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# CONTINUED FRACTIONS WITH ERROR ESTIMATES FOR SOME SPECIAL FUNCTIONS.

by

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## INTRODUCTION

The advent of digital computers has produced a surge of interesting techniques for computing functions, but many lack useful error estimates. There is therefore a case for re-examining the origins of the relevant mathematics and incorporating error estimates as an integral part of the basic structure.

How then can we relate the common functions to a directly computable form ? Linear relations in the form of continued fractions suggest themselves and have much to offer. Orthogonal polynomials originate in continued fractions, and useful error estimates have been developed for some truncated continued fractions. In this paper we develop the notion that rational functions derived from continued fractions complement orthogonal polynomials, have "built in error terms and afford a basis for much related mathematics.

First we develop the algebra for the interplay between two systems of linear equations ; one with polynomial solutions, the other with solutions that can be written as continued fractions and can therefore be approximated by rational functions. In particular we derive an expression for the error involved in this approximation.

After indicating the generality of the algebra, we illustrate it by focussing attention on some of the key results of special function theory. Specifically we deal with Laguerre, Hermite and Legendre functions in turn. We derive rational functions, with error estimates, that approximate to the functions of the second kind. Then to emphasise that our rational functions can be interpreted as transforms we deduce approximations to Bessel's function of general order, again with error estimates, from a continued fraction for its Laplace transform.

It is the underlying pattern that we wish to establish. In subsequent papers we will generalise our results to deal with a broad class of functions. The structure seems basic to linear problems and can be extended to some non-linear problems. As an introduction to special functions our approach has much to recommend it.

## 1. Three Term Linear Relations

We consider two types of linear relations in which the coefficients $c_i$ , $d_i$ are polynomials in x.

First type
$$g_1 = d_1 g$$
$$g_2 = d_2 g_1 + c_2 g$$
$$g_3 = d_3 g_2 + c_3 g_1$$
$$g_n = d_n g_{n-1} + c_n g_{n-2} \qquad (1.1)$$

Solving for $g_n$ in terms of g gives

$$g_n = D_n g, \qquad (1.2)$$

where $D_n$ (x)is a polynomial in x.

Second type
$$f_1 = d_1 f + c_1 (-1)$$
$$f_2 = d_2 f_1 + c_2 f$$
$$f_3 = d_3 f_2 + c_3 f_1$$
------
$$f_n = d_n f_{n-1} + C_n f_n-2 \qquad (1.3)$$

Solving these equations successively, we get

$$f_1 = D_1 f - C_1$$
$$f_2 = d_2 ( D_1 f - C_1) + c_2 f = D_2 f - C_2$$
$$f_3 = d_3 (D_2 f - C_2 ) + c_3 (D_1 f - C_1) = D_3 f - C_3$$

and in general

$$f_n = D_n f - C_n \qquad (1.4)$$

where we have defined the polynomials $C_n(x)$ and $D_n(x)$ to satisfy the same recurrence relations as $f_n$ . For $n \geq 1$

$$C_n = d_n C_{n-1} + C_n \ c_{n-2} \ C_0 = 0 \ , \ C_{-1} = 1 \tag{1.5}$$

$$D_n = d_n D_{n-1} \ + c_n \ D_{n-2} \ D_o = 1 \ , \ D_{-1} = 0 \ . \tag{1.6}$$

We notice that equations (1.1) can be added to equations (1.3) without altering the form of the latter.

To develop particular sets of equations (1.3), we will frequently be guided by results developed for continued fractions. Solving equations (1.3) for f, we will refer to f as the primary function,

$$f = \frac{c_1}{d_1 - \frac{f_1}{f}} = \frac{c_1}{d_1} + \frac{c_2}{d_2} + \frac{c_3}{d_3} + - - - + \frac{c_n}{d_n - \frac{f_n}{f_{n-1}}}. \tag{1.7}$$

At this stage we avoid discussion of convergence and whether when $n \rightarrow \infty$ this is strictly a continued fraction for f, by taking n finite. However we will use continued fraction (C.F.) terminology and refer to the right hand side terminated (chopped) after d as the $n^{th}$ convergent,

$$f_{o/n} = \frac{c_1}{d_1} + \frac{c_2}{d_2} + \frac{c_3}{d_3} + - - - + \frac{c_n}{d_n} \equiv \frac{C_n}{D_n} , \tag{1.8}$$

$f_o \equiv f$ and for clarity we write $f_{o/n}$ rather than $f_{/n}$ .

These convergents $f_{o/n}$ . are rational functions of x; that

$f_{o/n} \equiv \frac{C_n}{D_n}$ follows from (1.4). However (1.4) says more than this, for written

$$f - \frac{C_n}{D_n} = \frac{f_n}{D_n} \tag{1.9}$$

it expressed the error in approximating f by the $n^{th}$

convergent $f_{o/n}$ .

The relation (1.4) is crucial to much that follows. It is convenient to establish two further results from it at this stage

$$f_n = D_n f - C_n$$

$$f_i = D_i f - C_i$$

Eliminating f

$$D_n f_i - D_i f_n = C_n D_i - C_i D_n \tag{1.10}$$

Hence

$$\underline{f_i - \frac{C_n D_i - C_i D_n}{D_n} = \frac{D_i f_n}{D_n}} . \tag{1.11}$$

This generalises the result (1.9) and enables us to calculate convergents $f_{i/n}$ for $i \geq 0$ but of course less than n. The term on the right expresses the error involved in the approximation.

When $i = n - 1$ an important simplification of (1.10) takes place, we get

$$D_n f_{n-1} - D_{n-1} - f_n = C_n D_{n-1} - C_{n-1} D_n .$$

Now an elementary result in C.F theory which is readily deduced from (1.5) and (1.6) states

$$C_nD_{n-1} - C_{n-1} D_n = (-1)^{n-1} c_1c_2 -- c_n . \qquad (1. (1.12)$$

Defining $A_N = (-1)^{n-1} c_1c_2 -- c_n$ ,

$$\underline{D_n f_{n-1} , - D_{n-1} f_n = A_n .} \qquad (1. (1.13)$$

This holds for all n ≥ 1 and is an important relation between the two sets of equations (l.l) and (1.3) as it does not contain $C_n$ .

Writing i instead of n, it is worth observing this enables us to write any of the f. in continued fraction form

$$f_{i-1} = \cfrac{A_i}{D_i - D_{i-1}\cfrac{f_i}{f_{i-1}}} = \frac{A_i}{D_i} + \frac{D_{i-1}c_{i+1}}{d_{i+1}} + \frac{c_{i+2}}{d_{i+2}} - - - + \frac{c_n}{d_n -} \frac{f_n}{f_{n-1}} \qquad (1.14)$$

The notation $f_{i-1/n}$ is used to denote the convergent obtained by truncating the right hand side after $d_n$. With this notation the denominators of all the $f_{i/n}$ , are $D_n$ for 0 ≤ i ≤ n. Clearly we can readily deduce (1.14) knowing the relation (1.7) for the primary function f.

