_k(d,n)$ by the maximal cardinality of k-distance sets in the one dimension smaller space.

5.2 Proofs

We start with proving some auxiliary results, the first of which is a simple statement implied by the triangle inequality.

Lemma 5.9. *Let $S \subseteq \mathbb{R}^d$ a finite set and assume that the pairs of points $\{p_1, p_2\}$, $p_1, p_2 \in S$, are coloured in red and blue, such that the distance between the points in any blue pair is strictly more than 3 times as big as the distance between any red pair. If B is a largest blue clique in S then S can be partitioned into $|B|$ vertex-disjoint red cliques $R_1, \dots, R_{|B|}$ satisfying the following properties.*

1. *Each R_i shares exactly one vertex with B.*
2. *If $p \in R_i$, $q \in R_j$, $i \neq j$, then $\{p,q\}$ is blue.*

Proof. Take a largest blue clique $B = \{v_1, \dots, v_b\}$. Construct R_i by including in it v_i and all the points that form a red pair with v_i. By the triangle inequality each R_i is a red clique. Further, by the maximality of B, each point from S forms a red distance with at least one point in B, and thus $R_1, \dots, R_b$ cover S. Next, they are disjoint: if $v \in R_i \cap R_j$, then both $\{v, v_i\}$ and $\{v, v_j\}$ are red, which by triangle inequality implies that either $i = j$ or that $\{v_i, v_j\}$ is red (but the second possibility contradicts the definition of B). Finally, if $v \in R_i$, $w \in R_j$, $i \neq j$, then $\{v, w\}$ must be blue by the triangle inequality: otherwise

$\|v_i - v_j\| \leq \|v_i - v\| + \|v - w\| + \|w - v_j\|$, and if all the pairs on the right are red, then $\{v_i, v_j\}$ is red. □

The next lemma follows by a standard compactness argument.

Lemma 5.10. *Let $S_1, S_2, \dots$ be a sequence such that S_i is an ε_i-nearly k-distance set in $\mathbb{R}^{d'}$ with distances $1 \leq t_{i,1} < \dots < t_{i,k}$ and with $\varepsilon_i \to 0$. Then the following is true.*

(i) If there is a K such that $\sup_i \max_{1 \leq j < k} \frac{t_{i,j+1}}{t_{i,j}} \leq K$, then $\limsup_{i \to \infty} |S_i| \leq m_k(d')$. If additionally there is a $0 \leq d \leq d'$ such that for every i the set S_i is (p_i, d, ε_i)-flat for some $p_i \in S$, then $\limsup_{i \to \infty} |S_i| \leq m_k(d)$.

(ii) If there is a K such that $\sup_i \max_{1 \leq j < k} \frac{t_{i,j+1}}{t_{i,j}} \leq K$ and for some $1 \leq r \leq k-1$ we have $\lim_{i \to \infty} \frac{t_{i,r+1}}{t_{i,r}} = 1$, then $\limsup_{i \to \infty} |S_i| \leq m_{k-1}(d')$. If additionally for every i S_i is (p_i, d, ε_i)-flat for some $p_i \in S_i$, then $\limsup_{i \to \infty} |S_i| \leq m_{k-1}(d)$.

Proof. We only give details of the proof of (ii), the rest can be done similarly. We start with the first part of the statement. Take any sequence $S_1, S_2, \dots$, satisfying the conditions and scale each S_i by $\frac{1}{t_{i,1}}$. Abusing the notation, we denote the new sets by S_i as well. Then the condition $\sup_i \max_{1 \leq j < k} \frac{t_{i,j+1}}{t_{i,j}} \leq K$ implies that there is an absolute $R > 0$ such that each S_i is contained in a ball B of radius R. A volume argument implies that there exist an M_K such that $|S_i| \leq M_K$ for all i. Take an infinite subsequence of $S_1, S_2 \dots$ in which all sets have fixed cardinality $M \leq M_K$. Using the compactness of $\underbrace{B \times \ldots \times B}_{M \text{ times}}$ we can select out of it a subsequence $S_{i_1}, S_{i_2}, \dots$ that point-wise converges to the set $S := \{P_1, \dots, P_M\} \subset B$ with distances $T_1, \dots, T_k$, and where $T_j = \lim_{s \to \infty} \frac{t_{i_s,j}}{t_{i_s,1}}$. Note that S is indeed of cardinality M, since each S_i is separated. Due to the assumption $\lim_{i \to \infty} \frac{t_{i,r+1}}{t_{i,r}} = 1$ we have $T_{r+1} = T_r$, thus S is a $(k-1)$-distance set, and so $M = |S| \leq m_{k-1}(d')$.

Let us next show the second part of the statement. Taking the set S as above, we obtain that it must additionally be $(d, 0)$-flat, which means that S lies in an affine subspace of dimension d, thus $M = |S| \leq m_{k-1}(d)$. □

The statement below allows us to get a grip on $M_k(d)$.

Lemma 5.11. *For any $1 \leq k$, and $0 \leq d \leq d'$ we have*

$$N_k(d', d) \leq f(d, k) = \max \left\{ \prod_{i=1}^{s} m_{k_i}(d) \ : \ \sum_{i=1}^{s} k_i = k \right\}.$$

In particular $M_k(d) < \infty$.

Note the difference in the definition of $M'_k(d)$ and the function f above.

Proof. First note that f satisfies $f(d,k_1+k_2) \geq f(d,k_1)f(d,k_2)$ for any $1 \leq k_1,k_2$. For each d we induct on k.

Let S be an ε-nearly (d,α)-flat k-distant set in $\mathbb{R}^d$ with distances $1 \leq t_1 < \cdots < t_k$. We need to show that $|S| \leq f(d,k)$.

If $\frac{t_i}{t_{i-1}} \leq 3$ holds for every $1 < i \leq k$ (or if $k=1$), then by Lemma 5.10 (i) we have $|S| \leq m_k(d) \leq f(d,k)$. Otherwise, let i be the largest index such that $\frac{t_i}{t_{i-1}} > 3$. Colour a pair $\{p_1,p_2\}$ with $p_1,p_2 \in S$ blue if $\|p_1-p_2\| \geq t_i$ and with red otherwise. Let B be the largest blue clique in this colouring. By induction $|B| \leq f(d,k-i+1)$ if α and ε are sufficiently small. Next, by Lemma 5.9, S can be covered by $|B|$ vertex disjoint red cliques $R_1,\ldots,R_{|B|}$. By induction again, the cardinality of any red clique is at most $f(d,i-1)$, and thus

$$|S| \leq f(d,k-i+1)f(d,i-1) \leq f(d,k).$$

□

The next four statements describe some cases when β-flatness with respect to different subspaces can be "combined" into α-flatness with respect to a smaller-dimensional subspace.

Lemma 5.12. *For any $\alpha > 0$ there exists $\beta_0 > 0$ such that the following is true for every $\beta \leq \beta_0$. Let $\Lambda_1,\ldots,\Lambda_m \subseteq \mathbb{R}^d$ be subspaces of dimension $d-1$, and j be the smallest integer for which the set of their unit normal vectors $V := \{v_1,\ldots,v_m\}$ is (j,β^j)-flat with respect to some subspace Λ_B of dimension j. If the angle between a vector v and Λ_i is at most β^d for every $i \in [m]$, then the angle between v and $\Lambda := (\Lambda_B)^{\perp}$ is at most α.*

Since the proof of Lemma 5.12 is a technical calculation, we postpone it to the Appendix.

Lemma 5.12 immediately implies the following.

Corollary 5.13. *For any $\alpha > 0$ there exist $\beta_0 > 0$ such that the following is true for every $\beta \leq \beta_0$. Let $\Lambda_1,\ldots,\Lambda_m \subseteq \mathbb{R}^d$ be $(d-1)$-dimensional subspaces, and let j be the lowest dimension for which the set of their unit normal vectors $V = \{v_1,\ldots,v_m\}$ is (j,β^j)-flat with respect to a subspace Λ_B. For every $p \in S$ if S is $(p,d-1,\beta^d)$-flat with respect to Λ_i for every $i \in [m]$, then S is $(p,d-j,\alpha)$-flat with respect to $\Lambda = (\Lambda_B)^{\perp}$.*

We call two subspaces $\Lambda_1,\Lambda_2 \subset \mathbb{R}^d$ *almost-orthogonal,* if there exists a basis $\{v_1,\ldots,v_d\}$ of $\mathbb{R}^d$, where for some $0 \leq a \leq b \leq c \leq d$ the vectors $v_1,\ldots,v_b$ form an orthogonal basis of Λ_1, the vectors $v_{a+1}, \ldots, v_c$ form an orthogonal basis of Λ_2 such that the following is true. For any vector v from the subspace spanned by $\{v_1 \ldots, v_a\}$ the angle between v and Λ_2 lies in $[\pi/2 - 0.01, \pi/2 + 0.01]$, and for any vector w from the subspace spanned by $\{v_{b+1}, \ldots, v_c\}$ the angle between w and Λ_1 lies in $[\pi/2 - 0.01, \pi/2 + 0.01]$. The proof of the next lemma is a technical calculation, thus we postpone it to the Appendix.

Lemma 5.14. *If $\Lambda_1, \Lambda_2 \subset \mathbb{R}^d$ are almost-orthogonal subspaces and the angle between some vector v and each of the two subspaces Λ_1, Λ_2 is at most α for some $0 < \alpha \leq \frac{1}{3}$, then the angle between v and $\Lambda_1 \cap \Lambda_2$ is at most 10α.*

Lemma 5.15. *For any d' and $\alpha > 0$ there exist $K, \beta' > 0$ such that the following is true for any $\beta' \geq \beta > 0$. Let $S = B \cup R \subseteq \mathbb{R}^{d'}$ with $\{b\} = R \cap B$ be a separated set that satisfies the following conditions.*

1. *For any $r_1 \neq r_2 \in R$ and $b_1 \neq b_2 \in B$ we have $K\|r_1 - r_2\| \leq \|b_1 - b_2\|$.*
2. *For any $b_1 \neq b \in B$ there is a distance t such that $\|b_1 - r\| \in [t, t + \beta^{d'+1}]$ for any $r \in R$.*

Further, let j be the lowest dimension such that B is (b, j, β^j)-flat with respect to a subspace Λ_B of dimension j. Then putting $\Lambda := (\Lambda_B)^\perp$ we have the following.

(i) R is $(r, d' - j, \alpha)$-flat with respect to Λ for any $r \in R$.

(ii) If for some $r \in R$ and sufficiently small $\alpha' \leq \alpha$ the set S is (r, d, α')-flat with respect to a d-dimensional subspace Λ_r, then R is $(r, d - j, 10\alpha)$-flat with respect to $\Lambda \cap \Lambda_r$.

Proof. (i) Let $r_1, r_2 \in R$, $b_i \in B$ with $b \neq b_i$, and β_0 as in Corollary 5.13. If $\beta \leq \beta_0$ is sufficiently small and K is sufficiently large, then $\angle b_i r_1 r_2 \in [\frac{\pi}{2} - \frac{\beta^{d'}}{2}, \frac{\pi}{2} + \frac{\beta^{d'}}{2}]$, otherwise we would obtain $|\|b_i - r_i\| - \|b_i - r_2\|| > \beta^{d'+1}$, contradicting condition 2. Further, by condition 1, if K is sufficiently large, then $\angle b b_i r_1 \leq \frac{\beta^{d'}}{2}$. Thus, the angle of $r_1 - r_2$ and $b - b_i$ is contained in $[\angle b_i r_1 r_2 - \angle b b_i r_1, \angle b_i r_1 r_2 + \angle b b_i r_1] \subset [\frac{\pi}{2} - \beta^{d'}, \frac{\pi}{2} + \beta^{d'}]$. In other words, R is $(r_1, d-1, \beta^{d'})$-flat with respect to the $(d-1)$-dimensional subspace Λ_{b_i} whose normal vector is $b - b_i$. Since j is the lowest dimension such that $\{b - b_i : b_i \in B\}$ is (j, β^j)-flat with respect to a j-dimensional subspace Λ_B, by Corollary 5.13 we obtain that R is $(r_1, d' - j, \alpha)$-flat with respect to Λ.

(ii) It suffices to show that Λ_r and Λ are almost-orthogonal if K is sufficiently large and β is sufficiently small, since then the statement is a direct corollary of Lemma 5.14 applied to every vector $r_1 - r$ with $r_1 \in R \setminus \{r\}$. The almost-orthogonality follows from the fact that if α' is sufficiently small, we may assume that Λ_B is a subspace of Λ_r, and that R is $(r, d' - j, \alpha')$-flat with respect to Λ (the orthogonal complement of Λ_B) if β is sufficiently small and K is sufficiently large. □

The proof of the following lemma is a simple calculation, that we postpone again to the Appendix.

Lemma 5.16. *Let $S \subseteq \mathbb{R}^d$ be a set such that $\|p_1 - p_2\| \leq K\|q_1 - q_2\|$ holds for any p_1, p_2, q_1, q_2 with $q_1 \neq q_2$. Then if S is (p, j, α)-flat for some $p \in S$, then S is globally $(j, 20(K\alpha)^{1/2})$-flat.*

5.2.1 Fixed k: Proof of Theorem 5.5 (i)

We will prove that for any $d' \geq d \geq 1$ we have $A_k(d', d) = m_k(d)$ if d is sufficiently large compared to k. This is sufficient in view of (5.2). We induct on k. The case $k = 1$ is implied by Lemma 5.10 (i). Assume that the statement of Theorem 5.5 is true for $k' \leq k - 1$ with $d > D_{k-1}$. We shall prove the statement for k and $d > D_k$, where the quantity D_k is chosen later.

For an ε-nearly k-distance set S with distances $1 \leq t_1 < \cdots < t_k$ and $K > 0$, let $q_S(K)$ be the largest index $1 < i \leq k$ such that $\frac{t_i}{t_{i-1}} \geq K$ and, if $\max_i \frac{t_i}{t_{i-1}} < K$, then let $q_S(K) = 1$.

The proof for fixed k is by (backwards) induction on $k - i$, given in the form of the following lemma. The lemma applied with $i = 1$ implies the theorem. In the proof we will use the bound

$$\binom{d+1}{k} \leq m_k(d) \leq \binom{d+k}{k} \tag{5.5}$$

from [5].

Lemma 5.17. *If ε and α' are sufficiently small and $d > D_k$ then the following is true. For each $1 \leq i \leq k$ there are K_i with $K_{i-1} \geq K_i$ such that if S is an almost (d, α')-flat ε-nearly k-distance set in $\mathbb{R}^{d'}$ with $q_S(K_i) \geq i$, then $|S| \leq m_k(d)$.*

Proof. The proof is by induction on $k - i$. We start by showing that the statement is true for $i = k$ with some sufficiently large $K_k \geq 4$.

Assume $q_S(K_k) = k$ and colour each pair $\{p_1, p_2\}$ in S with blue if $\|p_1 - p_2\| \in [t_k, t_k + \varepsilon]$ and with red otherwise. Let B be the largest blue clique in S. Then S can be covered by $|B|$ disjoint red cliques $R_1, \dots, R_{|B|}$ by Lemma 5.9. Let R be a largest red clique and $R \cap B = \{b\}$. We will apply Lemma 5.15 to $R \cup B$ with a sufficiently small $\alpha \geq \alpha'$ to be chosen later to bound $|R|$.

