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THE GEOMETRIC PARADIGM IN COMPUTATIONAL ELASTO-PLASTICITY

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Abstract. Computational methods, for large displacements of continua in the elasto-plastic range, rely on the mathematical modeling of the nonlinear constitutive behavior. In last decades an increasing favor has been deserved to nonlinear models based on a chain decomposition of the deformation gradient. The troubles involved in a structural analysis based on this model are well-known and have not been overcome although many efforts devoted to this end. Our investigation towards a more satisfactory model starts from the new analysis of the rate elastic behavior performed in [1, 2] since the difficulties faced by previous formulations were the very motivation for the discard of rate constitutive models in elasto-plasticity [3]. The new definition of hypo-elasticity, the detection of simple integrability conditions and a new formulation of conservativeness, lead to a definition of rate elasticity suitable for an effective modeling of rate elasto-plastic constitutive behaviors [4]. The treatment is based on a geometric definition of spatial and material fields and on the statement of a geometric paradigm assessing the rules for comparison of material fields naturally provided by push-pull according to the relevant transformation. The rates involved in constitutive relations are Lie-derivatives of stress field and constitutive parameters. Geometric compatibility requires that elastic and plastic stretchings additively give the Lie-derivative of metric field. No privileged reference configuration is involved and no consequent multiplicative decomposition of deformation gradient is assumed. Computational methods are shown to be based on the pull-back of constitutive relations to a straightened out trajectory segment which plays the role of computation chamber wherein linear operations of differentiation and integration may be performed. Accordingly, finite elastic and plastic stretches are considered as purely computational tools with no physical interpretation in constitutive relations. Both 3-D and lower dimensional structural models, such as wires and membranes, may be analysed by a direct application of the theory. The outcome is a significant improvement of physical insight and computational effectiveness with respect to previous treatments of finite elasto-plasticity.

1 INTRODUCTION

To comply with principles of *Geometric Naturality* and *Dimensionality Independence* and with the dictates of the ensuing *Geometric Paradigm*, enunciated in [1, 2, 4], the elasticity model will be introduced with a rate formulation by a complete rephrasing of the original hypo-elastic model proposed by TRUESDELL [5]. The basic distinctive feature is that stress time-rate is defined in the natural way as LIE derivative of the stress field along the motion and that the constitutive law is inverted, so to provide the definition of elastic stretching. By virtue of this new definition, a simple and complete analysis of time-independence, integrability and conservativeness may be performed and this is a major merit of the geometric approach to elasticity. The consequent brand new analysis of rate constitutive relations, lead naturally to a clear, definite and computationally effective theory of elasto-plastic constitutive behavior in the non-linear range. The geometric treatment reveals that the rate formulation of the elasticity model, defined as a time-invariant and integrable hypo-elastic model, is self-proposing as natural and physically meaningful. Moreover, compatibility with a hyper-elastic behavior and conservation of elastic energy are ensured by conservation of mass and by directly verifiable integrability conditions. The key answer given by a proper geometric formulation concerns the way material tensors should be compared and differentiated in time along the trajectory. This requires a clear distinction between *spatial* and *material* tensor fields in continuum mechanics. The topic was first pointed out in [1] with reference to hypo-elasticity and then investigated in comprehensive geometrical terms in a space-time framework [2, 4]. Once the basic tools have been made available, the natural and geometrically consistent choice in elasto-plasticity is to formulate the rate constitutive behavior in terms of LIE-derivatives of the relevant material tensors along the trajectory, by performing an additive split of the total stretching into an elastic stretching and a plastic stretching. These latter are not LIE-derivatives of material fields, unless either the elastic or the plastic stretching vanishes. The hypo-elastic relation is governed by a constitutive operator which is non-linearly dependent on the stress tensor and provides the elastic stretching as a linear function of the stressing tensor. The visco-plastic flow rule is governed by a multi-valued operator. Indeed the current values of stress and tensorial internal state parameters determine a convex set of variation of the visco-plastic stretching. The evolution law for the internal parameters may be included in the visco-plastic flow by suitably enlarging the stress and the stretching spaces, for instance according to the treatment proposed in [6]. No privileged reference configuration nor mysterious intermediate states are needed. Finite plastic and elastic strains may however be conveniently introduced in reference linear spaces as purely computational tools to perform there, on suitable tensor fibers, basic linear operations such as time differentiations or integrations and approximate numerical evaluations. This computational trick is based on the peculiar properties of LIE derivatives under push-pull transformations, but no physical interpretation is given to finite plastic and elastic strains. The conclusions of the new theory restore credit to early computational choices of a rate description of elasto-plastic behavior, but with the decisive improvement that the rules of the game are now clearly assessed. The theory leads to a conceptually clean, methodologically definite model and to drastic simplifications of both theoretical and computational aspects of geometrically non-linearized elasto-visco-plasticity.