Constructing the Linear Relations.

Three term linear relations of the form (1.3) arise in many parts of Mathematics - in special function theory, oscillation problems, Markov processes and in eigenvalue problems. Such relations yield the rational functions $f_{i/n}$ directly, see (1.8) and (1.14). However almost any function whether defined explicitly or implicitly can be approximated by functions rational in one of its variables. The results are often conveniently expressed in a continued fraction form. Just as with polynomials, the coefficients of rational functions can be chosen to fit various properties of the function; they can be readily chosen to fit derivatives at a point or particular values of the function.

In this paper we will be primarily concerned with rational functions which fit derivatives of the function f at a regular point, usually the origin or infinity. These we will obtain by terminating the S fraction for f

$$\frac{P_o}{1}+\frac{P_1x}{1}+\frac{P_2x}{1}+------\frac{P_{2n-1}x}{1}\Big/_+ \qquad (1.15)$$

or its contracted form, the J fraction

$$\frac{P_o}{1+P_1x}-\frac{P_1P_2x}{1+(P_2+P_3)x}------\frac{P_{2n-1}P_{2n}x^2}{1+(P_{2n}+P_{2n+1})x}\Big/_- \qquad (1.16)$$

where the $p_i$ are chosen to fit successive derivatives of f, in this case at the origin.

The S fraction for a function is readily constructed from its Taylor series

$$f = a_0 + a_1x + a_2x^2 + - - -$$

by successively forming the following three term relations

$$f_1 = f - a_0 \qquad a_0 \equiv p_0 \;.$$

$$f_2 = f_1 + p_1\, x\, f \qquad (1.17)$$

$$f_3 = f_2 + p_2\, x\, f_1$$

---

the $p_i$. being chosen so that $f_i = 0(x^i)$.

For suppose $f = \frac{P_0}{1} + \frac{P_1 x}{1} + \frac{P_2 x}{1} + --- \frac{P_{n-1} x}{1} - \frac{f_n}{f_{n-1}}$

by induction $f_n = (-1)^n\, p_0\, P_1 - p_n\, x^n [1 + 0(x)]$ ,

then by (1.9) $f - \frac{C_n}{D_n} = (-1)^n\, P_Q P_1 - P_n\, x^n\, [\, 1 + 0(x)]$ . (1.18)

That the first n terms of $\frac{C_n}{D_n}$ agree with n terms of the Taylor series for f, follows by differentiating with respect to x n-1 times .

In practice when deriving the equations (1.17) the series for an $f_j$ can have additional leading terms (say r) zero, i.e. $f_j = 0(x^{r+i})$, these we would take advantage of in forming $f_{i+1}$ by setting

$$f_{i+1}\, f_i \,{}^{+}\, p_i\, {}_x{}^{r+1}\, f_{i+1} \qquad (1.19)$$

The powers of the terms in $f_{j+1}$ and of all subsequent $f_j$ are then increased by r. The resulting continued fractions are sometimes referred to as C fractions.

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* We do not draw Wall's distinction between S and C fractions, we attribute all such fractions to Stieltjes by calling them S fractions.

Although any Taylor series can be converted into an S fraction by the above process the resulting rational approximations will be most useful if the "behaviour of the function at infinity is borne in mind. Functions which behave as $0(x^{-\mu})$, $0 \leq \mu \leq 1$, at infinity, frequently can be approximated over much of the complex plane by an S fraction . While functions which behave as $0(x^{m-\mu})$ , m an integer and $0 \leq \mu \leq 1$, at infinity, require some prior adjustment. A simple technique, but not the only one, for making this adjustment is as follows:

When $m > 0$, we form the S fraction for the function

$$F = \frac{f-(a_0+a_1 x+a_2x^2+----+a_{m-1}x^{m-1})}{x^m}, \qquad (1.20)$$

when $m < 0$, we first form the series for the reciprocal of the function and then apply the technique for $m > 0$ to the reciprocal.

The rational functions produced by this technique can be identified with the Pade approximants for the function f as described in Wall [ p380].

Converting a series into an S fraction in the above manner usually improves the rate of convergence in the neighbourhood of the origin, and can extend the region of convergence beyond that for the series. In fact even divergent series can become convergent in the process see Wall [ § 93] or Stieltjes [ Oeuvres Vol.2] . Rapid convergence along a line in the complex plane can often be achieved by fitting the derivatives of f at two regular points using Murphy's M fraction, but we will not pursue such rational functions here.

Transforming Differential Equations.

Linear equations of the second type (1.3) can arise directly or as the result of operating on simple differential equations. The inter-relation between the two we believe is important and fundamental. To form a basis for further work, we will develop these inter-relations for some of the so-called special functions of mathematics.

We will require the following result. To invert our Laplace transforms it will be necessary to expand the convergents

$\frac{C_n}{D_n}$ of the terminated fraction

$$f_{o/n} = \frac{c_1}{d_1} + \frac{c_2}{d_2} + \; ---- + \frac{c_n}{d_n} \equiv \frac{C_n}{D_n}$$

in partial fractions.

Suppose $C_n$ and $D_n$ are polynomials in x, such that degree of $C_n$ < degree $D_n$ ,and that $D_n$ is of degree n and has only simple zeros $a_{r.n}$ r = 1,2 n.

Then $$\frac{C_n}{D_n} = \sum_{r=1}^{n} \frac{A_{rn}}{x-\alpha_{rn}} = \sum_{r=1}^{n} \frac{C_n(\alpha_{rn})}{D_n(\alpha_{rn})} \frac{1}{x-\alpha_{rn}} \tag{1.21}$$

But by (1.12)

$$C_n(\alpha_{rn}) D_{n-1}(\alpha_{rn}) = (-1)^{n-1} c_1 c_2 --- c_n , \tag{1.22}$$

therefore the partial fraction expansion can also be written

$$\frac{C_n}{D_n} = \sum_{r=1}^{n} \frac{(-1)^{n-1} c_1 c_2 --- c_n}{D_{n-1}(\alpha_{rn}) D_n'(\alpha_{rn})} \cdot \frac{1}{x - \alpha_{rn}}. \tag{1.23}$$

The importance of this expression is that it often simplifies further when the relation expression $D_n'$ in terms of Ds is known. It enables the partial fraction expansion of $f_{o/n}$ to be derived without calculating the numerators $C_n$.

