TR/16/84 November 1984

# Finite difference methods for a class of singular two-point boundary value problems.

E.H. Twizell

Summary A two-step finite difference method, which is $0(h^{1+\alpha})$ convergent, h being the step size of a uniform mesh and $\alpha$ a parameter with $0<\alpha<1$, is developed for the solution of the singular two-point boundary value problem $(x^{\alpha}y')' = f(x,y)$, $y(0) = A$, $y(1) = B$. The method is derived from *a* three-point recurrence relation and is seen to reduce to the classical second order method when $\alpha = 0$.

Employing a second grid of step size $\frac{1}{2}h$ and combining results for both grids is seen to give $0(h^{3+\alpha})$ convergence; introducing a third grid of step size $\frac{1}{3}$ h and combining results for all three grids is seen to give a method with $0(h^{5+\alpha})$ convergence.

The three methods are tested on two numerical examples from the literature.

*Subject Classifications:* AMS( MOS): 65L10; CR:5.17,

## 1. Introduction

In a paper published in 1982, Chawla and Katti [1] considered the two-point boundary value problem

$$(x^{\alpha}y'(x))' - f(x, y)\ ,\ 0 < x \leq 1$$
$$y(0) = A\ ,\ y(1) = B \qquad (1)$$

in which $0<\alpha< 1$ and A, B are finite constants. In the present paper it will be assumed that, for $x \in [0,1]$, the real valued, non-linear function f(x,y) is continuous, $\partial f / \partial y$ exists and is continuous and that $\partial f / \partial y > 0$.

In [1], Chawla and Katti constructed three two-step finite difference methods for the numerical solution of (1). Their methods were based on uniform and non-uniform meshes and, under certain conditions, the methods were shown to be $0(h^2)$ convergent. Chawla and Katti compared the convergence properties of their methods with those of Jamet [5] and Ciarlet et al [2] which were $0(h^{1-\alpha})$ and $0(h^{2-\alpha})$ respectively. Chawla and Katti also drew the attention of the reader to the methods of Gustafsson [3], Reddien [6] and Reddien and Schumaker [7] for the solution of singular two point boundary value problems but did not compare their numerical results with any of the authors they referenced.

In the present paper a finite difference method which is $0(h^{1+\alpha})$ convergent is derived first of all; the derivation of the method depends on a recurrence relation first used by the author in the numerical solution of the simple wave equation in [8], The finite difference method uses a step size h, but it is shown that computing results using a grid with step size $\frac{1}{2}h$ also, and taking a linear combination of the two sets of results, improves the convergence to $0(h^{3+\alpha})$. Using a third grid of step size $\frac{1}{3}h$, and taking a linear combination of the three sets of results, is seen to give $0(h^{5+\alpha})$ convergence.

## 2. The discretization and a recurrence relation

Suppose the independent variable x in (1) is incremented using a constant step size $h = 1/(N+1)$ where $N \geq 1$ is the number of interior points of the discretization of the interval $0<x<1$. The solution $y(x)$ will be computed at the points $x_m = mh$ ($m = 1,2,\ldots,N$) and the notation $y_m$ will be used to denote the solution of an approximating difference scheme at the grid point $x_m$ ($m = 1,2,\ldots,N$); clearly $y_o = A$ and $y_{N+1} = B$.

The function $y(x)$ may be shown to satisfy the two-step recurrence relation

$$- y(x-h) + \{\exp(hD) + \exp(-hD)\}\, y(x) - y(x+h) = 0 \qquad (2)$$

where $D = d/dx$ (see [8]). Replacing the exponential terms in (2) by rational approximants such as Pade approximants, necessitates the use of at least the second derivative of $y(x)$ at $x = 0$ when (2) is applied to $x_1$. There is, of course, a singularity in $y''(0)$ and so non-rational approximants to $\exp(\pm hD)$ must be used in (2) for the solution of problems such as (1).

To this end, the (0,k) Pade approximants, where $k \geq 2$ is an even integer, are appropriate (methods based on odd values of k have the same order as the previous even value); values of $k < 2$ lead to inconsistent numerical methods.

Using the (0,2) Pade approximant (the Taylor series of order 2) in (2) gives

$$-y(x-h) + (2 + h^2 D^2)\, y(x) - y(x+h) = 0 \qquad (3)$$

which results in the finite difference method

$$- y_{n-1} + \left(2y_n + h^{2-\alpha} f_n / n^{\alpha} - h\alpha\, y'_n / n\right) - y_{n+1} = 0 \qquad (4)$$

where $n = 1, 2.....N$ and $f_n = f(x_n, y_n)$.

