Applications in Engineering Science 1 (2020) 100006

Contents lists available at ScienceDirect

Applications in Engineering Science

journal homepage: www.elsevier.com/locate/apples

Moore-Gibson-Thompson thermoelasticity with two temperatures

Ramón Quintanilla

Departament de Matemàtiques, UPC C. Colom 11, Terrassa, Barcelona 08222, Spain

ARTICLE INFO

Keywords:
Moore-Gibson-Thompson heat conduction
Two temperatures
Exponential stability
Slow decay
Polynomial decay

ABSTRACT

In this note we propose the Moore-Gibson-Thompson heat conduction equation with two temperatures and prove the well posedness and the exponential decay of the solutions under suitable conditions on the constitutive parameters. Later we consider the extension to the Moore-Gibson-Thompson thermoelasticity with two temperatures and prove that we cannot expect for the exponential stability even in the one-dimensional case. This last result contrasts with the one obtained for the Moore-Gibson-Thompson thermoelasticity where the exponential decay was obtained. However we prove the polynomial decay of the solutions. The paper concludes by giving the main ideas to extend the theory for inhomogeneous and anisotropic materials.

1. Introduction

The Fourier formulation to describe heat conduction is widely used by scientists. In this situation, the heat flux vector is proportional to the gradient of the temperature. However, the combination of this proposition with the energy equation

$$c\,\dot{\theta} + \operatorname{div}\mathbf{q} = 0, \quad (c > 0) \tag{1}$$

predicts the instantaneous propagation of heat[1]. That means that every thermal perturbation is instantaneously felt at any point of the material regardless of the distance. This phenomenon is not well accepted from the physical point of view because it contradicts the *causality principle*. In order to overcome this drawback, several alternative proposals have been stated. In this sense we can recall the one proposed by Tzou where the heat flux and the gradient of the temperature have a delay in the constitutive equation (Tzou, 1995). In this case it is usual to speak of *phase-lag theories*. The constitutive equation is given by:

$$q_i(\mathbf{x}, t + \tau_1) = -k\theta_{,i}(\mathbf{x}, t + \tau_2), \qquad k > 0, \tag{2}$$

where τ_1 and τ_2 are the delay parameters which are assumed to be positive. The notation $\theta_{,i}$ means the derivative of θ with respect to the variable x_i, and from now on the repeated subscripts mean summation. The derivative with respect to the time is denoted using a dot over the function. This formulation suggests that the temperature gradient established across a material volume at position $\mathbf{x}$ and time $t + \tau_2$ results in a heat flux to flow at a different time $t + \tau_1$. These delays are usually understood in terms of the microstructure of the material.

Choudhuri (2007) suggested a generalization of Tzou's theory where the heat flux vector is assumed to be in the form:

$$q_i(\mathbf{x}, t + \tau_1) = -k_1\alpha_{,i}(\mathbf{x}, t + \tau_3) - k_2\theta_{,i}(\mathbf{x}, t + \tau_2), \tag{3}$$

where $\dot{\alpha} = \theta$. The variable α is called the *thermal displacement*, and was used by Green and Naghdi to propose their theories (Green and Naghdi, 1992; 1993). The new parameter τ_3 is again a delay parameter. Choudhuri's proposition is known as three-dual-phase-lag.

These two proposals have different derivations when the heat flux and the gradients of the temperature are approximated by the Taylor polynomials and one thinks that the proposal of Choudhuri tries to recover Green and Naghdi theories when different Taylor approximations are considered. This new approach gives rise to different equations (depending on the selected Taylor polynomial) to describe heat conduction that have been analyzed by many authors (see, for example, Abdallah, 2009; Borgmeyer et al., 2014; Hader et al., 2002; Miranville and Quintanilla, 2011; Quintanilla, 2002; Quintanilla, 2003; Quintanilla and Racke, 2006a; Quintanilla and Racke, 2006b; Quintanilla and Racke, 2007; Quintanilla and Racke, 2008; Quintanilla and Racke, 2015; Rukolaine, 2014; Zhang, 2009).

Unfortunately both proposals (Tzou and Choudhuri), lead to *ill-posed* problems in the sense of Hadamard. It has been shown that combining Eqs. (2) (or (3)) with the energy Eq. (1) leads to the existence of a sequence of elements in the point spectrum such that its real part tends to infinity (Dreher et al., 2009) and therefore the continuous dependence of solutions fails.

To obtain a heat conduction theory with delays but without such an explosive behavior, Quintanilla (2008, 2009) combined the delay parameters of Tzou and Choudhuri with the two-temperatures theory proposed by Chen and Gurtin (1968), Chen et al. (1968, 1969), Warren and Chen (1973). The constitutive equation reads

$$q_i(\mathbf{x}, t + \tau_1) = -k_1\beta_{,i}(\mathbf{x}, t + \tau_3) - k_2 T_{,i}(\mathbf{x}, t + \tau_2), \tag{4}$$

where $\alpha = \beta - a\Delta\beta$, $\theta = T - a\Delta T$ and a is a positive constant. In fact, this theory was extended to the thermoelastic context

E-mail address: ramon.quintanilla@upc.edu

[1] In the above equation $\mathbf{q} = (q_i)$ is the heat flux vector and θ is the temperature.

https://doi.org/10.1016/j.apples.2020.100006
Received 12 June 2020; Received in revised form 22 June 2020; Accepted 24 June 2020

(Quintanilla, 2008; 2009). To do so, one must assume the equation of motion

$$t_{ji,j} = \rho \ddot{u}_i, \tag{5}$$

the energy equation

$$T_0^* \dot{\eta} = -q_{i,i} \tag{6}$$

and the constitutive equations

$$\begin{aligned} t_{ji} &= 2\mu e_{ij} + \lambda e_{rr}\delta_{ij} + \beta^* \theta \delta_{ij} \\ \eta &= -\beta^* e_{ii} + c\theta \end{aligned} \tag{7}$$

where t_{ji} represents the stress tensor, η is the entropy, (u_i) is the displacement vector, e_{ij} is the strain tensor, λ and μ are the Lamé constants, T_0^* is the uniform reference temperature, and β^* is related with the thermal expansion constant and ρ and c are the mass density and the thermal capacity, respectively.