We now consider the three sets of functions Laguerre, Hermite and Legendre in turn. We construct rational functions which approximate the second kind functions and derive error estimates for these approximations. Our main objective is to display the close connections between the polynomial and the rational solutions. It is the structure, rather than new results, that we wish to exhibit.

## 2. Lagugrre Functions

The denominators of the C.F.

$$\frac{1}{s+1}-\frac{1^2}{s+3}-\frac{2^2}{s+5}-\text{----}-\frac{n^2}{s+2n+1-} \qquad (2.1)$$

satisfy the recurrence relation

$$D_{n+1} = (s+2n+1)d_{n-1} - n^2D_{n-1} \qquad (2.2)$$

and are (apart from a sign) the Laguerre polynomials

$$D_n(s) = L_n(-s) \ . \qquad (2.3)$$

The C.F. (2.1) is known to match the series expansion in $\frac{1}{s}$ of the Laplace transform of $\frac{1}{1+t}$ namely the integral

$$\int_o^\infty e^{-st}\frac{1}{1+t}dt = e^s\int_s^\infty \frac{e^{-u}}{u}du \ . \qquad (2.4)$$

Let us consider then the function $y=\frac{1}{1+t}$, with (2.1) in mind.

$$\frac{dy}{dt} + y = \frac{t}{(1+t)^2} = y_1 \qquad (2.5)$$

and defining $y_n(t) = \frac{n!t^n}{(1+t)^{n+1}}$ it is easily shown

$$y_{n+1} = \left[\frac{dy_n}{dt}+(2n+1)y_n\right]-n^2y_{n-1}. \qquad (2.6)$$

If we now Laplace transform the set of differential equations (2.5) and (2.6) we obtain a set of three term linear relations of the type (1.3). We denote Laplace transforms by the corresponding capital letter

$$Y_1 = (s + 1)Y - 1$$

$$Y_2 = (s + 3)Y_1 - Y$$

$$------ \qquad (2.7)$$

$$Y_{n+1} = (s + 2n+1)Y - n^2 Y_{n-1}$$

and hence

$$Y = \int_o^\infty e^{-st} \frac{1}{1+t} dt = \frac{1}{s+1-\frac{Y_1}{Y}}$$

$$= \frac{1}{s+1} - \frac{1^2}{s+3} - \frac{2^2}{s+5} - --- - \frac{n^2}{s+2n+1-\frac{Y_{n+1}}{Y_n}} \qquad (2.8)$$

By terminating this expression after n terms we can obtain rational functions which approximate Y with an error

$$Y - \frac{C_n(s)}{D_n(s)} = \frac{Y_n(s)}{D_n(s)} \qquad (2.9)$$

In addition using (1.14)

$$Y_i = \int_o^\infty e^{-st} \frac{i!t^i}{(1+t)i+1} dt = \frac{(i!)^2}{D_{i+1}}, - \frac{D_i(i+1)^2}{s+2i+3-} -- \frac{n^2}{s+2n+1} - \frac{Y_{n+1}}{Y_n}.$$

(2.10)

These approximations will be good for s large, and can be shown to converge as $n \to \infty$ for all s not lying in the interval $(-\infty, 0)$.

The integral for $Y_n$ can be recast in a variety of forms, we will take as standard the Stieltjes form (2.12) because of its direct relation to the series in 1/s and hence the moment problem.

As $\int_0^\infty e^{-px} L_n(x)\,dx = n!\dfrac{(p-1)^n}{p^{n+1}}$,

$$y_n(t) = \int_0^\infty e^{-(t+1)x} L_n(x)\,dx \tag{2.11}$$

$$\therefore y_n(s) = \int_0^\infty \frac{e^{-x} L_n(x)\,dx}{s+x}. \tag{2.12}$$

Now the $Y_n$ satisfy the same recurrence relation as the polynomials $D_n$ (s), and therefore $Y_n$ must be essentially a second solution of Laguerre's differential equation.

Consider $z_n(s) = e^{-s} Y_n(s) = n!\int_0^\infty e^{-s(1+t)} \dfrac{t^n}{(1+t)^{n+1}}\,dt$ (2.13)

then
$$s\frac{d^2z}{ds^2} + (1+s)\frac{dz}{ds} - nz$$
$$= n!\int_0^\infty e^{-s(1+t)}\left\{\frac{st^n}{(1+t)^{n-1}} - (1+s)\frac{t^n}{(1+t)^n} - n\frac{t^n}{(1+t)^{n+1}}\right\}dt$$
$$= n!\int_0^\infty e^{-s(1+t)}\left\{\frac{st^{n+1}}{(1+t)^n} - \frac{t^n}{(1+t)^n} - \frac{nt^n}{(1+t)^{n+1}}\right\}dt$$

Integration of the first term in the brackets by parts shows that the right hand side vanishes.
Hence the complete solution of the equation

is
$$s\frac{d^2z}{ds^2} + (1+s)\frac{dz}{ds} - nz = 0. \tag{2.14}$$

$$z = AD_n(s) + Be^{-s}y_n(s). \tag{2.15}$$

The two solutions correspond to the two systems of equations (1.1) and (1.3). For the usual form of Laguerre's differential equation simply replaces s by -s.

Associated Laguerre Functions

The appropriate generalisations of (2.7) and (2.8) for the associated Laguerre functions are

$$Y_{n+1}^{\alpha} = (s+\alpha+2n+1)\,Y_n^{\alpha} - n(n+\alpha)\,Y_{n-1}^{\alpha} \qquad (2.16)$$

and

$$Y^{\alpha} = \int_0^{\infty} \frac{e^{-st}}{(1+t)^{1+\alpha}}\,dt = \frac{1}{s+\alpha+1} - \frac{\alpha+1}{s+\alpha+3} - \frac{2(\alpha+2)}{s+\alpha+5} - - - \frac{n(\alpha+n)}{s+\alpha+2n+1 - \frac{Y_{n}^{\alpha}+1}{Y_n^{\alpha}}} \qquad (2.17)$$

where

$$Y_n^{\alpha} = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} \int_0^{\infty} e^{-st}\,\frac{t^n}{(1+t)}\,\frac{dt}{n+1+\alpha}, \quad D_n^{\alpha} = n!\,L_n^{\alpha}(-s). \qquad (2.18)$$

The error is approximating $Y^{\alpha}$ by the first n in terms of (2.17) is given by

$$Y^{\alpha} - \frac{C_n^{\alpha}}{D_n^{\alpha}} = \frac{Y_n^{\alpha}}{D_n^{\alpha}} \qquad (2.19)$$

This form for the error is amenable to analysis. We will now show the error behaves as

$$\sim \frac{2\Pi}{\Gamma(\alpha+1)}\,e^{s}\,s^{\alpha}\,e^{-4\sqrt{ks}} \qquad (2.20)$$

for n large, where $k = n + \frac{1}{2}(\alpha+1)$.