## 3. Development and analyses of the numerical methods

### *3.1 A low order method*

Replacing $y_n'$ in (4) by the numerical differentiation formula $y_n' = \frac{1}{2}h^{-1}$ $(-y_{n-1} + y_{n+1}) + 0(h^2)$ gives

$$-\left(1-\frac{\alpha}{2n}\right)y_{n-1} + \left(2y_n + \frac{h^{2-\alpha}}{n^{\alpha}}f_n\right) - \left(1-\frac{\alpha}{2n}\right)y_{n+1} = 0 \tag{5}$$

with n = 1,2,...,N. It is noted that, putting $\alpha = 0$ in (1) gives the special second order boundary value problem

$$y''(x) = f(x,y) \quad , \quad 0<x\leq 1$$
$$y(0) = A \ , \quad y(1) = B \tag{6}$$

and putting $\alpha = 0$ in (4) or (5) gives the classical second order method for the solution of (6) .

The local truncation error $t_n$ associated with (5) at the fixed point $X = x_n$ ( n = 1,2,..,N) has the form

$$t_n^{(1)} = -\frac{\alpha h^3}{6n}y'''(x_n) - \frac{h^4}{12}y^{(iv)}(x_n) - \frac{\alpha h^5}{120n}y^{(v)}(x_n) - \frac{h^6}{360}y^{(vi)}(x_n)$$
$$-\frac{\alpha h^7}{5040n}y^{(vii)}(x_n) - \frac{h^8}{20160}y^{(viii)}(x_n) - \ldots , \tag{7}$$

in which it is noted that, for $\alpha = 0$, (7) becomes the local truncation error of the classical second order method used for solving (6),

### *3. 2 Convergence*

The vector $\underline{y}(1) = \left[y_1^{(1)}, y_2^{(1)}, \ldots, y_N^{(1)}\right]^T$ , T denoting transpose, of computed values of y is obtained by solving an algebraic system of the form

$$A\underline{Y}^{(1)} + \frac{1}{2}\alpha B\underline{Y}^{(1)} + h^{2-\alpha}\ c\underline{f}(\underline{Y}^{(1)}) = \underline{r} \ , \tag{8}$$

where $A = (a_{ij})$, $B = (b_{ij})$ , $C = (c_{ij})$ (i,j = 1,2, . . . , N) are matrices given by

$a_{i,i-1}=-1(i=2,\ldots,N)$, $a_{i,i}=2(i=1,2,..,N)$, $a_{i,i+1}=-1(i=1,2,..,N-1)$,

$b_{i,i-1}=1/i\ (i=2,..,N)$, $b_{i,i+1}=-1/i\ (i=1,2,..,N-1)$,

$c_{i,i}=1/i^{\alpha}\ (i=1,2,..,N)$

with all other elements of A,B,C equal to zero, and $\underline{r}$ is the vector given by

$$\underline{r}=[(1-\frac{1}{2}\alpha)A, 0,\ldots,0,(\text{--------}\ 1+\frac{1}{2}\alpha/N)B]^{T}.$$

It is clear that the vector $\underline{y}=[y_1, y_2, \ldots, y_N]^{T}$ satisfies

$$A\underline{y}+\frac{1}{2}\alpha B\underline{y}+h^{2-\alpha}c\underline{f}(\underline{y})=\underline{r}+\underline{t}^{(1)}, \qquad (9)$$

where $\underline{t}^{(1)}=\left[t_1^{(1)}, t_2^{(1)},\ldots, t_N^{(1)}\right]^{T}$ is the vector of local truncation errors.

Defining $\underline{E}=\underline{y}-\underline{Y}^{(1)}$ to be the vector of discretization errors, it follows that $\underline{E}$ satisfies

$$\underline{f(y)}-\underline{f(Y^{(1)})}=F(\underline{y})\underline{E}, \qquad (10)$$

where $F=F(\underline{y})=\mathrm{diag}\{\partial f_i/\partial y_i\}$. It further follows that $\underline{E}$ satisfies

$$Q\underline{E}=\underline{t}^{(1)} \qquad (11)$$

where $Q=A+\frac{1}{2}\alpha B+h^{2-\alpha}CF$.