These new theories are currently under deep study (Banik and Kanoria, 2012; Ezzat et al., 2012; Leseduarte et al., 2017; Magaña et al., 2018; 2019; Mukhopadhyay et al., 2011; Quintanilla and Jordan, 2009).

If $\tau_2 = \tau_3 < \tau_1$ in equation (3) and the heat flux vector is approximated by[2]

$$q_i(\mathbf{x}, t + \tau^*) \approx q_i(\mathbf{x}, t) + \tau^* \dot{q}_i(\mathbf{x}, t), \quad \tau^* = \tau_1 - \tau_2,$$

one obtains the thermal formulation of the Moore-Gibson-Thompson equation (Quintanilla, 2019)[3](see also Conti et al., 2020, Conti et al.,Pellicer and Quintanilla, 2020). This equation has received a lot of attention in the last years (see among others Conejero et al., 2015; Dell'Oro et al., 2016; Dell'Oro and Pata, 2017; Kaltenbacher et al., 2011; Lasiecka and Wang, 2015; Pellicer and Said-Houari, 2017; Pellicer and Sola-Morales, 2019).

Therefore it is also natural to consider the equation obtained in a similar way, but in the case that we consider the Eq. (4) instead of Eq. (3). We note that in this case we obtain the equation

$$\tau c \dddot{\alpha} + c\ddot{\alpha} = k^* \Delta \beta + k \Delta T. \tag{8}$$

One thinks that it is suitable to denominate to this equation as Moore-Gibson-Thompson with two temperatures (in short MGT + 2TT).

The aim of this paper is double. First we want to study the stability/instability of the solutions to this equation. Second we will consider the thermoelastic one-dimensional problem. Therefore, the system of equations that we want to study is given by

$$\begin{aligned} t_x &= \rho \ddot{u} \\ T_0^* \dot{\eta} &= -q_x \end{aligned} \tag{9}$$

with the following constitutive equations:

$$\begin{aligned} t &= \mu u_x + \beta^* \theta \\ \eta &= -\beta^* u_x + c\theta \end{aligned} \tag{10}$$

and we will prove the stability of solutions, the slow decay and the polynomial decay.

The results proposed in this paper have a theoretical aspect, but they are relevant from the physical and the engineering point of view. We focus our attention to prove the well-posedness of problems as well as to obtain the rate of decay of the solutions of them. The term *well posed problem* comes from the definition of Jacques Hadamard (1865–1963) that believed that mathematical models for the description of physical phenomena should satisfy the existence and uniqueness of solutions and the continuous dependence with respect to the initial data. When a problem is not well posed in the sense of Hadamard the solutions are highly sensitive to changes. Small changes in the data of the problem provoke relevant differences in the behaviour of the solutions. In particular the numerical instabilities occur. Therefore well-posedness is a needed step to investigate numerical aspects in the analysis of a problem. On the other side the rate of decay of the solutions also plays a relevant rol from the engineering point of view. The damping of the system determines when we can despise the effects of a perturbation. For instance when the rate of decay is fast (exponential) the vibrations become very small after a short period of time and the consequences of the perturbation have a negligible impact. However, when the rate of decay is slow the vibrations can be noted for a large period of time and we cannot despise the effects before a *very* large period of time. Therefore we choose here two examples to illustrate this aspect. They correspond to the heat conduction of Moore-Gibson-Thompson with two temperatures and the one-dimensional thermoelasticity when the heat conduction is described by the same heat equation.

In the next section we obtain the suitable conditions on the constitutive parameters to guarantee the stability/instability of the solutions. In fact, we see that the decay for the case of the MGT + 2TT heat equation is controlled by an exponential. Later we show the stability of the system of the MGT + 2TT thermoelasticity, but we prove that in the one-dimensional case the decay is not controlled by an exponential. This last result is in contrast with the case of the MGT thermoelasticity. However we show the polynomial decay. The paper concludes by giving the main ideas to extend the theory for inhomogeneous and anisotropic materials.

It is worth saying that the existence and the exponential decay of the solutions of the purely thermal problem can be obtained from the reference (Kaltenbacher et al., 2011) in their study from an abstract point of view of the Moore-Gibson-Thompson equation. However, we believe that it is suitable to present our approach because we emphasize the form of the operators and the dissipation. Our proposition with respect the heat equation is an alternative approach to the problem. Furthermore, with the help of the approach to the heat equation problem the thermoelastic problem is easy.

2. Thermal problem

We consider equation (8) in a three-dimensional domain B whose boundary is smooth enough to apply the divergence theorem. To have a well-posed problem we need to introduce the initial conditions:

$$\beta(\mathbf{x}, 0) = \beta_0(\mathbf{x}), \quad \dot{\beta}(\mathbf{x}, 0) = T_0(\mathbf{x}), \quad \ddot{\beta}(\mathbf{x}, 0) = \psi_0(\mathbf{x}), \quad \mathbf{x} \in B, \tag{11}$$

and the homogeneous Dirichlet boundary conditions

$$\beta(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \partial B. \tag{12}$$

From now on, we assume that the constitutive constants satisfy:

(i) $c > 0$, $\tau > 0$, $k > 0$, $k^* > 0$.
(ii) $k > k^* \tau$.

The meaning of the positivity of c is clear. Assumptions (i) on k and k^* are the natural ones and they are related with the stability of solutions for type II/III theories. Also the positivity of τ is a standard requirement. Condition (ii) is usual in the study of the MGT equation to guarantee the stability of the solutions. We see here that this condition also works for the MGT + 2TT.

From the definition of α, it is clear that

$$\int_B \alpha^2 dV = \int_B \left(\beta^2 + 2a|\nabla\beta|^2 + a^2|\Delta\beta|^2 \right) dV$$

when we assume null Dirichlet boundary conditions. Therefore, taking into account the Poincaré inequality, we have

$$\int_B \alpha^2 dV \approx \int_B (|\nabla\beta|^2 + a|\Delta\beta|^2) dV.$$

We will transform our problem into an abstract problem involving a convenient Hilbert space. First, we note that $Id - a\Delta : \beta \to \beta - a\Delta\beta = \alpha$ is an isomorphism on $W^{2,2}(B) \cap W_0^{1,2}(B)$ and takes values in $L^2(B)$,

[2] From now on we omit the star.