Apply Lemma 5.15 with d, α and take $K_k \geq 2K, \varepsilon \leq \beta^{d'+1}$, where $K, \beta^{d'+1}$ are as in Lemma 5.15. Then the conditions 1,2 are satisfied automatically. Let j, Λ be as in the lemma. Thus R is $(d'-j, \alpha)$-flat with respect to Λ by Lemma 5.15 (i). Further, if for some $r \in R$ we have that $R \cup B \subseteq S$ is (r, d, α)-flat, then R is $(r, d-j, 10\alpha)$-flat by Lemma 5.15 (ii).

Since $R \cup B$ is (r, d, α)-flat for all but at most two $r \in R$, we obtain that R is almost $(d-j, 10\alpha)$-flat. Thus if α is sufficiently small we have $|R| \leq A_{k-1}(d', d-j)$. Note also that $|B| \leq m_1(j) = j+1$ by Lemma 5.10 (i) and the fact that α and ε are sufficiently small. These imply

$$|S| \leq |R_1| + \cdots + |R_{|B|}| \leq (j+1)A_{k-1}(d', d-j).$$

We separate two cases in order to bound $(j+1)A_{k-1}(d', d-j)$.

Case 1: $d - j \geq D_{k-1}$. In this case we obtain

$$(j+1)A_{k-1}(d', d-j) \leq (j+1)\binom{d-j+k-1}{k-1} \leq \binom{d+1}{k} \leq m_k(d),$$

where the first inequality is true by induction and by (5.5), and the second is true if d is sufficiently large.

Case 2: $d - j < D_{k-1}$. In this case we have $A_{k-1}(d', d-j) \leq 2 + N_{k-1}(d', d-j)$, which is according to Lemma 5.11 bounded by a constant C_k depending on k and D_{k-1} and hence only on k. Thus

$$(j+1)A_{k-1}(d', d-j) \leq (j+1)C_k \leq m_k(d),$$

if d is sufficiently large.

We now turn to the induction step. Assume that the statement holds for all $i' > i$, and let us prove it for i. Again colour a pair $\{p_1, p_2\}$ in S in blue if $\|p_1 - p_2\| \geq t_i$ and in red otherwise. Let B be the largest blue clique in S. Then S can be covered by $|B|$ red cliques $R_1, \dots, R_{|B|}$ as in Lemma 5.9.

We may assume that $q(K_{i+1}) \leq i$, otherwise we are done by induction. It implies that $\max_{i<j\leq k} \frac{t_j}{t_{j-1}} \leq K_{i+1}$ by the choice of i. Thus, by Lemma 5.10 (ii) we may assume that there exists a constant $c > 1$ such that the following is true for sufficiently small α, ε:

$$\text{if } \min_{i<j\leq k} \frac{t_j}{t_{j-1}} < c, \text{ then } |B| \leq m_{k-i}(d). \tag{5.6}$$

Set $K_i' = \max\left\{\frac{2}{c-1}, K_{i+1}\right\}$. We are ready to verify the statement of the lemma for sufficiently large $K_i > 2K_i'$. We separate two cases.

Case 1: $\min_{i<j\le k} \frac{t_j}{t_{j-1}} < c$. If R is the largest red clique then using (5.6), the induction hypothesis for fewer distances and (5.5) we have

$$|S| \le |B||R| \le m_{k-i}(d)A_{i-1}(d',d) \le \binom{k-i+d}{k-i}\binom{i-1+d}{i-1} < \binom{d+1}{k} \le m_k(d),$$

where the second to last inequality holds for all sufficiently large d.

Case 2: $\min_{i<j\le k} \frac{t_j}{t_{j-1}} \ge c$. Let R be the largest red clique and $R \cap B = \{b\}$.

Apply Lemma 5.15 with d, α, and take $K_i \ge 2K, \varepsilon \le \beta^{d'+1}$, where $K, \beta^{d'+1}$ are as in Lemma 5.15. Then condition 1 is satisfied automatically. Condition 2 is satisfied as well, as long as "all the distances from a point in B to R fall in one interval", i.e., as long as we can show that it is impossible to have $j_1 > j_2 \ge i$ and points $b \ne b' \in B$, $r_1, r_2 \in R$, such that $\|b'-r_1\| \in [t_{j_1}, t_{j_1}+\varepsilon]$, $\|b'-r_2\| \in [t_{j_2}, t_{j_2}+\varepsilon]$. If that would have been the case, then, by the triangle inequality $t_{j_1} \le \|b'-r_1\| \le \|b'-r_2\| + \|r_1-r_2\| \le t_{j_2} + t_{i-1} + 2\varepsilon$, but on the other hand $t_{j_1} - t_{j_2} \ge (c-1)t_i \ge (c-1)K_i t_{i-1} \ge (c-1)\frac{2}{c-1}t_{i-1} \ge 2t_{i-1} > t_{i-1} + 2\varepsilon$, a contradiction.

Let j, Λ be as in the lemma. Thus R is $(d'-j, \alpha)$-flat with respect to Λ by Lemma 5.15 (i). Further, if $R \cup B \subseteq S$ is (r, d, α)-flat for some $r \in R$, then R is $(r, d-j, 10\alpha)$-flat by Lemma 5.15 (ii).

Since $R \cup B$ is (r, d, α)-flat for all but at most two $r \in R$, we obtain that R is almost $(d-j, \alpha)$-flat. Thus if α is sufficiently small we have $|R| \le A_{i-1}(d', d-j)$. Note also that $\max_{i<j\le k} \frac{t_i}{t_{i-1}} \le K_{i+1}$, and thus, by making ε and $\alpha' \le \alpha$ sufficiently small, we get that $|B| \le m_{k-i+1}(j)$ by Lemma 5.10 (i). We obtain that

$$|S| \le |R||B| \le m_{k-i+1}(j)A_{i-1}(d', d-j).$$

We separate two cases in order to bound $m_{k-i+1}(j)A_{i-1}(d', d-j)$.

Case 2.1: $d - j \ge D_{i-1}$. In this case we obtain

$$m_{k-i+1}(j)A_{i-1}(d', d-j) \le m_{k-i+1}(j)m_{i-1}(d-j) \le \binom{j+k-i+1}{k-i+1}\binom{d-j+i-1}{i-1} \le \binom{d+1}{k} \le m_k(d),$$

where the first two inequalities are true by induction, and the third is true if d is sufficiently large.

Case 2.2: $d - j < D_{i-1}$ In this case we have $A_{i-1}(d', d-j) \leq 2 + N_{i-1}(d', d-j)$, which is by Lemma 5.11 bounded by a constant C_i depending on i and D_{i-1} and hence only on i. Thus

$$m_{k-i+1}(j)A_{i-1}(d', d-j) \leq m_{k-i+1}(j)C_i \leq m_k(d)$$

if d is sufficiently large. □

5.2.2 $k = 2$ and $k = 3$: Proof of Theorem 5.5 (ii)

We will prove that for any $d' \geq d$ and $k = 2, 3$ we have $A_k(d', d) = m_k(d)$. This is sufficient in view of (5.2).

Let us first prove that $A_2(d', d) = m_2(d)$. Let $\varepsilon, \alpha > 0$ be sufficiently small and S be an almost (d, α)-flat ε-nearly 2-distance set in $\mathbb{R}^{d'}$ with distances $t_1 < t_2$. Then S is (p, d, α)-flat with respect to some d-dimensional subspace Λ_p for all but at most two $p \in S$. Let $K > 3$ be a sufficiently large constant to be specified later. We may assume that $\frac{t_2}{t_1} \geq K$, otherwise we have $|S| \leq m_2(d)$ by Lemma 5.10 (i). Colour a pair $\{p_1, p_2\}$ $(p_1, p_2 \in S)$ with blue if $\|p_1 - p_2\| \geq t_2$ and with red otherwise. Let B be the largest blue clique in S. Then S can be partitioned into $|B|$ red cliques $R_1, \dots, R_{|B|}$ as in Lemma 5.9.

Let j be the dimension of the affine subspace Λ_B spanned by B. Note that, since B is an almost 1-distance set, B is very close to a regular simplex, and hence there is an absolute $\gamma > 0$ such that, for any $b \in B$, the set B is not $(b, j-1, \gamma)$-flat if ε is sufficiently small. We apply Lemma 5.15 to $R \cup B$, where R is a red clique, and obtain that R is $(d' - j, \alpha)$-flat. Moreover, since S is (p, d, α)-flat for all but at most two $p \in S$, for all but at most two (say R_i and R_l) red cliques R we have that R is $(p, d-j, 10\alpha)$-flat for some $p \in R$, and $|R_i| + |R_l| \leq 2$. Now Lemma 5.10 (i) implies that for a red clique $R \neq R_i, R_l$ we have $|R| \leq m_1(d-j) = d - j + 1$.

Noting further that $|B| = j + 1$ we obtain

$$|S| = |R_1| + \cdots + |R_{|B|}| \leq \max\{(j+1)(d-j+1), j(d-j+1) + 2\}.$$

Then either $d = j$ or $(j+1)(d-j+1) \geq j(d-j+1) + 2$ holds. In the first case, we have $|S| \leq d+2 \leq \binom{d+1}{2} \leq m_2(d)$ if $d \geq 3$, and $|S| \leq d+2 \leq m_2(d)$ for $d = 1, 2$ since $m_2(1) = 3$ and $m_2(2) = 5$. In the second case $|S| \leq (j+1)(d-j+1) \leq \left(\frac{d+2}{2}\right)^2 \leq \binom{d+1}{2} \leq m_2(d)$ if $d \geq 4$, and $|S| \leq (j+1)(d-j+1) \leq m_2(d)$ if $d = 2, 3$ since $m_2(2) = 5$ and $m_2(3) = 6$ (see Table 5.1).

Next we prove $A_3(d', d) = m_3(d)$. Let $\varepsilon, \alpha > 0$ be sufficiently small and S be an ε-nearly almost (d, α)-flat 3-distance set in $\mathbb{R}^{d'}$ with distances $t_1 < t_2 < t_3$. Let $K > 3$ be a sufficiently large constant. We may assume that $\frac{t_2}{t_1} \geq K$ or $\frac{t_3}{t_2} \geq K$ holds, otherwise we have $|S| \leq m_3(d)$ by Lemma 5.10 (i). We will analyse these two cases.

Case 1: $\frac{t_3}{t_2} \geq K$. Colour a pair $\{p_1, p_2\}$ $(p_1, p_2 \in S)$ with blue if $\|p_1 - p_2\| \geq t_3$ and with red otherwise. Let B be the largest blue clique in S. Then S can be covered by $|B|$ red cliques $R_1, \dots, R_{|B|}$ as in Lemma 5.9. Let j be the dimension of the affine subspace Λ_B spanned by B. Note that, since B is an almost 1-distance set, B is very close to a regular simplex, and hence there is an absolute constant $\gamma > 0$ such that, for any $b \in B$, the set B is not $(b, j-1, \gamma)$-flat if ε is sufficiently small. We apply Lemma 5.15 (i) to $R \cup B$, where R is a red clique, and obtain that R is $(d' - j, \alpha)$-flat. Moreover, S is (p, d, α)-flat for all but at most two $p \in S$, and thus, by Lemma 5.15 (ii), each red clique R is almost $(p, d-j, 10\alpha)$-flat.

Using that $|B| = j + 1$ we obtain

$$|S| = |R_1| + \cdots + |R_{|B|}| \leq (j+1) A_2(d', d-j, 10\alpha).$$

For sufficiently small α and a red clique R, we have $|R| \leq 2$ if $d = j$. If $j < d$ then $|R| \leq m_2(d-j)$ by the $k = 2$ case of the theorem. In the first case, $|S| \leq 2(d+1)$. For $d \geq 4$ we have $|S| \leq 2(d+1) \leq \binom{d+1}{3} \leq m_3(d)$ and for $d = 1, 2, 3$ we have $|S| \leq 2(d+1) \leq m_3(d)$ given that $m_3(1) = 4$, $m_3(2) = 7$ and $m_3(3) = 12$ (see Table 5.1). In the second case, for $d \geq 9$ we have $(j+1)m_2(d-j) \leq (j+1)\binom{d-j+2}{2} \leq \binom{d+1}{3} \leq m_3(d)$. For $d \leq 8$, using the known values and bounds of $m_2(d)$ and $m_3(d)$, we check in the Appendix that

$$(j+1)m_2(d-j) \leq m_3(d). \tag{5.7}$$

Case 2: $\frac{t_2}{t_1} \geq K > \frac{t_3}{t_2}$. Colour a pair $\{p_1, p_2\}$ $(p_1, p_2 \in S)$ with blue if $\|p_1 - p_2\| \geq t_2$ and with red otherwise. Let B be the largest blue clique in S. Using Lemma 5.9, partition the set S into $|B|$ red cliques $R_1, \dots, R_{|B|}$.

Case 2.1: $\frac{t_3}{t_2} > 1 + \frac{2}{K}$. Let R be one of the red cliques and $R \cap B = \{b\}$. Apply Lemma 5.15 to $R \cup B$ with a sufficiently small $\beta \leq \beta'$. Let j be the smallest number such that B is (b, j, β^j)-flat. Note that $j \leq d$ and that, for any $p_1, p_2 \in R$, $b \neq b_1 \in B$, if $|p_1 - b_1| \in [t_i, t_i + \varepsilon]$ and $|p_2 - b_1| \in [t_l, t_l + \varepsilon]$, then by the triangle inequality $l = i$. By Lemma 5.15 (i) we obtain that R is $(d' - j, \alpha)$-flat. Moreover if for some $p \in R$ we have that S is (p, d, α)-flat, then R is $(p, d-j, 10\alpha)$-flat by Lemma 5.15 (ii). Further, note that the same is true for any red clique R with the same j. Indeed, since $\frac{t_3}{t_2} < K$, Lemma 5.16

implies that if β is sufficiently small, then there is a j such that B is (b_1, j, β^j)-flat for any $b_1 \in B$, but it is not $(b_1, j-1, \beta^{j-1})$-flat for any $b_1 \in B$. This and Lemma 5.10 (i) imply that for all but at most two red cliques R we have $|R| \leq m_1(d-j) = d-j+1$. Moreover, if the two potential exceptions are R_i, R_l, then $|R_i| + |R_l| \leq 2$.

Note that $|B| \leq m_2(j)$. We obtain

$$|S| \leq |R_1| + \cdots + |R_{|B|}| \leq \max\{m_2(j)(d-j+1), (m_2(j)-1)(d-j+1)+2\}.$$

Then either $d = j$ or $m_2(j)(d-j+1) \geq (m_2(j)-1)(d-j+1)+2$ holds. In the first case, we have $|S| \leq m_2(d)+1 \leq \binom{d+2}{2}+1 \leq \binom{d+1}{3} \leq m_3(d)$ if $d \geq 6$ and $|S| \leq m_2(d)+1 \leq m_3(d)$ for $1 \leq g \leq 5$ since $m_2(1) = 3$, $m_2(2) = 5$, $m_2(3) = 6$, $m_2(4) = 10$, $m_2(5) = 16$ and $m_3(1) = 4$, $m_3(2) = 7$, $m_3(3) = 12$, $m_3(4) = 16$, $m_3(5) \geq 24$ (see Table 5.1). Finally, in the second case we do the same analysis as in the end of Case 1.