The matrix $Q=(q_{ij})(i,j=1,2,\ldots,N)$ is clearly tridiagonal with

$$\begin{aligned} q_{i,i-1} &= -1+\frac{1}{2}\alpha/i \quad (i=2,\ldots,N), \\ q_{i,i} &= 2+h^{2-\alpha}F_i/i^{\alpha} \quad (i=1,2,\ldots,N), \\ q_{i,i+1} &= -1-\frac{1}{2}\alpha/i \quad (i=1,2,\ldots,N-1), \end{aligned} \qquad (12)$$

where $F_i=\partial f_i/\partial y_i>0$. It is noted from (12) that $q_{i,i-1}\neq 0\ (i=2,\ldots,N)$ and that $q_{i,i+1}\neq 0\ (i=1,2,\ldots,N-1)$; Q is therefore irreducible [4; p.359]. It is further noted from (12) that $q_{ij}\leq 0$ for $i\neq j\ (i,j=1,2,..,N)$ and that

$$S_1=1-\frac{1}{2}\alpha+h^{2-\alpha}F_1>h^{2-\alpha}F_1>0,$$

$$S_k=h^{2-\alpha}F_k/k^{\alpha}>0, \text{ for } k=2,..,N-1,$$

$$S_N=1+\frac{1}{2}\alpha/N+h^{2-\alpha}F_N/N^{\alpha}>h^{2-\alpha}F_N/N^{\alpha}>0$$

where $S_k$ $(k = 1,2,,..,N)$ is the sum of the elements of row k in Q. The matrix Q is therefore monotone; its inverse matrix $Q^{-1} = \left(q_{ij}^{-1}\right)(i, j = 1,2,..., N)$ exists and $q_{ij}^{-1} \geq 0(i, j = 1,2,..., N)$.

Now let $F^* = \min(\partial f / \partial y)$ ; then $F^* > 0$ and

$$S_k \geq h^{2-\alpha} F^*/ k^\alpha \geq h^{2-\alpha} F^*/ N^\alpha > 0 \quad , \quad k = 1, 2.....N \quad .$$

Also let $\underline{S} = [ S_1, S_2,... , S_N]^T$ , and $\underline{Z} = [ 1, 1, . . . , 1 ]^T$ ; then $\underline{S} = Q\underline{Z}$ is the vector of row sums of Q and $Q^{-1} S = Z$. It follows that

$$1 = \sum_{k=1}^{N} q_{ik}^{-1} s_k \geq h^{2-\alpha} F^* N^{-\alpha} \sum_{k=1}^{N} q_{ik}^{-1}, \text{ for all } i = 1,2,..., N,$$

$$= h^{2-\alpha} F^* N^{-\alpha} \|Q\|, \quad \text{since } q_{ik}^{-1} \geq 0.$$

(The norm referred to is the $L_\infty$ norm and from this point onwards the sub-script will be omitted). Thus $\|Q^{-1}\| \leq N^\alpha/(h^{2-\alpha}F^*)$ and, therefore, from (11) from (11)

$$\|\underline{E}\| \leq \|Q^{-1}\| \cdot \left\|t^{(1)}\right\| \leq \frac{h^{3\alpha}}{6} \cdot \frac{N^\alpha}{h^{2-\alpha} F^*} = 0\left(h^{1+\alpha}\right) \qquad (14)$$

Equation (5) is therefore $0(h^{1+\alpha})$ convergent for $0<\alpha< 1$ and it is obvious from (7) and (13) that $\|\underline{E}\| = 0(h^2)$ when $\alpha= 0$.

3.3 *Improving accuracy and order of convergence*

The local truncation error of the finite difference method (5) is $0(h^3)$ for (1) and $0(h^4)$ for (6) the same as the methods of Chawla and Katti [1]. The order of convergence of (5) is not, however, competitive with the method of Jamet [5] and of the method of Ciarlet et al [2] for $\alpha>\frac{1}{2}$.

Suppose, now, that a second, finer, grid of step size $\frac{1}{2}h$ is used. The interval $0<x< 1$ is now divided into 2N+2 subintervals each of width $\frac{1}{2}h$ and the points $x_n$ $(n =1,2,...,N)$ of the coarse grid used in §3.1 are named $x_m$ $( m = 2,4 ,.., 2N)$ with respect to the fine grid.