2 CONSTITUTIVE LAWS

Let E be a four-dimensional events manifold in which a nonlinear trajectory manifold $\mathcal{T}$, possibly lower dimensional, is immersed [4]. In a theoretical framework suitable for investigating a sufficiently general class of material behaviour for engineering applications, a constitutive law may be defined in terms of a constitutive operator $\mathbf{H}_{\mathcal{T}}$, as follows.

Definition 2.1 (Constitutive laws) *A constitutive operator $\mathbf{H}_{\mathcal{T}}$, in a body motion detected by an* Euclid *observer, is a possibly multivalued material tensor bundle morphism whose domain and codomain are* Whitney *products*[1] *of material tensor bundles.*

The *tensor bundle morphism* requirement means that material tensors in the domain and codomain of the constitutive map should have the same base point in the tangent trajectory bundle, that is, they should be evaluated at a common event (i.e. a pair particle position, time instant) in the trajectory. The covariance (or geometric) paradigm [1, 2, 4] allows to compare the constitutive laws at displaced configurations of a body. Invariance of the constitutive law is the property that material tensor fields related by the constitutive law must be still related:

- either when transformed by push according to the relevant material displacement map $\boldsymbol{\varphi}_{\tau,t}$, and this is the meaning of *time invariance* (TI) of the constitutive law,
- or when transformed by push according to a relative motion between observers, and this is the meaning of *frame invariance* (FI) of the constitutive law, which is named *material frame indifference* (MFI) when changes of Euclid observers are considered.

These definitions modify the standard TI and FI definitions as enunciated in [8] and quoted in subsequent literature, e.g. [9, 10, 11, 12, 13, 14, 15]. The change is from an invariance property to a property of variance by push [1, 2, 4]. In standard statements equality between constitutive responses measured by different observers is imposed without taking into account that constitutive operators to be compared do not have the same domain and codomain. This improper identification is responsible for the wrong assertion that FI implies isotropy, see [8] formula 99.5.

[1] The Whitney product of two tensor bundles $(\mathbb{N}, \boldsymbol{\pi}_{\mathbb{M},\mathbb{N}}, \mathbb{M})$ and $(\mathbb{H}, \boldsymbol{\pi}_{\mathbb{M},\mathbb{H}}, \mathbb{M})$, over the same base manifold $\mathbb{M}$, is the linear bundle defined by [7]:

$$\mathbb{N} \times_{\mathbb{M}} \mathbb{H} := \{ (\mathbf{n}, \mathbf{h}) \in \mathbb{N} \times \mathbb{H} \mid \boldsymbol{\pi}_{\mathbb{M},\mathbb{N}}(\mathbf{n}) = \boldsymbol{\pi}_{\mathbb{M},\mathbb{H}}(\mathbf{h}) \} .$$

Definition 2.2 (Constitutive time invariance) *According to the covariance paradigm, time invariance of the constitutive operator means that, along the motion:*

$$\mathbf{H}_{\mathcal{T},\tau} = \boldsymbol{\varphi}_{\tau,t}\!\uparrow\!\mathbf{H}_{\mathcal{T},t}\,,$$

for any time instants $\tau, t \in I$ *. Explicitly the condition writes:*

$$\mathbf{H}_{\mathcal{T},\tau}(\boldsymbol{\varphi}_{\tau,t}\!\uparrow\!\mathbf{s}_{\mathcal{T},t}) = (\boldsymbol{\varphi}_{\tau,t}\!\uparrow\!\mathbf{H}_{\mathcal{T},t})(\boldsymbol{\varphi}_{\tau,t}\!\uparrow\!\mathbf{s}_{\mathcal{T},t}) = \boldsymbol{\varphi}_{\tau,t}\!\uparrow\!(\mathbf{H}_{\mathcal{T},t}(\mathbf{s}_{\mathcal{T},t}))\,.$$

This means that time-invariant material tensor fields, fulfilling the constitutive relation at time $t \in I$, are still related by the law at time $\tau \in I$.

2.1 HYPO-ELASTICITY

The hypo-elastic constitutive law is properly formulated as a linear dependence of the elastic stretching upon the stressing, by means of a stress dependent compliance constitutive operator, as follows [1].