Error Analysis

The form of the error (2.19) is such that we can develop fairly good asymptotic estimates of the error as functions of n over the whole complex plane of the argument s.

It is not difficult to show that the error $-\frac{Y_n^\alpha(s)}{D_n^\alpha(s)}$, in approximating

$$\int_0^\infty \frac{e - st}{(1 + t)1 + \alpha} dt \quad \text{by} \quad \frac{C_n^\alpha(s)}{D_n^\alpha(s)} \tag{2.21}$$

satisfies the differential equation

$$s\frac{d}{ds}\left[\frac{Y_n^\alpha}{D_n^\alpha}\right] - (s+\alpha)\left[\frac{Y_n^\alpha}{D_n^\alpha}\right] = -\frac{An+1}{D_n^\alpha(s)2}, \tag{2.22}$$

where $A_{n+1} = \frac{n!\Gamma(\alpha+n+1}{\Gamma(\alpha+1)}$

Integrating this equation gives

$$s^{-\alpha}e^{-s}\frac{Y_n^\alpha}{D_n^\alpha} = A_{n+1}\int_s^\infty \frac{u^{-(\alpha+1)}e^{-u}}{\left[D_n^\alpha(u)\right]^2} du, \tag{2.23}$$

where $D_n^a(u) = u^n + n(\alpha+n)u^{n-1} + --- + \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)}$

$$= n!L_n^\alpha(-u) = \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)} {}_1F_1(-n;\alpha+1;-u).$$

A suitable asymptotic formula for approximating $D_n^\alpha$ for large n is according to Slater [ 4,*4*,15 and 4.4.31] with $k = \frac{b}{2} - a$,

$${}_1F_1(a;d;x) = \Gamma(b)e^{\frac{1}{2}x}(kx)^{\frac{1}{4}-\frac{1}{2}b}\Pi^{-\frac{1}{2}}\cos(2\sqrt{kx} - (\frac{b}{2} - \frac{1}{4}\Pi)\{1 + 0(|k|^{-\frac{1}{2}})\} \tag{2.24}$$

as $a \to -\infty$, $0 < \arg(kx) < 2\Pi$ and for any b.

i.e. $D_n^\alpha(u) = \Gamma(\alpha+n+1)e^{-\frac{1}{2}u}(xue^{i\Pi})^{-\frac{1}{4}-\frac{1}{2}\alpha}\Pi^{-\frac{1}{2}}\cosh(2\sqrt{ku} + \varphi i)\{1 + 0(|k|^{-\frac{1}{2}})\}$ (2.25)

for $|\arg(ku)| < \Pi$, putting $\varphi = (\frac{\alpha}{2} + \frac{1}{4})\Pi\Pi$

Substituting this in the integral of (2.23), an estimate for the error is

$$\frac{Y_n^\alpha(s)}{D_n^\alpha(s)} \sim \frac{\Pi\Gamma(n+1)}{\Gamma(\alpha+1)\Gamma)\Gamma+n+1)} e^s s^\alpha \int_s^\infty \frac{u^{-(\alpha+1)}du}{(kue^{i\Pi})^{-\frac{1}{2}-\alpha}\cosh^2(2\sqrt{ku}+\varphi i)}$$

$$= \frac{\Pi\Gamma(n+1)k^{\frac{1}{2}+\alpha}}{\Gamma(\alpha+1)\Gamma)\Gamma+n+1)} e^s s^\alpha e 2i\phi \int_s^\infty \frac{u^{-\frac{1}{2}}du}{\cosh^2(2\sqrt{ku}+\varphi i)}$$

$$\therefore \frac{Y_n^\alpha(s)}{D_n^\alpha(s)} \sim \frac{\Pi}{\Gamma(\alpha+1)} e^s s^\alpha 2i\phi[1-\tanh(2\sqrt{ks}+\phi i 0]\left\{1+0\left(\frac{1}{|k|\frac{1}{2}}\right)\right\} \tag{2.26}$$

for $|\arg(ks)| < \Pi$ and for any $\alpha, k = n + ½(+ 1)$ .

For s not near the singularities of $D_n^\alpha(s)$, this simplifies to

$$\frac{Y_n^\alpha(s)}{D_n^\alpha(s)} = \frac{2\Pi}{\Gamma(\alpha+1)} e^s s^\alpha e^{-4\sqrt{ks}}\left\{1+0\left(\frac{1}{|k|^{\frac{1}{2}}}\right)\right\} \tag{2.27}$$

This result (2.27), in essence seems to have been first derived by Luke [6.7]; however the method and the result (2.26) are quite different.

The estimates given by (2.27) are good even for small values of n. Table A records the values of some convergents of (2.8) with their error estimates (2.27 with $\alpha = 0$). For values of s not too close to the negative real axis the results suggest that the estimates given by (2.27) are adequate.

3. Hermite Functions.

From the function $\frac{1}{1+t}$ we developed naturally both Laguerre polynomials and rational functions. Let us now repeat the process for the lunction $e^{-t^2}$,this will lead us to Hermite functions. In fact the development is a little more natural with $e^{-t^2/2}$, so consider

$$y = e^{-t^2/2} \text{ and define } y_n - t^n e^{-t^2/2}. \tag{3.1}$$

Then $$y_1 = te^{-t^2/2} = -\frac{dy}{dt} \tag{3.2}$$

$$\frac{dy_n}{dt} = nt^{n-1} e^{-t^2/2} - t^{n+1}e^{-t^2/2}, \text{so that}$$

$$y_{n+1} = -\frac{dy}{dt}n + ny_{n-1} \tag{3.3}$$

If we now Laplace transform these differential equations, we get

$$Y_1 = - sY + 1$$

$$Y_2 = - sY_1 + Y$$

$$\text{-----------} \tag{3.4}$$

$$Y_{n+1} = - sY_n + nY_{n-1}$$

and hence

$$Y = \int_0^\infty e^{-st} e^{-t^2/2}\, dt = \underline{\frac{1}{s+}\,\frac{1}{s+}\,\frac{2}{s+} - - - \frac{n-1}{s}\,{}^{+}\frac{Y_n}{Y_{n-1}}}\,. \tag{3.5}$$