The finite difference method (5) becomes

$$-\left(1-\frac{\alpha}{2m}\right)y_{m-1}+\left(2y_m+\frac{h^{2-\alpha}}{2^{2-\alpha}m^{\alpha}}f_m\right)-\left(1+\frac{\alpha}{2m}\right)y_{m+1}=0 \qquad (15)$$

with m= 1,2,...,2N+1. The local truncation error associated with (15) at some fixed grid point X =nh on the coarse grid is given by

$$t_n^{(2)}=-\frac{\alpha h^3}{96n}y'''(x_n)-\frac{h^4}{192}y^{(iv)}(x_n)-\frac{\alpha h^5}{7680n}y^{(v)}(x_n)-\frac{h^6}{23040}y^{(vi)}(x_n)$$

$$-\frac{\alpha h^7}{1290240n}y^{(vii)}(x_n)-\frac{h^8}{5160960}y^{(viii)}(x_n)-\ldots \qquad (16)$$

with n = 1,2,...,N, since this fixed point $X = x_n = nh$ on the coarse grid is the point X = 2n(½h) on the fine grid.

Suppose that $\underline{U}=[U_1, U_2, \ldots, U_{2N+1}]^T$ is the vector of computed values of y obtained using (15) with m= 1,2,...,2N+1, and define the vector $\underline{Y}^{(2)}$ to be $\underline{Y}^{(2)}=\left[Y_1^{(2)}, Y_2^{(2)}, \ldots, Y_N^{(2)}\right]^T=[U_2, U_4, \ldots U_{2N}]^T$. This vector has N components and gives second approximations to y(x) at the N points $x_n$ (n =1,2,...,N) of the original (coarse) grid used in §3.1.

Consider now the linear combination

$$a\underline{Y}^{(2)}+(1-a)\underline{Y}^{(1)}, \qquad (17)$$

where a is some parameter. The local truncation error associated with (17) is found to be $0(h^5)$ with leading terms

$$\frac{\alpha h^5}{2400n}y^{(v)}(x_n)+\frac{h^6}{7200}y^{(vi)}(x_n)+\ldots. \qquad (18)$$

when $a=\frac{16}{15}$. The coarse-to-fine grid extrapolation just described has therefore produced two extra orders of accuracy for the singular boundary value problem (1) and (as a result of putting $\alpha = 0$ in (18)) for the special problem (6).

The orders of convergence of $\underline{Y}^{(1)}$ and $\underline{Y}^{(2)}$ are both $0(h^{1+\alpha})$ but it is easy to see that the order of convergence of the solution obtained from (17) is $0(h^{3+\alpha})$ since the local truncation error of (17) is $0(h^5)$. The order of convergence of (17) for the special problem (6) is $0(h^4)$.

*3.4 Further improvements*

Introduce, now, a third grid of step size $\frac{1}{3}h$. The interval $0<x<1$ is thus divided into $3N+3$ subintervals, each of width $\frac{1}{3}h$, and the points $x_n$ ($n=1,2,.,.,N$) of the coarse grid used in §3.1, which were coincident with the points $x_m$ ($m=2,4,\ldots,2N$) of §3.3, are now named $x_s$ ($s=3,6,..,3N$) with respect to the third grid.

The finite difference method (5) is now written

$$-\left(1-\frac{\alpha}{2s}\right)y_{s-1}+\left(2y_s+\frac{h^{2-\alpha}}{3^{2-\alpha}s^{\alpha}}f_s\right)-\left(1+\frac{\alpha}{2s}\right)y_{s+1}=0 \qquad (19)$$

with $s=1,2,...,3N+3$. The local truncation error associated with (19) at the fixed point $X=nh$ on the original, coarse grid of §3.1 is given by

$$t_n^{(3)}=-\frac{\alpha h^3}{486n}y'''(x_n)-\frac{h^4}{972}y^{(iv)}(x_n)-\frac{\alpha h^5}{87480n}y^{(v)}(x_n)-\frac{h^6}{262440}y^{(vi)}(x_n)$$
$$-\frac{\alpha h^7}{33067440n}y^{(vii)}(x_n)-\frac{h^8}{132269760}y^{(viii)}(x_n)-\ldots \qquad (20)$$

with $n=1,2,\ldots,N$ since the grid point $X=x_n=nh$ on the coarse grid of §3.1 is the point $X=3n(^1/_3\,h)$ on the new grid.