Definition 2.3 (Hypo-elasticity) *A hypo-elastic constitutive model is expressed by the law:*

$$\mathbf{el}_{\mathcal{T}} = \mathbf{H}_{\mathcal{T}}(\boldsymbol{\sigma}_{\mathcal{T}}) \cdot \dot{\boldsymbol{\sigma}}_{\mathcal{T}}\,,$$

where $\mathbf{el}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}\,; \mathrm{Cov}(\mathbb{V}\mathcal{T}))$ *is the covariant elastic stretching and the stressing* $\dot{\boldsymbol{\sigma}}_{\mathcal{T}} := \mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\boldsymbol{\sigma}_{\mathcal{T}}$ *is the* LIE *derivative of the contravariant stress tensor* $\boldsymbol{\sigma}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}\,; \mathrm{Con}(\mathbb{V}\mathcal{T}))$ *, along the motion. A purely elastic process occurs when the elastic stretching is equal to the total stretching, i.e.* $\mathbf{el}_{\mathcal{T}} := \boldsymbol{\varepsilon}_{\mathcal{T}} = \frac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\mathbf{g}_{\mathcal{T}}$ *, where* $\mathbf{g}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}, \mathrm{Sym}(\mathbb{V}\mathcal{T}))$ *is the material metric [2].*

Lemma 2.1 (Integrability) *The hypo-elastic compliance operator is integrable if and only if the constitutive operator* $\mathbf{H}_{\mathcal{T}}$ *and its fiber derivative (i.e. the derivative taken at a fixed event in the trajectory) are symmetric:*

$$\langle d_F\mathbf{H}_{\mathcal{T}}(\boldsymbol{\sigma}_{\mathcal{T}}) \cdot \delta\boldsymbol{\sigma}_{\mathcal{T}} \cdot \delta_1\boldsymbol{\sigma}_{\mathcal{T}}, \delta_2\boldsymbol{\sigma}_{\mathcal{T}} \rangle = \langle d_F\mathbf{H}_{\mathcal{T}}(\boldsymbol{\sigma}_{\mathcal{T}}) \cdot \delta\boldsymbol{\sigma}_{\mathcal{T}} \cdot \delta_2\boldsymbol{\sigma}_{\mathcal{T}}, \delta_1\boldsymbol{\sigma}_{\mathcal{T}} \rangle\,,$$

$$\langle \mathbf{H}_{\mathcal{T}}(\boldsymbol{\sigma}_{\mathcal{T}}) \cdot \delta_1\boldsymbol{\sigma}_{\mathcal{T}}, \delta_2\boldsymbol{\sigma}_{\mathcal{T}} \rangle = \langle \mathbf{H}_{\mathcal{T}}(\boldsymbol{\sigma}_{\mathcal{T}}) \cdot \delta_2\boldsymbol{\sigma}_{\mathcal{T}}, \delta_1\boldsymbol{\sigma}_{\mathcal{T}} \rangle\,,$$

for all $\delta\boldsymbol{\sigma}_{\mathcal{T}}, \delta_1\boldsymbol{\sigma}_{\mathcal{T}}, \delta_2\boldsymbol{\sigma}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}, \mathrm{Sym}^*(\mathbb{V}\mathcal{T}))$ *. The fiber-derivative* d_F *is taken by holding the base point fixed.*

The former condition ensures CAUCHY integrability, stating the existence of a stretching-valued stress potential $\boldsymbol{\Phi}_{\mathcal{T}} \in \mathrm{C}^1(\mathrm{Con}(\mathbb{V}\mathcal{T})\,; \mathrm{Cov}(\mathbb{V}\mathcal{T}))$ such that $d_F\boldsymbol{\Phi}_{\mathcal{T}} = \mathbf{H}_{\mathcal{T}}$. Both conditions ensure GREEN integrability, stating existence of a scalar-valued stress potential $E^*_{\mathcal{T}} \in \mathrm{C}^1(\mathrm{Con}(\mathbb{V}\mathcal{T})\,; \mathrm{Fun}(\mathbb{V}\mathcal{T}))$ such that:

$$d^2_F E^*_{\mathcal{T}} = d_F\boldsymbol{\Phi}_{\mathcal{T}} = \mathbf{H}_{\mathcal{T}}\,.$$

Integrability of a time-invariant hypo-elastic constitutive operator, at a given time, implies integrability at every time.