The equations (3.4) yield (3-5), but if we wish to relate them to our standard form (1.3) we must replace $Y_n$ by $(-1)^n Y_n$ . Thus we can obtain rational functions which approximate Y with an error

$$Y = \frac{Cn(s)}{Dn(s)} = \frac{(-1)^n Y_n(s)}{D_n(s)} \tag{3.6}$$

by terminating (3.5) after n terms. We can also readily obtain rational functions which approximate the integral

$$Y_n = \int_0^\infty e^{-st}\, t^n\, e^{-t2/2}\, dt. \tag{3.7}$$

The denominators of (3.5) truncated are polynomials $D_n$ satisfying the recurrence relation

$$D_{n+1} = sD_n + nD_{n-1}. \tag{3.8}$$

These polynomials are related to the Hermite polynomials $\overline{H}_n(x)$ by

$$\overline{H}_n(x) \equiv (-i)^n D_n(ix) \tag{3.9}$$

where the $\overline{H}_n(x)$ are defined by the generating function

$$e^{xt - t^2/2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}, \; = \sum_{r=0}^{\infty} y_r(t)\frac{x^r}{r!}. \tag{3.10}$$

We must now relate the $Y_n$ to the second solution of the differential equation satisfied by D . We verify that $z = e^{-s^2/2}Y_n$ is a solution of the differential equation (essentially Hermite's equation s = ix) satisfied by $D_n$

$$\frac{d^2z}{ds^2} + s\frac{dz}{ds} - nz = 0. \tag{3.11}$$

So that the complete solution of this equation is

$$z = ADn(s) + Be^{-s^2/2}Y_n(s). \tag{3.12}$$

Define then $z = e^{-s^2/2}y_n = \int_0^\infty t^n e^{-\frac{1}{2}(t+s)2}\, dt,$ (3.13)

$$\frac{d^2z}{ds^2} + s\frac{dz}{dz} - nz = \int_0^\infty t\left\{(t+s)^2 - 1 - s(t+s) - n\right\} e^{-\frac{1}{2}(t+s)2}\, dt$$

$$= \int_0^\infty tn\,\{t^2 + ts - (n+1)\}\, e^{-\frac{1}{2}(t+s)2}\, dt$$

$$= e^{-s^2/2}\{Y_{n+2} + sY_{n+1} - (n+1)Y_n\}$$

$$= 0$$

by the recurrence relations (3.4); a simple interesting proof

The integrals $Y_n$ are of course closely related to the error function, Dawson's integral, Fresnel integrals etc. Here we will content ourselves with establishing the link between the $Y_n$ and their Stieltjes form.

Using the expression $\overline{H}_n(x) \equiv (-1)^n e^{x^2/2} \frac{d_n}{dx^n}(e^{-x^2/2})$, it readily follows that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e - \frac{(u+x)^2}{2} \overline{H}_n(x) dx = (-u)^n \,. \tag{3.14}$$

So putting u = + it , we obtain the Fourier relation

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-itx} e^{-x^2/2} \overline{H}_n(x) dx = (-it)^n e^{-t^2/2} \tag{3.15}$$

and formally, Laplace transforming this relation, we get

$$Y_n(s) = \int_0^{\infty} e^{-st} t^n \; e - t^2/2 \, dt = \frac{i^n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e - x^2/2 \, \overline{H}_n(x) dx}{s + ix.} \tag{3.16}$$

or in terms of $D_n$

$$Y_n(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-x^2/2} D_n(ix) dx}{s + ix} \tag{3.17}$$

Again we obtain an integral relation between the two solutions.

This type of relation is clarified in the next example.

The C.F. obtained from (3.5) by letting $n \to \infty$ converges to Y for

all s not on the imaginary axis, the reason is highlighted by (3.17).

## Error Analysis

The Herraite functions and Laguerre functions are of course closely related. In fact the Hermite polynomials can be expressed as Laguerre polynomials with parameters $\alpha = \pm \frac{1}{2}$, see Szego [5.6.1 ]

$$H_{2n}(x) = (-1)^n 2^{2n} n! L_n^{\left(-\frac{1}{2}\right)}(x^2), \qquad H_{2n+1}(x) = (-1)^n 2^{2n+1} n! x L_n^{\left(-\frac{1}{2}\right)}(x^2) \tag{3.18}$$

and $\overline{H}_n(x) = 2^{-n/2} H_n(x/\sqrt{2})$.

Using these relations we can deduce the error (3.6) directly from (2.26). Writing (3.5) in our standard notation, $f_n = (-1)^n Y_n$

$$f = \frac{1}{s+}\,\frac{1}{s+}\,\frac{2}{s+} \text{------} \frac{\frac{2n-1}{s}}{\frac{-f_{2n}}{f_{2n-1}}}$$

the error for the even convergents is

$$f - \frac{C_{2n}}{D_{2n}} = \frac{f_{2n}}{D_{2n}} \tag{3.19}$$

Where

$$e^{-s^2/2}\,\frac{f_{2n}}{D_{2n}} = A_{2n+1}\int_s^{+\infty} \frac{e^{-\zeta^2/2}}{D_{2n}^2(\zeta\zeta}\,d\xi \tag{3.20}$$

The sign of the upper limit being taken the same as the sign of Rs.

$$\text{Now } D_{2n}(\zeta) = i^{2n}\,\overline{H}_{2n}(-i\zeta) = \left(-\frac{1}{2}\right)^n H_{2n}\left(-\frac{i\zeta}{\sqrt{2}}\right) = 2^n\, n!\, L_n^{\left(-\frac{1}{2}\right)}(-\zeta^2/2), \tag{3.21}$$

on substituting for Rs $> 0, \zeta = \sqrt{2u}$, we get

$$e^{-s^2/2}\,\frac{f_{2n}}{D_{2n}} = \frac{(2n)!}{\sqrt{2}.\,2^{2n}}\int_{s^2/2}^{\infty} \frac{e^{-u}\,u^{-\frac{1}{2}}}{[n!\,L_n^{(-\frac{1}{2})}(-u)]^2}\,du. \tag{3.22}$$

Apart from a change in the lower limit we evaluated this integral in the previous section; we conclude the error in approximating f with the even convergents is, for Rs > 0,

$$\frac{f_{2n}}{D_{2n}} = \sqrt{\frac{\pi}{2}}\,e^{s^2/2}(1 - \tanh(\sqrt{2(n+\frac{1}{4})}s\{1 + 0(n^{-\frac{1}{2}})\}. \tag{3.23}$$

The error for the odd convergents is, for R s > 0 ,

$$\frac{f_{2n+1}}{D_{2n+1}} = -\sqrt{\frac{\pi}{2}e^{s^2/2}}\,(1 - \tanh(\sqrt{2(n+\frac{3}{4})s + \frac{\pi}{2}i}))\{1 + 0(n^{-\frac{1}{2}})\} \qquad (3.24)$$

When R s < 0 change the signs of these expressions for the error and the sign of s.

