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LOCAL MESH REFINEMENT WITH FINITE ELEMENTS FOR ELLIPTIC PROBLEMS

by

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1. Introduction

When a finite element technique is used for the solution of a two dimensional elliptic boundary value problem, P, the region of definition Ω of the problem is divided in a number of non-overlapping elements. In this paper the region Ω is rectangular, the elements are all rectangles and we consider a technique of local mesh refinement.

A weak formulation of the problem P is constructed and it is the solution of this weak problem, the generalized solution of P lying in a larger space, W, rather than the strong solution of P, which is sought. In the Galerkin technique an approximation U(x,y) to the generalized solution u(x,y) is constructed from a finite-dimensional space S^h, which is usually a subspace of W, where h is a parameter designating the size of the elements. The key step in the successful application of the Galerkin method is the construction of S^h which generally consists of functions which are piecewise polynomial over Ω. In each element the approximating function is derived from an interpolation function which interpolates the values of u, and frequently also certain derivatives of u, at nodes in the element.

Let the interpolant in each element have the form

$$\tilde{u}(x,y) = \sum_{i=1}^{k} u_i \phi_i(x,y) , \qquad (1.1)$$

where the Φ_i. are the cardinal basis functions of the interpolation with respect to u_i, the function and derivative values at the nodal points. The approximating function then has in each element the form

$$U(x,y) = \sum_{i=1}^{k} U_i \phi_i(x,y) , . \qquad (1.2)$$

where now the U_i. are the corresponding values of U and certain of its derivatives at the nodal points of the element. The unknown

values of U_i are determined by the Galerkin method. The functions ϕ_i considered over the totality of elements generate the finite dimensional space S^h .

The global piecewise polynomial approximating function must satisfy certain continuity requirements across interelement boundaries in order that S^h be a subspace of W. This is the conforming condition. For Poisson type problems the conforming condition is that $S^h \subset C^o(\bar{\Omega})$, whilst for biharmonic problems it is that $S^h \subset C^1(\bar{\Omega})$; see e.g. Zlamal [10].

When a standard rectangular mesh is taken together with bilinear interpolation to the function values at the four corners of each element, the global approximating function is in C^o and the conforming condition for Poisson problems is satisfied. In this case the Lagrange basis functions in each element are the linear pyramid functions of the finite element method. However, if the mesh is refined locally about some point 0, as for example in Figure 1, mid-side nodes are introduced and special interpolants must be derived for use in these five-node rectangle elements so that the global approximating function may again be in C^o. Such interpolants are derived in Section 2.