3 ELASTICITY AND HYPER-ELASTICITY

Definition 3.1 (Elasticity) *An elastic (resp. hyper-elastic) constitutive model is a time-invariant and* CAUCHY *(resp.* GREEN*) integrable hypo-elastic model, so that:*

$$\mathbf{el}_{\mathcal{T}} = d_F \boldsymbol{\Phi}_{\mathcal{T}}(\boldsymbol{\sigma}_{\mathcal{T}}) \cdot \dot{\boldsymbol{\sigma}}_{\mathcal{T}} \,, \qquad (\mathbf{el}_{\mathcal{T}} = d_F^2 E^*_{\mathcal{T}}(\boldsymbol{\sigma}_{\mathcal{T}}) \cdot \dot{\boldsymbol{\sigma}}_{\mathcal{T}}) \,,$$

with time-invariance expressed by:

$$\boldsymbol{\Phi}_{\mathcal{T},\tau} = \boldsymbol{\varphi}_{\mathcal{T},t} \!\uparrow\! \boldsymbol{\Phi}_{\mathcal{T},t} \,, \qquad (E^*_{\mathcal{T},\tau} = \boldsymbol{\varphi}_{\tau,t} \!\uparrow\! E^*_{\mathcal{T},t}) \,.$$

Denoting by $\rho_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}\,; \mathrm{FUN}(\mathbb{V}\mathcal{T}))$ the scalar density field, the material mass form is defined by $\mathbf{m}_{\mathcal{T}} := \rho_{\mathcal{T}} \boldsymbol{\mu}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}\,; \mathrm{VOL}(\mathbb{V}\mathcal{T}))$, where $\boldsymbol{\mu}_{\mathcal{T}}$ is the material volume form [2]. The next result shows that conservation of mass and GREEN's integrability of the hypo-elastic operator imply conservation of the mechanical energy [4].

Proposition 3.1 (Conservativeness) *The constitutive operator of a hypo-elastic material, which is hyper-elastic when expressed in terms of the* KIRCHHOFF *stress tensor, is conservative, that is:*

$$\int_I \int_{\boldsymbol{\Omega}_t} \langle \boldsymbol{\sigma}_{\mathcal{T},t}, \mathbf{el}_{\mathcal{T},t} \rangle \, \mathbf{m}_{\mathcal{T},t} \, dt = 0 \,,$$

for any covariantly closed stress path, i.e. any path such that its pull-back to any fixed reference placement is a cycle, a property expressed by the condition:

$$\boldsymbol{\sigma}_{\mathcal{T},t_2} = \boldsymbol{\varphi}_{t_2,t_1} \!\uparrow\! \boldsymbol{\sigma}_{\mathcal{T},t_1} \,, \quad I = [t_1, t_2] \,.$$

3.1 Computational issues

In computational algorithms of the static evolution of an elastic structure undergoing large displacements, the equilibrium process is approximated by finite step incremental solutions in time. To underline the decisive role of a referential formulation, let us briefly describe the iterative scheme leading to the solution of the elastostatic problem in a finite time step. The *control process* is described by a time-parametrized curve $\mathbf{c} \in \mathrm{C}^1(I\,; \mathrm{C})$ in a control set C. The force acting on a body at time $\tau \in I$ is assumed to depend on the control point $\mathbf{c}(\tau)$ and on the displacement $\boldsymbol{\varphi}_{\tau,t}$ from a given placement $\boldsymbol{\Omega}_t$, so that we may write

$$\mathbf{f}(\tau) = \mathbf{F}(\mathbf{c}(\tau), \boldsymbol{\varphi}_{\tau,t}) : \boldsymbol{\varphi}_{\tau,t}(\boldsymbol{\Omega}_t) \mapsto \mathbb{T}^*(\boldsymbol{\varphi}_{\tau,t}(\boldsymbol{\Omega}_t)) \,.$$

An iterative trial and error procedure is then formulated as follows.

1. The start point is an equilibrium solution at time $t_1 \in I$ under a force system

   $$\mathbf{f}(t_1) = \mathbf{F}(\mathbf{c}(t_1), \mathrm{ID}_{\boldsymbol{\Omega}_{t_1}}) : \boldsymbol{\Omega}_{t_1} \mapsto \mathbb{T}^*(\boldsymbol{\Omega}_{t_1})$$

   and a stress field $\boldsymbol{\sigma}(t_1)$ fulfilling the virtual power variational equality

   $$\langle \mathbf{f}(t_1), \delta\mathbf{v} \rangle = \int_{\boldsymbol{\Omega}(t_1)} \langle \boldsymbol{\sigma}(t_1), \boldsymbol{\varepsilon}(\delta\mathbf{v}) \rangle \, \mathbf{m} \,.$$