Case 2.2: $\frac{t_3}{t_2} \leq 1 + \frac{2}{K}$. For a sufficiently small $\beta > 0$ let j be the lowest dimension such that B is (j, β^j)-flat. Then by Lemma 5.16 we may assume that j is the lowest dimension for any $b_1 \in B$ such that B is $({}_1b, j, \beta^j)$-flat. By Lemma 5.10 (ii), if $\frac{2}{K}$ is sufficiently small, then $|B| \leq m_1(d) = d+1$. Further, any red clique R is almost (d, α)-flat, thus either there is no $p \in R$ for which R is (p, d, α)-flat, or by Lemma 5.10 (i) we have $|R| \leq m_1(d) = d+1$. These two imply $|R| \leq \max\{2, m_1(d)\} \leq m_1(d) = d+1$. We obtain $|S| = |R_1| + \cdots + |R_{|B|}| \leq (d+1)(d+1) \leq (d+1)^2 \leq m_3(d)$ if $d \geq 7$. Indeed, it follows from $m_3(8) \geq 121 \geq (8+1)^2$, $m_3(7) \geq 65 \geq (7+1)^2$, and $m_3(d) \geq \binom{d+1}{3} \geq (d+1)^2$, where the first inequality is by 5.5. and the second is true if if $d \geq 9$. Therefore, in the rest of the proof we may assume that $d \leq 6$.

Case 2.2.1: $t_1 \geq K^{0.1}(t_3 - t_2)$**.** First we will show that in this case any red clique R is $(d-j+1, \alpha)$-flat, provided β is sufficiently small and K is sufficiently large.

Let $R \cap B = \{b\}$ and B be (b, j, β^j)-flat with respect to Λ_B. Further let $v, w \in R$ and for $b \neq b_1 \in B$ let S_2, S_3 be spheres centred at b_1 and of radii t_2 and t_3 respectively (see Figure 2). Then w is ε-close to one of them, w.l.o.g. to S_2, and v is ε-close to S_2 or S_3. Let Λ_1 be the $(d'-1)$-dimensional subspace through orthogonal to $b_1 - w$. If v is ε-close to S_2, then for some absolute constant c_1 the vector $v - w$ has an angle at most c_1/K with Λ_1. If v is ε-close to S_3, then, since $|v-w| \in [t_1, t_1+\varepsilon]$ and $t_1 \geq K^{0.1}(t_3 - t_2)$, and because the radius of S_3 is much bigger than t_1, $v - w$ has an angle at most $c_2/K^{0.1}$ with Λ_1, where c_2 is some absolute constant. Thus we can conclude that if K is sufficiently

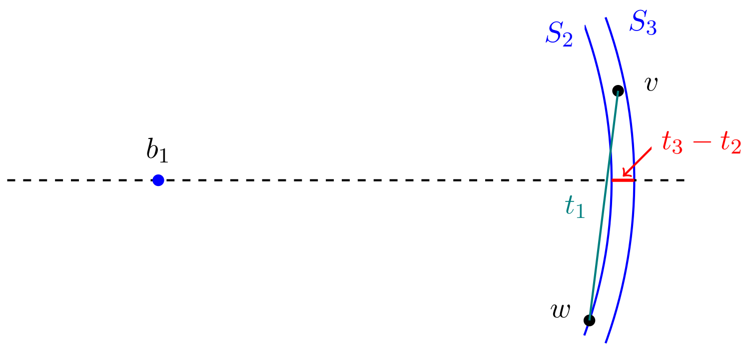

Figure 5.2

large, then $v - w$ has an angle at most β^{d+1} with Λ_1.

Since the above conclusion is true for any $b \neq b_1 \in B$, Corollary 5.13 implies that R is $(b, d' - j, 3\alpha)$-flat with respect to $\Lambda = (\Lambda_B)^{\perp}$. Moreover, if S is (r, d, α)-flat with respect to Λ_r for some $r \in R$, then Lemma 5.14 implies that R is $(r, d - j, 10\alpha)$-flat: indeed, the subspaces Λ and Λ_r are almost orthogonal. Thus either there is no $r \in R$ for which R is (r, d, α)-flat, in which case $|R| \leq 2$, or by Lemma 5.10 (i) we have $|R| \leq m_1(d - j)$. These two imply $|R| \leq \max\{2, m_1(d - j)\}$. We obtain that $|S| \leq (j + 1)(d - j + 1) \leq m_3(d)$, where the second inequality was already checked in the previous cases.

Case 2.2.2: $t_1 \leq K^{0.1}(t_3 - t_2)$. For $i = 1, \ldots, |B|$, put $\{b_i\} := B \cap R_i$ and, for $j = 2, 3$, let $S_j(i)$ be the sphere of radius t_j with centre in b_i. We need the following claim.

Claim 5.18. *Assume that for some $i \neq l \in [|B|]$ there are points from R_i in the ε-neighbourhoods of both $S_2(l)$ and $S_3(l)$. Then R_l is contained in the ε-neighbourhood of either $S_2(i)$ or $S_3(i)$.*

Proof. Assume the contrary. We may assume that $|b_i - b_l| \in [t_2, t_2 + \varepsilon]$ (the case with t_2 replaced by t_3 is treated similarly). Then there are points $v \in R_i$, $w \in R_l$ such that v is in the ε-neighbourhood of $S_3(l)$, and w is in the ε-neighbourhood of $S_3(i)$ (see Figure 3). Let v', w' denote the projections of v, w on the line e passing through b_i and b_l, and let u_i and u_l denote the points of intersection of e and spheres $S_3(l)$, $S_3(i)$ respectively. Note that $\|u_i - u_l\| \geq t_3 + (t_3 - t_2)$.

We claim that $\|u_i - v'\|, \|u_l - w'\| \leq (t_3 - t_2)/10$. This would imply that $\|w - v\| \geq \|w' - v'\| \geq \|u_i - u_l\| - \frac{2}{10}(t_3 - t_2) > t_3 + \varepsilon$, which is a contradiction. Let us only prove

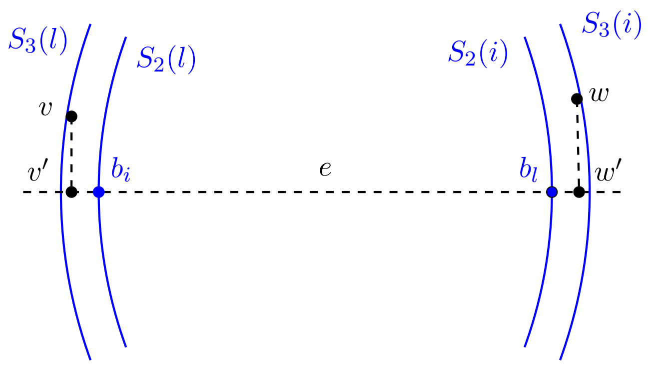

Figure 5.3

$\|u_l - w'\| \leq (t_3 - t_2)/10$, since the other inequality is proved in the same way. Due to our condition on t_2, we have $\|u_l - w\| \leq \|u_l - b_l\| + \|b_l - w\| \leq 2t_i + 2\varepsilon \leq 3K^{0.1}(t_3 - t_2)$. Since we have $t_3 - t_2 \leq 2t_3/K$, and w lies in the ε-neighbourhood of $S_3(i)$, the angle γ between $w - u_l$ and the line e satisfies $2\cos\gamma = \frac{\|u_l - w\|}{\|w - b_i\|} \leq \frac{3K^{0.1}(t_3 - t_2)}{t_3} \leq 3K^{-0.9}$. Therefore, $\|u_l - w'\| = \|u_l - w\| \cos\gamma \leq 3/K^{0.8} < (t_3 - t_2)/10$ for sufficiently large K. □

Since each of B and $R_1, \dots, R_{|B|}$ are nearly-regular simplices, the following lemma is applicable to S. Note that the third condition in its formulation is satisfied for S due to Claim 5.18. The conclusion of the lemma is that $|S| \leq m_3(d)$, which finishes the proof.

Lemma 5.19. *Let $d \leq 6$ and $S = \bigcup_{i=1}^{|B|} R_i \subseteq \mathbb{R}^{d'}$ be an almost (d, α)-flat set for which the following is true.*

- *B is the vertex set of a nearly-regular simplex of dimension j.*
- *Each R_i is the vertex set of a nearly-regular simplex (of dimension at most d) such that any edge-length in B is at least K times larger than any edge-length R_i.*
- *For every pair $b_i, b_\ell \in B$, one of R_i, R_ℓ, say R_i, lies in the ε-neighbourhood of the sphere $S_j(\ell)$ for $j = 2$ or 3, and the other (i.e., R_ℓ) lies in the ε-neighbourhood of $S_2(i) \cup S_3(i)$.*

Then $|S| \leq m_3(d)$ if α is sufficiently small.

Proof. Let Λ_B be j-flat spanned by B. Assign an ordered pair (ρ_1, ρ_2) to each ordered pair (i, ℓ), $i \neq \ell$, if R_i can be covered by the ε-neighbourhood of ρ_1 spheres out of $S_2(l), S_3(\ell)$, and R_ℓ by the ε-neighbourhood of ρ_2 spheres out of $S_2(i), S_3(i)$. By Claim 5.18 we have $(\rho_1, \rho_2) \in \{(1,1),(2,1),(1,2)\}$. If there are $m(i)$ indices $l^1, \ldots, l^{m(i)} \in B \setminus \{i\}$ such that we assigned $(1,2)$ or $(1,1)$ to (i, ℓ), then R_i is contained in the intersection of the ε-neighbourhood of $m(i)$ spheres of radii t_2 or t_3 (and having centres in $b_{\ell^1}, \ldots, b_{\ell^{m(i)}}$). Let Λ_i' denote the m_i-dimensional subspace spanned by the vectors $b_i - b_{\ell^s}$, $s = 1, \ldots, m(i)$. Corollary 5.13 implies that R_i is $(r, d' - m(i), \alpha)$-flat with respect to $\Lambda_i := (\Lambda_i')^\perp$ for each $r \in R_i$, provided ε is sufficiently small. Moreover, if for $r \in R_i$ we have that S is (r, d, α)-flat with respect to Λ_r, then Λ_i and Λ_r are almost orthogonal (if K is sufficiently large and ε sufficiently small). Hence, by Lemma 5.14 we obtain that R_i is almost $(d - m(i), 10\alpha)$-flat. Moreover, since each pair of vertices contributes at least 1 to one of $m(i)$, we remark that $\sum_{i=1}^{|B|} m(i) \geq \binom{|B|}{2} = \binom{j+1}{2}$.

Recall that S is (r, d, α)-flat for all but at most 2 of its vertices. Thus, for all but at most two (say, R_1 or R_1, R_2) sets R_i we have $|R_i| \leq d - m(i) + 1$ by Lemma 5.10 (i). If in all R_i there is an r such that R is (r, d, α)-flat, then we obtain

$$|S| = \sum_{i=1}^{|B|} |R_i| \leq (j+1)(d+1) - \sum_{i=1}^{|B|} m(i) \leq (j+1)(d+1) - \binom{j+1}{2} \leq m_3(d). \tag{5.8}$$

Otherwise, repeating the same argument for $S' := \bigcup_{i=2}^{|B|} R_i$ or for $S'' := \bigcup_{i=3}^{|B|} R_i$ and using $|R_1| \leq 2$ or $|R_1| + |R_2| \leq 2$ we obtain

$$|S| = \sum_{i=1}^{|B|} |R_i| \leq j(d+1) - \binom{j}{2} + 2 \leq m_3(d). \tag{5.9}$$

In both (5.8) and (5.9) the last inequality follows from $m_3(2) = 7$, $m_3(3) = 12$, $m_3(4) = 16$, $m_3(5) \geq 24$, $m_3(6) \geq 40$ (see Table 5.1). □

5.2.3 Fixed d: Proof of Theorem 5.6

We start with introducing the following spherical version of $N_k(d)$. Let $NS_k(d)$ denote the largest number M such that for any $\alpha, \varepsilon > 0$ there is an ε-nearly $(d-1, \alpha)$-flat k-distance set of cardinality M on a $(d-1)$-sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$. Note that $NS_k(d) \leq N_k(d)$ holds for any d.

To see that $M_k(d) = \Omega\left(k^d\right)$ and $N_k(d) = \Omega\left(k^{d-1}\right)$ consider the product of k-distance sets in $\mathbb{R}$ as in the examples in Section 5.1.1. For the lower bound it is sufficient to prove $N_k(d+1) = O\left(k^d\right)$, since any set in $\mathbb{R}^d$ is 0-flat in $\mathbb{R}^{d+1}$.

First we prove $NS_k(d+1) = O\left(k^{d-1}\right)$. We induct on d. The statement is clearly true for $d = 1$. Assuming it is true for $d' < d$, we prove it for d.

Let $\alpha, \varepsilon > 0$ be sufficiently small and T be a (d, α)-flat ε-nearly k-distance set on a sphere $\mathbb{S}^d$ in $\mathbb{R}^{d+1}$ with distances $1 \leq t_1 < \cdots < t_k$, and let $v \in T$. Define $T_i := T \cap S_\varepsilon(v, t_i)$, where $S_\varepsilon(v, t_i)$ is the ε-neighbourhood of the sphere $S(v, t_i)$ of radius t_i centred at v. Then $T = \bigcup_{i=1}^k T_i$ and each T_i is a (d, α)-flat ε-nearly k-distance set, contained in the ε-neighbourhood of the $(d-1)$-sphere $\mathbb{S}^d \cap S(v, t_i)$.

Moving each point of T_i by a distance at most ε, we obtain a $(d, 2\alpha)$-flat 3ε-nearly k-distance set T_i' on the sphere $S(v, t_i)$ with $|T_i| = |T_i'|$. If T_i' is $(p, d, 2\alpha)$-flat with respect to some d-dimensional subspace Λ_p, and Λ is the subspace containing $S(v, t_i)$, then Λ_p and Λ are almost orthogonal. Hence T_i' is $(p, d-1, 20\alpha)$-flat by Lemma 5.14. Thus $|T_i| = |T_i'| \leq NS_k(d) = O(k^{d-2})$ by induction, and overall we obtain $|T| = 1 + \sum_{i=1}^k |T_i| \leq k \cdot O(k^{d-2}) = O(k^{d-1})$.

We now turn to the proof of $N_k(d+1) = O\left(k^d\right)$. For sufficiently small ε and α, let T be a (d, α)-flat ε-nearly k-distance set in $\mathbb{R}^{d+1}$ with distances $1 \leq t_1 < \cdots < t_k$, and let $v \in T$. Define $T_i = T \cap S_\varepsilon(v, t_i)$, where where $S_\varepsilon(v, t_i)$ is the ε-neighbourhood of the sphere $S(v, t_i)$ of radius t_i centred at v. Then $T = \bigcup_{i=1}^k T_i$, and each T_i is a (d, α)-flat ε-nearly k-distance set, contained in the ε-neighbourhood of the $(d-1)$-sphere $\mathbb{S}^d \cap S(v, t_i)$. Similarly as in the first half of the proof, we obtain $|T_i| \leq NS_k(d+1) = O(k^{d-1})$ by induction, and overall we obtain $|T| = 1 + \sum_{i=1}^k |T_i| \leq k \cdot O(k^{d-1}) = O(k^d)$.

5.2.4 Many nearly-equal distances: Proof of Theorem 5.8

Let $\ell := A_k(d) + 1$ and $\alpha, \varepsilon > 0$ be fixed such that there exists no almost α-flat ε-nearly k-distance set in $\mathbb{R}^d$ of cardinality ℓ. Assume on the contrary that for every $c > 0$ and n_0 there is an $n \geq n_0$, there are k distances $t_1 < \cdots \leq t_k$ and a set $S'' \subset \mathbb{R}^d$ of n points for which

$$\left| \left\{ (p, q) \in S'' \times S'' \, : \, \|p - q\| \in [t_i, t_i + cn^{1/d}] \text{ for some } i \in [k] \right\} \right| > T(n, A_k(d)).$$

Our goal is to derive a contradiction by constructing an almost α-flat ε-nearly k-distance set of cardinality ℓ.