[3] This procedure is equivalent to introduce a relaxation parameter for the type III heat equation extending the Cattaneo-Maxwell proposition for the Fourier law. Details can be found in Quintanilla (2019)

where $W^{2,2}(B)$, $W_0^{1,2}(B)$ and $L^2(B)$ are the usual Hilbert spaces. We shall denote by $\Phi(\alpha)=\beta$ the inverse operator. Therefore, the L^2 norm of α is equivalent to the $W^{2,2}$ norm of β.

We work in the Hilbert space

$$\mathcal{H}=L^2(B)\times L^2(B)\times L^2(B). \tag{13}$$

To propose a synthetic expression to the above problem, we define the following operators:

$$A^*(\alpha)=\frac{k^*}{c\tau}\Delta\Phi(\alpha),\quad B^*(\theta)=\frac{k}{c\tau}\Delta\Phi(\theta),\quad C(\phi)=-\frac{1}{\tau}\phi. \tag{14}$$

Hence, our problem can be written as

$$\frac{dU}{dt}=\mathcal{A}U,\quad U(0)=(\alpha_0,\theta_0,\phi_0), \tag{15}$$

where $\alpha_0=\beta_0-a\Delta\beta_0$, $\theta_0=T_0-a\Delta T_0$, $\phi_0=\psi_0-a\Delta\psi_0$ and

$$\mathcal{A}=\begin{pmatrix} 0 & Id & 0\\ 0 & 0 & Id\\ A^* & B^* & C\end{pmatrix}.$$

We will prove that $\mathcal{A}$ generates a contractive semigroup. We first note that the domain $\mathcal{D}$ of $\mathcal{A}$ agrees with the Hilbert space and then it is dense.

We consider the following inner product in $\mathcal{H}$

$$\begin{aligned}\langle U,V\rangle &= \frac{1}{2}\int_B\left(c(\theta+\tau\phi)\overline{(\theta^*+\tau\phi^*)}+k^*(\beta_{,i}+\tau T_{,i})\overline{(\gamma^*_{,i}+\tau T^*_{,i})}+\tau K T_{,i}\overline{T}^*_{,i}\right)dv\\ &+\frac{1}{2}\int_B\left(k^*a(\beta_{,ii}+\tau T_{,ii})\overline{(\gamma^*_{,jj}+\tau T^*_{,jj})}+\tau a K T_{,ii}\overline{T}^*_{,jj}\right)dv,\end{aligned} \tag{16}$$

where here and from now on we denote $K=k-\tau k^*$, $U=(\alpha,\theta,\phi)$ and $V=(\alpha^*,\theta^*,\phi^*)$ where $\alpha=\beta-a\Delta\beta$, $\alpha^*=\gamma^*-a\Delta\gamma^*$, $\theta=T-a\Delta T$, $\theta^*=T^*-a\Delta T^*$ and the bar means the conjugated complex. It is worth noting that this inner product defines a norm which is equivalent to the usual one in the Hilbert space.

Lemma 2.1. *The following inequality*

$$\mathrm{Re}\langle\mathcal{A}U,U\rangle\le 0, \tag{17}$$

is satisfied for every $U\in\mathcal{D}$.

Proof. Straight calculations give

$$\mathrm{Re}\langle\mathcal{A}U,U\rangle=-\frac{K}{2}\int_B(T_{,i}T_{,i}+aT_{,ii}T_{,jj})dv \tag{18}$$

In view of the condition (ii) the lemma is proved. □

Lemma 2.2. *Zero belongs to the resolvent of the operator $\mathcal{A}$. In short $0\in\rho(\mathcal{A})$.*

Proof. let $(f_1,f_2,f_3)\in\mathcal{H}$. We must prove that the system

$$\theta=f_1,\quad \phi=f_2,\quad A^*\alpha+B^*\theta+C\phi=f_3,$$

has a solution. After substitution we obtain the equation:

$$A^*\alpha=f_3-B^*f_1-Cf_2.$$

We first note that the right hand side of this equation is in $L^2(B)$. As the operators B^*, C are bounded and $-A^*$ is bounded and coercive in L^2 this equation has a solution. Then, the lemma is proved. □

In view of the Lumer-Phillips corollary to the Hille-Yosida theorem we have:

Theorem 2.3. *The operator $\mathcal{A}$ generates a contractive semigroup.*

A consequence of the previous theorem is the following result:

Theorem 2.4. *For any $U(0)\in\mathcal{H}$, there exists a unique solution to our problem such that $U(t)\in C^1([0,t_1],\mathcal{D})$.*

Moreover, we know that the continuous dependence of solutions on initial data and supply terms (in case they were assumed) could also been obtained. Therefore our problem is well posed in the sense of Hadamard. Even more, the solutions are stable in the sense that the energy of the system does not increase.

The spectral analysis of the equation may give some information on the behavior of the solutions with respect to the time. In fact, we have the following:

Remark 2.5. We have assumed that (ii) holds. If we do impose that this assumption holds we can obtain the instability of solutions. In fact, let us assume that there are solutions of the form $\beta(\mathbf{x},t)=\exp(\omega t)\eta_n(\mathbf{x})$ where $\eta_n(\mathbf{x})$ is an eigenfunction of the Laplace operator with null boundary conditions. We see that ω will satisfy the equation

$$\tau c(1+\lambda_n a)x^3+c(1+\lambda_n a)x^2+k\lambda_n x+k^*\lambda_n=0.$$

By the Hurwitz rule the necessary and sufficient condition to guarantee that the solutions of this equation are on the left hand side of the complex plane is that the coefficients are positive and that

$$\tau c(1+\lambda_n a)k^*\lambda_n<c(1+\lambda_n a)k\lambda_n.$$

But this is equivalent to assume (ii). Therefore if (ii) fails to be true there exist elements at the point spectrum which are on the right hand side of the complex plane and the instability of solutions is proved.

