TR/21 December.1972.

A CLASS OF ALGORITHMS FOR RATIONAL APPROXIMATION OF FUNCTIONS FORMALLY DEFINED BY POWER SERIES

by

J.A, MURPHY and M.R. O'DONOHOE.

ABSTRACT

Corresponding sequence algorithms are defined and shown to exist for a wide range of corresponding continued fractions. Particular examples of these algorithms are given, including an algorithm for forming Pade approximants, and an error analysis is given in one case.

1. Continued Fractions

1.1 Corresponding Fractions

We denote a continued fraction by

$$\frac{u_1}{v_1} + \frac{u_2}{v_2} = + \dots + \frac{u_n}{v_n} \dots\dots \tag{1.1}$$

where u_n and v_n are numbers, real or complex. In this work we shall consider u_n and v_n to be polynomials in the complex variable z. The nth convergent of (1.1) is

$$\frac{U_n(z)}{V_n(z)} = \frac{u_1(z)}{v_1(z)} + \frac{u_2(z)}{v_2(z)} + \dots + \frac{u_n(z)}{v_n(z)} \tag{1.2}$$

where U_n and V_n both satisfy the recurrence formula

$$W_n = u_n W_{n-2} + v_n W_{n-1} \tag{1.3}$$

with initial values $U_0 = 0, U_1 = u_1$ and $V_0 = 1$, $V_1 = v_1$.

Clearly, $U_n(z)$ and $V_n(z)$ are also polynomials so that (1.2) is a rational approximation to (1.1).

We consider the function $f_0(z)$ formally defined by the power series expansion

$$f_0(z) = a_0 + a_1 z + a_2 z^2 + \dots, \tag{1.4}$$

convergent for $|z| < R$.

The continued fraction (1.1) is said to correspond to the power series (1.4) if

$$f_0(z) - \frac{U_n(z)}{V_n(z)} = 0(s^{\sigma(n)}), \tag{1.5}$$

for $|z| < R$, where $\sigma(n) > n$ for $n = 1,2,3,\ldots\ldots$

This is a slight generalisation of the usual definition for corresponding fractions. Under this definition, given $\sigma(n)$ for all n, there may be many different continued fractions which correspond to the same power series but in the remainder of this work we shall use the notation $f_o(z)$ to refer to the power series (1.4) or to any of its corresponding fractions.

We now establish a form of continued fraction which satisfies the definition (1.5). Most of the widely studied corresponding fractions are particular forms of the fraction

$$f_0(z) = \frac{p_1(z)}{q_1(z)} + \frac{z^v(1)p_2(z)}{q_2(z)} + \frac{z^v(2)p_3(z)}{q_3(z)} + \ldots + \frac{z^v(n-1)p_n(2)}{q_n(z)} + \ldots \qquad (1.6)$$

where $\{v(n)\}$ is a sequence of positive integers and $P_n(z)$, $q_n(z)$ are polynomials such that $p_n(0) \neq 0$, $q_n(o) \neq 0$ and $P_n(z)$, $q_n(z)$ are both of degree $v(n)-1$ at most. Without loss of generality, we normalise (1.6) by setting $q_n(0) = 1$ for all n. We also impose the following restrictions:

(i) The sum of the numbers of non-zero coefficients in $P_n(z)$ and $q_n(z)$ shall he $v(n)$.

(ii) If the polynomial $r_n(z) = p_n(z) + q_n(z)$ has degree $\mu(n) \leq v(n)-1$ then all the $\mu(n) + 1$ coefficients of $r_n(z)$ shall be non-zero.

If $P_n(z)/Q_n(z)$ is the nth convergent of (1.6) then we must prove that

$$f_0(z) - \frac{p_n(z)}{Q_n(z)} = 0(z^{\sigma(n)}) \tag{1.7}$$

where

$$\sigma(n) = \sum_{i=1}^{n} \nu(i). \tag{1.8}$$

Now, (1.7) may be written

$$Q_n(z)f_0(z) - p_n(z) = z\sigma(n)s_n(z) \tag{1.9}$$

where S_n (z) is some power series of the form (1. 4). The identity (1.9) may be proved by induction. We first assume that (1.9) holds for both n - 1 and n, and using (1.3) we have

$$Q_{n+1}f_0 - P_{n+1} = z^{v(n)}P_{n+1}(Q_{n-1}f_0 - P_{n-1}) + q_{n+1}(Q_nf_0 - P_n).$$

By our hypothesis we have

$$Q_{n+1}f_0 - p_{n+1} = z^{v(n)}P_{n+1}z^{\sigma(n-1)}s_{n-1} + q_{n+1}z^{\sigma(n)}s_n.$$

Using (1.8) this gives

$$Q_{n+1}f_0 - P_{n+1} = r_{n+1}z^{\sigma(n)}(s_{n-1} + s_n). \tag{1.10}$$

Since, by restriction (ii), none of the coefficients of r_{n+1} (z) are missing it can be shown that we may choose the v(n+l) non-zero coefficients of $p_{n+1}(z)$ and $q_{n+1}(z)$ such that the first v(n+l) terms of. r_{n+1} (s_{n+1} + S_n) vanish.