Suppose that $\underline{V}$ - $[V_1, V_2, \ldots, V_{3N+3}]$ is the vector of computed values of y obtained using (19) with $s=1,2,\ldots,3N+3$, and define the vector $\underline{Y}^{(3)}$ to be $\underline{Y}^{(3)}=\left[Y_1^{(3)},Y_2^{(3)},\ldots,Y_N^{(3)}\right]^T=\left[V_3,V_6,\ldots,V_{3N}\right]^T$. The N-component (vector $\underline{Y}^{(3)}$ gives third approximations to $y(x)$ at the N points $x_n$ ($n=1,2,..,N$) of the coarse grid.

The vectors $\underline{Y}^{(1)}$, $\underline{Y}^{(2)}$, $\underline{Y}^{(3)}$ all have $0(h^3)$ local truncation errors but the linear combination

$$a\,\underline{Y}^{(3)}+b\,\underline{Y}^{(2)}+(1-a-b)\,\underline{Y}^{(1)}, \qquad (21)$$

in which a and b are parameters, may be shown to have $0(h^7)$ local truncation error with leading terms

$$-\frac{\alpha h^7}{2540160n}y^{(vii)}(x_n)-\frac{h^8}{101606400}y^{(viii)}(x_n)-\ldots \qquad (22)$$

when $a = \frac{729}{560}$ and $b = -\frac{32}{105}$ $\left(\text{and, thus, } 1 - a - b = \frac{1}{336}\right)$.

This three-grid extrapolation has produced four extra orders of accuracy for both the singular and special boundary value problems given by (1) and (6). It is easy to see that the order of convergence of the three-grid method is $0(h^{5+\alpha})$ for the singular problem (1) and $0(h^5)$ for the special problem (6).

*3,5 Alternative higher order formulations*

In deriving the low order method (5) the familiar $0(h^2)$ replacement to y' was used in (4). Using, instead, the replacements

$$y_1' = h^{-1}\left(-\frac{1}{4}A - \frac{5}{6}y_1 + \frac{3}{2}y_2 - \frac{1}{2}y_3 + \frac{1}{12}y_4\right) + 0\left(h^4\right) \tag{23}$$

$$y_n' = h^{-1}\left(\frac{1}{12}y_{n-2} - \frac{2}{3}y_{n-1} + \frac{2}{3}y_{n+1} - \frac{1}{12}y_{n+2}\right) + 0\left(h^4\right),\ n = 2,3,\ldots,\ N-1, \tag{24}$$

$$y_N' = h^{-1}\left(-\frac{1}{12}y_{N-3} + \frac{1}{2}y_{N-2} - \frac{3}{2}y_{N-1} + \frac{5}{6}y_N + \frac{1}{4}B\right) + 0\left(h^4\right) \tag{25}$$

leads to a numerical method for which $t_n^{(1)} = -\frac{1}{12}h^4 y_n^{(iv)} + 0\left(h^5\right)$ n=1,2,...N).

For a in the range $0<\alpha<1$, this represents an improvement in accuracy compared with the low-order method (5), but not for the special problem (6).

The price to pay for this slight improvement in accuracy is the loss of tridiagonality of the matrix Q in (11) and of the Jacobian of the left hand side of (8). The presence of the term in $y_4$ in (23) and the term in $y_{N-3}$ in (25) indicate that Q and the Jacobian are not even uindiagonal. These features of the method indicate that it is not an economic alternative to (5) and this criticism is even more applicable when the two-grid formulation, which has $0(h^5)$ local truncation error, is discussed.

It was noted in §2 that (0,k) Pade approximants are suitable for use in the recurrence relation (2), because of the singularity in y"(0). To this end, the (0,4) Pade approximant is a feasibility; equation (2) becomes

$$-y(x-h) + \left(2+h^2D^2 + \frac{1}{12}h^4 D^4\right) y(x) - y(x+h) = 0 \tag{26}$$

which results in another finite difference method for which the matrix Q has more than five non-zero diagonals. This method has $0(h^5)$ local truncation error and is clearly not an economic alternative to the two—grid formulation of (5) for the solution of (1).