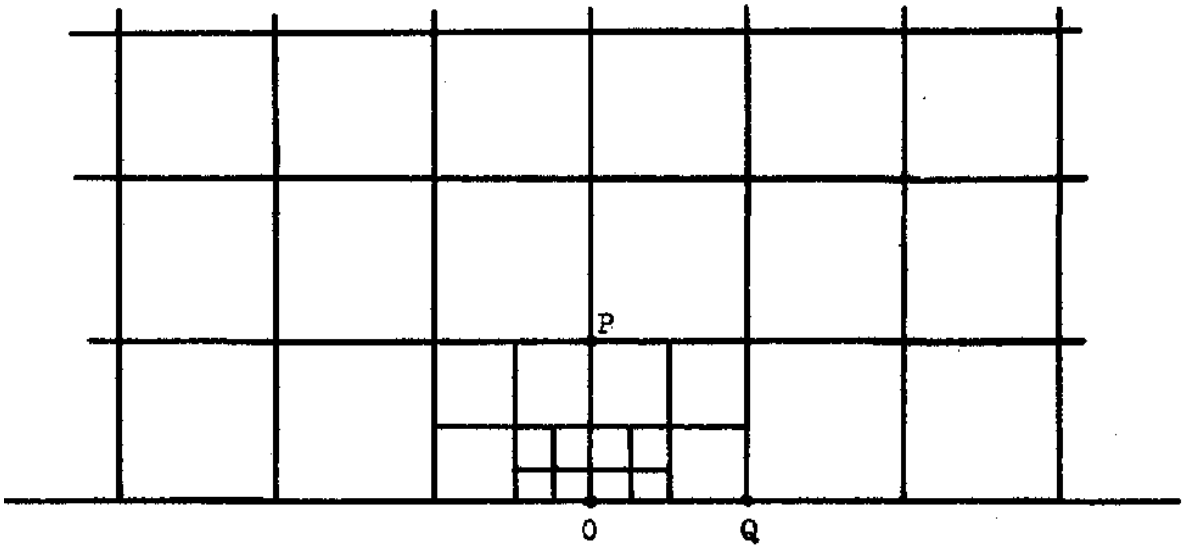

Figure 1

For biharmonic problems, when a standard rectangular mesh is used, together with a bicubic interpolant in each element to the values of $u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ and $\frac{\partial^2 u}{\partial x \partial y}$ at each of the four corners, the global approximating function will be in C1 . Again, if the mesh is refined locally, special interpolants are necessary in the five-node elements for the global approximating function to he in C^1 . These are derived in Section 3.

In Section 4 a Galerkin procedure is described briefly for a problem involving Laplace's equation, and in Section 5 numerical results are given.

2. C^o approximating functions

We consider the unit square with vertices at (0,0), (1,0), (1,1) and (0,1). The linear interpolant to the values $U(0,0)$ and $U(1,0)$ along [0,1] can be written as

$$U(x,0) = (1-x)\ U(0,0) + x\ U(1,0) \ . \qquad (2.1)$$

By using tensor products we see immediately that the bilinear interpolant to $U(0,0)$, $U(1,0)$, $U(1,1)$ and $U(0,1)$ over the square is

$$U(x,y) = (1-x)(1-y)U(0,0)+x(1-y)U(1,0)+xyU(1,1)+(1-x)yU(0,1),$$

$$= \sum_{i=1}^{4} U_i \ \phi_i \ (x,y) \ . \qquad (2.2)$$

The ϕ_i. are the basis functions referred to in Section 1, and use of (2.2) in each element of a regular mesh produces a C° approximating function.

When the mesh is refined locally by successive halving of the mesh length, as in Figure 1, mid-side nodes are introduced

and situations as in Figure 2 arise. An obvious approach for

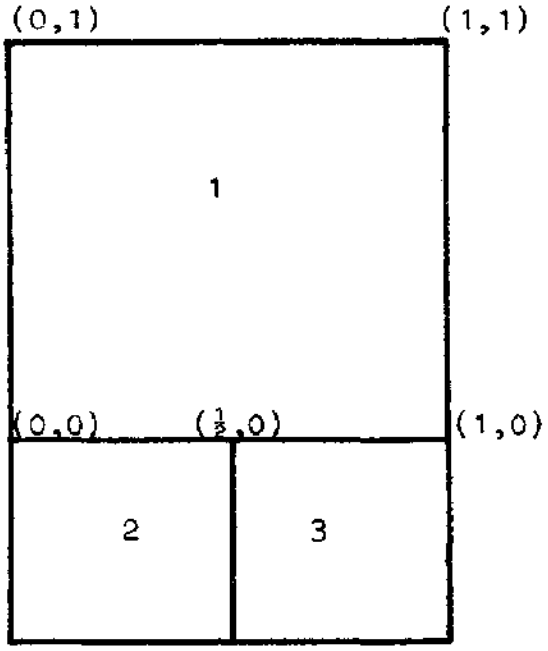

Figure 2

dealing with the mid-side node $(\frac{1}{2},0)$ is to take (2.2) in element 1, which thus gives the value at $(\frac{1}{2},0)$ as the linear interpolant "between (0,0) and (1,0). This can then be used directly when (2.2), suitably scaled;, is applied in elements 2 and 3. Thus the unknown value at $(\frac{1}{2},0)$ is not introduced in element 1. However, the effect of this scheme is to spread the domain of influence of the coarse mesh into the region of fine mesh. The effect of the refinement is therefore reduced; see e.g. Wait and Mitchell [5] where this procedure is adopted. In order to avoid this we choose the alternative scheme given below.