In the proof, we shall use a hierarchy of "small" constants given below. We say that $\mu \ll \nu$ if μ is a certain (positive, but typically quickly tending to 0) function, depending on ν

only. Thus, the arrows indicate the order of choosing the parameters from the right to the left below. (For consistency, one only needs to check that every condition we impose on a constant in the hierarchy only depends on the constants that are to the right from it and is of the form "it is sufficiently small compared to some of the constants to the right").

$$1/n_0 \ll c \ll c_1 \ll 1/C \ll 1/m \ll 1/M, \delta, c_2, \nu \ll 1/d, 1/k, \alpha, \varepsilon \tag{5.10}$$

We recommend the reader to refer to this chain of dependencies throughout the proof.

We need the following simple claim.

Claim 5.20. *For any $k \geq 0$, we have $N_k(d) < A_{k+1}(d)$.*

Proof. Take a construction S of a $(d-1,\mu)$-flat μ-nearly k-distance set in $\mathbb{R}^d$ with all distances of order at least $1/\mu$. Pick any $x \in S$, and let Λ be a $(d-1)$-dimensional subspace such that S is $(x, d-1, \mu)$-flat with respect to Λ. Let $y \in \mathbb{R}^d$ be a point at distance 1 apart from x such that $x - y$ is orthogonal to Λ. Then it is easy to see that $S \cup \{y\}$ is an almost $(d-1, 3\mu)$-flat 3μ-nearly $k+1$-distance set in $\mathbb{R}^d$. □

Using this claim, we may assume that $t_1 \geq c_2 n^{1/d}$. Indeed, assume the contrary. Since S'' is separated, a volume argument implies that, for each vertex $v \in S''$, the number of vertices in S'' at distance at most $c_2 n^{1/d}$ from v is at most $(4c_2)^d n$. Thus, removing all edges from G'' that correspond to such distances, we only remove at most $(4c_2)^d n^2$ edges. At the same time, we reduce the size of the set of possible intervals by 1. Hence we apply Theorem 5.7 with ν playing the role of ε and using the hierarchy (5.10) we obtain

$$\begin{aligned} M_k(d,n) &\leq (4c_2)^d n^2 + M_{k-1}(d,n) \\ &\leq (4c_2)^d n^2 + \frac{n^2}{2}\left(1 - \frac{1}{N_{k-1}(d)} + \nu\right) \leq \frac{n^2}{2}\left(1 - \frac{1}{A_k(d)}\right). \end{aligned}$$

We note here that in the proof of Theorem 5.7, this step of the proof is automatic since the removal of edges corresponding to small distances only change the potential value of ε.

Our first goal is to obtain a sufficiently structured subset of S''. We need the following result of Erdős.

Theorem 5.21 ([17]). *Every n-vertex graph with at least $T(n, \ell-1)+1$ edges contains an edge that is contained in $\delta n^{\ell-2}$ cliques of size ℓ, where $\delta > 0$ is a constant that depends only on ℓ.*

Consider the graph $G'' = (S'', E)$, where the set of edges consist of all pairs of points $p_1, p_2 \in S''$ that satisfy

$$\|p_1 - p_2\| \in \bigcup_{i=1}^{k} [t_i, t_i + cn^{1/d}].$$

Using the theorem above, we shall prove the following lemma.

Lemma 5.22. *For any fixed m, there exists a choice of $c_1 = c_1(m)$ such that G'' contains a complete ℓ-partite subgraph $K_{1,1,m,\ldots,m}$ with the distances between any two of its vertices strictly bigger than $c_1 n^{1/d}$.*

Proof. We construct this multipartite graph in three steps.

Step 1. Using Theorem 5.21, we find an edge $e = \{v_1, v_2\}$ that is contained in $\delta n^{\ell-2}$ cliques of size ℓ. Let E'' be the set of those edges of the ℓ-cliques, that are not incident to v_1 or v_2, and F be the set of $\ell - 2$-tuples formed by the $\ell - 2$ vertices of the cliques that are different from v_1 and v_2. The vertices of e form the first two parts of the multipartite graph. In what follows, we shall work with the graph G'' induced on $S'' \setminus \{v_1, v_2\}$ by E''.

Step 2. We select a set S_H of C vertices of G'' at random, and define a hypergraph H' on S_H as follows. Recall that $c_1 \ll 1/C \ll \delta, 1/\ell, 1/m$ (see (5.10); the exact dependency of C on δ, m and of c_1 on C shall be clear later) and consider the induced subgraph $G' := G''[S_H]$. S'' is separated, and hence a volume argument implies that any vertex in $S'' \setminus \{v_1, v_2\}$ is at distance strictly bigger than $c_1 n^{1/d}$ from all but at most $(4c_1)^d n$ vertices of $S'' \setminus \{v_1, v_2\}$. The number of vertices in $S'' \setminus \{v_1, v_2\}$ is $n - 2$, and so by the union bound the following is true.

(i) With probability at least $1 - \binom{C}{2}(4c_1)^d n/(n-2) > 1 - c_1$, every pair of vertices in S_H is at distance bigger than $c_1 n^{1/d}$ from each other.

Indeed, the total number of pairs of vertices is $\binom{C}{2}$, and for each pair the probability that it is at distance $\le c_1 n^{1/d}$ is at most $(4c_1)^d n/(n-2)$. The inequality in (i) is possible to satisfy by fixing ℓ, C and choosing c_1 to be sufficiently small.

Next, we consider the $(\ell - 2)$-uniform hypergraph $H'' = (S'' \setminus \{v_1, v_2\}, F)$. The following is an easy consequence of a Markov inequality-type argument.

(ii) With probability at least $\delta/2$, the edge density of the hypergraph $H' = H''[S_H]$ is at least $\delta/2$.

Indeed, the average density of cliques should be the same as of H'', i.e., at least δ. But if (ii) does not hold, then the average density is at most $(1-\delta/2)\cdot\delta/2+\delta/2\cdot 1 = \delta-\delta^2/4 < \delta$, a contradiction.

If we choose $c_1 < \delta/2$ then with positive probability both the property in (i) and in (ii) hold. Pick a subset $S_H \subseteq S \setminus \{v_1, v_2\}$ that satisfies both.

Step 3. We apply the following hypergraph generalisation of the Kővári–Sós–Turán due to Erdős.

Theorem 5.23 ([16])**.** *For any $\ell \geq 4$, $m \geq 1$, $\delta > 0$ there is a constant $C(\ell, m, \delta)$ such that the following holds for any $C \geq C(\ell, m, \delta)$. Any $(\ell-2)$-uniform hypergraph on C vertices of edge density at least $\frac{\delta}{2}$ contains a copy of a complete $(\ell-2)$-partite $(\ell-2)$-uniform hypergraph with parts of size m.*

Applying the theorem to the $(\ell-2)$-hypergraph H', we obtain a complete $(\ell-2)$-partite $(\ell-2)$-uniform hypergraph with parts of size m. This complete multipartite hypergraph corresponds to a complete $(\ell-2)$-partite graph in G with parts of size m and with all distances between points being at least $c_1 n^{1/d}$. Together with the edge e, this gives the desired ℓ-partite subgraph $K_{1,1,m,\ldots,m}$. □

Let the ℓ parts of the $K_{1,1,m\ldots,m}$ in G'' be $S'_1, \ldots, S'_\ell$, with $S_1 = \{v_1\}$, $S_2 = \{v_2\}$ and $|S_3| = \cdots = |S_l| = m$, further set $S' = S_1 \cup \cdots \cup S_l$. S' has much more structure than the original set S''. However, for any fixed $i, j \in [\ell]$ $(i \neq j)$ distances from several intervals from $[t_1, t_1 + cn^{1/d}], \ldots, [t_n, t_n + cn^{1/d}]$ may appear between the vertices of S'_i and S'_j. To reduce it to one interval between any two parts, we will do a second "preprocessing" step using the following version of the Kővári–Sós–Turán theorem.

Theorem 5.24 ([44])**.** *For any $\zeta > 0$ and $r \geq 1$ there exists n_0, such that for any $n \geq n_0$ we have the following. Any graph on n vertices with at least $\zeta\binom{n}{2}$ edges contains $K_{r,r}$ as a subgraph.*

Take S' and set $i := 1$. Then do the following procedure.

1. Set $j := i + 1$. If $i = 1, j = 2$, set $j := j + 1$.
2. Take the subgraph of G' induced between S'_i and S'_j. Choose an index $f = f(i, j) \in [k]$ such that

$$\left|\left\{(v_i, v_j) : v_i \in S'_i, v_j \in S'_j, |v_i - v_j| \in [t_f, t_f + cn^{1/d}]\right\}\right| \geq \frac{m^\sigma}{k},$$

where $\sigma = 1$ if $i \in \{1, 2\}$ and $\sigma = 2$ otherwise. Set G_{ij} to be the graph between S_i' and S_j' with the set of edges specified in the displayed formula above.

3. If $i \in \{1, 2\}$, let S_i'' be the set of neighbours of p_i in G_{ij}. If $i \notin \{1, 2\}$ apply Theorem 5.24 to G_{ij} and find sets $S_i'' \subset S_i'$, $S_j'' \subset S_j'$, each of size $1 \ll m' \ll m$, such that the graph G_{ij} between S_i'' and S_j'' is complete bipartite.

5. Set $S_i' := S_i''$, $S_j' := S_j''$, $m := m'$, $j := j + 1$. If $j \leq k$ then go to Step 2. If $j > k$ then set $i := i + 1$. If $i \geq k$, then terminate, otherwise go to Step 1.

Clearly, if m in the beginning of the procedure was large enough, then at the end of the procedure m is still larger than some sufficiently large M. By running a procedure similar to the one above, we can shrink the parts S_i's further such that for any $p_i \in S_i$ and $p_j, q_j \in S_j$ ($j \notin \{1, 2\}$) the angle $\angle p_j p_i q_j$ is at most α. If M is sufficiently large (see the hierarchy (5.10)), then at the end of this second procedure each S_i ($i \notin \{1, 2\}$) has at least 2 points. Thus we obtain a subset $S \subset S'$, such that $G := G''[S]$ is complete multipartite with parts $S_1, \ldots, S_\ell$ such that $|S_1| = |S_2| = 1$ and $|S_3| = \cdots = |S_\ell| = 2$, moreover for any two parts S_i, S_j there is an $f(i, j) \in [k]$ such that

$$\text{for any } p_i \in S_i,\ p_j, q_j \in S_j \text{ we have } \|p_i - p_j\| \in [t_{f(i,j)}, t_{f(i,j)} + cn^{1/d}] \text{ and } \angle p_j p_i q_j \leq \alpha. \quad (5.11)$$

For each $3 \leq i \leq \ell$ let $S_i = \{p_i, q_i\}$. We will show that $P = \frac{1}{c_2 n^{1/d}}\{p_1, \ldots, p_\ell\}$ is an almost α-flat ε-nearly k-distance set and obtain the desired contradiction.

First, we show that P is an ε-nearly k-distance set with distances $1 \leq t_1' < \cdots < t_k'$, where $t_i' = \frac{t_i}{c_2 n^{1/d}}$.

Indeed, note that $t_1 \geq c_2 n^{1/d}$, therefore $t_1' = \frac{t_1}{c_2 n^{1/d}} \geq 1$. Further, by (5.11) for any $p_i \in S_i$ and $p_j \in S_j$ we have $\|p_i - p_j\| \in [t_{f(i,j)}, t_{f(i,j)} + cn^{1/d}]$, which implies that $\frac{1}{c_2 n^{1/d}}\|p_i - p_j\| \in [t'_{f(i,j)}, t'_{f(i,j)} + c/c_2] = [t'_{f(i,j)}, t'_{f(i,j)} + \varepsilon]$.

Second, we show that P is almost α-flat. We prove that $\angle q_i p_i p_j \in [\frac{\pi}{2} - \alpha, \frac{\pi}{2} + \alpha]$ for $j \neq i$ and $i \notin \{1, 2\}$. Take the point q_i' on the line through p_i, p_j such that $\|q_i - p_j\| = \|q_i' - p_j\|$. Then, first, $\angle q_i q_i' p_j \in [(\pi - \alpha)/2, \pi/2]$ since $\angle q_i p_j p_i \leq \alpha$ and triangle $q_i q_i' p_j$ is equilateral and, second, $\|q_i' - p_i\| \leq cn^{1/d}$. Since $\|q_i - p_i\| \geq c_1 n^{1/d}$, we may assume that $\angle q_i' q_i p_i \leq \alpha/2$, thus $\angle q_i p_i p_j \in [(\pi - \alpha)/2 - \angle q_i' q_i p_i, \pi/2 + \angle q_i' q_i p_i] \subset [\pi/2 - \alpha, \pi/2 + \alpha]$. Hence for every $i \notin \{1, 2\}$ we can choose Λ_{p_i} to be the $(d-1)$-dimensional subspace orthogonal to $p_i - q_i$. This finishes the proof of Theorem 5.8.

5.2.5 Many nearly-equal distances: Proof of Theorem 5.7

First we prove the lower bound.

Let $\alpha, \varepsilon > 0$ be sufficiently small, and $t_1 > 2n^2$. Consider a $(d-1,\alpha)$-flat ε-nearly k-distance set $S' \subseteq \mathbb{R}^d$ with distances $t_1 \leq \cdots \leq t_k$ of cardinality $N_k(d)$. For each $v \in S'$ let Λ_v be a hyperplane with normal vector m_v of unit length such that for any $w \in S'$ the angle of $v - w$ and Λ_v is at most α. For simplicity assume that $N_k(d) | n$. Replace each point $v \in S'$ with an arithmetic progression $A_v = \left\{v + tm_v : t \in \{1, \ldots, \frac{n}{N_k(d)}\}\right\}$.

If $|\cos(\frac{\pi}{2} - \alpha)| < \frac{2}{n}$, then the distances between any point from A_v and any point from A_w $(v \neq w)$ is within 1/2 from the distance between v and w. The set $S = \bigcup_{v \in S'} A_v$ has cardinality n, and the graph with edges between its points that are at a distance closer than 1 to a distance in the set S' is a complete $N_k(d)$-partite graph with equal parts. By definition its number of edges is $T(n, N_k(d))$. This argument can easily be modified to deal with the case when $N_k(d) \nmid n$.

As the proof of the upper bound on $M_k(d,n)$ is very similar to those of Theorem 5.8 we only sketch it, pointing out the differences.

Let $\ell := N_k(d) + 1$ and $\alpha, \varepsilon > 0$ be fixed such that there exists no $(d-1,\alpha)$-flat ε-nearly k-distance set in $\mathbb{R}^d$ of cardinality ℓ. Assume on the contrary that for every $c > 0$ and n_0 there is an $n \geq n_0$, there are k distances $t_1 < \cdots \leq t_k$ and a set $S'' \subset \mathbb{R}^d$ of n points for which

$$\left| \left\{ (p,q) \in S'' \times S'' \,:\, \|p - q\| \in [t_i, t_i + cn^{1/d}] \text{ for some } i \in [k] \right\} \right| > T(N_k(d), n) + \gamma n^2.$$

Our goal is to derive a contradiction by constructing an a $(d-1,\alpha)$-flat ε-nearly k-distance set of cardinality ℓ.

After including γ in the hierarchy of constants on the same level as α, the proof is the same as that of (5.8) up to the point of Lemma 5.22. Instead of Lemma 5.22 we will use the following.