We can then write (1.10) in the form

$$Q_{n+1}f_0 - p_{n+1} = z^{\sigma(n+1)}s_{n+1}$$

so that (1.9) holds for n+1 provided that it holds for n-1 and n. If we choose $\sigma(0) = 0$ then the result holds trivially for $n = 0$ so that to complete the proof we need only verify (1.9) for n=1. In this case we have

$$Q_1 f_0 - P_1 \equiv q_1 f_0 - P_1.$$

Once again restrictions (i) and (ii) are clearly sufficient to ensure that we can choose the $v(1)$ coefficients of $p_1(z)$ and $q_1(z)$ so that the first $v(1)$ terms vanish. Thus we have proved that the successive convergents of the continued fraction (1.6) correspond to $\sigma(1)$, $\sigma(2)$, $\sigma(3)$ . . . . terms of the power series (1. 4).

1.2 The Corresponding Sequence

We consider a function $f_0(z)$ formally defined by the power series (1.4). We state without proof that if a corresponding fraction (1.1) exists then it may be represented by the set of recurrence relations

$$\left.\begin{array}{lcl} f_1(z) = u_1(z) & - & v_1(z)f_0(z) \\ f_2(z) = u_2(z)f_0(z) & - & v_2(z)f_1(z) \\ f_3(z) = u_3(z)f_1(z) & - & v_3(z)f_2(z) \\ \quad - - - - - & & \\ \quad - - - - - & & \\ f_n(z) = u_n(z)f_{n-2}(z) & - & v_n(z)f_{n-1}(z) \\ \quad - - - - - & & \\ \quad - - - - - & & \end{array}\right\} \qquad (1.11)$$

where the sequence $[f_n(z)]$ is a sequence of functions which, like $f_0(z)$, may be represented either as series or as corresponding fractions. We shall refer to $[f_n(z)]$ as the corresponding sequence associated with the continued fraction (1.1) and we will use the series representations of the corresponding sequence to form algorithms to compute the coefficients of the continued fraction (1.6).

The basic similarity transformation of continued fractions is defined by

$$f_0(z) = \frac{k_1u_1}{k_1v_1} + \frac{k_1k_2u_2}{k_2v_2} + \frac{k_2k_3u_3}{k_3v_3} + \ldots + \frac{k_{n-1}k_nu_n}{k_nv_n} + \ldots \tag{1.12}$$

where k_1, k_2, k_3, ... are arbitrary non-zero numbers. Although the value of the continued fraction does not change under this transformation the corresponding sequence is altered. The transformation is equivalent to multiplying the nth equation of the set (1.11) by K_n, where $k_n = \prod_{r=1}^{n} k_r$, and forming the new corresponding sequence $\{F_n\}$, where

$$\left.\begin{aligned} F_0 &= f_0 \;, \\ F_n &= k_n f_n \;. \end{aligned}\right\} \tag{1.13}$$

Under this transformation the nth equation of (1 . 11) becomes

$$F_n(z) = k_n\{k_{n-1}\, u_n(z)\, F_{n-2}(z) - v_n(z)\, F_{n-1}(z)\} \tag{1.14}$$

We shall use this transformation to modify some of the algorithms we shall develop in Section 2.

2. The Corresponding Sequence Algorithms

2.1 The General Algorithm.

We first apply result (1.14) to the continued fraction (1.6) to obtain

$$F_n^{(z)} = k_n \{ k_{n-1} z^{v(n-1)} p_n^{(z)} F_{n-2}^{(z)} - q_n^{(z)} F_{n-1}^{(z)} \} \tag{2.1}$$

for $n = 1,2,3, \ldots$ where $k_o = v(0) = F_{-1}(z) = 1$. If we write

$$F_n(z) = z^{\sigma(n)}(b_0^{(n)} + b_1^{(n)} z + b_2^{(n)} z^2 + \ldots.. + b_r^{(n)} z^r + \ldots.) \tag{2.2}$$

then we can equate coefficients of powers of z in (2.1) and, in general, obtain

$$b_r^{(n)} = k_n E^{v(n)} \{k_{n-1} p_n (E^{-1}) b_r^{(n-2)} - q_n (E^{-1}) b_r^{(n-1)}\} . \tag{2.3}$$