A suitably scaled form of (2.2) is used in each of the elements 2 and 3. This function in element 2 interpolates U(0,0) and $U(\frac{1}{2},0)$, whilst at the same time it is linear on $L_{12} = \{(x,y); 0 \leqq x \leqq \frac{1}{2}, y=0\}$. Similarly in 3 the function interpolates $U(\frac{1}{2},0)$ and U(1,0) and is linear on $L_{13} \equiv \{(x,y); \frac{1}{2} \leqq x \leqq 1, y=0\}$. As there is a node at $(\frac{1}{2},0)$ in element 1,

the interpolant (2.2) has to be adapted so that in 1 it will interpolate the value $U(\frac{1}{2},0)$ as well as the values at the four corners of the element, whilst being linear on L_{12} and L_{13}. The piecewise polynomial function over the union of the elements 1,2 and 3 will then again be in C^o.

The technique is to consider separately the two rectangles

$$R_1 \equiv \{(x,y)\,;\, 0 \leqq x \leqq \frac{1}{2},\quad 0 \leqq y \leqq 1\} \tag{2.3}$$

and

$$R_2 = \{(x,y);\ \frac{1}{2} \leqq x \leqq 1,\quad 0 \leqq y \leqq 1\}\ . \tag{2.4}$$

In R_1 the interpolating function is

$$U(x,y) = (1-2x)(1-y)U(0,0) + 2x(1-y)U(\frac{1}{2},0) + 2xy\,U(\frac{1}{2},1) + (1-2x)yU(0,1). \tag{2.5}$$

However, the point $(2,1)$ is not a node of the element 1, and so the value $U(\frac{1}{2},1)$ is eliminated using the continuity of the approximating function across $\{(x,y);\ 0 \leqq x \leqq 1, y = 1\}$ by the substitution of

$$U(\frac{1}{2},1) = \frac{1}{2}(U(0,1) + U(1,1)).$$

Thus for $(x,y)\ \varepsilon\ R_1$

$$U(x,y) = (1-2x)(1-y)U(0,0) + 2x(1-y)U(\frac{1}{2},0) + xy\,U(1,1) + y(1-x)U(0,1). \tag{2.6}$$

A similar technique is adopted in R_2 so that in element 1

$$U(x,y) = y(1-x)U(0,1) + xy\,U(1,1) + \begin{cases} (1-2x)(1-y)U(0,0) + 2x(1-y)U(\frac{1}{2},0), & 0 \leqq x \leqq \frac{1}{2}, \\ 2(1-x)(1-y)U(\frac{1}{2},0) + (2x-1)(1-y)U(1,0), & \frac{1}{2} \leqq x \leqq 1. \end{cases} \tag{2.7}$$

The function U(x,y) in (2.7) is bilinear in R_1 and R_2 and continuous across $x=\frac{1}{2}$, $0 \leqq y \leqq 1$. Use of trial functions of the type (2.7) in the five node elements together with bilinear trial functions in the standard elements will ensure that the resulting global approximating function is in C^o.

Note that, in terms of "+ functions" commonly used in splines, it is possible to write

$$\begin{aligned} U(x,y) = {} & y(1-x)U(0,1) + xy\,U(1,1) \\ & + (1-2x)(1-y)U(0,0) + 2x(1-y)U(\tfrac{1}{2},0) \\ & + (1-y)(2x-1)_+ \quad [U(0,0) - 2U(\tfrac{1}{2},0) + U(1,0)] \ , \end{aligned} \tag{2.8}$$

where

$$(2x-1)_+ = \begin{cases} (2x-1) \ , & x \geqq \frac{1}{2} \ , \\ 0 \ , & x \leqq \frac{1}{2} \ . \end{cases}$$