2. An initial guess of the displacement $\boldsymbol{\varphi}_{t_2,t_1}$ corresponding to the update of the input control point from $\mathbf{c}(t_1)$ to $\mathbf{c}(t_2)$ may be obtained by computing the force system $\mathbf{F}(\mathbf{c}(t_2), \mathrm{ID}_{\boldsymbol{\Omega}_{t_1}}) : \boldsymbol{\Omega}_{t_1} \mapsto \mathbb{T}^*(\boldsymbol{\Omega}_{t_1})$ solution of the linear elastostatic problem

$$\langle \mathbf{F}(\mathbf{c}(t_2), \mathrm{ID}_{\boldsymbol{\Omega}_{t_1}}), \delta\mathbf{v} \rangle = \int_{\boldsymbol{\Omega}(t_1)} \langle \mathbf{H}(\boldsymbol{\sigma}(t_1)) \cdot \boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\delta\mathbf{v}) \rangle \, \mathbf{m} \, .$$

3. Setting $\boldsymbol{\varphi}_{t_2,t_1}(\mathbf{x}) = \mathbf{u}(\mathbf{x}) + \mathbf{x}$ for any $\mathbf{x} \in \boldsymbol{\Omega}_{t_1}$, the control algorithm provides the trial force system

$$\mathbf{f}(t_2) = \mathbf{F}(\mathbf{c}(t_2), \boldsymbol{\varphi}_{t_2,t_1}) : \boldsymbol{\varphi}_{t_2,t_1}(\boldsymbol{\Omega}(t_1)) \mapsto \mathbb{T}^*(\boldsymbol{\varphi}_{t_2,t_1}(\boldsymbol{\Omega}(t_1))) \, .$$

   The trial referential finite step elastic strain is given by

$$\mathbf{el}_{\mathrm{REF}}(t_2, t_1) = \boldsymbol{\varphi}_{\mathrm{REF}}(t_1) {\downarrow} \tfrac{1}{2} (\boldsymbol{\varphi}_{t_2,t_1} {\downarrow} \mathbf{g}_{\mathcal{T}} - \mathbf{g}_{\mathcal{T}}) \, .$$

   The updated stress in the trial placement $\boldsymbol{\Omega}(t_2)$ is evaluated by

$$\boldsymbol{\sigma}(t_2) = \boldsymbol{\varphi}_{\mathrm{REF}}(t_2) {\uparrow} \big(\boldsymbol{\sigma}_{\mathrm{REF}}(t_1) + \boldsymbol{\Phi}^{-1}_{\mathrm{REF}} \big(\mathbf{el}_{\mathrm{REF}}(t_2, t_1) \big) \big) \, ,$$

   and the related elastic response is provided by the virtual power principle

$$\langle \mathbf{r}(t_2), \delta\mathbf{v} \rangle = \int_{\boldsymbol{\Omega}(t_2)} \langle \boldsymbol{\sigma}(t_2), \boldsymbol{\varepsilon}(\delta\mathbf{v}) \rangle \, \mathbf{m} \, .$$

4. The residual force gap

$$\mathbf{f}(t_2) - \mathbf{r}(t_2) : \boldsymbol{\varphi}_{t_2,t_1}(\boldsymbol{\Omega}(t_1)) \mapsto \mathbb{T}^*(\boldsymbol{\varphi}_{t_2,t_1}(\boldsymbol{\Omega}(t_1))) \, ,$$

   is then applied to perform a correction to the previous guess concerning the displacement $\boldsymbol{\varphi}_{t_2,t_1}$ by solving the linear elastostatic problem

$$\langle \mathbf{f}(t_2) - \mathbf{r}(t_2), \delta\mathbf{v} \rangle = \int_{\boldsymbol{\varphi}_{t_2,t_1}(\boldsymbol{\Omega}(t_1))} \langle \mathbf{H}(\boldsymbol{\sigma}(t_2)) \cdot \boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\delta\mathbf{v}) \rangle \, \mathbf{m} \, ,$$

   and then performing the replacement $\boldsymbol{\varphi}_{t_2,t_1}(\mathbf{x}) \mapsto \mathbf{u}(\mathbf{x}) + \boldsymbol{\varphi}_{t_2,t_1}(\mathbf{x})$ for any $\mathbf{x} \in \boldsymbol{\Omega}_{t_1}$.