The shift operator E is defined by

$$E^m b_r^{(n)} = b_{r+m}^{(n)} \tag{2.4}$$

and we choose $b_r^{(n)} = 0$ for $r < 0$, $b_r^{(-1)} = 1$, and $b_r^{(-1)} = 0$ for $r \neq 0$. The expression (2.3) then holds for $n = 1,2,3,\ldots.$ and $r = -v(n), -v(n)+1, \ldots.-2, -1,0,1,2,3,\ldots.$ and we note that, in particular $b_r^{(0)} = a_r$ for $r = 0,1,2,3\ldots\ldots$ The equation (2.3) summarises an algorithm for obtaining the coefficients of the continued fraction (1.6) from the sequence $\{a_r\}$, We call this algorithm the corresponding sequence algorithm, or CS algorithm, for the continued fraction (1.6).

Although not generally true, the equations summarised by (2.3) often form a triangular system so that the problem of solution is simple. The class of continued fractions for which this is true includes most of the widely studied corresponding fractions. Many of these fractions are such that $p_n(z)$ is a constant, λ_n say, for all n. In this case we also define the modified CS algorithm in which we choose

$$k_n = \lambda_{n+1}^{-1} \tag{2.5}$$

for n= 1,2,3, ... and we adjust our series such that $a_0 = \lambda_1 = 1$. The modified CS algorithm may then be summarized by

$$b_r^{(n)} = k_n E^{v(n)} \left\{ b_r^{(n-2)} - q_n (E^{-1}) b_r^{(n-1)} \right\} \tag{2.6}$$

We also note that we need not store the values $\left\{ b_0^{(n)} \right\}$ as they are all unity.

The essential difference between the alternative algorithms is that the sequence $\{k_n\}$ is arbitrarily chosen in the ordinary algorithm (2.3) but is computed in the modified algorithm (2.6). However, the modified algorithm is usually simpler for computational purposes. In the remaining sections we discuss particular examples of CS algorithms.

2.2 S-Fractions and Padé Approximants

The continued fraction

$$f_0(z) = \frac{c_0}{1+}\,\frac{c_1 z}{1+}\,\frac{c_2 z}{1+}\,\cdots + \frac{c_n z}{1+}\,\cdots \tag{2.7}$$

is called an S-fraction if the sequence $\{c_n\}$ is chosen such that the fraction corresponds, term for term, with the power series (1.4). Such a fraction exists provided that the Hankel determinants

$$H_{2n} = \begin{vmatrix} a_0 & a_1 & \cdots a_n \\ a_1 & a_2 & \\ - - - - - & & \\ - - - - - & & \\ a_n & a_{n+1} & \cdots a_{2n} \end{vmatrix} \text{ and, } H_{2n+1} = \begin{vmatrix} a_1 & a_2 & \cdots a_{n+1} \\ a_2 & a_3 & \cdots a_{n+2} \\ - - - - - & & \\ - - - - - & & \\ a_{n+1} & a_{n+2} & \cdots a_{2n+1} \end{vmatrix} \tag{2.8}$$

are non-zero for $n= 0,1,2,3, \ldots\ldots$ We shall assume this condition is satisfied and to simplify the computation we adjust our series so that $a_0 = c_0 = 1$.

Choosing $k_n = 1$ for all n and applying result (2.3) to the continued fraction (2.7) we obtain the CS algorithm

$$b_r^{(n)} = c_{n-1} b_{r+1}^{(n-2)} - b_{r+1}^{(n-1)} \tag{2.9}$$

for all r, n or, written in full,

$$\begin{cases} c_0 = 1 \\ b_r^{(1)} = -a_{r+1}\,, \; r = 0,1,2,3, \ldots, \\ c_n = \dfrac{b_0^{(n)}}{b_0^{(n-1)}}\,, \; n = 1,2,3, \ldots, \\ b_r^{(n)} = c_{n-1} b_{r+1}^{(n-2)} - b_{r+1}^{(n-1)} \quad r = 0,1,2,3,\ldots, \; n = 2,3,4 \ldots \end{cases}$$