3. C^1 Approximating Functions

The notation

$$U_{1,0}(x,y) \equiv \frac{\partial U(x,y)}{\partial x}, \ U_{0,1}(x,y) \equiv \frac{\partial U(x,y)}{\partial y}, \ U_{1,1}(x,y) = \frac{\partial^2 U(x,y)}{\partial x \partial y}$$

is adopted, and with this notation the cubic interpolant to U(0,0), U(1,0), $U_{1,0}(0,0)$, $U_{1,0}(0,1)$ over [0,1] can be written

$$U(x,0) = \phi_1(x)U(0,0) + \phi_2(x)U_{1,0}(0,0) + \phi_3(x)U(1,0) + \phi_4(x)U_{1,0}(1,0) \ , \tag{3.1}$$

where

$$\left.\begin{aligned} \phi_1(t) &= (t-1)^2(2t+1), \\ \phi_2(t) &= (t-1)^2 t, \\ \phi_3(t) &\equiv \phi_1(1-t) = t^2(-2t+3) \ , \\ \phi_4(t) &\equiv -\phi_2(1-t) = t^2(t-1). \end{aligned}\right\} \tag{3.2}$$

The ϕ's are the cardinal basis functions for Hermite interpolation and it is noted that, if $\phi'(t) = d\phi(t)/dt$,

$$\phi_1(0) = \phi_3(1). = \phi_2(0) = \phi_4'(1) = 1 \ ,$$

$$\phi_2(0) = \phi_3(0) = \phi_4(0) = 0 \ ,$$

$$\phi_1(0) = \phi_3(0) = \phi_4(0) = 0 \ ,$$

$$\phi'_1(0) = \phi'_3(0) = \phi'_4(0) = 0 \ ,$$

$$\phi'_1(1) = \phi'_2(1) = \phi'_3(1) = 0 \ .$$

Taking tensor products we obtain the bicubic interpolant to

$$Z \equiv \{U(x_i,y_i),\ U_{1,0}(x_i, y_i),U_{0,1}(x_i,y.),\ U_{1,1}(x_i,y_i)\} \ , \qquad (3.3)$$

at respectively each of the four points $(x_i\, y_j) = (0,0),(1,0),(1,1),(0,1)$ over the unit square as

$$\begin{aligned} U(x,y) = {} & \phi_1(x)\ [\phi_1(y)U(0,0) + \phi_2(y)U_{0.1}(0,0) + \phi_3(y)U(0,l) + \phi_4(y)U_{0,1},(0,1) \\ & + \phi_2(x)[\ \phi_1(y)U_1,0(0) + \phi_2(y)U_{1,1}(0,0) + \phi_3(y)U_{1,0}(0,1) + \phi_4(y)U_{1,1}(0,1)] \\ & + \phi_3(x)\ [\phi_1(y)U(1,0) + \phi_2(y)U_{0,1}\,(1,0) + \phi_3\ (y)U(l,l) + {}_4(y)U_{0,1}\,(1,1)] \\ & + \phi_4(x)[\ \phi_1(y)U_{\ 1,\ 0}(1,0) + \phi_2(y)U_{1,1}(1,0) + \phi_3(y)U_{1,0}(1,1) + \phi_4(y)U_{1,1}(1,1)], \end{aligned}$$

(3.4)

where the ϕ's are as in (3.2). Use of (3.4) as the trial function in. each element of a standard rectangular mesh, together with the specifying of Z as in (3.3) at each node, will produce a C^1 approximating function. However, we wish to refine the mesh as in Figure 1, whilst retaining C^1 continuity in the global approximating function. Referring again to the situation as in Figure 2, a special trial function is thus needed in elements such as 1. Following Section 2 we split the element 1 into the two rectangles R_1 and R_2 of (2.3) and (2.4). The bicubic interpolant to the vales of Z at the