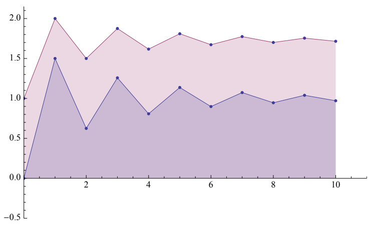
The procedure is iterated on the new guess until the ratio between a suitable norm of the force gap and of the trial force system becomes less than a prescribed tolerance, thus reaching the approximated fixed point of the algorithm. The next time-step is then performed starting at the solution placement $\boldsymbol{\Omega}(t_2) = \boldsymbol{\varphi}_{t_2,t_1}(\boldsymbol{\Omega}(t_1))$ under the force system $\mathbf{f}(t_2) = \mathbf{F}(\mathbf{c}(t_2), \boldsymbol{\varphi}_{t_2,t_1}) : \boldsymbol{\Omega}_{t_2} \mapsto \mathbb{T}^*(\boldsymbol{\Omega}_{t_2})$. The hyper-elastic model can be extended to nonlocal models presented in [16, 17].

4 Elastic extension of a wire

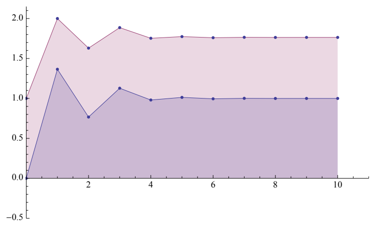
Let us consider, as a simple example, a wire of initial length L under the action of an axial force increment $\mathbf{F}$. The elastic constitutive relation writes $\mathbf{El} := \mathbf{H}^{\mathrm{Mix}}(\mathbf{K}) \cdot \dot{\mathbf{K}}$ with $\mathbf{El}$ mixed elastic stretching, $\mathbf{K}$ is mixed KIRCHHOFF stress and $\dot{\mathbf{K}} := \dot{\boldsymbol{\sigma}} \circ \mathbf{g}_{\mathcal{T}}$ mixed alteration of the KIRCHHOFF stressing $\dot{\boldsymbol{\sigma}}$. The hypo-elastic constitutive operator is given by

$$\mathbf{H}^{\mathrm{Mix}}(\mathbf{K}) := \frac{1}{2\,\mu}\,\mathbb{I} - \frac{\nu}{E}\,\mathbf{I} \otimes \mathbf{I}\,.$$

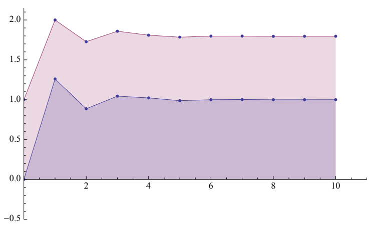
The referential mixed elastic stretching is given by the pull-back formula $\mathbf{El}_{\mathrm{REF}} = T\boldsymbol{\varphi}_{\mathrm{REF}}^{-1} \circ \mathbf{El} \circ T\boldsymbol{\varphi}_{\mathrm{REF}}$. The force increment is equal to 1 and the initial linearized response is equal to 1.5 for a POISSON ratio $\nu = 0.00$. The initial length is equal to 1 and is doubled by the first linearized estimate. Convergence features of the algorithm are shown in fig.1 for the values $\nu = 0.00, 0.30, 0.40, 0.49$. The physically significant example is the one with POISSON ratio $\nu = 0.49$, which is appropriate for a rubber wire.[2]



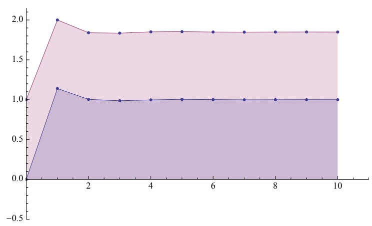
(a) ν =0.00



(b) ν =0.30



(c) ν =0.40



(d) ν =0.49

Figure 1: Elastic extension of a wire.

Upper diagrams represent the length while lower diagrams represent the elastic response, as the iterative algorithm described in Sect.3.1 proceeds.

5 ELASTO-PLASTICITY

Once that the hypo-elastic model has been properly formulated and the right conditions for time invariance and conservativeness have been assessed, the rate model of elasto-visco-plastic constitutive behavior, which is of primary applicative interest in NLCM, may be

[2]Computations and graphics implemented with Wolfram Mathematica 8.