The C.F.(3.5) is matching the divergent series

$$\frac{1}{s} - \frac{1}{s^3} + \frac{1.3}{s^5} - \frac{1.3.5}{s^7} + ----------$$

so that it is perhaps not surprising to find the error estimate denominated by $e^{s^{2/2}}$ . Neither (3-5) nor the error estimates are particularly useful. A rather more powerful C.F. for calculating the error function, and related functions, is considered in TR/26 [12.10] It matches series like

$$\frac{e^{-z^2}}{z}\int_0^z e^{t^2}dt = \sum_{r=0}^{\infty}\frac{(-2)^r z^{2r}}{1.3---(2r-1)}. \qquad (3.25)$$

4. Legend-re Functions.

We expect an integral relation to exist between the two solutions of a second order system. Let us use this to derive the recurrence relation for the rational solution from that for the polynomial solution.

The Legendre polynomials $P_n(\mu)$ satisfy the recurrence relation

$$(n+1)\ P_{n+1}(\mu)\ -\ (2n+1)\ \mu P_n(\mu)\ +\ nP_{n-1}(\mu)\ =0. \tag{4.1}$$

Defining the two sided transform

$$y_n(x) = \frac{1}{2}\int_{-1}^{1} e^{x\mu} P_n(\mu)d\mu \tag{4.2}$$

and applying it to (4.1) we obtain the set of differential equations

$$(n+1)Y_{n+1} - (2n+1)\frac{dy_n}{dx} + nY_{n-1} = 0. \tag{4.3}$$

In fact $$Yn(x) = x^n\left(\frac{1}{x}\frac{d}{dx}\right)^n \frac{\sinh x}{x} = \sqrt{\frac{\pi}{2x}} I_{n+\frac{1}{2}}(x) \tag{4.4}$$

and is a modified spherical Bessel function of the first kind.

Laplace transforming (4.3), $y = \dfrac{\sinh x}{x}$ and the initial equation is $y_1 - y = 0$, yields the set of equations:

$$\begin{aligned} Y_1 - sY + 1 &= 0 \\ 2Y_2 - 3sY_1 + Y &= 0 \end{aligned} \tag{4.5}$$

--------

$$(n+1)\ Y_{n+1}\ -(2n+1)sY_n + nY_{n-1} = 0$$

and hence

$$Y(s) = \int_0^\infty e^{-sx}\frac{\sinh x}{x}dx = \frac{1}{2}\log\left(\frac{s+1}{s-1}\right)$$

$$= \frac{1}{s-}\,\frac{1}{3s-}\,\frac{2^2}{5s-}\,\frac{3^2}{7s-}\,-----\,\frac{n^2}{(2n+1)s-}(n+1)\frac{Y_{n+1}}{Y_n} \tag{4.6}$$

where the denominators $D_n(s) = n!P_n(s)$. (4.7)

Now $$\mathrm{yn}(x) = \frac{1}{2}\int_{-1}^{1} e^{x\mu} P_n(\mu)\,d\mu \tag{4.8}$$

so that $$Y_n(s) \equiv \mathcal{L}\,Y_n(x) = \tfrac{1}{2}\int_{-1}^{1} \frac{P_n(\mu)\,d\mu}{s-\mu} \tag{4.9}$$

which by Neumann's formula is $Q_n(s)$, the Legendre function of the second kind.

By terminating (4.6) and using the result (1.14) we can obtain rational functions for $Y_n(s)$. As (4.6) matches the series, in $\frac{1}{s}$ of $Y(s)$, we expect these to give good approximations in the complex plane for s large and in fact as $n \to \infty$. (4.6) will converge everywhere to $Y(s)$ outside of a cut from [-1,1] along the real axis.

$$\overline{-1 \qquad 0 \quad 1}$$

The error in approximating $Y(s)$ by the first n terms of (4.6) is

$$Y(s) - \frac{C_n(s)}{D_n(s)} = \frac{n!\,Y_n(s)}{D_n(s)} = \frac{Q_n(s)}{P_n(s)}\;. \tag{4.10}$$

It is readily estimated from the integral, see for example Sneddon [18.12],

$$\frac{Q_n(s)}{P_n(s)} = \int_s^\infty \frac{du}{(u^2-1)P_n(u)]^2}$$

$$= \int_{\cos^{-1}s}^{\infty} \frac{d\xi}{\sinh\zeta \mathrm{in}_n(u)]^2}. \tag{4.11}$$

But for $R\,\zeta > 0$, i.e. not on the cut [- 1 ,1], Copson [p.284] gives

$$P_n(\cos\zeta) = \sqrt{\frac{2}{n\pi\sinh\zeta}}\, e^{\pi/4}\left\{\cosh[\,n + \frac{1}{2})\zeta - \frac{\pi}{4} i] + 0\left(\frac{1}{n}\right)\right\} \tag{4.12}$$

and hence the error

$$\frac{Q_n(s)}{P_n(s)} = \frac{\pi}{2i}\left[1 - \tanh[(\,n + \frac{1}{2})\zeta - \frac{\pi}{4} i\right] + 0\left(\frac{T}{n}\right). \tag{4.13}$$

For s not near the cut[-1,1], it will often be sufficient to take

$$\frac{Q_n(s)}{P_n(S)} \sim \frac{\pi}{[s+\sqrt{s^2-1}]}(2n+1), \qquad (4.14)$$

or for |s[ sufficiently large simply $\frac{\pi}{(2s)} 2n+1$ (4.15)

For s on the cut, when $\varepsilon \le \theta \le \pi - \varepsilon$ and $s = \cos\theta$ ,

$$\frac{Q_n(\cos\theta)}{P_n(\cos\theta)} \sim -\frac{\pi}{2}\tan[(n+\frac{1}{2})\theta - \pi/4] \qquad (4.16)$$

see Abramowitz & Segun [8.10.7].

Further we can obtain approximations to the spherical Bessel functions

$y_n(x) = \sqrt{\frac{\pi}{2x}} I_{n+\frac{1}{2}}$ by inverting the rational functions for $Y_n$ he

form of the error we can obtain immediately from (4.14).

since $\mathcal{L}\frac{I_v(x)}{x} = \frac{1}{v}\{s-\sqrt{s^2-1})^v$, the error in approximating y(x) by

$\mathcal{L}^{-1} \frac{C_n(s)}{D_n(s)}$ is

$$\mathcal{L}^{-1}\frac{Q_n}{P_n} \sim \pi(2n+1)\frac{I2n+1(x)}{x} = \frac{\pi x^{2n}}{2^{2n+1}(2n)!}[1+0(x^2)] \qquad (4.17)$$

for $|x|$ small.