four vertices of R_1 is, from (3.4),

$$u(x.y) = \phi_1(2x)[\phi_1(y)U(0,0) + \phi_2(y)U_{0,1}(0,0) + \phi_3(y)U(0,1) + \phi_4(y)U_{0,1}(0,1)]$$
$$+ \frac{1}{2}\phi_2(2x)[\phi_1(y)U_{1,0}(0,0) + \phi_2(y)U_{1,1}(0,0) + \phi_3(y)U_{1,0}(0,1) + \phi_4(y)U_{1,1}(0.1)]$$
$$+ \phi_3(2x)[\phi_1(y)U(\frac{1}{2},0) + \phi_2(y)U_{0,1}(\frac{1}{2},0) + \phi_3(y)U(\frac{1}{2},1) + \phi_4(y)U_{0,1}(\frac{1}{2},1)]$$
$$+ \frac{1}{2}\phi_4(2x)[\phi_1(y)U_{1,0}(\frac{1}{2},0) + \phi_2(y)U_{1.1}(\frac{1}{2},0) + \phi_3(y)U_{1.0}(\frac{1}{2},1) + \phi_4(y)U_{1,1}(\frac{1}{2},1)]. \quad (3.5)$$

The right hand side of (3.5) involves values of Z at the point $(\frac{1}{2},1)$, which is not a node of the discretization. In order that these values may be eliminated interpolants having the form (3.1) are used on $\{(x,y) ; 0 \leqq x \leqq 1, y = 1\}$, so that

$$U(\frac{1}{2},1) = \phi_1(\frac{1}{2})U(0,1) + \phi_2(\frac{1}{2})U_{1,0}(0,1) + \phi_3(\frac{1}{2})U(1,1) + \phi_4(\frac{1}{2})U_{1,0}(1,1), \quad (3.6)$$

$$U_{0,1}(\frac{1}{2},1) = \phi_1(\frac{1}{2})U_{0,1}(0,1) + \phi_2(\frac{1}{2})U_{1,1}(0,1) + \phi_3(\frac{1}{2})U_{0,1}(1,1) + \phi_4(\frac{1}{2})U_{1,1}(1,1), \quad (3.7)$$

$$U_{1,0}(\frac{1}{2},1) = \phi_1(\frac{1}{2})U(0,1) + \phi_2(\frac{1}{2})U_{1,0}(0,1) + \phi_3(\frac{1}{2})U(1,1) + \phi_4(\frac{1}{2})U_{1,0}(1,1), \quad (3.8)$$

$$U_{1,1}(\frac{1}{2},1) = \phi_1(\frac{1}{2})U_{0,1}(0,1) + \phi_2(\frac{1}{2})U_{1,1}(0,1) + \phi_3(\frac{1}{2})U_{0,1}(1,1) + \phi_4(\frac{1}{2})U_{1,1}(1,1). \quad (3.9)$$

But

$$\phi_1(\frac{1}{2}) = \phi_3(\frac{1}{2}) = \frac{1}{2}, \qquad \phi_2(\frac{1}{2}) = -\phi_4(\frac{1}{2}) - \frac{1}{8},$$

$$\phi'_1(\frac{1}{2}) = -\frac{3}{2}. \qquad \phi'_2(\frac{1}{2}) = \phi_4(\frac{1}{2}) = -\frac{1}{4}, \qquad \phi_3(\frac{1}{2}) = \frac{3}{2}.$$

Substitution of these values into (3.6) - (3.9) and the subsequent substitution of the resulting expressions for Z at $(\frac{1}{2},1)$ in (3.5)