readily described by the relations:

$$\begin{cases} \boldsymbol{\varepsilon}_{\mathcal{T}} = \mathbf{el}_{\mathcal{T}} + \mathbf{pl}_{\mathcal{T}} \,, \\ \mathbf{el}_{\mathcal{T}} = d_F^2 E_{\mathcal{T}}^*(\boldsymbol{\sigma}_{\mathcal{T}}) \cdot \dot{\boldsymbol{\sigma}}_{\mathcal{T}} \,, \\ \mathbf{pl}_{\mathcal{T}} \in \partial_F \mathcal{F}_{\mathcal{T}}(\boldsymbol{\sigma}_{\mathcal{T}}) \,, \end{cases}$$

with $\mathcal{F}_{\mathcal{T}} \subset \mathrm{F{\scriptsize UN}}(\mathbb{V}\mathcal{T})$ a fiberwise subdifferentiable convex potential [18]. These constitutive relations are in fact extensions of the classical formula introduced, with reference to visco-elasticity, by JAMES CLERK-MAXWELL in [19]. Neither the *elastic stretching* $\mathbf{el}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}, \mathrm{S{\scriptsize YM}}(\mathbb{V}\mathcal{T}))$ nor the *plastic stretching* $\mathbf{pl}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}, \mathrm{S{\scriptsize YM}}(\mathbb{V}\mathcal{T}))$ are convective time derivatives of a material field. The elastic and plastic stretchings should then not be denoted by a superimposed dot, as usually made in literature. The issue will be further addressed in Sect. 6. Elasto-plasticity is modeled by assuming that the stress potential is the indicator function of the convex set of admissible stresses $\mathcal{K}_{\mathcal{T}} \subset \mathrm{S{\scriptsize YM}}^*(\mathbb{V}\mathcal{T})$, so that:

$$\partial_F \mathcal{F}_{\mathcal{T}}(\boldsymbol{\sigma}_{\mathcal{T}}) = \mathcal{N}_{\mathcal{K}_{\mathcal{T}}}(\boldsymbol{\sigma}_{\mathcal{T}}) \,,$$

where $\mathcal{N}_{\mathcal{K}_{\mathcal{T}}}(\boldsymbol{\sigma}_{\mathcal{T}})$ is the outward normal cone to $\boldsymbol{\sigma}_{\mathcal{T}} \in \mathcal{K}_{\mathcal{T}}$. The visco-plastic constitutive relation specializes then into the plastic flow rule:

$$\mathbf{pl}_{\mathcal{T}} \in \mathcal{N}_{\mathcal{K}_{\mathcal{T}}}(\boldsymbol{\sigma}_{\mathcal{T}}) \,.$$

A nonlocal model of elastoplasticity has been analysed in [20, 21]. Moreover a local elastoplastic behavior and a nonlocal damage model in the strain space has been addressed in [22, 23].

5.1 Computational issues

By a pull-back procedure the elasto-visco-plastic constitutive relations may be formulated in terms of material tensor fields defined in a fixed placement $\boldsymbol{\Omega}_{\mathrm{REF}}$. Setting

$$\begin{aligned} \mathbf{el}_{\mathcal{T},t}^{\mathrm{REF}} &= \boldsymbol{\varphi}_{t,\mathrm{REF}}{\downarrow}\mathbf{el}_{\mathcal{T},t} \,, \quad & \boldsymbol{\sigma}_{\mathcal{T},t}^{\mathrm{REF}} &= \boldsymbol{\varphi}_{t,\mathrm{REF}}{\downarrow}\boldsymbol{\sigma}_{\mathcal{T},t} \,, \\ \mathbf{pl}_{\mathcal{T},t}^{\mathrm{REF}} &= \boldsymbol{\varphi}_{t,\mathrm{REF}}{\downarrow}\mathbf{pl}_{\mathcal{T},t} \,, \quad & \mathcal{F}_{\mathcal{T},t}^{\mathrm{REF}} &= \boldsymbol{\varphi}_{t,\mathrm{REF}}{\downarrow}\mathcal{F}_{\mathcal{T},t} \,, \end{aligned}$$

with $\boldsymbol{\varphi}_{t,\mathrm{REF}} : \boldsymbol{\Omega}_{\mathrm{REF}} \mapsto \boldsymbol{\Omega}_t$, we get:

$$\begin{cases} \partial_{\tau=t}\, \boldsymbol{\varepsilon}_{\mathcal{T},\tau,t}^{\mathrm{REF}} = \mathbf{el}_{\mathcal{T},t}^{\mathrm{REF}} + \mathbf{pl}_{\mathcal{T},t}^{\mathrm{REF}} \,, \\ \qquad \mathbf{el}_{\mathcal{T},t}^{\mathrm{REF}} = \partial_{\tau=t}\, d_F E_{\mathrm{REF}}^*(\boldsymbol{\sigma}_{\mathcal{T},\tau}^{\mathrm{REF}}) \,, \\ \qquad \mathbf{pl}_{\boldsymbol{\varphi},t}^{\mathrm{REF}} \in \partial_F \mathcal{F}_{\mathcal{T},t}^{\mathrm{REF}}(\boldsymbol{\sigma}_{\mathcal{T},t}^{\mathrm{REF}}) \,. \end{cases}$$