Lemma 5.25. *For any fixed m there exists a choice $c_1 = c_1(m,\gamma)$ such that G'' contains a complete ℓ-partite subgraph $K_{m,\ldots,m}$ such that the distance between any two of its vertices is bigger than $c_1 n^{1/d}$.*

The proof of Lemma 5.25 is very similar to the proof of Lemma 5.22, except that instead of Theorem 5.21 we use a result of Erdős and Simonovits [23] about the supersaturation of ℓ-cliques. (And then work with ℓ-uniform hypergraphs instead of $\ell - 2$.) Therefore we only give an outline of the proof.

Theorem 5.26 ([23]). *For any $\ell, \gamma > 0$ there is a δ such that if a graph G on n vertices has at least $T(n, \ell) + \gamma n^2$ edges, then it contains at least δn^ℓ cliques of size ℓ.*

Sketch of proof of Lemma 5.25. We construct this multipartite graph in three steps.

Step 1. Using Theorem 5.26, we find δn^ℓ cliques of size ℓ. Let E'' be the set of the ℓ-cliques, and F be the set of the ℓ-tuples. In what follows, we shall work with the graph G'' induced on S'' by E''.

Step 2. Select C vertices of G'' at random, where $c_1 \ll 1/C \ll \delta, 1/\ell, 1/m$. Denote by S_H the set of C vertices that we chose and consider the induced subgraph $G' := G''[S_H]$. A similar calculation as in the proof of Lemma 5.22 implies the following.

> (i) With probability at least $> 1 - c_1$, every pair of vertices in S_H is at distance bigger than $c_1 n^{1/d}$ from each other.

Next, we consider the ℓ-uniform hypergraph $H'' = (S'', F)$. As before we obtain the following.

> (ii) With probability at least $\delta/2$, the edge density of the hypergraph $H' = H''[S_H]$ is $\delta/2$.

If we choose $c_1 < \delta/2$ then with positive probability both the property in (i) and in (ii) hold. Pick a subset $S_H \subseteq S$ that satisfies both.

Step 3. Applying Theorem 5.23 to the ℓ-hypergraph H', we obtain a complete ℓ-partite ℓ-uniform hypergraph with parts of size m. This complete multipartite hypergraph corresponds to a complete ℓ-partite graph in G with parts of size m and with all distances between points being at least $c_1 n^{1/d}$. □

Let the ℓ parts of the $K_{m,\dots,m}$ in G'' be $S'_1, \dots, S'_\ell$, with $|S_1| = \dots = |S_l| = m$ further set $S' = S_1 \cup \dots \cup S_l$. Running a similar procedure as before we obtain a subset $S \subset S'$, such that $G := G''[S]$ is complete multipartite with parts $S_1, \dots, S_\ell$ with $|S_1| = \dots = |S_\ell| = 2$. Moreover, for any two parts S_i, S_j there is an $f(i,j) \in [k]$ such that

$$\text{for any } p_i \in S_i,\ p_j, q_j \in S_j \text{ we have } \|p_i - p_j\| \in [t_{f(i,j)}, t_{f(i,j)} + cn^{1/d}]$$
$$\text{and } \angle p_j p_i q_j \le \alpha.$$

For each $1 \leq i \leq \ell$ let $S_i = \{p_i, q_i\}$. Then we can show that $P = \frac{1}{c_2 n^{1/d}}\{p_1, \ldots, p_\ell\}$ is a $(d-1, \alpha)$-flat ε-nearly k-distance set, and obtain a contradiction.

5.3 Concluding remarks

Let us list some of the intriguing open problems that arose in our studies. One important step forward would be to get rid of the (almost-)flatness in the relationship between nearly k-distance sets and the quantity $M_k(d, n)$ that appears in Theorems 5.7 and 5.8. In particular, it would be desirable to prove the first equality in Conjecture 5.3 and, more generally, show the following.

Problem 5.27. *Show that* $A_k(d+1, d) = N_k(d+1) = M_k(d)$ *holds for any* k, d.

In fact, even showing the first equality would imply that the value of $M_k(d, n)$ for large n is determined *exactly* by the value of $N_k(d+1)$.

Another interesting question that looks approachable is to determine the value of $M_k(d)$ on the part of the spectrum opposite to that of Theorem 5.5: for any fixed d and k sufficiently large. Note that the order of magnitude of $M_k(d)$ in this regime is easy to find, as it is shown in Theorem 5.6.

Problem 5.28. *Determine* $M_k(d)$ *for any fixed* d *and sufficiently large* k.

If resolved, then with some effort it would most likely be possible to determine the value of $M_k(d, n)$ for large n in this regime as well.

5.A Appendix

Proof of Lemma 5.12. Indirectly, assume that for any β_0 there is a $\beta < \beta_0$ such that the angle between v and Λ is larger than α. We will show that then V is $(j-1, \beta^{j-1})$-flat with respect to $\Lambda_B \cap v^\perp$, contradicting the minimality of j. We may assume that $\|v\| = 1$. Let $\{b_1, \dots, b_d\}$ be an orthonormal basis of $\mathbb{R}^d$, where additionally $\{b_1, \dots, b_{j-1}\}$ is an (orthonormal) basis of $\Lambda_B \cap v^\perp$, $\{b_1, \dots, b_j\}$ is a basis of Λ_B, and $\{b_{j+1}, \dots, b_d\}$ is a basis of Λ.

Then v can be written as $v = y_j b_j + \dots + y_d b_d$, where $|y_j| > \sin\alpha$, since v forms an angle larger than α with Λ. Next, any $v_i \in V$ can be written as $\gamma_1 b_1 + \dots + \gamma_d b_d$, where

$$\gamma_1^2 + \dots + \gamma_j^2 \geq \cos^2(\beta^j),$$

since v_i has an angle at most β^j with Λ_B. Further, we have

$$|\langle v_i, v\rangle| = |\gamma_j y_j + \gamma_{j+1} y_{j+1} \cdots + \gamma_d y_d| \leq \beta^d,$$

since the angle of v and v_i is in $[\frac{\pi}{2} - \beta^d, \frac{\pi}{2} + \beta^d]$. By the Cauchy-Schwarz inequality we have

$$|\gamma_{j+1} y_{j+1} + \dots + \gamma_d y_d| \leq \|v\| \sqrt{\gamma_{j+1}^2 + \ldots + \gamma_d^2} \leq \sqrt{1 - \cos^2(\beta^j)} = \sin(\beta^j) \leq \beta^j.$$

Combining the previous two inequalities, we get

$$|\gamma_j y_j| \leq |\gamma_j y_j + \dots + \gamma_d y_d| + |\gamma_{j+1} y_{j+1} \cdots + \gamma_d y_d| \leq 2\beta^j.$$

This implies $|\gamma_j| = |\gamma_j y_j| / |y_j| \leq 2\beta^j / \sin\alpha < \beta^{j-0.5}$, if $\beta < \frac{\sin^2\alpha}{4}$ and thus if β is sufficiently small, then

$$|\gamma_1|^2 + \dots + |\gamma_{j-1}|^2 \geq \cos^2(\beta^j) - \beta^{2j-1} \geq \cos^2(\beta^{j-1}),$$

where the last inequality follows from the fact that $\cos\gamma = 1 - (\frac{1}{2} + o(1))\gamma^2$ for small γ. This means that the angle between v_i and $\Lambda_B \cap v^\perp$ is at most β^{j-1}. Since this is valid for any $i = 1, \dots, m$, we conclude that V is $(j-1, \beta^{j-1})$-flat with respect to $\Lambda_B \cap v^\perp$, a contradiction. □

Proof of Lemma 5.14. We may assume that $\|v\| = 1$, and will use the notation from the definition above. First note that the length of the projection of v on $\Lambda_2^\perp$ is at most $\sin\alpha \leq \alpha$.

Next we prove that the length of the projection of v on the subspace spanned by $\{v_1, \ldots, v_a\}$ is at most 2α. Indeed, if it is of length larger than 2α then, using the fact that this projection forms an angle in $[\pi/2 - 0.01, \pi/2 + 0.01]$ with Λ_2, we get that the projection of v on $(\Lambda_2)^{\perp}$ has length larger than α, and thus the angle that v forms with Λ_2 is larger than α, contradicting the first observation.

Noting further that the length of the projection of v on $\Lambda_1^{\perp}$ is at most $\sin\alpha \le \alpha$, we obtain that the projection of v on $\Lambda_1 \cap \Lambda_2^{\perp}$ has length at most $2\alpha + \alpha = 3\alpha$. Indeed, this follows since $\Lambda_1 \cap \Lambda_2^{\perp}$ is the subspace spanned by the union of $\{v_1, \ldots, v_a\}$ and $\Lambda_1^{\perp}$, further the subspace spanned by $\{v_1, \ldots, v_a\}$ and $\Lambda_1^{\perp}$ are orthogonal. We conclude that v forms an angle at most $\arcsin(3\alpha) < 10\alpha$ with $\Lambda_1 \cap \Lambda_2$. □

Proof of Lemma 5.16. Let S be (p, j, α)-flat with respect to Λ and let $q \in S$ $(q \ne p)$. We will show that for any $r \in S$ $(r \ne p, q)$ there is a vector $v \in \Lambda$ such that the angle between $q - r$ and v is at most $20(K\alpha)^{1/2}$, implying that S is $(q, j, 20(K\alpha)^{1/2})$-flat.

Let $v_q, v_r \in \Lambda$ such that the angle between $p - q$ and v_q and the angle between $p - r$ and v_r is 2α, further $\|p - q\| = \|v_q\|$ and $\|p - r\| = \|v_r\|$. We will show that the angle β between $v = v_q - v_r \in \Lambda$ and $q - r$ is at most $20(K\alpha)^{1/2}$, which finishes the proof.

Let $\beta_1 = \angle qrq'$ and $\beta_2 = \angle rq'r'$. Then $\beta \le \beta_1 + \beta_2$ thus it is sufficient to show that $\beta_1, \beta_2 \le 10(K\alpha)^{1/2}$. We will prove it for β_2, for β_1 it can be done similarly. By the low of cosines we have

$$\cos\beta_2 = \frac{\|q' - r\|^2 + \|q' - r'\|^2 - \|r - r^2\|}{2\|q' - r\|\|q' - r'\|}.$$

Using $\|q - r\| - \|q - q'\| \le \|q' - r\| \le \|q - r\| + \|q - q'\|$, $\|q - r\| - \|q - q'\| - \|r - r'\| \le \|q' - r'\| \le \|q - r\| + \|q - q'\| + \|r - r'\|$, $\|q - q'\| = 2\sin\alpha\|p - q\| \le 2\alpha K\|q - r\|$, $\|r - r'\| = 2\sin\alpha\|p - r\| \le 2\alpha K\|q - r\|$, and denoting $\|q - r\| = z$, these imply

$$1 - \cos\beta_2 \le 1 - \frac{2(z - 4\alpha Kz)^2 - 4(\alpha Kz)^2}{2(z + 4\alpha Kz)^2} \le 25\alpha K.$$

Combining with $\frac{\beta_2^2}{4} \le 1 - \cos\beta_2$ we obtain $\beta_2 \le 10(\alpha K)^{1/2}$. □

Proof of (5.7). Using the known values of $m_2(d)$ and bounds on $m_3(d)$, we obtain the following.

$d = 8$: $\max\{(j+1)m_2(d-j) : j = 0, \ldots, 8\} = \max\{45, 2\cdot 29, 3\cdot 27, 4\cdot 16, 5\cdot 10, 6\cdot 6, 7\cdot 5, 8\cdot 3, 9\cdot 1\} = 81 \le 121 \le m_3(8)$;

$d=7$: $\max\{(j+1)m_2(d-j) : j=0,\dots,7\} = \max\{29, 2\cdot 27, 3\cdot 16, 4\cdot 10, 5\cdot 6, 6\cdot 5, 7\cdot 3, 8\cdot 1\} = 54 \le 65 \le m_3(7)$;

$d=6$: $\max\{(j+1)m_2(d-j) : j=0,\dots,6\} = \max\{27, 2\cdot 16, 3\cdot 10, 4\cdot 6, 5\cdot 5, 6\cdot 3, 7\cdot 1\} = 32 \le 40 \le m_3(6)$;

$d=5$: $\max\{(j+1)m_2(d-j) : j=0,\dots,5\} = \max\{16, 2\cdot 10, 3\cdot 6, 4\cdot 6, 5\cdot 3, 6\cdot 1\} = 24 \le m_3(5)$;

$d=4$: $\max\{(j+1)m_2(d-j) : j=0,\dots,4\} = \max\{10, 2\cdot 6, 3\cdot 5, 4\cdot 3, 5\cdot 1\} = 15 \le 16 \le m_3(4)$;

$d=3$: $\max\{(j+1)m_2(d-j) : j=0,\dots,3\} = \max\{6, 2\cdot 5, 3\cdot 3, 4\cdot 1\} = 10 \le 12 \le m_3(3)$;

$d=2$: $\max\{(j+1)m_2(d-j) : j=0,1,2\} = \max\{5, 2\cdot 3, 3\cdot 1\} = 6 \le 7 \le m_3(2)$;

$d=1$: $\max\{(j+1)m_2(d-j) : j=0,1\} = \max\{3, 2\cdot 1\} = 3 \le 4 \le m_3(1)$. □

Chapter 6

Equilateral sets

6.1 Introduction

In a normed space $(X, \|\cdot\|)$ a set $S \subseteq X$ is called *c-equilateral* if $\|x-y\| = c$ for all distinct $x, y \in S$. S is called *equilateral* if it is c-equilateral for some $c > 0$. The *equilateral number* $e(X)$ of X is the cardinality of the largest equilateral set of X.

The norm $\|\cdot\|_\infty$ of $x \in \mathbb{R}^d$ is defined as $\|x\|_\infty = \max_{1\le i\le d} |x_i|$, and ℓ_∞^d denotes the normed space $(\mathbb{R}^d, \|\cdot\|_\infty)$.

We prove lower bounds on the equilateral number of subspaces of ℓ_∞^d of small codimension.

Theorem 6.1. *Let X be a $(d-k)$-dimensional subspace of ℓ_∞^d. Then*

$$e(X) \ge \frac{2^{d-k}}{(d-k)^k}, \tag{6.1}$$

$$e(X) \ge 1 + \frac{1}{2^{k-1}} \sum_{r=1}^{\ell} \binom{d-k\ell}{r} \text{ for every } 1 \le \ell \le d/(k+1), \text{ and} \tag{6.2}$$

$$e(X) \ge 1 + \sum_{r=1}^{\ell} \binom{d-2k\ell}{r} \text{ for every } 1 \le \ell \le d/(2k+1). \tag{6.3}$$

According to Petty's conjecture [55], for every normed space X of dimension d we have $e(X) \ge d+1$. As a corollary of inequality (6.3) we confirm the conjecture, if the unit ball of X is a polytope with few facets.

Corollary 6.2. *Let P be an origin-symmetric convex polytope in $\mathbb{R}^n$ having at most $\frac{4n}{3} - \frac{1+\sqrt{n+9}}{6} = \frac{4n}{3} - o(n)$ opposite pairs of facets. If X is a n-dimensional normed space with P as a unit ball, then $e(X) \geq n+1$.*

For two d-dimensional normed spaces X,Y we denote by $d_{BM}(X,Y) = \inf_T\{\|T\|\|T^{-1}\|\}$ their *Banach-Mazur distance*, where the infimum is taken over all linear isomorphisms $T : X \to Y$. We prove the following.