Also, from (2.6) we obtain the modified CS algorithm by choosing $k_n = c_n^{-1}$ so that

$$b_r^{(n)} = k_n \left\{ b_{r+1}^{(n-2)} - b_{r+1}^{(n-1)} \right\} \tag{2.10}$$

for all r, n. We note that the values $\left\{ b_r^{(n)} \right\}$ are not the same in each algorithm. In computational form the modified algorithm is

$$\begin{cases} c_0 = 1, \quad c_1 = -a_1, \\ b_r^{(1)} = -a_{r+1}, \; r = 1,2,3, \; ...., \\ c_n = b_1^{(n-2)} - b_1^{(n-1)}, \quad n = 2,3,4, \; ...., \\ b_r^{(n)} = \dfrac{1}{c_n} \left\{ b_{r+1}^{(n-2)} - b_{r+1}^{(n-1)} \right\} \; r = 1,2,3, \; ..., \; n = 2,3,4, \; .... \end{cases}$$

As an example we perform each algorithm on the power series

$$e^{-z} = 1 - + \frac{1}{2} z^2 - \frac{1}{6} z^3 + \frac{1}{24} z^4 - \frac{1}{120} z^5 + ...$$

$$
\begin{array}{c|c|cccccc}
a_5 & -\frac{1}{120} & & & & & \\
a_4 & \frac{1}{24} & \frac{1}{120} & & & & \\
a_3 & -\frac{1}{6} & -\frac{1}{24} & \frac{1}{30} & & & \\
a_2 & \frac{1}{2} & \frac{1}{6} & -\frac{1}{8} & -\frac{1}{80} & & \\
a_1 & -1 & -\frac{1}{2} & \frac{1}{3} & \frac{1}{24} & -\frac{1}{120} & \\
a_0 & 1 & 1 & -\frac{1}{2} & -\frac{1}{12} & \frac{1}{72} & \frac{1}{720} \\
\hline
 & 1 & 1 & -\frac{1}{2} & \frac{1}{6} & -\frac{1}{6} & \frac{1}{10} \\
 & c_0 & c_1 & c_2 & c_3 & c_4 & c_5
\end{array}
$$

ordinary CS algorithm

$$
\begin{array}{c|c|cccccc}
a_5 & -\frac{1}{120} & & & & & \\
a_4 & \frac{1}{24} & \frac{1}{120} & & & & \\
a_3 & -\frac{1}{6} & -\frac{1}{24} & -\frac{1}{15} & & & \\
a_2 & \frac{1}{2} & \frac{1}{6} & \frac{1}{4} & \frac{3}{20} & & \\
a_1 & -1 & -\frac{1}{2} & -\frac{2}{3} & -\frac{1}{2} & -\frac{3}{5} & \\
\hline
a_0 & 1 & 1 & -\frac{1}{2} & \frac{1}{6} & -\frac{1}{6} & \frac{1}{10} \\
 & c_0 & c_1 & c_2 & c_3 & c_4 & c_5
\end{array}
$$

ModifiedCS algorithm

Each algorithm indicates the S-fraction expansion

$$
e^{-z} = \frac{1}{1} + \frac{z}{1} - \frac{\frac{1}{2}z}{1} + \frac{\frac{1}{6}z}{1} - \frac{\frac{1}{6}z}{1} + \frac{\frac{1}{10}z}{1} - \ldots. \ -
$$

The modified algorithm (2. 10) commends itself for hand calculation as it is simple to use and easy to remember. Also, the coefficients $\left\{b_r^{(n)}\right\}$ in the ordinary algorithm (2.9) become small very quickly and are more inconvenient to use than those of (2.10).

We now consider the effect of rounding errors in the series coefficients $[a_n|$ on the continued fraction coefficients $\{c_n\}$. As the effect of any particular choice of - the constants $\{k_n\}$ is ultimately cancelled out, the build-up of rounding errors in the continued fraction coefficients is independent of these constants. Consequently the modified algorithm is equivalent to the ordinary algorithm in this respect, so we perform a numerical error analysis on the modified algorithm only.