leads to

$$
\begin{aligned}
U(x,y) &= \phi_1(y)[\Phi_1(2x)U(0,0) + \tfrac{1}{2}\phi_2(2x)U_{1,0}(0,0) + \phi_3(2x)U(\tfrac{1}{2},0) + \tfrac{1}{2}\phi_4(x)U_{1,0}(\tfrac{1}{2},0)] \\
&+ \phi_2(y)[\phi_1(2x)U_{1,0}(0,0) + \tfrac{1}{2}{}_2(2x)U_{1,1}(0.0) + \phi_3(2x)U_{1,0}(\tfrac{1}{2},0) \\
&\quad + \tfrac{1}{2}\phi_4(2x)\,U_{1,1}(\tfrac{1}{2},0)] \\
&+ \phi_3(y)[\{\phi_1(2x) + \tfrac{1}{2}\phi_3 + {}_3(2x) - \tfrac{3}{4}\phi_4(2x)\}\,U(0,1) \\
&\quad + \{\tfrac{1}{2}\phi_2(2x) + \tfrac{1}{8}\phi_3(2x) - \tfrac{1}{8}\phi_4(2x)\}U_{1,0}(0,1)] \\
&\quad + \{\tfrac{1}{2}\phi_3(2x) + \tfrac{3}{4}\phi_4(2x)\}U(1,1) - \tfrac{1}{8}\{\phi_3(2x) + \phi_4(2x)\}U_{1,0}(1,1)] \\
&+ \phi_4(y)[\{\phi_1(2x) + \tfrac{1}{2}\phi_3(2x) - \tfrac{3}{4}\phi_4(2x)\}U_{0,1}(0,1) \\
&\quad + \{\tfrac{1}{2}\phi_2(2x) + \tfrac{1}{8}\phi_3(2x) + \tfrac{1}{8}\phi_4(2x)\}U_{1,1}(0,1) \\
&\quad + \{\tfrac{1}{2}\phi_3(2x) + \tfrac{3}{4}\phi_4(2x)\}U_{0,1}(1,1) - \tfrac{1}{8}\{\phi_3(2x) + \phi_4(2x)\}U_{1,1}(1,1)]
\end{aligned}
\tag{3.10}
$$

for $0 \leqq x \leqq \frac{1}{2}$, $\qquad 0 \leqq y \leqq 1.$

An expression of a similar form to that in (3.10) is obtained for the interpolant in $R_2 \equiv \{(x.y);\ \frac{1}{2} \leqq x \leqq 1, 0 \leqq y \leqq 1\}$.

This taken together with the interpolant (3-10) produces in element 1 an interpolant which is C^1 and which is cubic on the sides, the top and each of the halves of the bottom of the element. Incorporation of this into the space of piecewise bicubic functions will ensure that the resulting global approximating function is in C^1 .

4. Galerkin Method For Model Problem

We consider the problem in which u(x,y) satisfies

$$\left.\begin{aligned} -\Delta[u(x,y)] &= 0 \ , && (x,y)\ \varepsilon\ \Omega \ , \\ u(x,y) &= 500 \ , && (x,y)\ \varepsilon\ \overline{BC}, \\ u(x,y) &= 0 \ , && (x,y)\ \varepsilon\ \overline{EO} \ , \\ \frac{\partial u(x,y)}{\partial y} &= 0 \ , && (x,y)\ \varepsilon\ \overline{OB} \cup \overline{CD} \ , \\ \frac{\partial u(x,y)}{\partial x} &= 0 \ , && (x,y)\ \varepsilon\ \overline{DE} \ , \end{aligned}\right\} \qquad (4.1)$$

where Ω is the rectangular region OBCDEO of Figure 3, in which EO=OB=BC=0.5.