Now, we will show the exponential decay of the solutions for our problem when (ii) holds.

To prove the exponential decay, we recall the characterization stated in the book of Liu and Zheng (1999).

Theorem 2.6. *Let $S(t)=\{e^{\mathcal{A}t}\}_{t\ge0}$ be a C_0-semigroup of contractions on a Hilbert space. Then $S(t)$ is exponentially stable if and only if the following two conditions are satisfied:*

(i) $i\mathbb{R}\subset\rho(\mathcal{A})$,
(ii) $\overline{\lim}_{|\lambda|\to\infty}\|(i\lambda\mathcal{I}-\mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})}<\infty$.

Lemma 2.7. *The operator $\mathcal{A}$ satisfies $i\mathbb{R}\subset\rho(\mathcal{A})$.*

Proof. We here follow the arguments given in the book of (Liu and Zheng (1999), page 25). Let us assume that the intersection of the imaginary axis and the spectrum is non-empty. Therefore, there exist a sequence of real numbers λ_n with $\lambda_n\to\varpi$, $|\lambda_n|<|\varpi|$ and a sequence of vectors $U_n=(\alpha_n,\theta_n,\phi_n)$ in $\mathcal{D}(\mathcal{A})$ and with unit norm such that

$$\|(i\lambda_n\mathcal{I}-\mathcal{A})U_n\|\to 0. \tag{19}$$

In our case, writing this condition term by term we get

$$i\lambda_n\alpha_n-\theta_n\to 0 \text{ in } L^2, \tag{20}$$

$$i\lambda_n\theta_n-\phi_n\to 0 \text{ in } L^2, \tag{21}$$

$$i\lambda_n\phi_n-A^*\alpha_n-B^*\theta_n-C\phi_n\to 0 \text{ in } L^2. \tag{22}$$

In view of the dissipative term for the operator, we see that

$$\theta_n\to 0 \text{ in } L^2. \tag{23}$$

From (20) we also see that $\alpha_n\to 0$ in L^2.

We now want to see that ϕ_n tends to zero in L^2. To this end we multiply (21) by ϕ_n to see that

$$i\langle\theta_n,\lambda_n\phi_n\rangle-||\phi_n||^2\to 0. \tag{24}$$

The convergence of ϕ_n will be guaranteed whenever we show that $\langle\theta_n,\lambda_n\phi_n\rangle\to 0$. From (22) we see

$$ic\tau\langle\theta_n,\lambda_n\phi_n\rangle=\langle\theta_n,A^*\alpha_n+B^*\theta_n+C\phi_n\rangle \tag{25}$$

We see that the right hand side tends to zero because A^*, B^* and C are bounded. Therefore the convergence of ϕ_n to zero follows in L^2. Then we arrive to a contradiction and the lemma is proved. □

Lemma 2.8. *The operator $\mathcal{A}$ satisfies*

$$\overline{\lim_{|\lambda|\to\infty}} \|(i\lambda\mathcal{I}-\mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Proof. The proof also follows a contradiction argument. Suppose that the thesis is not true. Therefore there exist a sequence of real numbers λ_n such that $|\lambda_n| \to \infty$ and a sequence of unit vectors in $\mathcal{D}(\mathcal{A})$ in such a way that (19) holds. Again, conditions (20)–(22) still hold. Now we can use a similar argument to the one used in the proof of the previous lemma because the key point is that λ_n does no tend to zero. □

The two previous lemmas give rise to the following result.

Theorem 2.9. *The C_0-semigroup $S(t) = \{e^{\mathcal{A}t}\}_{t\geq 0}$ is exponentially stable. That is, there exist two positive constants M and α such that $\|S(t)U\| \leq M\|U\|e^{-\alpha t}$.*

Proof. The proof is a direct consequence of the two previous Lemmas and the Theorem 2.6. □

3. One dimensional thermoelastic theory

We consider now the three-dimensional isotropic and homogeneous thermoelastic materials. In this situation the field equations become[4]

$$\rho\ddot{u}_i = \mu u_{i,jj} + (\lambda+\mu)u_{j,ji} + \beta^*\theta_{,i} \tag{26}$$

$$c\ddot{\alpha} + c\tau\dddot{\alpha} = \beta^*(\dot{u}_{i,i} + \tau\ddot{u}_{i,i}) + kT_{,jj} + k^*\beta_{,jj} \tag{27}$$

It is not difficult to prove the existence and uniqueness of solutions under homogeneous Dirichlet boundary conditions by adapting the semigroup arguments to this new situation. In fact if we define the energy of the system

$$\begin{aligned} E(t) = {} & \frac{1}{2}\int_B \big(\rho\dot{\hat{u}}_i\dot{\hat{u}}_i + \mu\hat{u}_{i,j}\hat{u}_{i,j} + (\lambda+\mu)\hat{u}_{i,i}\hat{u}_{j,j}\big)dv \\ & + \frac{1}{2}\int_B \big(c(\theta+\tau\phi)^2 + k^*(\beta_{,i}+\tau T_{,i})(\beta_{,i}+\tau T_{,i}) + \tau K T_{,i}T_{,i}\big)dv \\ & + \frac{1}{2}\int_B \big((ak^*(\beta_{,ii}+\tau T_{,ii})^2 + \tau a K(T_{,ii})^2\big)dv, \end{aligned} \tag{28}$$

where $\hat{f} = f + \tau\dot{f}$, we obtain that

$$E(t) + F(t) = E(0), \tag{29}$$

where

$$F(t) = \int_0^t \int_B K(T_{,i}T_{,i} + aT_{,ii}T_{,jj})dv. \tag{30}$$

Therefore if we assume that ρ, μ and $\lambda+\mu$ are positive, we obtain the stability of the solutions of the system. However we cannot expect exponential decay of solutions for this problem. To show this claim we concentrate our attention to the one-dimensional problem. That is, we consider the system

$$\rho\ddot{u} = \mu u_{xx} + \beta^*(\dot{\alpha}_x + \tau\ddot{\alpha}_x) \tag{31}$$

$$c\ddot{\alpha} + c\tau\dddot{\alpha} = \beta^*\dot{u}_x + kT_{xx} + k^*\beta_{xx}, \tag{32}$$

where we have drop the hats to simplify the notation.