Theorem 6.3. *Let X be an $(d-k)$-dimensional subspace of ℓ_∞^d, and Y be an $(d-k)$-dimensional normed space such that $d_{BM}(X,Y) \leq 1 + \frac{\ell}{2(d-2k-\ell k-1)}$ for some integer $1 \leq \ell \leq \frac{d-2k}{k}$. Then $e(Y) \geq d - k(2+\ell)$.*

6.2 Norms with polytopal unit ball and small codimension

We recall the following well known fact to prove Corollary 6.2. (For a proof, see for example [4].)

Lemma 6.4. *Any centrally symmetric convex n-polytope with $f \geq n$ opposite pairs of facets is a n-dimensional section of the f-dimensional cube.*

Proof of Corollary 6.2. By Lemma 6.4, P can be obtained as an n-dimensional section of the $\left(\frac{4n}{3} - \frac{1+\sqrt{8n+9}}{6}\right)$-dimensional cube. Choose $d = \frac{4n}{3} - \frac{1+\sqrt{8n+9}}{6}$, $\ell = 2$, $k = \frac{n}{3} - \frac{n+\sqrt{8n+9}}{6}$, and apply inequality (6.3) from Theorem 6.1. This yields $e(X) \geq n+1$. $\square$

To confirm Petty's conjecture for subspaces of ℓ_∞^d of codimension 2 and 3 when $d \geq 9$ and respectively $d \geq 15$, apply inequality (6.2) from Theorem 6.1 with $\ell = 2$.

6.3 Large equilateral sets

Notation

In this chapter, we denote the i-th coordinate of a vector $\mathbf{a} \in \mathbb{R}^d$ by a^i. We treat vectors by default as column vectors. We denote by $2^{[d]}$ the set of all subsets of $[d]$, and by $\binom{S}{\leq m}$ the set of all non-empty subsets of S of cardinality at most m. $\mathbf{0}$ denotes the vector $(0,\ldots,0) \in \mathbb{R}^d$. For two vectors $\mathbf{a}$ and $\mathbf{b}$, let $\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^d a^i b^i$ be their scalar product.

Idea of the constructions

For two vectors $\mathbf{x}, \mathbf{y} \in X$ we have $\|\mathbf{x} - \mathbf{y}\|_\infty = c$ if and only if the following hold.

$$\text{There is an } 1 \leq i \leq d \text{ such that } |x^i - y^i| = c, \text{ and} \tag{6.4}$$

$$|x^i - y^i| \leq c \text{ for all } 1 \leq i \leq d. \tag{6.5}$$

In our constructions of c-equilateral sets $S \subseteq X$, we split the index set $[d]$ of the coordinates into two parts $[d] = N_1 \cup N_2$. In the first part N_1, we choose all the coordinates from the set $\{0, 1, -1\}$, so that for each pair from S there will be an index in N_1 for which (6.4) holds with $c = 1$ or 2, and (6.5) is not violated by any index in N_1. We use N_2 to ensure that all of the points we choose are indeed in the subspace X. For each vector, this will lead to a system of linear equations. The main difficulty will be to choose the values of the coordinates in N_1 so that the coordinates in N_2, obtained as a solution to those systems of linear equations, do not violate (6.5).

Proof of Theorem 6.1

For vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^k$ let $B(\mathbf{v}_1, \ldots, \mathbf{v}_k) \in \mathbb{R}^{k \times k}$ be the matrix whose i-th column is $\mathbf{v}_i$. For a matrix $B \in \mathbb{R}^{k \times k}$, a vector $\mathbf{v} \in \mathbb{R}^k$ and an index $i \in [k]$, we denote by $B(i, \mathbf{v})$ the matrix obtained from B by replacing its i-th column by $\mathbf{v}$.

Let $\{\mathbf{a}_i : 1 \leq i \leq k\}$ be a set of k linearly independent vectors in $\mathbb{R}^d$ spanning $X^\perp$. That is, $\mathbf{x} \in X$ if and only if $\mathbf{a}_i \cdot \mathbf{x} = 0$ for all $1 \leq i \leq k$. Further, let $A \in \mathbb{R}^{k \times d}$ be the matrix whose i-th row is $\mathbf{a}_i^T$, and let $\mathbf{b}_j = (a_1^j, \ldots, a_k^j)$ be the j-th column of A. For $I \subseteq [d]$ and for $\boldsymbol{\sigma} \in \{\pm 1\}^d$ let $\mathbf{b}_I = \sum_{i \in I} \mathbf{b}_i$ and $\mathbf{b}_{I,\boldsymbol{\sigma}} = \sum_{i \in I} \sigma^i \mathbf{b}_i$.

Proof of (6.1). We will construct a 2-equilateral set of cardinality $\frac{2^{d-k}}{(d-k)^k}$. Let $B = B(\mathbf{b}_{d-k+1}, \mathbf{b}_{d-k+2}, \ldots, \mathbf{b}_d)$. We may assume without loss of generality that $|\mathrm{det} B| \geq |\mathrm{det} B(\mathbf{b}_{i_1}, \ldots, \mathbf{b}_{i_k})|$ for all possible choices of $i_1, \ldots, i_k \in [d]$. The vectors $\{\mathbf{a}_i : i \in [k]\}$ are linearly independent, hence $\mathrm{det} B \neq 0$. The first part of the indices (N_1) now will be $[d-k]$, and for these indices we choose coordinates from the set $\{1, -1\}$. For $J \subseteq [d-k]$ we define the first $n-k$ coordinates of the vector $\mathbf{w}(J) \in \mathbb{R}^d$ as

$$w(J)^i = \begin{cases} 1 & \text{if } i \in J \\ -1 & \text{if } i \in [d-k] \setminus J. \end{cases}$$

To ensure that $\mathbf{w}(J) \in X$ we must have $A\mathbf{w}(J) = \mathbf{0}$. This means $(w(J)^{d-k+1}, \ldots, w(J)^d)$ is a solution of

$$B\mathbf{x} = \mathbf{b}_{[d-k]\setminus J} - \mathbf{b}_J. \tag{6.6}$$

By Cramer's rule $\mathbf{x} = (x^1, \ldots, x^k)$ with

$$x^i = \frac{\det B(i, \mathbf{b}_{[d-k]\setminus J} - \mathbf{b}_J)}{\det B}$$

is a solution of (6.6). Thus we obtain that $\mathbf{w}(J)$, defined by

$$w(J)^i = \begin{cases} 1 & \text{if } i \in J \\ -1 & \text{if } i \in [d-k] \setminus J \\ \frac{\det B(i-d+k, \mathbf{b}_{[d-k]\setminus J} - \mathbf{b}_J)}{\det B} & \text{if } i \in [d] \setminus [d-k], \end{cases}$$

is in X. By the multilinearity of the determinant we have

$$\det B(i-d+k, \mathbf{b}_{[d-k]\setminus J} - \mathbf{b}_J) = \sum_{j \in [d-k]\setminus J} \det B\left(i-d+k, \mathbf{b}_j\right) - \sum_{j \in J} \det B\left(i-d+k, \mathbf{b}_j\right).$$

Thus by the maximality of $|\det B|$ and by the triangle inequality:

$$\left|\det B(i-d+k, \mathbf{b}_{[d-k]\setminus J} - \mathbf{b}_J)\right| \leq (d-k)|\det B|.$$

This implies that for each J and $i \in [d] \setminus [d-k]$ we have $-(d-k) \leq w(J)^i \leq d-k$.

Consider the set $W = \left\{\mathbf{w}(J) : J \in 2^{[d-k]}\right\}$. W is not necessarily 2-equilateral, because for $J_1, J_2 \in 2^{[d-k]}$ and for $i \in [d] \setminus [d-k]$ we only have that $|w(J_1)^i - w(J_2)^i| \leq 2(d-k)$. However, we can find a 2-equilateral subset of W that has large cardinality, as follows.

First we split W into $d-k$ parts such that if $w(J_1)$ and $w(J_2)$ are in the same part, then $\left|w(J_1)^{d-k+1} - w(J_2)^{d-k+1}\right| \leq 2$, and keep the largest part. Then we split the part we kept into two parts again similarly, based on $w(J)^{d-k+2}$, and keep the largest part. We continue in the same manner for $w(J)^{d-k+3}, \ldots, w(J)^d$.

More formally, for each vector $\mathbf{s} \in \{-(d-k), (-d-k)+2, \ldots, d-k-2\}^k = T^k$ let $W(\mathbf{s})$ be the set of those vectors $\mathbf{w}(J)$ for which

$$w(J)^{d-k+i} \in [s^i, s^i + 2] \text{ for every } i \in k.$$

We have $W \subseteq \bigcup_{\mathbf{s} \in T^k} W(\mathbf{s})$, and hence there is an $\mathbf{s}$ for which $|W(\mathbf{s})| \geq \frac{2^{d-k}}{(d-k)^k}$.

It is not hard to check that $W(\mathbf{s})$ is 2-equilateral. Indeed, for every $J_1, J_2 \in W(\mathbf{s})$, we have $|w^i(J_1) - w^i(J_2)| \leq 2$ for $i \in [d] \setminus [d-k]$ by the definition of $W(\mathbf{s})$, and for $i \in [d-k]$ by the definition of $\mathbf{w}(J)$. Further, by the definition of $\mathbf{w}(J)$ there is an index $j \in [d-k]$ for which $\{w(J_1)^j, w(J_2)^j\} = \{1, -1\}$ (assuming $J_1 \neq J_2$). $\square$

Proof of (6.2). Fix some $1 \leq \ell \leq d/(k+1)$. We will construct a 1-equilateral set of cardinality $\frac{1}{2^{k-1}} \sum_{1\leq r\leq \ell} \binom{d-k\ell}{r} + 1$. Let $I_1, \ldots, I_k \subseteq \binom{[d]}{\leq \ell}$ and $\boldsymbol{\sigma} \in \{\pm 1\}^d$ be such that the determinant of $B = B(\mathbf{b}_{I_1,\boldsymbol{\sigma}}, \ldots, \mathbf{b}_{I_k,\boldsymbol{\sigma}})$, is maximal among all possible choices of k disjoint $I_1, \cdots I_k \binom{[d]}{\leq \ell}$ and $\sigma \in \{\pm 1\}^d$. Note that $\det B > 0$ since the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_k$ are linearly independent. Let $I = \bigcup_{i\in[k]} I_i$ and $|I| = m$. By re-ordering the coordinates, we may assume that $I = [d] \setminus [n-m]$.

The first part of the indices now will be $[d-m]$, and for these indices we choose all the coordinates from the set $\{-1, 0, 1\}$. For a set $J \in \binom{[d-m]}{\leq \ell}$ we define the first $d-m$ coordinates of the vector $\mathbf{w}(J) \in \mathbb{R}^d$ as

$$w(J)^i = \begin{cases} -\sigma^i & \text{if } i \in J \\ 0 & \text{if } i \in [d-m] \setminus J. \end{cases}$$

To ensure that $\mathbf{w}(J) \in X$ we must have $A\mathbf{w}(J) = \mathbf{0}$. That means $(w(J)^{d-m+1}, \ldots, w(J)^d)$ has to be a solution of

$$B(\mathbf{b}_{d-m+1}, \mathbf{b}_{d-m+2}, \ldots, \mathbf{b}_d)\mathbf{x} = \mathbf{b}_{J,\sigma}. \tag{6.7}$$

We will find a solution of (6.7) of a specific form, where for each $j \in [k]$, if $i_1, i_2 \in I_j$, then $\sigma^{i_1} x^{i_1} = \sigma^{i_2} x^{i_2}$. For this, let $\mathbf{y} = (y^1, y^2, \ldots, y^k)$ be a solution of

$$B\mathbf{y} = \mathbf{b}_{J,\sigma},$$

and for each $j \in [k]$ and $i \in I_j$ let $x^i = \sigma^i y^j$. Then $(x^{d-m+1}, \ldots, x^d)$ is a solution of (6.7), and by Cramer's rule we have $y^j = \frac{\det B(j, \mathbf{b}_{J,\sigma})}{\det B}$. Thus we obtained that $\mathbf{w}(J)$, defined as

$$w(J)^i = \begin{cases} -\sigma^i & \text{if } i \in J \\ 0 & \text{if } i \in [d-m] \setminus J \\ \frac{\sigma^i \det B(j, \mathbf{b}_{J,\sigma})}{\det B} & \text{if } i \in I_j \text{ for some } j \in [k], \end{cases}$$

is in X. Note that $B(j, \mathbf{b}_{J,\sigma}) = B(\mathbf{b}_{J_1,\sigma}, \ldots, \mathbf{b}_{J_k,\sigma})$ for some disjoint sets $J_1, \ldots, J_k$, hence by the maximality of $\det B$ we have

$$|w(J)^i| \leq 1 \text{ for each } 1 \leq i \leq d. \tag{6.8}$$

Consider the set $W = \left\{\mathbf{w}(J) : J \in \binom{[d-m]}{\leq \ell}\right\}$. W is not a 1-equilateral set, because for $J_1, J_2 \in \binom{[d-m]}{\leq \ell}$ and for some $i_1 \in I_1 \cup \cdots \cup I_{k-1}$ we only know that $w(J_1)^i, w(J_2)^i \in [-1, 1]$, and thus $|w(J_1)^i - w(J_2)^i| \leq 2$. However we can find a 1-equilateral subset of W that has large cardinality.

First note that we may assume that for any $j \in [d-m]$ we have $\det B(k, \sigma^j \mathbf{b}_j) \geq 0$. Indeed, we can ensure this by changing the first $d-m$ coordinates of $\boldsymbol{\sigma}$ if necessary.[1] This we may do, since in the definition of B we only used the last m coordinates of $\boldsymbol{\sigma}$. Together with the multilinearity of the determinant, this implies that for for $i \in I_k$ we have

$$\sigma^i w(J)^i = \frac{\det B(k, \mathbf{b}_{J,\sigma})}{\det B} = \frac{\det B(k, \sum_{j\in J} \sigma^j \mathbf{b}_j)}{\det B} = \frac{\sum_{j \in J} \det B(k, \sigma^j \mathbf{b}_j)}{\det B} \geq 0. \tag{6.9}$$

Next we split W into two parts such that if $\mathbf{w}(J_1)$ and $\mathbf{w}(J_2)$ are in the same part, then for $i \in I_1$, $\mathbf{w}(J_1)^i$ and $\mathbf{w}(J_2)^i$ have the same sign, and we keep the largest part. Then we split that part into two parts again similarly, based on I_2, and keep the largest part. We continue in the same manner for $I_3, \dots, I_{k-1}$.