We consider rounding errors $\in_r^{(n)}$ in $b_r^{(n)}$ and η_n inn c_n and, by substitution in (2.10), we get

$$b_r^{(n)} + \in_r^{(n-2)} = \frac{\left\{b_{r+1}^{(n-2)} + \in_{r+1}^{(n-2)}\right\} - \left\{b_{r+1}^{(n-1)} + \in_{r+1}^{(n-1)}\right\}}{c_n + \eta_n}$$

so that, for small errors,

$$\in_r^{(n)} \doteq \frac{1}{c_n}\left\{\in_{r+1}^{(n-2)} - \in_{r+1}^{n-1}\right\} - \frac{\eta_n}{c_n}\left\{b_{r+1}^{(n-2)} - b_{r+1}^{(n-1)}\right\}. \qquad (2.11)$$

In computational form this is

$$
\begin{cases}
\eta_1 \doteq -\epsilon_1^{(o)} \\[2ex]
\epsilon_r^{(1)} \doteq -\left\{ \dfrac{\epsilon_{r+1}^{(o)}}{c_1} + \dfrac{\epsilon_1^{(o)} a_{r+1}}{c_1^2} \right\}, \qquad r = 1,2,3,\ldots, \\[2ex]
\eta_n \doteq \epsilon_1^{(n-2)} - \epsilon_1^{(n-1)}, \ n = 2,3,4,\ldots, \\[2ex]
\epsilon_r^{(n)} \doteq \dfrac{1}{c_n}\left\{\epsilon_{r+1}^{(n-2)} - \epsilon_{r+1}^{(n-1)}\right\} - \dfrac{\eta_n}{c_n^2}\left\{b_{r+1}^{(n-2)} - b_{r+1}^{(n-1)}\right\}
\end{cases}
$$

$$r = 1,2,3, \ \ldots \quad n = 2,3,4, \ \ldots$$

We now perform two error analyses for the example e^{-z} already given. First, we suppose an error e in each a_n.

$\epsilon_5^{(0)}$	ϵ					
$\epsilon_4^{(0)}$	ϵ	$-\frac{119}{120}\epsilon$				
$\epsilon_3^{(0)}$	ϵ	$-\frac{25}{24}\epsilon$	$-\frac{259}{60}\epsilon$			
$\epsilon_2^{(0)}$	ϵ	$-\frac{5}{6}$	$-\frac{17}{6}\epsilon$	$\frac{147}{10}\epsilon$		
$\epsilon_1^{(0)}$	ϵ	$-\frac{3}{6}$	-7ϵ	$\frac{57}{2}\epsilon$	233ϵ	
0		ϵ	$\frac{5}{2}\epsilon$	$\frac{11}{2}\epsilon$	$-\frac{71}{2}\epsilon$	$-\frac{409}{2}\epsilon$
		η_1	η_2	η_3	η_4	η_5

Now, we suppose an error $(-1)^n \epsilon$ in each a_n

$$
\begin{array}{ll|lllll}
\in_5^{(o)} & -\in & & & & & \\
\in_4^{(o)} & \in & \frac{119}{120}\in & & & & \\
\in_3^{(o)} & -\in & -\frac{23}{24}\in & \frac{1}{20}\in & & & \\
\in_2^{(o)} & \in & \frac{5}{6}\in & -\frac{1}{6}\in & -\frac{53}{10}\in & & \\
\in_1^{(o)} & -\in & -\frac{1}{2} & \frac{1}{3}\in & \frac{7}{2}\in & -\frac{97}{5} & \\
\hline
 & 0 & \in & -\frac{1}{2}\in & -\frac{5}{6}\in & -\frac{19}{6}\in & \frac{229}{10}\in \\
 & & \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5
\end{array}
$$

Given a maximum rounding error $|\epsilon|$, these examples may be used to estimate the number of terms for which the continued fraction expansion may be computed with a desired degree of accuracy. Clearly, if implemented on a computer the algorithm will accurately evaluate more continued fraction coefficients if higher precision working is used.

We now define the [M / N] Padé approximant to the series (1.4) to be $A_M(z) / B_N(z)$ where $A_M(z)$, $B_N(z)$ are polynomials of degree M,N respectively such that

$$B_N(z)\, f_o(z) - A_M(z) = 0(z^{M+N+1}). \tag{2.12}$$

The sequence of Padé approximants

$$\begin{array}{llllll}
[L-1/0] & [L/0] & & & & \\
 & [L/1] & [L+1/1] & & & \\
 & & [L+1/2] & [L+2/2] & \ldots\ldots\ldots & \\
 & & & & \ldots\ldots\ldots\ldots &
\end{array}$$

is given by the successive convergents of the corresponding fraction

$$f_0(z) = a_0 + a_1 z + a_2 z^2 + \ldots.. + a_{L-1} z^{L-1} + a_L z^L \left\{ \frac{1}{1+} \frac{c_1^{(L)}}{1+} \frac{c_2^{(L)} z}{1+} \ldots. \right\} \tag{2.13}$$

and the sequence

$$\begin{array}{llll}
[0/L-1] & & & \\
[0/L] & [1/L] & & \\
 & [1/L+1] & [2/L+1] & \\
 & & [2/L+2] & \ldots\ldots\ldots \\
 & & & \ldots\ldots\ldots\ldots
\end{array}$$

is given by the successive convergents of the corresponding fraction

$$f_0(z) = \cfrac{1}{d_0 + d_1 z + \ldots.d_{L-1} z^{L-1} + d_L z^L \left\{ \frac{1}{1+} \frac{g_1^{(L)} z}{1+} \frac{g_2^{(L)} z}{1+} \right\}} \tag{2.14}$$