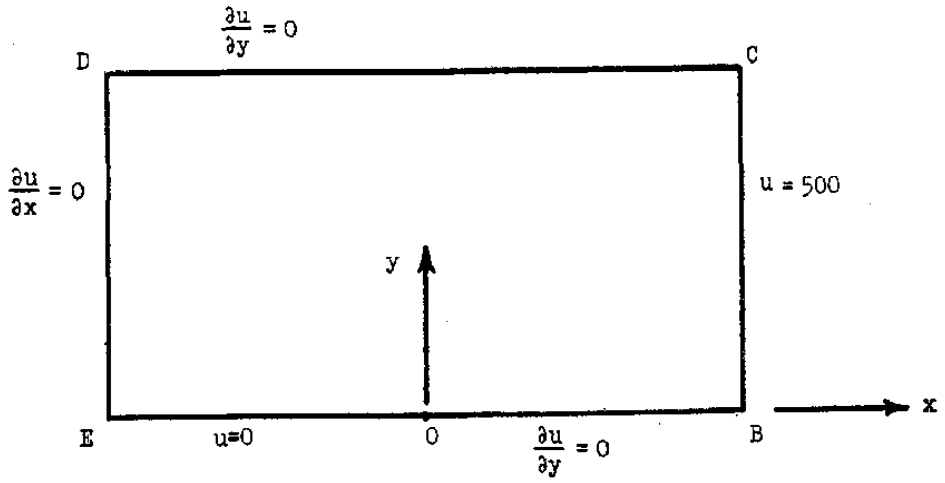

Figure 3

The problem(4.1) is derived using symmetry from a well known problem in a rectangle containing a slit which has been much studied; see for example Whiteman [6], [7] and Wait and Mitchell [5] . We define the two disjoint parts of the boundary $\partial\Omega_1 \equiv \overline{BC} \cup \overline{EO}$, $\partial\Omega_2 \equiv \overline{OB} \cup \overline{CD} \cup \overline{DE}$ and let $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ with $\overline{\Omega} \equiv \Omega \cup \partial\Omega$.

Let $W^1{}_2(\Omega)$ be the Sobolev space of functions which together with their generalized derivatives of order one are in $L_2(\Omega)$. The subspace of functions in $W^1{}_2(\Omega)$ which satisfy a homogeneous boundary condition on $\partial\Omega_1$ is written $W^1{}_2(\Omega) \cap (\partial\Omega_1)_0$; that is for v e $W^1{}_2(\Omega) \cap (\partial\Omega_1)_0$,

$V \varepsilon W^1{}_2(\Omega)$ and $v = 0$ on $\partial\Omega_1$.

The weak problem corresponding to (4.1) is ;

find $u \varepsilon f + W^1{}_2(\Omega)$ such that

$$a(u,v) = 0 \qquad \forall \quad v \varepsilon W^1{}_2(\Omega) \cap (\partial\Omega_1)_0 \ , \tag{4.2}$$

where $f \varepsilon W^1{}_2(\overline{\Omega})$ with $f=500$ on $\overline{BC}$ and $f=0$ on $\overline{EO}$

The notation $u \varepsilon f + W^1{}_2(\Omega)$ means that $u = f+v$, where $v \varepsilon W^1{}_2(\Omega) \cap (\partial\Omega_1)_0$.

In (4.2) the bilinear functional $a(u,v)$ is defined as

$$a(u,v) = \iint_{\Omega} \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) dx \ dy \quad \forall \ u, v \ \varepsilon \ W_2^1(\Omega) .$$

The energy norm $||v||_E$ is defined by

$$||v||_E = (a(v.v)^{\frac{1}{2}} . \tag{4.3}$$

The region Ω is discretized into rectangular elements of generic length h so that there are m internal nodes, n nodes on $\partial\Omega_1$ and p nodes on $\partial\Omega_2$. The global approximating function $U \varepsilon S^h$ is written as

$$U(x,y) = \sum_{i=1}^{m+p} U_i \ B_i(x,y) + \sum_{j=1}^{n} f_j^h \ C_j(x,y) \ , \tag{4.4}$$

where the f^h_j. are the known values of $u = U$ at the nodes of $\partial\Omega_1$ and the B_i and C_j are formed from the Φ_i of (1,2). The B_i and C_j. are linear pyramid basis functions at respectively the nodal points of Ω U $\partial\Omega_2$ and the nodal points of $\partial\Omega_1$