## 4. Numerical experiments

The methods developed in §§3.1, 3.3,3.4 were tested on the two problems used in the paper by Chawla and Katti [1]. These are

*Problem 1*

$$(x^{\alpha}y')' = \beta x^{\alpha+\beta-2}(\alpha + \beta - 1 + \beta x^{\beta})y \quad , \qquad y(0) = 1, \quad y(1) = e \quad ,$$

for which the theoretical solution is

$$y(x) = \exp(x^{\beta}),$$

and

*Problem 2*

$$(x^{\alpha}y')' = \beta x^{\alpha+\beta-2}\{\beta x^{\beta}e^{y} - (\alpha + \beta - 1)\}/(4+x^{\beta}) \, ,$$

$$y(0) = \log_e(0.25) \quad , \quad y(1) = \log_e(0.2) \, ,$$

for which the theoretical solution is

$$y(x) = \log_e\{1/(4+x^{\beta})\} \, .$$

Problem 1 is linear while Problem 2 is non-linear, both exhibiting a singularity at $x = 0$.

Exactly the same numerical experiments were carried out as were undertaken by Chawla and Katti [1] and the computed results were obtained using a 32K BBC model BD microcomputer. The parameter pair $(\alpha, \beta)$ were given the values (0.5,4.0), (0.5,5.0), (0.75,3.75), (0.75,4.75) and h was given the values $2^{-k}$ $(k = 4,5,6,7)$ so that N = 15,31,63,127, respectively.

The solution vectors $\underline{Y}^{(1)}$, $\underline{U}$, $\underline{V}$ for Problem 1 were obtained using the tridiagonal solver described in most texts on ordinary or partial differential equations (see, for instance, Twizell [9; p.20]) and the vectors $\underline{Y}^{(2)}$ and $\underline{Y}^{(3)}$ were computed using (17) with $a = \frac{16}{15}$ and (21) with $a = \frac{729}{560}$, $b = -\frac{32}{105}$, respectively.

Denoting by $T_1, T_2, T_3$, respectively, the methods with solution vectors $\underline{Y}^{(1)}$, $\underline{Y}^{(2)}$, $\underline{Y}^{(3)}$, the error norms $\| \underline{y} - \underline{Y}^{(i)} \|$, $i = 1,2,3$, for each parameter pair $(\alpha, \beta)$ and each value of N are given in Table1 for Problem 1. Also contained in Table1 are the corresponding error norms of Chawla and Katti [1] (in some cases Chawla and Katti quoted error norms for more than one method; the figures quoted in Table1 are their minimum error norms).

Comparison of the four columns of error norms in Table1 shows that all of the methods $T_1$, $T_2$, $T_3$ behave as predicted in §3, with results for any one method improving by approximately the same ratio as N is successively (roughly) doubled. Method $T_1$, which is $0(h^{1+\alpha})$ convergent, is seen to give better results that the best $0(h^2)$ convergent method of Chawla and Katti [1] and is easier and more economic to implement.

The solution vectors $\underline{Y}^{(1)}$, $\underline{U}$, $\underline{V}$ for Problem 2 were obtained using the Newton-Raphson method for a non-linear algebraic system. It was found for all experiments that at most three iterations were required to give convergence of the solution vectors to three significant figures. The vectors $\underline{Y}^{(2)}$ and $\underline{Y}^{(3)}$ were computed as described above.

The error norms for $T_1, T_2, T_3$ for each numerical experiment are given in Table 2 for Problem2, together with the corresponding, best error norms of Chawla and Katti. As with Problem 1, method $T_1$ is seen to give better numerical results than the most accurate method of Chawla and Katti [1], though the local truncation errors of each are of the same order . Overall for Problem2, methods $T_1, T_2, T_3$ behave as indicated in §3 with error moduli decreasing by the same factor as the step size is successively halved.

References

1. Chawla, M.M., Katti, C,P,: Finite difference methods and their convergence for a class of singular two point boundary value problems- Numer, Math. 39, 341-350 (1982)

2. Ciarlet, P.G., Natterer, F., Varga, R.S.: Numerical methods of high order accuracy for singular nonlinear boundary value problems, Numer. Math, 15, 87-99 (1970)

3. Gustafsson, B.: A numerical method for solving singular boundary value problems, Numer, Math. 21, 328-344 (1973)

4. Henrici, P.; Discrete variable methods in ordinary differential equations, New York: John Wiley & Sons 1962

5. Jamet, P.: On the convergence of finite difference approximations to one-dimensional singular boundary value problems, Numer, Math.14, 355-378 (1970)

6. Reddien,G.W.: Projection methods and singular two point boundary value problems. Numer. Math. 21, 193-205 (1973)

7. Reddien, G.W., Schumaker, L.L.: On a collocation method for singular two point boundary value problems. Numer. Math. 25, 427-432 (1976)