By suitable choice of L, we can express any Padé approximant as a convergent of one of the fractions (2.13) and (2.14). In (2.13) the first (L + 1) coefficients are identical to those of the series (1.4) and the coefficients $c_1^{(L)}, c_2^{(L)}, c_3^{(L)}, \ldots$ may be obtained by applying the modified CS algorithm to the sequence $a_{L+1}/a_L, a_{L+2}/a_L, a_{L+3}/a_L, \ldots$. In (2.14) the series $d_0 + d_1 z + d_2 z^2 + \ldots$ is the power series of the reciprocal of $f_o(z)$ and its coefficients may be computed from the recurrence relation

$$d_n = -d_0 \sum_{r=1}^{n} d_{n-r}\, a_r \tag{2.15}$$

for n = 1,2,3, ... and where $d = a_o^{-1}$. The coefficients $g_1^{(L)}, g_2^{(L)}, g_3^{(L)}, \ldots$ are then obtained by applying the modified CS algorithm to the sequence d_{L+1}/d_L, d_{L+2}/d_L, d_{L+3}/d_L, ...

In particular, we find that the [2/ 3] Pade approximant is given by

$$\frac{1}{d_0 + d_1 z +}\;\frac{d_2 z^2}{1 +}\;\frac{g_1(2)z}{1 +}\;\frac{g_2(2)z}{1 +}\;\frac{g_3^{(2)} z}{1}$$

As an example we shall consider the [2/3] Padé approximant to e^{-z} . We already know the reciprocal series

$$e^z = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5 + \ldots..$$

so that we have

$$\frac{d_3}{d_2} = \frac{1}{3}, \frac{d_4}{d_2} = \frac{1}{12}, \frac{d_5}{d_2} = \frac{1}{60}$$

and we apply the modified CS algorithm to these values

$$\begin{array}{cc|ccc}
d_5/d_2 & \frac{1}{60} & & & \\
d_4/d_2 & \frac{1}{12} & \frac{1}{20} & & \\
d_3/d_2 & \frac{1}{3} & \frac{1}{4} & \frac{2}{5} & \\
\hline
 & 1 & -\frac{1}{3} & \frac{1}{12} & -\frac{3}{20} \\
 & & g_1^{(2)} & g_2^{(2)} & g_3^{(2)}
\end{array}$$

Thus, the [2/3] Padé approximant to e^{-z} is

$$e^{-z} = \cfrac{1}{1 + z + \cfrac{\frac{1}{2} z^2}{1 - \cfrac{\frac{1}{3} z}{1 + \cfrac{\frac{1}{12} z}{1 - \frac{3}{20} z}}}} + 0(z^6).$$

This is one of the simplest methods for obtaining a Padé approximant and is easily accomplished by a minimum of hand calculation.

It is interesting to compare this algorithm with that of Longman (1971). Longman's algorithm computes the coefficients of Padé approximants in the more usual rational form. The advantage of the continued fraction form is that, by computing just one more coefficient, we can progress from one approximant to another. Whilst Longman's algorithm is more useful for computing the whole Padé table, we can use the CS algorithm to calculate high order approximants without computing the whole of the preceding table. As fewer computational steps are necessary we may suppose that there is less build-up of rounding error with the CS algorithm.

2.3 J-Fractions.

The continued fraction

$$f_0(z) = \frac{c_0}{1 + d_0 z +} \frac{c_1 z^2}{1 + d_1 z +} \frac{c_2 z^2}{1 + d_2 z +} \cdots\cdots \frac{C_n Z^2}{1 + d_n z +} \cdots\cdots \qquad (2.16)$$

is called a J-fraction and corresponds to the power series (1. 4), successive convergents corresponding to 2,4,6 ....terms of the series. If c_0 =1 the modified CS algorithm for this fraction is

$$b_r^{(n)} = k_n \left\{ b_{r+2}^{(n-2)} - b_{r+2}^{(n-1)} - d_{n-1} b_{r+1}^{(n-1)} \right\} \qquad (2.17)$$