In the Galerkin procedure we seek U as in (4.4) such that

$$a(U, B_k) = 0 \ , \qquad k = 1, 2, \ ... \ , m+p. \tag{4. 5}$$

Equations (4.5) are the normal equations for the solution of the unknown coefficients U_i. Bounds on the error in the Galerkin solution U, having the form

$$||u-U||_E \leq Kh|u|_2 \quad , \tag{4.6}$$

where

$$|u|_2 = \left\{ \left\| \frac{\partial^2 u}{\partial x^2} \right\|^2_{L_2(\Omega)} + \left\| \frac{\partial^2 u}{\partial x \partial y} \right\|^2_{L_2(\Omega)} + \left\| \frac{\partial^2 u}{\partial y^2} \right\|^2_{L_2(\Omega)} \right\} \quad ,$$

are well known; see e.g. Ciarlet and Raviart [3]. For problems containing boundary singularities, for example due to re-entrant corners such as in the problem in the slit rectangle which is equivalent to (4.1), the second derivatives of u are not in L_2. However, error bounds involving $h^{\pi/a}$ for problems containing re-entrant corners with interior angles $a > \pi$ have been derived; see Babuska and Aziz [l,p.274].

The effect of the singularity is to reduce the accuracy of the Galerkin solution, particularly in the neighbourhood of the re-entrant corner. It is shown by Babuska [2] that with "proper" refinement of the elements around the corners the effect of the singularity can be removed. For ease and automation of computation we now use the local refinement scheme of Section 2, which is based on successive halving of the mesh length to obtain accurate Galerkin solutions to the problem (4. 1).

5. Numerical Results

In applying the Galerkin technique of Section 4 to the problem (4.1) we use square elements and take the basis functions

$B_i(x,y)$ and $C_j(x,y)$ of (4.4) to be pyramids in the appropriate elements. Thus in each square of a standard mesh of length h the local trial function has the form (2.2). For the value h= 1/14 the results obtained at three specific points are given in Figure 4. For comparison accurate results obtained using a conformal transformation method (CTM) due to Whiteman and Papamichael [9] are also included in Figure 4. It is seen that, as expected, the greatest inaccuracies occur in the neighbourhood of 0.

The refinement scheme of Figure 1 is now used with the trial functions (2.4) in five node elements. We note that each level of refinement introduces 8 new equations into the system (4,5). The ordering of the nodes is chosen so that the inclusion of the extra equations is performed automatically. This is achieved by ordering peripherally about the singularity. In each case the matrix of coefficients is symmetric and positive definite, and full advantage is taken of the band structure. The three sample points $P \equiv (0, 1/14)$ and $Q \equiv (1/14, 0)$ near to 0 and $R \equiv (-3/7, 3/7)$ remote from 0, with the origin of co-ordinates at 0, are again chosen and results at these points are given in Figure 4 for levels of refinement ranging from 1 to 8 with the original mesh length h again taken as $1/14$.

It is seen that with continued local mesh refinement the stage has been reached where the Galerkin solutions are more accurate near the singularity than they are at a point in Ω remote from 0. The effect of the singularity on the numerical solution has thus been neutralized by the refinement. The error at points on the coarse mesh is due to the coarse mesh spacing.

An alternative technique to local mesh refinement is to include in the space S^h singular functions having the form of the singularity, as has been done by Fix [4] and Whiteman [8] . In order to do this one

has of course to know the form of the singularity. Further the augmentation of S^h poses difficult computational problems. We feel that local mesh refinement is a viable alternative.

Number of Levels of Refinement	Value at U (x, *y*) at			Number of Linear Equations
	P≡(0,$^1/_{14}$)	Q≡($^1/_{14}$,0)	R≡($^3/_7$,$^3/_7$)	
0 (h=1/14)	97.05	147.05	88.73	104
1	99.61	150.52	89.78	112
2	101.62	153.39	90.31	120
3	102.72	154.92	90.57	128
4	103.27	155.69	90.70	136
5	103.54	156.07	90.78	144
6	103.68	156.26	90.80	152
7	103.75	156.36	90.82	160
8	103.78	156.40	90.83	168
CTM [9] Results	103.77	156.48	91. 34	-

Figure 4

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