The approximations will be good even for moderately large x.
Some values of convergents of (4.6) with estimates of the error (4.14) are listed in Table B. These indicate the effectiveness of (4.6) in approximating the Legendre functions, and the reliability of the error estimate for s not near the cut [-1,1]. Table C shows some approximations to $\sqrt{\frac{\pi}{2x}} I_{n+\frac{1}{2}}$ x) obtained by inverting the tenth convergent of (4.6). Table C should be considered in conjunction with Table D.

Lender's Associated Functions.

the result (4.6), The solutions of Legendre's associated equation

$$(1-s^2)\frac{d^2Y}{ds^2}-2s\frac{dy}{ds}+\left\{n(n+1-\frac{m^2}{1-s^2}\right\}Y=0 \tag{4.18}$$

are often defined in terms of ordinary Legendre functions; and in particular for the second kind functions

$$Y_n^m(s)=(s^2-1)^{m/2}\frac{d^mY_n}{ds^m}. \tag{4.19}$$

Using this result the generalisation of the recurrence relation (4.5) is readily obtained, and is

$$(n+1-m)Y_{n+1}^m-(2n+1)sY_n^m+(n+m)Y_{n-1}^m=0. \tag{4.20}$$

This relation holds even for complex m and n. However restricting n to be an integer, leading m arbitrary, and analysising the n = 0 case, we were able to relate the solutions of (4.18) to the Continued fraction

$$\frac{(s+1)^m-(s-1)^m}{(s+1)^m+(s-1)^m}=\frac{m}{s+}\,\frac{m^2-1}{3s+}\,\frac{m^2-2^2}{5s+}-\frac{m^2-n^2}{(2n+1)s+}\;-----\tag{4.21}$$

which is a generalisation of (4.6). In fact (4.6)is obtained if we let $m \to 0$.

To avoid considerable algebra, we will now use our conclusions to obtain the solution of (4.18) directly. First we write

$$\left(\frac{s+1}{s-1}\right)^m=1+\frac{2m}{s-1+}\,\frac{m^2-1}{3s+}\,\frac{m^2-2^2}{5s+}---\frac{m^2-m^2}{(2n+1)s+}\tag{4.22}$$

and denote its partial numerators and denominators by $D_n^m(s)$ and

$D_n^{-m}$ respectively.$D_n^m(s)=(-1)^nD_n^{-m}(-s)$.

Now the two solutions of (4.18) for n = 0 are

$$\left(.\frac{s+1}{s-1}\right)^{m/2} \text{and} \left(\frac{s-1}{s+1}\right)^{m/2}.$$

Putting $Y=\left(\frac{s+1}{s-1}\right)^{m/2} w$ in (4.8) we obtain

$$(1-s^2)\frac{d^2w}{ds^2}-2(s-m)\frac{dw}{ds}+n(n+1)w=0 \tag{4.23}$$

a simple generalisation of Legeridre' s equation. This equation has a polynomial solution, in fact $D_n^{-m}(s)$.

Similarly, putting $Y=\left(\frac{s-1}{s+1}\right)^{m/2} w^*$ in (4.18) gives

$$(1-s^2)\frac{d^2w^*}{ds^2}-2(s+m)\frac{dw^*}{ds}+n(n+1)w^*=0 \tag{4.24}$$

in this case the polynomial solution is $D_n^m(s)$.

The complete solution of (4.18) , for n an integer and m≠ an integer is,

$$\underline{Y_n^m = AD_n^{-m}(s)\left(\frac{s+1}{s-1}\right)^{m/2} + BD_n^m(s)\left(\frac{s-1}{s+1}\right)^{m/2}}, \tag{4.25}$$

where the D s satisfy the same recurrence relation

$$D_{n+1}(s) = (2n + 1)sD_n + (m^2-n^2)\ D_{n-1}, \tag{4.26}$$

the initial values being

$$D_0^{-m}=1=D_0^m, D_1^{-m}=s-m, D_1^m=s+m.$$

The case m = o has already been discussed under Legendre functions, for other integer values of m the appropriate limiting form of (4.21) must be considered.

In addition to approximating functions directly, frequently our rational functions can be interpreted as transforms. To emphasise this point we derive approximations for Bessel functions of the first kind from a C.F. for their Laplace transform. Again we deduce useful error estimates.

The clue on how to attack Bessel functions lies in (4.4) and the subsequent analysis in which the Laplace transform of

$$y_n(x) = \sqrt{\frac{\pi}{2\times} I_{n+\frac{1}{2}}(x)}$$

is approximated by rational functions.

5. Bessel Functions.

The modified Bessel functions $I_v$ (x) satisfy the recurrence relation

$$I_{v+n+1} = 2\frac{dI_{v+n}}{dx} - I_{v+n-1} \tag{5.1}$$

where n is a positive integer.

First we remove the facfcpr $x^v$ from $I_{v+n}$ and take $: z_n^v \equiv \frac{I_{v+n}}{x^v}$.
Unless we wish to make the v dependence explicit we will

write $z_n$ without the v. We obtain

$$z_{n+1} = \frac{2}{x^v}\frac{d}{dx}[x^v z_n] - z_{n-1}$$

$$= 2\frac{dz_n}{dz} + \frac{2v}{x}z_n - z_{n-1}.$$

Now $2\frac{(v+n)}{x}z_n = z_{n-1} - z_{n+1}$, eliminating $\frac{z_n}{x}$ we get

$$(2v+n)z_{n+1} = 2(v+n)\frac{dz_n}{dx} - nz_{n-1} \tag{5.2}$$

(When v = ½ , this reduces to the relation (4.3) for modified spherical Bessel functions.)