for all r, n. In computational form this algorithm is

$$\begin{cases} c_0 = 1, \quad d_0 = -a_1 \quad , \\ c_1 = -a_2 + a_1^2 \quad , \\ b_r^{(1)} = \dfrac{1}{c_1} \{-a_{r+2} + a_1 a_{r+1}\} \ , \ r = 1,2,3, \ \ldots.., \\ d_n = b_1^{(n-1)} - b_1^{(n)} \ , \ n = 1,2,3,\ldots. ., \\ d_n = b_1^{(n-1)} - b_1^{(n)}, \ n = 1,2,3,\ldots. , \\ c_n = b_2^{(n-2} - b_2^{(n-1)} - d_{n-1} b_1^{(n-1)}, \ n = 2,3,4,\ldots. , \\ b_r^{(n)} = \dfrac{1}{c_n} \{b_{r+2}^{(n-2)} - b_{r+2}^{(n-1)} - b_{r+1}^{(n-1)}\} \quad , \end{cases}$$

r= 1,2,3 . . . . . .n =2,3,4, . . .

If we replace z by 1/z in (2.16) and use the basic similarity transformation we obtain the alternative form of the J-fraction

$$f_0(z) = \frac{c_0}{d_0 + z +}\,\frac{c_1}{d_1 + z +}\,\frac{c_2}{d_2 + z +}\,\cdots\cdots + \frac{c_n}{d_n + z +}\,\cdots\cdots \tag{2.18}$$

This fraction corresponds to the asymptotic series

$$f_0(z) = \frac{a_0}{z} + \frac{a_1}{z^2} + \frac{a_2}{z^3} + \cdots\cdots \tag{2.19}$$

for $|z|$ large. Clearly, if we choose $a_o = c_o = 1$ then the algorithm (2.17) may also be applied to the fraction (2.18).

2.4. M-Fractions.

The continued fraction

$$f_0(z) = \frac{c_0}{1 + d_0 z +}\,\frac{c_1 z}{1 + d_1 z +}\,\frac{c_2 z}{1 + d_2 z +}\,\cdots\,\frac{c_n z}{+\ 1 + d_n z +}\,\cdots \tag{2.20}$$

is called an M-fraction [see McCabe (1971)] if its coefficients are chosen such that it corresponds simultaneously to the power series (1.4) and to the asymptotic series

$$f_0(z) = \frac{\alpha_0}{z} + \frac{\alpha_1}{z^2} + \frac{\alpha_2}{z^3} + \cdots \tag{2.21}$$

for $|z|$ large, successive convergents of (2.20) corresponding to 1,2,3, ... terms of each series. The ordinary CS algorithm for converting the series (1.4) to the M-fraction (2.20)

is

$$b_r^{(n)} = c_{n-1} b_{r+1}^{(n-2)} - b_{r+1}^{(n-1)} - d_{n-1} b^{(n-1)})_r \tag{2.22}$$

for all r,n. Using the basic similarity transformation

we see that the fraction (2.20) may be written

$$f_0(z) = \frac{c_0 \frac{1}{z}}{d_0 + \frac{1}{z} + d_1 + \frac{1}{z} + d_n + \frac{1}{z} +} \frac{c_1 \frac{1}{z}}{} \frac{C_2 \frac{1}{z}}{} \cdots\cdots \frac{c_n \frac{1}{z}}{+ d_n + \frac{1}{2} +} \cdots\cdots \tag{2.23}$$

By writing the fraction in this form we see that there is

also a CS algorithm for converting the series (2. 21) to

the fraction (2. 23). This algorithm is

$$\beta_r^{(n)} = c_{n-1} \beta_{r+1}^{(n-2)} - d_{n-1} \beta_{r+1}^{(n-1)} - \beta_r^{(n-1)} \tag{2.24}$$

for all r,n where the values $\{ \beta_r^{(n)} \}$ are analogous to the

values $\{ b_r^{(n)} \}$ in (2. 22). The relations (2. 22) and (2. 24)

taken together form the CS algorithm for the M-fraction (2. 20).