More formally, for each vector $\mathbf{s} \in \{\pm 1\}^{k-1}$ let $W(\mathbf{s}) \subseteq W$ be the set of those vectors $\mathbf{w}(J) \in W$ for which

$$s^j w(J)^i \sigma^i \geq 0 \text{ for each } i \in I_1 \cup \dots \cup I_{k-1}, \text{ where } j \in [k-1] \text{ is such that } i \in I_j.$$

Then $\bigcup_{\mathbf{s}\in\{\pm1\}^{k-1}} W(\mathbf{s})$ is a partition of W, hence there is an $\mathbf{s}$ for which $|W(\mathbf{s})| \geq \frac{1}{2^{k-1}}|W| = \frac{1}{2^{k-1}} \sum_{1\leq r \leq \ell} \binom{d-m}{r} \geq \frac{1}{2^{k-1}} \sum_{1 \leq r \leq \ell} \binom{d-k\ell}{r}$. $W(\mathbf{s})$ is a 1-equilateral set, because for any two vectors $\mathbf{w}_1, \mathbf{w}_2 \in W(\mathbf{s})$, there is an index $i \in [d-m]$ for which either $\{w_1^i, w_2^i\} = \{0, -1\}$ or $\{w_1^i, w_2^i\} = \{0, 1\}$, and for all $i \in [d]$ we have $|w_1^i - w_2^i| \leq 1$ by (6.8), by the definition of $W(\mathbf{s})$ and by (6.9). Finally, it is not hard to see that we can add $\mathbf{0}$ to $W(\mathbf{s})$. Thus $W(\mathbf{s}) \cup \{\mathbf{0}\}$ is a 1-equilateral set of the promised cardinality. □

Proof of (6.3). Fix some $1 \leq \ell \leq d/(2k+1)$ and let $N = d - 2k\ell$. We will construct a 1-equilateral set of cardinality $\sum_{1\leq r\leq \ell} \binom{N}{r} + 1$. For $1 \leq i \leq 2\ell$ let

$$U_i = (i-1)k + [k] = \{(i-1)k+1, (i-1)k+2, \dots, ik\}$$

and

$$B_i = B(\mathbf{b}_{N+(i-1)k+1}, \mathbf{b}_{N+(i-1)k+2} \dots, \mathbf{b}_{N+ik}).$$

By working from 2ℓ down to 1, we may assume without loss of generality that for $1 \leq i \leq 2\ell$

$$|\det B_i| \geq |\det B_i(j, \mathbf{b}_r)| \text{ for all } j \in [k] \text{ and } r \leq N + (i-1)k. \tag{6.10}$$

Assume now that

$$|\det B_i| > 0 \text{ for all } 1 \leq i \leq 2\ell. \tag{6.11}$$

[1] This is the only reason why we took the maximum also over σ at the beginning of the proof.

We will handle the case when this assumption does not hold at the end of the proof.

The first part of the indices now will be $[N]$. We will obtain vectors (denoted by $\mathbf{y}(J)$) whose coordinates corresponding to the first part are from the set $\{0,-1\}$, and whose coordinates from the second part have absolute value at most $\frac{1}{2}$. We do not construct them directly, but as the sum of some other vectors $\mathbf{w}(J,i), \mathbf{z}(J,i) \in X$, whose coordinates in the first part are from $\{0, -\frac{1}{2}\}$.

For a set $J = \{j_1, \ldots, j_{|J|}\} \in \binom{[N]}{\leq \ell}$ with $j_1 < \cdots < j_{|J|}$, and for $1 \leq i \leq |J|$ let us define the first N coordinates of $\mathbf{w}(J,i) \in \mathbb{R}^d$ and $\mathbf{z}(J,i) \in \mathbb{R}^d$ as

$$w(J,i)^j = z(J,i)^j = \begin{cases} -\frac{1}{2} & \text{if } j = j_i \\ 0 & \text{if } j \in [N] \setminus \{j_i\}. \end{cases}$$

To ensure that $\mathbf{w}(J,i)$ and $\mathbf{z}(J,i)$ are in X, we must have $A\mathbf{w}(J,i) = A\mathbf{z}(J,i) = \mathbf{0}$. Hence both $\big(w(J,i)^{N+1}, w(J,i)^{N+2} \ldots, w(J,i)^d\big)$ and $\big(z(J,i)^{N+1}, z(J,i)^{N+2} \ldots, z(J,i)^d\big)$ are solutions of

$$B\mathbf{x} = \frac{1}{2}\mathbf{b}_{j_i}, \tag{6.12}$$

where $B = (\mathbf{b}_{N+1}, \mathbf{b}_{N+2}, \ldots, \mathbf{b}_d)$.

By Cramer's rule we have that $\mathbf{x} = (x^1, x^2 \ldots, x^{2k\ell})$ with

$$x^j = \begin{cases} 0 & \text{if } j \in [2k\ell] \setminus U_{2i} \\ \frac{\det B_{2i}\left(j-(2i-1)k, \frac{1}{2}\mathbf{b}_{j_i}\right)}{\det B_{2i}} & \text{if } j \in U_{2i} \end{cases}$$

is a solution of (6.12).

We obtain that $\mathbf{w}(J,i)$ defined as

$$w(J,i)^j = \begin{cases} -\frac{1}{2} & \text{if } j = j_i \\ 0 & \text{if } j \in [d] \setminus (\{j_i\} \cup (N + U_{2i})) \\ \frac{\det B_{2i}\left(j-N-(2i-1)k, \frac{1}{2}\mathbf{b}_{j_i}\right)}{\det B_{2i}} & \text{if } j \in N + U_{2i} \end{cases}$$

is in X.

Similarly, by Cramer's rule we have that $\mathbf{x} = (x^1, x^2 \ldots, x^{2k\ell})$ with

$$x^j = \begin{cases} 0 & \text{if } j \in [2k\ell] \setminus U_{2i-1} \\ \frac{\det B_{2i-1}\left(j-(2i-2)k, \frac{1}{2}\mathbf{b}_{j_i}\right)}{\det B_{2i-1}} & \text{if } j \in U_{2i-1} \end{cases}$$

is a solution of (6.12).

We obtain that $\mathbf{z}(J,i)$ defined as

$$z(J,i)^j = \begin{cases} -\frac{1}{2} & \text{if } j = j_i \\ 0 & \text{if } j \in [d] \setminus (\{j_i\} \cup (N + U_{2i-1})) \\ \frac{\det B_{2i-1}\left(j-N-(2i-2)k, \frac{1}{2}\mathbf{b}_{j_i}\right)}{\det B_{2i-1}} & \text{if } j \in N + U_{2i-1} \end{cases}$$

is in X.

Therefore $\mathbf{y}(J) = \sum_{1 \le i \le |J|} (\mathbf{w}(J,i) + \mathbf{z}(J,i)) \in X$. Note that by assumption (6.10) and by the multilinearity of the determinant we have $|w(J,i)^j|, |z(J,i)^j| \le \frac{1}{2}$ for all $1 \le j \le n$. It is not hard to check that by the construction we have

$$\begin{aligned} y(J)^i &= -1 && \text{if } i \in J, \\ y(J)^i &= 0 && \text{if } i \in [N] \setminus J, \\ |y(J)^i| &\le \frac{1}{2} && \text{if } i \in [d] \setminus [N]. \end{aligned}$$

Thus, for any two distinct $J_1, J_2 \in \binom{[N]}{\le \ell}$, there is an $i \in [N]$ with $\{y(J_1)^i, y(J_2)^i\} = \{0, -1\}$, and for all $1 \le i \le n$ we have $|y(J_1)^i - y(J_2)^i| \le 1$. This means $||\mathbf{y}(J_1) - \mathbf{y}(J_2)||_\infty = 1$, and $\left\{\mathbf{y}(J) : J \in \binom{[N]}{\le \ell}\right\} \cup \{\mathbf{0}\}$ is a 1-equilateral set of cardinality $\sum_{1 \le r \le \ell} \binom{N}{r} + 1$.

To finish the proof it is only left to handle the case when assumption (6.11) does not hold. For $S = \{s_1, \dots, s_r\} \subseteq [d]$ with $s_1 < \dots < s_r$ and $T = \{t_1, \dots, t_m\} \subseteq [k]$ with $t_1 < \dots < t_m$ let

$$B(S,T) = \begin{pmatrix} b_{s_1}^{t_1} & \dots & b_{s_r}^{t_1} \\ \vdots & \ddots & \vdots \\ b_{s_1}^{t_m} & \dots & b_{s_r}^{t_m} \end{pmatrix} \in \mathbb{R}^{r \times m}.$$

We recursively define $m_i \in \mathbb{N}$, $B_i \in \mathbb{R}^{m_i \times m_i}$ for $i \in [2\ell] \cup \{0\}$, and $M_i \in \mathbb{N}$ for $i \in [2\ell]$ as follows. Let $m_1 = k$, $M_0 = 0$, $M_1 = m_1$ and $B_1 = B([d] \setminus [d - m_1], [k])$. By changing the order of A, we may assume that

$$|\det B_1| \ge |\det B(S, [k])| \text{ for all } S \in \binom{[d]}{m_1}. \tag{6.13}$$

Assume now that we have already defined m_{i-1}, M_{i-1} and B_{i-1}. If $m_{i-1} > 0$, then let $m_i = \operatorname{rank} B([d - M_{i-1}], [k])$, otherwise let $m_i = 0$. If $m_i > 0$, then let $S_i \subseteq \binom{[k]}{m_i}$ such that $\operatorname{rank} B([d - M_{i-1}], S_i) = m_i$, and let $B_i = B([d - M_{i-1}] \setminus [d - M_i], S_i)$. Further, let $M_i = M_{i-1} + m_i = \sum_{r \le i} m_r$. If $m_i > 0$, then again, by re-indexing the first $d - M_{i-1}$ columns of A, we may assume that

$$|\det B_i| \ge |\det B(S, S_i)| \text{ for all } S \subseteq \binom{[d - M_{i-1}]}{m_i}. \tag{6.14}$$

Finally define $\mathbf{b}_j(i) = B(\{j\}, S_i) \in \mathbb{R}^{m_i}$.

We now redefine U_i as

$$U_i = [d - M_{i-1}] \setminus [d - M_i],$$

and redefine $\mathbf{w}(J,i)$ and $\mathbf{z}(J,i)$ as

$$w(J,i)^j = \begin{cases} -\frac{1}{2} & \text{if } j = j_i \\ 0 & \text{if } j \in [d] \setminus (\{j_i\} \cup U_{2i}) \\ \frac{\det B_{2i}\left(j-n+M_{2i}, \frac{1}{2}\mathbf{b}_{j_i}(2i)\right)}{\det B_{2i}} & \text{if } j \in U_{2i}, \end{cases}$$

and

$$z(J,i)^j = \begin{cases} -\frac{1}{2} & \text{if } j = j_i \\ 0 & \text{if } j \in [d] \setminus (\{j_i\} \cup U_{2i-1}) \\ \frac{\det B_{2i-1}\left(j-n+M_{2i-1}, \frac{1}{2}\mathbf{b}_{j_i}(2i-1)\right)}{\det B_{2i-1}} & \text{if } j \in U_{2i-1}. \end{cases}$$

If $m_{2i} = 0$ ($m_{2i-1} = 0$), then $w(J,i)^j = 0$ ($z(J,i)^j = 0$) for every $j \neq j_i$, since $U_{2i} = \emptyset$ ($U_{2i-1} = \emptyset$). Further, $m_i = \operatorname{rank} B([d - M_{i-1}], [k]) = \operatorname{rank} B([d - M_{i-1}] \setminus [d - M_i], S_i)$ implies $\operatorname{span}\left\{\left(a_j^1, \dots, a_j^{d-M_{i-1}}\right) : j \in [k]\right\} = \operatorname{span}\left\{\left(a_j^1, \dots, a_j^{d-M_{i-1}}\right) : j \in S_i\right\}$. This means that if $\mathbf{v} \in \mathbb{R}^d$ is a vector for which $v^j = 0$ if $j > d - M_{i-1}$, then $\mathbf{v} \cdot \mathbf{a_j} = 0$ for all $j \in S_i$ implies $\mathbf{v} \in X$.

Therefore $\mathbf{w}(J,i), \mathbf{z}(J,i) \in X$ for all i, J, thus $\mathbf{y}(J) = \sum_{1 \leq i \leq |J|} (\mathbf{w}(J,i) + \mathbf{z}(J,i)) \in X$. By (6.13), (6.14), and by the multilinearity of the determinant we have $|w(J,i)^j|, |z(J,i)^j| \leq \frac{1}{2}$ for all $1 \leq j \leq d$. The argument that was used under assumption (6.11) now gives that $\left\{y(J) : J \in \binom{[d-2k\ell]}{\leq \ell}\right\} \cup \{\mathbf{0}\}$ is a 1-equilateral set of cardinality $\sum_{1 \leq r \leq \ell} \binom{d-2k\ell}{r} + 1 = \sum_{1 \leq r \leq \ell} \binom{N}{r} + 1$. □

6.4 Equilateral sets in spaces close to subspaces of ℓ_∞^d

The construction we use is similar to the one from [66]. Let us fix $1 \leq \ell \leq \frac{d-2k}{k}$, and let $N = d - k(2 + \ell)$, and $c = \frac{\ell}{2(N-1)} > 0$. We assume that the linear structure of Y is identified with the linear structure of X, and the norm $||\cdot||_Y$ of Y satisfies

$$||x||_Y \leq ||x||_\infty \leq (1 + c)||x||_Y$$

for each $x \in X$. Let $M = \{(i,j) : 1 \leq i < j \leq N\}$. For every $\boldsymbol{\varepsilon} = \left(\varepsilon_j^i\right)_{(i,j)\in M} \in [0,c]^M$ and $j \in N$ we will define a vector $\mathbf{p}_j(\boldsymbol{\varepsilon}) \in \mathbb{R}^d \in Y$ such that

$$p_j(\varepsilon)^i = -1 \quad \text{if } i = j, \tag{6.15}$$

$$p_j(\varepsilon)^i = \varepsilon_j^i \quad \text{if } i < j, \tag{6.16}$$

$$p_j(\varepsilon)^i = 0 \quad \text{if } i \in [N] \setminus [j], \tag{6.17}$$

$$\left|p_j(\varepsilon)^i\right| \leq \frac{1}{2} \quad \text{if } i \in [n] \setminus [N]. \tag{6.18}$$

Conditions (6.15)−(6.18) imply that $||\mathbf{p}_s(\boldsymbol{\varepsilon}) - \mathbf{p}_t(\boldsymbol{\varepsilon})||_\infty = 1 + \varepsilon_t^s$ for every $1 \leq s < t \leq N$. Define $\varphi : [0,c]^M \to \mathbb{R}^M$ by

$$\varphi_j^i(\boldsymbol{\varepsilon}) = 1 + \varepsilon_j^i - \|\mathbf{p}_i(\boldsymbol{\varepsilon}) - \mathbf{p}_j(\boldsymbol{\varepsilon})\|_Y,$$

for every $1 \leq i < j \leq N$. From

$$\begin{aligned} 0 = 1 + \varepsilon_j^i - \|\mathbf{p}_i(\boldsymbol{\varepsilon}) - \mathbf{p}_j(\boldsymbol{\varepsilon})\|_\infty \leq \varphi_j^i(\boldsymbol{\varepsilon}) &= 1 + \varepsilon_j^i - \|\mathbf{p}_i(\boldsymbol{\varepsilon}) - \mathbf{p}_j(\boldsymbol{\varepsilon})\|_Y \\ &\leq 1 + \varepsilon_j^i - (1+c)^{-1}\|\mathbf{p}_i(\boldsymbol{\varepsilon}) - \mathbf{p}_j(\boldsymbol{\varepsilon})\|_\infty \leq c, \end{aligned}$$

it follows that the image of φ is contained in $[0,c]^M$. Since φ is continuous, by Brouwer's fixed point theorem [7] φ has a fixed point $\boldsymbol{\varepsilon}_0 \in [0,c]^M$. Then $\{\mathbf{p}_j(\boldsymbol{\varepsilon}_0) : j \in [N]\}$ is a 1-equilateral set in Y of cardinality $N = d - k(2+\ell)$.

To finish the proof, we only have to find vectors $\mathbf{p}_j(\boldsymbol{\varepsilon})$ that satisfy conditions (6.15)−(6.18). We construct them in a similar way as the equilateral sets in the proof of Theorem 6.1.