We consider these equations in the interval $[0,\pi]$ and assuming homogeneous Dirichlet boundary conditions for the displacement and homogeneous Neumann boundary conditions for β. We could find solutions of the form

$$u = A\exp(\omega t)\sin nx, \quad \beta = B\exp(\omega t)\cos nx, \tag{33}$$

whenever

$$A(\rho\omega^2 + \mu n^2) - B(1+n^2a)\beta^*\omega(1+\tau\omega) = 0, \tag{34}$$

and

$$A\beta^* n\omega + B\big[c(1+n^2a)\omega^2(1+\tau\omega) + n^2(k\omega + k^*)\big] = 0. \tag{35}$$

To guarantee the existence of nontrivial solutions we need to impose that

$$\begin{aligned} & (\rho\omega^2 + \mu n^2)\big[c(1+n^2a)\omega^2(1+\tau\omega) + n^2(k\omega+k^*)\big] \\ & \quad + (\beta^*)^2 n\omega(1+n^2a)\omega(1+\tau\omega) = 0. \end{aligned} \tag{36}$$

We can write

$$\omega^5 + p_1\omega^4 + p_2\omega^3 + p_3\omega^2 + p_4\omega + p_5 = 0, \tag{37}$$

where

$$p_1 = \tau^{-1}, \quad p_2 = \frac{\mu n^2}{\rho} + \frac{(\beta^*)^2 n}{\rho c} + \frac{kn^2}{c\tau(1+n^2a)} \tag{38}$$

$$p_3 = \frac{\mu n^2}{\rho\tau} + \frac{(\beta^*)^2 n}{\rho c\tau} + \frac{k^* n^2}{c\tau(1+n^2a)}, \tag{39}$$

$$p_4 = \frac{\mu k n^4}{\rho c\tau(1+n^2a)}, \quad p_5 = \frac{\mu k^* n^4}{\rho c\tau(1+n^2a)} \tag{40}$$

We want to see that there are solutions to the equation $p(\omega) = 0$ as near as we want to the imaginary axis. This fact will be shown if the polynomial $p(\omega-\epsilon)$ has a root with positive real part for every ϵ as small as we want, but positive.

We have the polynomial

$$x^5 + q_1x^4 + q_2x^3 + q_3x^2 + q_4x + q_5,$$

where

$$q_1 = p_1 - 5\epsilon, \quad q_2 = p_2 + 10\epsilon^2 - 4\epsilon p_1$$

$$q_3 = p_3 - 10\epsilon^3 + 6\epsilon^2 p_1 - 3\epsilon p_2$$

$$q_4 = p_4 + 5\epsilon^4 - 4\epsilon^3 p_1 + 3\epsilon^2 p_2 - 2\epsilon p_3$$

$$q_5 = p_5 - \epsilon^5 + \epsilon^4 p_1 - \epsilon^3 p_2 + \epsilon^2 p_3 - \epsilon p_4$$

We use the Hurwitz theorem that says that the necessary and sufficient condition to guarantee that the solutions of the equation

$$x^5 + q_1x^4 + q_2x^3 + q_3x^2 + q_4x + q_5 = 0, \tag{41}$$

have negative real part is:

$$\Lambda_1 = q_1 > 0, \Lambda_2 = \det\begin{pmatrix} q_1 & 1 \\ q_3 & q_2 \end{pmatrix} > 0, \Lambda_3 = \det\begin{pmatrix} q_1 & 1 & 0 \\ q_3 & q_2 & q_1 \\ q_5 & q_4 & q_3 \end{pmatrix} > 0, \tag{42}$$

$$\Lambda_4 = \det\begin{pmatrix} q_1 & 1 & 0 & 0 \\ q_3 & q_2 & q_1 & 1 \\ q_5 & q_4 & q_3 & q_2 \\ 0 & 0 & q_5 & q_4 \end{pmatrix} > 0, \text{ and}$$

$$\Lambda_5 = \det\begin{pmatrix} q_1 & 1 & 0 & 0 & 0 \\ q_3 & q_2 & q_1 & 1 & 0 \\ q_5 & q_4 & q_3 & q_2 & q_1 \\ 0 & 0 & q_5 & q_4 & q_3 \\ 0 & 0 & 0 & 0 & q_5 \end{pmatrix} > 0. \tag{43}$$

We will study Λ_2,

$$\Lambda_2 = q_1q_2 - q_3 = p_1p_2 - p_3 - \epsilon(2p_2 + 4p_1^2) + 24\epsilon^2 p_1 - 40\epsilon^3. \tag{44}$$

It is clear that

$$p_1p_2 - p_3 = \frac{(k\tau^{-1} - k^*)n^2}{c\tau(1+n^2a)}, \tag{45}$$

which is bounded as well as $24\epsilon^2 p_1 - 40\epsilon^3$. However for every ϵ as small as we want (but positive) we can select n large enough to guarantee

[4] From now on we assume that $T_0^* = 1$ to simplify the calculations.

that $\epsilon(2p_2 + 4p_1^2)$ becomes unbounded and therefore $q_1q_2 - q_3 < 0$. This argument shows that the solutions of our system decay in a slow way, or, in other words, that a uniform rate of decay of exponential type for all the solutions can not be obtained. We also note that from the analysis proposed here it is clear that the result can be obtained for any finite interval.

We have proved that:

Theorem 3.1. *The solutions of the problem determined by the (31, 32) with homogeneous Dirichlet condition for the displacement and homogeneous Neumann boundary condition for the thermal variable do not decay in a uniform exponential way.*

It is worth comparing this situation with the case of the Moore-Gibson-Thompson thermoelasticity where the exponential decay was proved for the one dimensional case (Quintanilla, 2019). That is, in the present case a combination of a hyperbolic equation with two temperatures does not imply the exponential decay even in the one dimensional thermoelastic case. We emphasize that this is not the first time that we observe this result, because a similar quality was obtained in the references (Leseduarte et al., 2017; Magaña et al., 2019).