8. Twizell, E.H.: An explicit difference method for the wave equation with extended stability range. BI T 19 , 378-383 (1979)

9. Twizell, E.H.: Computational methods for partial differential equations, Chichester: Ellis Horwood/John Wiley & Sons 1984

Table 1 Maximum error moduli in computed results for Problem 1.

| | | | Method | | | |
|---|---|---|---|---|---|---|
| α | ß | N | $T_1$ | $T_2$ | $T_3$ | Chawla and Katti [1] |
| 0.5 | 4.0 | 15 | 1.1E-2 | 2.2E-3 | 7.9E-4 | 1.2E-2 |
| | | 31 | 2.7E-3 | 5.5E-4 | 2.0E-4 | 3.0E-3 |
| | | 63 | 6.9E-4 | 1.4E-4 | 4.9E-5 | 7.3E-4 |
| | | 127 | 1.7E-4 | 3.4E-5 | 1.2E-5 | 1.8E-4 |
| | 5.0 | 15 | 1.7E-2 | 3.6E-3 | 1.3E-3 | 1.8E-2 |
| | | 31 | 4.5E-3 | 9.1E-4 | 3.3E-4 | 4.7E-3 |
| | | 63 | 1.1E-3 | 2.3E-4 | 8.1E-5 | 1.2E-3 |
| | | 127 | 2.8E-4 | 5.7E-5 | 2.0E-5 | 3.0E-4 |
| 0.75 | 3.75 | 15 | 1.0E-2 | 2.1E-3 | 7.4E-4 | 1.2E-2 |
| | | 31 | 2.6E-3 | 5.2E-4 | 1.8E-4 | 2.9E-3 |
| | | 63 | 6.4E-4 | 1.3E-4 | 4.6E-5 | 7.2E-4 |
| | | 127 | 1.6E-4 | 3.2E-5 | 1.5E-5 | 1.8E-4 |
| | 4.75 | 15 | 1.7E-2 | 3.4E-3 | 1.2E-3 | 1.8E-2 |
| | | 31 | 4.2E-3 | 8.6E-4 | 3.1E-4 | 4.6E-3 |
| | | 63 | 1.1E-3 | 2.2E-4 | 7.7E-5 | 1.2E-3 |
| | | 127 | 2.7E-4 | 5.4E-5 | 1.9E-5 | 2.9E-4 |

Table 2 Maximum error moduli in computed results for Problem 2

| | | | Method | | | |
|---|---|---|---|---|---|---|
| $\alpha$ | $\beta$ | N | $T_1$ | $T_2$ | $T_3$ | Chawla and Katti [1] |
| 0.5 | 4.0 | 15 | 3.2E-4 | 6.6E-5 | 2.4E-5 | 3.9E-4 |
| | | 31 | 8.2E-5 | 1.7E-5 | 6.1E-6 | 9.7E-5 |
| | | 63 | 2.1E-5 | 4.1E-5 | 1,4E-6 | 2.4E-5 |
| | | 127 | 5.2E-6 | 3.3E-7 | 1.0E-7 | 6.1E-6 |
| | 5.0 | 15 | 6.3E-4 | 1.3E-4 | 4.6E-5 | 7.5E-4 |
| | | 31 | 1.6E-4 | 3.3E-5 | 1.2E-5 | 1.9E-4 |
| | | 63 | 4.0E-5 | 8.2E-6 | 2.9E-6 | 4.7E-5 |
| | | 127 | 1.0E-5 | 1,2E-6 | 3.7E-7 | 1. 2E-5 |
| 0.75 | 3.75 | 15 | 3.8E-4 | 8.2E-5 | 3.0E-5 | 6.2E-4 |
| | | 31 | 1.0E-4 | 2.1E-5 | 7.8E-6 | 1.6E-4 |
| | | 63 | 2.6E-5 | 5.3E-6 | 1.7E-6 | 3.9E-5 |
| | | 127 | 6.6E-6 | 1.0E-6 | 9.9E-7 | 9.7E-6 |
| | 4.75 | 15 | 7.2E-4 | 1.5E-4 | 5.5E-5 | 1.0E-3 |
| | | 31 | 1.9E-4 | 3.8E-5 | 1.4E-5 | 2.6E-4 |
| | | 63 | 4.8E-5 | 9.7E-6 | 3.5E-6 | 6.4E-5 |
| | | 127 | 1.2E-5 | 1.7E-6 | 1. 1E-6 | 1.6E-5 |