For $1 \leq i \leq 2+\ell$ let

$$U_i = (i-1)k + [k],$$

and

$$B_i = B(\mathbf{b}_{d-ik+1}, \mathbf{b}_{d-ik+2}, \ldots, \mathbf{b}_{d-(i-1)k}).$$

By working from $2+\ell$ down to 1 we may assume without loss of generality that for $1 \leq i \leq 2+\ell$

$$|\det B_i| \geq |\det B(\mathbf{b}_{i_1}, \ldots, \mathbf{b}_{i_k})| \tag{6.19}$$

for all choices of $1 \leq i_1 < \cdots < i_k \leq d-(i-1)k$ for which $|\{i_1, \ldots, i_k\} \cap ([d] \setminus [d-2\ell])| \leq 1$. Assume now that

$$|\det B_i| > 0 \text{ for all } 1 \leq i \leq 2+\ell.$$

We can handle the case when this assumption does not hold in a similar way as the case in the proof of inequality (6.3) in Theorem 6.1 when assumption (6.11) did not hold. Therefore we omit the details.

We construct $\mathbf{p}_j(\boldsymbol{\varepsilon})$ as a sum of $2+\ell$ other vectors $\mathbf{p}_j(\boldsymbol{\varepsilon},1),\mathbf{p}_j(\boldsymbol{\varepsilon},2),\dots,\mathbf{p}_j(\boldsymbol{\varepsilon},2+\ell)$, where $\mathbf{p}_j(\boldsymbol{\varepsilon},1)$ is defined as follows.

For $m\in\{1,2\}$ let

$$p_j(\boldsymbol{\varepsilon},m)^i=\begin{cases}-\frac{1}{2} & \text{if } i=j\\ 0 & \text{if } i\in[d]\setminus(\{j\}\cup N+U_m)\\ \frac{\det B_m\left(i-N-km,\frac{1}{2}\mathbf{b}_j\right)}{\det B_m} & \text{if } i\in N+U_m,\end{cases}$$

and for $m\in\{3,\dots,2+\ell\}$ let

$$p_j(\boldsymbol{\varepsilon},m)^i=\begin{cases}\frac{\varepsilon_j^i}{\ell} & \text{if } i<j\\ 0 & \text{if } i\in[d]\setminus([j-1]\cup N+U_m)\\ \frac{\det B_m(i-N-km,-\mathbf{s}(\boldsymbol{\varepsilon},j))}{\det B_m} & \text{if } i\in N+U_m.\end{cases}$$

where $\mathbf{s}(\boldsymbol{\varepsilon},j)=\sum_{r<j}\frac{\varepsilon_j^r}{\ell}\mathbf{b}_r$. As before, by Cramer's rule we have $p_j(\varepsilon)(m)\in Y$ for all $m\in[2+\ell]$, and thus $\mathbf{p}_j(\boldsymbol{\varepsilon})=\sum_{m\in[2+\ell]}\mathbf{p}_j(\boldsymbol{\varepsilon},m)\in Y$. It follows immediately that $\mathbf{p}_j(\boldsymbol{\varepsilon})$ satisfies conditions (6.15)−(6.17).

Further, by the multilinearity of the determinant, (6.19), and the triangle inequality for $m\in\{1,2\}$ we have

$$|p_j(\boldsymbol{\varepsilon},m)^i|=\left|\frac{\det B_m\left(i-N-km,\frac{1}{2}\mathbf{b}_j\right)}{\det B_m}\right|\le\frac{1}{2}.$$

and for every $m\in\{3,\dots,2+l\}$ we have

$$\begin{aligned}\left|p_j(\boldsymbol{\varepsilon},m)^i\right|&=\left|\frac{\det B_m(i-N-km,-\mathbf{s}(\boldsymbol{\varepsilon},j))}{\det B_m}\right|\le\left|\sum_{r<j}\frac{\varepsilon_j^r}{\ell}\frac{\det B_m\left(i-N-km,-\mathbf{b}_r\right)}{\det B_m}\right|\\&\le\sum_{r<j}\frac{\varepsilon_j^r}{\ell}\left|\frac{\det B_m\left(i-N-km,-\mathbf{b}_r\right)}{\det B_m}\right|\le\sum_{r<j}\frac{\varepsilon_j^r}{\ell}\le(N-1)\frac{c}{\ell}=\frac{1}{2}.\end{aligned}$$

This implies that condition (6.18) holds for $\mathbf{p}_j(\boldsymbol{\varepsilon})$ as well, finishing the proof.

Bibliography

[1] P. K. Agarwal, E. Nevo, J. Pach, R. Pinchasi, M. Sharir, and S. Smorodinsky, *Lenses in arrangements of pseudo-circles and their applications*, J. ACM **51** (2004), no. 2, 139–186.

[2] N. Alon and A. Kupavskii, *Two notions of unit distance graphs*, J. Combin. Theory Ser. A **125** (2014), 1–17.

[3] B. Aronov and M. Sharir, *Cutting circles into pseudo-segments and improved bounds for incidences*, Discrete Comput. Geom. **28** (2002), no. 4, 475–490.

[4] K. Ball, *An elementary introduction to modern convex geometry*, Flavors of geometry, Math. Sci. Res. Inst. Publ., vol. 31, Cambridge Univ. Press, Cambridge, 1997, pp. 1–58.

[5] E. Bannai, E. Bannai, and D. Stanton, *An upper bound for the cardinality of an s-distance subset in real Euclidean space, II*, Combinatorica **3** (1983), no. 2, 147–152.

[6] P. Brass, *On equilateral simplices in normed spaces*, Beiträge Algebra Geom. **40** (1999), no. 2, 303–307.

[7] A. Browder, *Mathematical analysis*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1996, An introduction.

[8] J. Chaffee and M. Noble, *Dimension 4 and dimension 5 graphs with minimum edge set.*, Australas. J. Combin. **64** (2016), 327–333.

[9] ———, *A dimension 6 graph with minimum edge-set*, Graphs Combin. **33** (2017), 1565–1576.

[10] N. G. de Bruijn and P. Erdös, *A colour problem for infinite graphs and a problem in the theory of relations*, Indag. Math. **13** (1951), 369–373.

[11] A. D. N. J. de Grey, *The chromatic number of the plane is at least 5*, Geombinatorics **28** (2018), 18–31.

[12] A. D. N. J. de Grey, *Polymath proposal: finding simpler unit distance graphs of chromatic number 5*, `https://polymathprojects.org/2018/04/10/polymath-proposal-finding-simpler-unit-distance-graphs-of-chromatic-number-5/`, 2018.

[13] B. V. Dekster, *Simplexes with prescribed edge lengths in Minkowski and Banach spaces*, Acta Math. Hungar. **86** (2000), no. 4, 343–358.

[14] P. Erdős, *On sets of distances of* n *points*, Amer. Math. Monthly **53** (1946), 248–250.

[15] ———, *On sets of distances of* n *points in Euclidean space*, Magyar Tud. Akad. Mat. Kutato Int. Közl. **5** (1960), 165–169.

[16] ———, *On extremal problems of graphs and generalized graphs*, Israel J. Math. **2** (1964), 183–190.

[17] ———, *On the number of complete subgraphs and circuits contained in graphs*, Časopis Pěst. Mat. **94** (1969), 290–296.

[18] P. Erdős, R. L. Graham, P. Montgomery, B. L. Rothschild, J. Spencer, and E. G. Straus, *Euclidean Ramsey theorems. I*, Journal of Combinatorial Theory, Series A **14** (1973), no. 3, 341 – 363.

[19] P. Erdős, R. L. Graham, P. Montgomery, B. L. Rothschild, J. Spencer, and E. G. Straus, *Euclidean Ramsey theorems. III*, Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. I, North-Holland, Amsterdam, 1975, pp. 559–583.

[20] P. Erdős, E. Makai, and J. Pach, *Nearly equal distances in the plane*, Combin. Probab. Comput. **2** (1993), no. 4, 401–408.

[21] ———, *Two nearly equal distances in* $\mathbb{R}^d$, 2019, arXiv:1901.01055.

[22] P. Erdős, E. Makai, J. Pach, and J. Spencer, *Gaps in difference sets, and the graph of nearly equal distances*, Applied geometry and discrete mathematics: The Victor Klee Festschrift, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., vol. 4, Amer. Math. Soc., Providence, RI, 1991, pp. 265–273.

[23] P. Erdős and M. Simonovits, *Supersaturated graphs and hypergraphs*, Combinatorica **3** (1983), no. 2, 181–192.

[24] P. Erdős, F. Harary, and W. T. Tutte, *On the dimension of a graph*, Mathematika **12** (1965), no. 2, 118–122.

[25] P. Erdős and M. Simonovits, *On the chromatic number of geometric graphs*, Ars Combin. **9** (1980), 229–246.

[26] Szöllősi F. and P. R. J. Östergård, *Constructions of maximum few-distance sets in Euclidean spaces*, Electron. J. Combin. **27** (2018).

[27] [fedja], *Comment on to the problem "Almost monochromatic point set" on MathOverflow*, `https://mathoverflow.net/q/300604`, 2018.

[28] N. Frankl, *Large equilateral sets in subspaces of ℓ_∞^n of small codimension*, 2020, arXiv:2005.04256.

[29] N. Frankl, T Hubai, and D. Pálvölgyi, *Almost-monochromatic sets and the chromatic number of the plane*, 2019, arXiv:1912.02604.

[30] N. Frankl and A. Kupavskii, *Almost sharp bounds on the number of discrete chains in the plane*, 2019, arXiv:1912.00224.

[31] ———, *Nearly k-distance sets*, 2019, arXiv:1906.02574.

[32] N. Frankl, A. Kupavskii, and K. J. Swanepoel, *Embedding graphs in Euclidean space*, J. Combin. Theory Ser. A **171** (2020), 105146.

[33] S Gołab, *Some metric problems of the geometry of Minkowski*, Trav. Acad. Mines Cracovie **6** (1932), 1–79.

[34] R. L. Graham, *Euclidean Ramsey theory*, Handbook of Discrete and Computational Geometry, Third Edition (J. O'Rourke J. E. Goodman and Cs. D. Tóth, eds.), Chapman and Hall/CRC, 2017, pp. 281–297.

[35] R. L. Graham, B. L. Rothschild, and J.H. Spencer, *Ramsey Theory*, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons, Inc., New York, 1990.

[36] B. Grünbaum, *Convex polytopes*, second ed., Graduate Texts in Mathematics, vol. 221, Springer-Verlag, New York, 2003.

[37] L. Guth and N. H. Katz, *On the Erdős distinct distances problem in the plane*, Ann. of Math. (2) **181** (2015), no. 1, 155–190.

[38] A. W. Hales and R. I. Jewett, *Regularity and positional games*, Trans. Amer. Math. Soc. **106** (1963), 222–229.

[39] R. F. House, *A 4-dimensional graph has at least 9 edges*, Discrete Math. **313** (2013), no. 18, 1783–1789.

[40] T. Hubai, *Comment in "polymath16, fourth thread: Applying the probabilistic method"*, `https://dustingmixon.wordpress.com/2018/05/05/polymath16-fourth-thread-applying-the-probabilistic-method/#comment-4391`, 2018.

[41] H. Kaplan, J. Matoušek, Z. Safernová, and M. Sharir, *Unit distances in three dimensions*, Comb. Probab. Comput. **21** (2012), no. 4, 597–610.

[42] L. M. Kelly, *Elementary Problems and Solutions: Solutions: E735*, Amer. Math. Monthly **54** (1947), no. 4, 227–229.

[43] P. J. Kelly, *A property of Minkowskian circles*, Amer. Math. Monthly **57** (1950), 677–678.

[44] T. Kővári, V. T. Sós, and P. Turán, *On a problem of K. Zarankiewicz*, Colloq. Math. **3** (1954), 50–57.

[45] T. Kobos, *Equilateral dimension of certain classes of normed spaces*, Numer. Funct. Anal. Optim. **35** (2014), no. 10, 1340–1358.

[46] A. Kupavskii, A. M. Raigorodskii, and M. V. Titova, *New bounds for the distance Ramsey number*, Discrete Math. **313** (2013), no. 22, 2566–2574.

[47] L. Lovász, *On decomposition of graphs*, Studia Sci. Math. Hungar. **1** (1966), no. 273, 238.

[48] L. Lovász, M. Saks, and A. Schrijver, *Orthogonal representations and connectivity of graphs*, Linear Algebra Appl. **114** (1989), 439–454.

[49] ______, *A correction: Orthogonal representations and connectivity of graphs*, Linear Algebra Appl. **313** (2000), no. 1-3, 101–105.

[50] E. Makai, J. Pach, and J. Spencer, *New results on the distribution of distances determined by separated point sets*, Paul Erdős and his mathematics, II, Bolyai Soc. Math. Stud., vol. 11, János Bolyai Math. Soc., Budapest, 2002, pp. 499–511.

[51] V. V. Makeev, *Equilateral simplices in normed 4-space*, Zapiski Nauchnykh Seminarov POMI **329** (2005), 88–91.

[52] A. Marcus and G. Tardos, *Intersection reverse sequences and geometric applications*, J. Combin. Theory Ser. A **113** (2006), no. 4, 675–691.

[53] J. Pach and M. Sharir, *Geometric incidences*, Towards a theory of geometric graphs, Contemp. Math., vol. 342, Amer. Math. Soc., Providence, RI, 2004, pp. 185–223.

[54] E. A. Palsson, S. Senger, and A. Sheffer, *On the number of discrete chains*, 2019, arXiv:1902.08259.

[55] C. M. Petty, *Equilateral sets in Minkowski spaces*, Proc. Amer. Math. Soc. **29** (1971), 369–374.

[56] D. Pálvölgyi [domotorp], *Comment in "polymath16, fourth thread: Applying the probabilistic method"*, `https://dustingmixon.wordpress.com/2018/05/05/polymath16-fourth-thread-applying-the-probabilistic-method/#comment-4306`, 2018.

[57] ———, *Proposing the problem "Almost monochromatic point sets" on MathOverflow*, `https://mathoverflow.net/q/300604`, 2018.

[58] I. Rivin, *Answer to the question "Existence of Rational Orthogonal Matrices" on MathOverflow*, `https://mathoverflow.net/a/90072/932`, 2012.

[59] M. Schaefer, *Realizability of graphs and linkages*, Thirty essays on geometric graph theory, Springer, New York, 2013, pp. 461–482.

[60] A. Soifer, *The mathematical coloring book*, Springer-Verlag New York, 2009.

[61] P. S. Soltan, *Analogues of regular simplexes in normed spaces*, Dokl. Akad. Nauk SSSR **222** (1975), no. 6, 1303–1305.

[62] J. Solymosi and V. H. Vu, *Near optimal bounds for the Erdős distinct distances problem in high dimensions*, Combinatorica **28** (2008), no. 1, 113–125.

[63] J. Spencer, E. Szemerédi, and W. Trotter, *Unit distances in the Euclidean plane*, Graph theory and combinatorics, Academic Press, London, 1984, pp. 293–303.

[64] K. J. Swanepoel, *Unit distances and diameters in Euclidean spaces*, Discrete Comput. Geom. **41** (2009), no. 1, 1–27.

[65] ———, *Combinatorial distance geometry in normed spaces*, New Trends in Intuitive Geometry, Springer, 2018, pp. 407–458.

[66] K. J. Swanepoel and R. Villa, *A lower bound for the equilateral number of normed spaces*, Proc. Amer. Math. Soc. **136** (2008), no. 1, 127–131.

[67] B. L. van der Waerden, *Beweis einer Baudetschen Vermutung*, Nieuw Arch. Wiskunde **15** (1927), 212–216.

[68] J. Zahl, *An improved bound on the number of point-surface incidences in three dimensions*, Contrib. Discrete Math. **8** (2013), no. 1, 100–121.

[69] ———, *Breaking the 3/2 barrier for unit distances in three dimensions*, Int. Math. Res. Notices **2019** (2019), no. 20, 6235–6284.