To finish this section we will prove that the solutions of the problem determined by the system (31,32) with the initial conditions $u(x,0) = u_0(x)$, $\dot{u}(x,0) = v_0(x)$ and (11) with the boundary conditions

$$u(0,t) = u(\pi,t) = \beta(0,t) = \beta(\pi,t) = 0, \tag{46}$$

decay at least as $t^{1/2}$. To this end we first consider the Hilbert space $\mathcal{H} = H_0^1 \times L^2 \times L^2 \times L^2 \times L^2$. If $U = (u, v, \alpha, \theta, \phi)$ and $V = (u^*, v^*, \alpha^*, \theta^*, \phi^*)$, where $\alpha = \beta - a\beta_{xx}$, $\alpha^* = \gamma^* - a\gamma^*_{xx}$, $\theta = T - aT_{xx}$, $\theta^* = T^* - aT^*_{xx}$ we can define the inner product

$$\begin{aligned}\langle U, V\rangle = \frac{1}{2}\int_B \Big(&\rho v\overline{v^*} + \mu u_x\overline{u^*_x} + c(\theta + \tau\phi)\overline{(\theta^* + \tau\phi^*)} \\ &+ k^*(\beta_x + \tau T_x)\overline{(\gamma^*_x + \tau T^*_x)} + \tau K T_x\overline{T}^*_x\Big)dx \\ &+ \frac{1}{2}\int_B \Big(k^* a(\beta_{xx} + \tau T_{xx})\overline{(\gamma^*_{xx} + \tau T^*_{xx})} + \tau a K T_{xx}\overline{T}^*_{xx}\Big)dx,\end{aligned} \tag{47}$$

This product is equivalent to the usual one in $\mathcal{H}$.

We can write our problem in the form

$$\frac{dU}{dt} = \mathcal{A}U, \quad U(0) = (u_0, v_0, \alpha_0, \theta_0, \phi_0), \tag{48}$$

where α_0, θ_0 and ϕ_0 are defined as in second section. Our operator $\mathcal{A}$ can be written as

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 & 0 \\ M & 0 & 0 & N & L \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & P & A^* & B^* & C \end{pmatrix},$$

where $Mu = \mu\rho^{-1}u_{xx}$, $N\theta = \beta^*\rho^{-1}\theta_x$, $N\phi = \tau\beta^*\rho^{-1}\phi_x$, $Pv = \beta^*(c\tau)^{-1} v_x$ and A^*, B^* and C are the restriction of the operators defined at section two in the one dimensional case.

We note that the domain of the operator is determined by the elements of the Hilbert space such that $v \in H_0^1$ and $Mu + N\theta + L\phi \in L^2$. It is clear that is a dense subspace. At the same time we also have

$$\mathrm{Re}\langle \mathcal{A}U, U\rangle = -\frac{K}{2}\int_B (|T_x|^2 + a|T_{xx}|^2)dx \le 0. \tag{49}$$

It is also easy to prove that zero belongs to the resolvent of the operator. If we consider $(f_1, f_2, f_3, f_4, f_5) \in \mathcal{A}$ we need to solve the system

$$v = f_1, \quad \theta = f_3, \quad \phi = f_4$$

$$Mu + N\theta + L\phi = f_2, \quad Pv + A^*\alpha + B^*\theta + C\phi = f_5.$$

We have v, θ and ϕ and we need to solve the equations

$$Mu = f_2 - Nf_3 - LF_4, \quad A^*\alpha = f_5 - Pf_1 - B^*f_3 - Cf_4.$$

Now it is transparent the existence of this solutions for u and α and we have proved:

Theorem 3.2. *The operator $\mathcal{A}$ generates a contractive semigroup. And for any $U(0)$ in the domain of the operator, there exists a unique solution to our problem such that $U(t) \in C^1([0, t_1], \mathcal{D})$*[5].

We now prove the decay estimate:

Theorem 3.3. *Our semigroup is polynomially stable of order 1/2. That is, for every $U(0)$ in the domain of the operator there exists a constant (independent of the initial data) such that $\|S(t)U\| \le C\|U(0)\| t^{-1/2}$.*

Proof. To show the result we will prove that the imaginary axis in included in the resolvent of the operator and that(see Borichev and Tomilov, 2010)

$\overline{\lim}_{|\lambda|\to\infty} \lambda^{-2}\|(i\lambda\mathcal{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$

First we assume that the imaginary axis is not included at the resolvent. Hence it will exist a sequence $\lambda_n \to \lambda$ and a unit norm sequence $U_n = (u_n, v_n, \alpha_n, \theta_n, \phi_n)$ such that

$\|(i\lambda_n\mathcal{I} - \mathcal{A})U_n\|_{\mathcal{L}(\mathcal{H})} \to 0.$

In this case, we have

$$i\lambda_n u_n - v_n \to 0 \text{ in } H^1, \tag{50}$$

$$i\lambda_n v_n - Mu_n - N\theta_n - L\phi_n \to 0 \text{ in } L^2, \tag{51}$$

$$i\lambda_n \alpha_n - \theta_n \to 0 \text{ in } L^2, \tag{52}$$

$$i\lambda_n \theta_n - \phi_n \to 0 \text{ in } L^2, \tag{53}$$

$$i\lambda_n \phi_n - Pv_n - A^*\alpha_n - B^*\theta_n - C\phi_n \to 0 \text{ in } L^2. \tag{54}$$

In view of the dissipation we see that θ_n goes to zero in L^2 and hence α_n and ϕ_n also tend to zero because the origin is not on the resolvent. It follows that $Pv_n \to 0$ in L^2 which implies that $v_n \to 0$ in H^1 and then also u_n tends to zero at the same norm. We arrive to a contradiction.