In particular for n = 0 $\frac{I_{v+1}}{x^v} = \frac{d}{dx}\left\{\frac{I_v}{x_v}\right\}$, (5.3)

Where $$\frac{I_v}{x^v} = \frac{1}{2^v r(v+1)}\sum_{r=0}^{\infty}\frac{(x/2)^{2r}}{r!(v+1)_r}. \tag{5.4}$$

Taking the Laplace transform of the relations (5.2) and (5.3) and

Denoting $\mathcal{L}\left\{\frac{I_{v+n}}{x^v}\right\}$by $z_n^v$ produces the following linear equations

$$z_1 = sZ_0 - \frac{1}{2^v r(v+1)}$$

$$(2v+1)\ Z_2 = 2(v+1)sZ_1 - Z$$

$$\text{---------} \tag{5.5}$$

$$(2v+n)\ Z_{n+1} = 2(v+n)sZ_n - nZ_{n-1}$$

which can be written

$$z \equiv \mathcal{L}\left\{\frac{I_v}{x_v}\right\} = \frac{{}^1\!/_2{}^{v}\ r(v+1)^{-1}}{s - \frac{z_1}{z}} \qquad (5.6)$$

$$z^{v}\ r\,(v+1)\mathcal{L}\left\{\frac{I_v}{x^{v}}\right\} = \frac{2v}{2vs-}\,\frac{2}{2(v+1)s-}\,\frac{2(2v+1)}{2(v+2)s-}\,---\,\frac{n(2v+n-1)}{2(v+n)s-}(2v+n)\frac{z_{n+1}}{z_n}.$$

Clearly v cannot be a negative integer. It is readily verified that the expression on the right progressively matches terms in the series in $\frac{1}{s}$ of the transform. Terminating (5.6) will give rational functions which, when expanded by partial fractions and inverted, will give approximations to Bessel functions I . We must of course discuss the nature of the function that is being approximated. But first we should observe that the denominator polynomials are essentially the Gegenbauer polynomials.

The denominator polynomials satisfy the recurrence relation

$$D^{v}_{n+1} = 2(v+n)n^{v}_{n} - n(2v+n-1)D^{v}_{n-1} \qquad (5.7)$$

and the generating function for them is

$$(1+t^2-2st)^{-v} = \sum_{n=0}^{\infty} D^{v}_{n}(s)\frac{t^n}{n!} \qquad (5.8)$$

and thus $D^{s}_{n}(s) = n!G^{v}_{n}(s)$, where $G^{v}{}_{n}(s)$ the Gegenbauer polynomials*.

To establish the nature of the functions and the restrictions on v, consider the effect of the substitution $y=x^{v}z$ on Bessel's equation,

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2+v^2)y = 0. \qquad (5.\ 9)$$

we get

$$x\frac{d^2z}{dx^2} + (2v+1)\frac{dz}{dx} - xz = 0. \qquad (5.10)$$

* The Gegenbauer polynomials are usually denoted by $C_n{}^{v}(s)$.

Laplace transforming this equation, not worrying unduly about initial conditions, gives the first order equation

$$(s^2-1)\frac{dz}{ds}+(1-2v)sZ=2vA \qquad (5.11)$$

whose solution is

$$z=A2v(s^2-1)^{v-\frac{1}{2}}\int^s \frac{ds}{(s^2-1)^{\frac{1}{2}+v}}+B(s^2-1)^{v-\frac{1}{2}} \qquad (5.12)$$

One solution of this equation for all v (≠ negative integer) is $\mathcal{L}\frac{I_v}{x^v}$, the second solution is, for v ≠ integer,

$$\mathcal{L}\left\{\frac{I_{-v}(x)}{x^v}\right\}=2^v\frac{r(1-2v)}{r(1-2v)}\frac{1}{(s^2-1)^{\frac{1}{2}v}} \qquad Rv<\frac{1}{2}, \qquad (5.13)$$

and when v = 0 it is

$$\mathcal{L}\{k_0(x)\}=\frac{\cos^{-1}s}{(s^2-1^{\frac{1}{2}}} \qquad s>1. \qquad (5.14)$$

When v = 0 (5.6) exists and is simply an expression for $\frac{1}{(s^2-1)}\frac{1}{2}\equiv\mathcal{L}I_0$, a result used very effectively by Murphy .[3.1].

Notice with $v=\frac{1}{2}$ that (5.6) reduces to (4.6).

The pattern we have established suggests that $z_n^v(s)$ and $D_n^v(s)$ will be the solutions, apart from a factor, of a second order differential equation. This equation and its complete solution will be of interest in their own right. But the connection with Bessel's equation is important ,so for completeness let us start with Bessel's equation, we take $-\frac{1}{2}\le Rv<\frac{1}{2}$,

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-\left[x^2+(v+n)^2\right]y=0. \qquad (5.15)$$

Again putting y = $x^v$z and taking the Laplace transform, we obtain the differential equation

$$(s^2-1)\frac{d^2z}{ds^2}+(3-2v)s\frac{dZ}{ds}+(n+1)(1-2v-n)z=0 \qquad (5.16)$$

The equation (5.11) is just a particular case of this equation, and its solution (5.12) indicates the further substitution that will lead us to the differential equation with the polynomial solutions $D_n^v(s)$.

Putting W = $(s^2 - 1)^{\lambda}$Z, and choosing $\lambda = \frac{1}{2} - v$, (5.16) becomes

$$(s^2-1)\frac{d^2w}{ds^2}+(2v+1)s\frac{dW}{ds}-n(n+2v)w=0 \qquad (5.17)$$

and the complete solution of this equation is

$$\underline{W = AD_n^v(s) + B(s^2-1)^{\frac{1}{2}-v} Z_n^v(s)} \qquad v \neq 0. \qquad (5.18)$$

v = 0 must be treated separately.

There is clearly a strong link between (5.17) and the equations that arose in discussing Legendre's associated equation, we will deal with this generalisation in another paper. Although we have developed the above in terms of the normally accepted parameter v, in many ways λ is a more natural parameter.

We will round off this discussion of Bessel functions by obtaining the Stieltjes integral form for $\mathcal{L}\left\{\frac{I_{v+n}}{x^v}\right\}$ and so establish the integral relation between the two solutions of (5.16). Now

$$\frac{I_v}{x^v} = \frac{1}{2^v\sqrt{\pi}\Gamma(v+\frac{1}{2})}\int_{-1}^{1} e^{xt}(1-t^2)^{v-\frac{1}{2}}dt, \qquad Rv > -\frac{1}{2} \qquad (5.19)$$

using (5.3), then inductively the recurrence relation (5.2) we soon establish that

$$z_n^v \equiv \frac{I_{v+n}}{x^v} = \frac{1}{2^v\sqrt{\pi}\Gamma(v+\frac{1}{2})}\cdot\frac{\Gamma(2v)}{\Gamma(2v+n)}\int_{-1}^{1} e^{xt}(1-t^2)^{v-\frac{1}{2}} D_n^v(t)\, dt$$

and hence that

$$\underline{\mathcal{L}\left\{\frac{I_{v+n}}{x^v}\right\} = c\int_{-1}^{1}\frac{(1-t^2)^{v-\frac{1}{2}}D_n^v(t)dt}{s-t},} \qquad Rv > -\frac{1}{2} \qquad (5.20)$$

where C = $\dfrac{\Gamma(2v)}{2^v\sqrt{\pi}\Gamma(v+\frac{1}{2})\Gamma(2v+n)}$.