Written in computational form this algorithm

is

$$
\begin{cases}
c_0 \quad a_0, d_0 = \dfrac{a_0}{\alpha_0}, \\
b_r^{(1)} = -\{a_{r+1} + d_0 a_r\}, r = 0,1,2,3,\ldots, \\
\beta_r^{(1)} = -\{d_0 \alpha_{r+1} + \alpha_r\}, r = 0,1,2,3,\ldots, \\
c_n = \dfrac{b_0^{(n)}}{b_0^{(n-1)}}, n = 1,2,3,\ldots,n \\
d_n = c_n \dfrac{\beta_0^{(n-1)}}{\beta_0^{(n)}}, n = 1,2,3,\ldots, \\
b_r^{(n)} = c_{n-1} b_{r+1}^{(n-2)} - b_{r+1}^{(n-1)} - d_{n-1} b_r^{(n-1)}, r = 0,1,2,3,\ldots n = 2,3,4,\ldots, \\
\beta_r^{(n)} = c_{n-1} \beta_{r+1}^{(n-2)} - d_{n-1} \beta_{r+1}^{(n-1)} - \beta_r^{(n-1)}, r = 0,1,2,3,\ldots n = 2,3,4,\ldots,
\end{cases}
$$

we can form a modified CS algorithm but as we cannot choose $a_o = \alpha_o = 1$ without rescaling z the resulting algorithm is no simpler than that above.

We now give a simple numerical example to illustrate a suitable layout for hand computation. We consider Dawson's integral

$$f_0(z) = e^{-2} \int_0^z e^{t^2} dt$$

which has the two series expansions

$$f_0(z) = 1 - \frac{z}{3} + \frac{z^2}{15} - \frac{z^3}{105} + \frac{z^4}{945} - \dots$$

for $|z|$ small and

$$f_0(z) = \frac{1}{z} + \frac{1}{z^2} + \frac{3}{z^3} + \frac{15}{z^4} + \frac{105}{z^5} + \dots.$$

for $|z|$ large.

$$\begin{array}{c|c|cccc}
a_4 & \frac{1}{945} & & & & \\
a_3 & -\frac{1}{105} & \frac{8}{945} & & \text{b-array} & \\
a_2 & \frac{1}{15} & -\frac{2}{35} & \frac{16}{945} & & \\
a_1 & -\frac{1}{3} & \frac{4}{15} & -\frac{8}{105} & \frac{64}{4725} & \\
a_0 & 1 & -\frac{2}{3} & \frac{8}{45} & -\frac{16}{525} & \frac{128}{33075} \\
\hline
 & 1 & -\frac{2}{3} & -\frac{4}{15} & -\frac{6}{35} & -\frac{8}{63} \\
 & c_0 & c_1 & c_2 & c_3 & c_4
\end{array}$$

$$\begin{array}{c|c|cccc}
\alpha_4 & 105 & & & & \\
\alpha_3 & 15 & -120 & & \beta - \text{array} & \\
\alpha_2 & 3 & -18 & 48 & & \\
\alpha_1 & 1 & -4 & -8 & -\frac{64}{5} & \\
\alpha_0 & 1 & -2 & \frac{8}{3} & -\frac{16}{5} & \frac{128}{35} \\
\hline
 & 1 & -\frac{1}{3} & -\frac{1}{5} & -\frac{1}{7} & -\frac{1}{9} \\
 & d_0 & d_1 & d_2 & d_3 & d_4
\end{array}$$

The resulting M-fraction is thus

$$f_0(z) = \frac{1}{1+z-}\;\frac{\frac{2}{3z}}{1+\frac{1}{3z}-}\;\frac{\frac{4}{15}z}{1+\frac{1}{5}z-}\;\frac{\frac{6}{35}z}{1+\frac{1}{7}z-}\;\frac{\frac{8}{63}z}{1+\frac{1}{9}z-}\;\cdots\cdots$$

or, using the basic similarity transformation,

$$f_0(z) = 1+z-\;\frac{2z}{3+z-}\;\frac{4z}{5+z-}\;\frac{6z}{7+z-}\;\frac{8z}{9+z-}\;\cdots\cdots$$

3. Conclusion

We have shown how CS algorithms may be applied in the case of S-fractions, Padé approxiaants, J- fractions and M- fractions and indicated the broad class or continued fractions for which such an algorithm exists. Many of these algorithms are simple and require a minimal number of computational steps. They appear to provide one of the most straightforward and accurate means for obtaining continued fraction approximations to functions defined by power series.

REFERENCES

LONGMAN, I. M. 1971 Computation of the Padé Table, Intern. J. Computer Math. Section B, Vol. 3, 53- 64.

McCABE, J. H. 1971 Continued Fraction Methods, with Applications to First Order Linear Differential Equations. Ph. D. Thesis, Brunel University.

WALL, H.S. 1948 Analytic Theory of Continued Fractions. Chelsea.

ACKNOWLEDGMENT

Since this report was written we understand from Mr. D.M.Drew of this Department that, independently, he has used a similar method to that given in this paper. In work that is to be published shortly he has applied the technique to a wider range of problems.