We now prove the asymptotic condition. Assume that it is not true. We also obtain the existence of a sequence $\lambda_n \to \infty$ and a unit norm sequence $U_n = (u_n, v_n, \alpha_n, \theta_n, \phi_n)$ such that

$$\lambda_n^2(i\lambda_n u_n - v_n) \to 0 \text{ in } H^1, \tag{55}$$

$$\lambda_n^2(i\lambda_n v_n - Mu_n - N\theta_n - L\phi_n) \to 0 \text{ in } L^2, \tag{56}$$

$$\lambda_n^2(i\lambda_n \alpha_n - \theta_n) \to 0 \text{ in } L^2, \tag{57}$$

$$\lambda_n^2(i\lambda_n \theta_n - \phi_n) \to 0 \text{ in } L^2, \tag{58}$$

$$\lambda_n^2(i\lambda_n \phi_n - Pv_n - A^*\alpha_n - B^*\theta_n - C\phi_n) \to 0 \text{ in } L^2. \tag{59}$$

Dissipation inequality implies that $\lambda_n\theta_n \to 0$ in L^2 and therefore α_n and ϕ_n also tend to zero in L^2. From the last convergence we see that $\lambda_n^{-1}Pv_n \to 0$ in L^2 which implies that $u_n \to 0$ in H^1 and we also obtain that $v_n \to 0$ in L^2 which finish the proof of the theorem. □

We point out that the analysis for Neumann boundary conditions for the temperature can be done in a similar way (see for instance Leseduarte et al., 2017)

[5] The corresponding result for the three dimensional case can be obtained after the natural extension of the proposed arguments.

4. Inhomogeneous case

It is possible to extend the previous analysis to inhomogeneous materials. We recall that in this case the equations are given by:

$$q_i + \tau\dot{q}_i = m_{ij}(\mathbf{x})(k^*\beta_{,j} + kT_{,j}), \tag{60}$$

$$\alpha = \beta - a(m_{ij}(\mathbf{x})\beta_{,i})_{,j}, \quad \theta = T - a(m_{ij}(\mathbf{x})T_{,i})_{,j}, \tag{61}$$

where $m_{ij}(\mathbf{x})$ is a symmetric positive definite matrix.

In this case the heat equation becomes

$$\tau c(\mathbf{x})\dddot{\alpha} + c(\mathbf{x})\ddot{\alpha} = (m_{ij}(\mathbf{x})(k^*\beta_{,j} + kT_{,j}))_{,i}. \tag{62}$$

We now assume that (i) and (ii) hold as well as that $m_{ij}(\mathbf{x})$ is positive definite. That is, there exists a positive constant C such that

$$m_{ij}(\mathbf{x})\xi_i\xi_j \geq C\xi_j\xi_j \tag{63}$$

for every vector (ξ_j) and for every point. The energy of the system determined by this new equation with the initial and boundary conditions proposed in section two reads

$$\begin{aligned} E(t) &= \frac{1}{2}\int_B \Big(c(\mathbf{x})(\theta + \tau\dot{\theta})^2 + k^* m_{ij}(\mathbf{x})(\beta_{,i} + \tau T_{,i})(\beta_{,j} + \tau T_{,j}) + \tau K m_{ij}(\mathbf{x})T_{,i}T_{,j}\Big)dv \\ &+ \frac{1}{2}\int_B \Big(k^* a(m_{ij}(\mathbf{x})(\beta_{,i} + \tau T_{,i})_{,j})^2 + K\tau a\big((m_{ij}(\mathbf{x})T_{,i})_{,j}\big)^2\Big)dv \\ &+ \int_0^t \int_B K\Big(m_{ij}(\mathbf{x})T_{,i}T_{,j} + a\big((m_{ij}(\mathbf{x})T_{,i})_{,j}\big)^2\Big)dv ds = E(0). \end{aligned} \tag{64}$$

The exponential stability can be obtained following a similar argument as the one we used in section two.

The problem can also be extended to the thermoelastic situation assuming that

$$t_{ji} = C_{ijkl}(\mathbf{x})u_{k,l} + \beta^*_{ij}(\mathbf{x})\theta, \quad \eta = -\beta^*_{ij}(\mathbf{x})u_{i,j} + c(\mathbf{x})\theta. \tag{65}$$

After combination of these equations with the equation of motion, the energy equation and the constitutive equation proposed in this section for the heat flux vector we obtain a new system of the MGT+2TT thermoelasticity. Assuming, as ususal, that the elasticity tensor is positive definite and symmetric tensor the existence and uniqueness of the solutions can be proved.

Acknowledgments

The investigations reported in this paper were supported by project “Análisis Matemático de Problemas de la Termomecánica” (MTM2016-74934-P), (AEI/FEDER, UE) of the Spanish Ministry of Economy and Competitiveness. This work is also part of the project “Análisis Matemático Aplicado a la Termomecánica”(PID2019-105118GB-I00), of the Spanish Ministry of Science Innovation and Universities.

References

Abdallah, I.A., 2009. Dual phase lag heat conduction and thermoelastic properties of a semi-infinite medium induced by ultrashort pulsed laser. Prog. Phys. 3, 60–63.

Banik, S., Kanoria, M., 2012. Effects of three-phase-lag on two temperatures generalized thermoelasticity for an infinite medium with a spherical cavity. Appl. Math. Mech. 33, 483–498.

Borgmeyer, K., Quintanilla, R., Racke, R., 2014. Phase-lag heat conduction: decay rates for limit problems and well-posedness. J. Evol. Equ. 14, 863–884.

Borichev, A., Tomilov, Y., 2010. Optimal polynomial decay of functions and operators semigroups. Math. Ann. 347, 455–478.

Chen, P.J., Gurtin, M.E., 1968. On a theory of heat involving two temperatures. J. Appl. Math. Phys. 19, 614–627.

Chen, P.J., Gurtin, M.E., Williams, W.O., 1968. A note on non-simple heat conduction. J. Appl. Math. Phys. 19, 969–970.

Chen, P.J., Gurtin, M.E., Williams, W.O., 1969. On the thermodynamics of non-simple materials with two temperatures. J. Appl. Math. Phys. 20, 107–112 .

Choudhuri, S.K.R., 2007. On a thermoelastic three-phase-lag model. J. Therm. Stresses 30, 231–238.

Conejero, J.A., Lizama, C., Ródenas, F., 2015. Chaotic behaviour of the solutions of the Moore- Gibson-Thompson equation. Appl. Math. Inf. Sci. 9, 2233–2238.

Conti, M., Pata, V., Pellicer, M., Quintanilla, R., 2020. On the analyticity of the MGT-viscoelastic plate with heat conduction. J. Differ. Equ. 269, 7862–7880.

Conti, M., Pata, V., Quintanilla, R., 2019. Thermoelasticity of Moore-Gibson-Thompson type with history dependence in temperature. Asymp. Anal. doi:10.3233/ASY-191576.

Dell'Oro, F., Lasiecka, I., Pata, V., 2016. The moore-gibson-thompson equation with memory in the critical case. J. Differ. Equ. 261, 4188–4222.

Dell'Oro, F., Pata, V., 2017. On the Moore-Gibson-Thompson equation and its relation to linear viscoelasticity. Appl. Math. Optim. 76, 641–655.

Dreher, M., Quintanilla, R., Racke, R., 2009. Ill-posed problems in thermomechanics. Appl. Math. Lett. 22, 1374–1379.

Ezzat, M.A., El-Karamany, A.S., Ezzat, S.M., 2012. Two-temperature theory in magneto-thermoelasticity with fractional order dual-phase-lag heat transfer. Nucl. Eng. Des. 252, 267–277.

Green, A.E., Naghdi, P.M., 1992. On undamped heat waves in an elastic solid. J. Therm. Stresses 15, 253–264.

Green, A.E., Naghdi, P.M., 1993. Thermoelasticity without energy dissipation. J. Elast. 31, 189–208.

Hader, M.A., Al-Nimr, M.A., Nabah, B.A.A., 2002. The dual-phase-lag heat conduction model in thin slabs under a fluctuating volumetric thermal disturbance. Int. J. Thermophys. 23, 1669–1680.

Kaltenbacher, B., Lasiecka, I., Marchand, R., 2011. Wellposedness and exponential decay rates for the Moore-Gibson-Thompson equation arising in high intensity ultrasound. Control Cybern. 40, 971–988.

Lasiecka, I., Wang, X., 2015. Moore-gibson-thompson equation with memory, part II: general decay of energy. J. Differ. Equ. 259, 7610–7635.

Leseduarte, M.C., Quintanilla, R., Racke, R., 2017. On (non-)exponential decay in generalized thermoelasticity with two temperatures. Appl. Math. Lett. 70, 18–25.

Liu, Z., Zheng, S., 1999. Semigroups associated with dissipative systems. Chapman & Hall/CRC Research Notes in Mathematics, vol. 398. Chapman & Hall/CRC, Boca Raton, FL.

Magaña, A., Miranville, A., Quintanilla, R., 2018. On the stability in phase-lag heat conduction with two temperatures. J. Evol. Equ. 18, 1697–1712.

Magaña, A., Miranville, A., Quintanilla, R., 2019. On the time decay in phase-lag thermoelasticity with two temperatures. Electron. Res. Arch. 27, 7–19.

Miranville, A., Quintanilla, R., 2011. A phase-field model based on a three-phase-lag heat conduction. Appl. Math. Optim. 63, 133–150.

Mukhopadhyay, S., Prasad, R., Kumar, R., 2011. On the theory of two-temperature thermoelaticity with two phase-lags. J. Therm. Stresses 34, 352–365.

Pellicer, M., Quintanilla, R., 2020. On uniqueness and instability for some thermomechanical problems involving the Moore-Gibson-Thompson equation. J. Appl. Math. Phys. 71, 84.

Pellicer, M., Said-Houari, B., 2017. Wellposedness and decay rates for the cauchy problem of the Moore-Gibson-Thompson equation arising in high intensity ultrasound. Appl. Math. Optim. 1–32.

Pellicer, M., Sola-Morales, J., 2019. Optimal scalar products in the moore-gibson-thompson equation. Evolution Equations and control Theory 8, 203–220.

Quintanilla, R., 2002. Exponential stability in the dual-phase-lag heat conduction theory. J. Non-Equilib. Thermodyn. 27, 217–227.

Quintanilla, R., 2003. A condition on the delay parameters in the one-dimensional dual-phase-lag thermoelastic theory. J. Therm. Stresses 26, 713–721.

Quintanilla, R., 2008. A well-posed problem for the dual-phase-lag heat conduction. J. Therm. Stresses 31, 260–269.

Quintanilla, R., 2009. A well-posed problem for the three-dual-phase-lag heat conduction. J. Therm. Stresses 32, 1270–1278.

Quintanilla, R., 2019. Moore-Gibson-Thompson thermoelasticity. Math. Mech. Solids 24, 4020–4031.

Quintanilla, R., Jordan, P.M., 2009. A note on the two-temperature theory with dual-phase-lag decay: some exact solutions. Mech. Res. Commun. 36, 796–803.

Quintanilla, R., Racke, R., 2006a. A note on stability of dual-phase-lag heat conduction. Int. J. Heat Mass Transf. 49, 1209–1213.

Quintanilla, R., Racke, R., 2006b. Qualitative aspects in dual-phase-lag thermoelasticity. SIAM J. Appl. Math. 66, 977–1001.

Quintanilla, R., Racke, R., 2007. Qualitative aspects in dual-phase-lag heat conduction. Proc. R. Soc. Lond. A 463, 659–674.

Quintanilla, R., Racke, R., 2008. A note on stability in three-phase-lag heat conduction. Int. J. Heat Mass Transf. 51, 24–29.

Quintanilla, R., Racke, R., 2015. Spatial behavior in phase-lag heat conduction. Differ. Integr. Equ. 28, 291–308.

Rukolaine, S.A., 2014. Unphysical effects of the dual-phase-lag model of heat conduction. Int. J. Heat Mass Transf. 78, 58–63.

Tzou, D.Y., 1995. A unified approach for heat conduction from macro to micro-scales. ASME J. Heat Transf. 117, 8–16.

Warren, W.E., Chen, P.J., 1973. Wave propagation in two temperatures theory of thermoelaticity. Acta Mech. 16, 83–117.

Zhang, Y., 2009. Generalized dual-phase lag bioheat equations based on nonequilibrium heat transfer in living biological tissues. Int. J. Heat Mass Transf. 52, 4829–4834.