In an evolution process, the computations of the pull-back of the stress fields are conveniently carried out by a discrete time integration scheme and by an iterative algorithm, for the solution at each time step of the non-linear discrete constitutive relation, on the basis of trial estimates of the elastic stretching evaluated at a fixed placement [24, 25].In conclusions, the constitutive relations of elasto-visco-plastic behavior, in the non-linear geometric range, differ by the ones pertaining to the linearized theory just because the LIE derivatives of the material metric tensor and of the stress tensor are approximated, in the linearized theory, by partial time derivatives at a fixed point in space. An expression in terms of partial time derivatives may be got also in the non-linear geometric range by pulling back the constitutive relations to a fixed configuration. Numerical solution algorithms are then analogous to the ones adopted in the linearized theory, but with the additional task of taking care of the fact that the unknown differential of the displacement map, from a fixed to the current configuration, is involved in the expression of the referential stress.

6 FINITE ELASTIC AND PLASTIC STRAINS

Given a plastic stretching field on the trajectory, a finite plastic strain from an initial time t_o to the current time t, may be defined by integration, in the time interval $I = [t_o, t]$, of the plastic stretching field pulled-back from the current to a fixed configuration along the displacement $\boldsymbol{\varphi}_{t,\mathrm{REF}} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,; \boldsymbol{\Omega}_{\mathrm{REF}})$:

$$\mathbf{pl}^{\mathrm{REF}}_{\mathcal{T},I} := \int_I \boldsymbol{\varphi}_{t,\mathrm{REF}}{\downarrow}\mathbf{pl}_{\mathcal{T},t}\, dt \in \mathrm{C}^1(\boldsymbol{\Omega}_{\mathrm{REF}}\,; \mathrm{Cov}(\mathbb{T}\boldsymbol{\Omega}_{\mathrm{REF}}))\,.$$

This operation defines a tensor field in the fixed configuration and, by push forward, also a tensor field in the current configuration. By performing a time-differentiation in the fixed configuration and then a push forward from the fixed to the current configuration, the plastic stretching field is recovered:

$$\boldsymbol{\varphi}_{t,\mathrm{REF}}{\downarrow}\mathbf{pl}_{\mathcal{T},t} = \partial_{\tau=t} \int_{t_o}^{\tau} \boldsymbol{\varphi}_{\tau,\mathrm{REF}}{\downarrow}\mathbf{pl}_{\mathcal{T},\tau}\, d\tau \in \mathrm{C}^1(\boldsymbol{\Omega}_{\mathrm{REF}}\,; \mathrm{Cov}(\mathbb{T}\boldsymbol{\Omega}_{\mathrm{REF}}))\,.$$

Anyway this fact does not motivate a definition of the plastic stretching as convective time derivative of the plastic finite strain, because the latter was not introduced independently of the notion of plastic stretching. The same occurs for elastic stretching fields and corresponding finite elastic strain fields. In fact the usefulness of finite plastic and elastic strains in the description of mechanical behaviors is rather questionable, the only directly definable finite strain being the total strain, which is expressed in terms of material displacements and of material metric tensors by:

$$\boldsymbol{\varepsilon}^{\mathrm{REF}}_{\mathcal{T},I} := \tfrac{1}{2} \int_I \partial_{\tau=t}\, \boldsymbol{\varphi}_{\tau,\mathrm{REF}}{\downarrow}\mathbf{g}_{\mathcal{T},\tau}\, dt = \tfrac{1}{2}\boldsymbol{\varphi}_{t_2,\mathrm{REF}}{\downarrow}\mathbf{g}_{\mathcal{T},t_2} - \tfrac{1}{2}\boldsymbol{\varphi}_{t_1,\mathrm{REF}}{\downarrow}\mathbf{g}_{\mathcal{T},t_1}\,.$$

7 CONCLUSIONS

The findings of the geometric approach compel to perform a revisitation of theoretical contributions to non-linear constitutive laws adopted in most recent geometrically non-linear formulations of elasto-plasticity, elasto-visco-plasticity, poro-elasticity, poro-plasticity, phase transformations, growth of biological tissues, and to carry out a consequent modification of relevant computational procedures. The results restore credit to early computational choices of a rate description of elasto-plastic behavior, but with the decisive improvement that the rules of the game are now clearly assessed also in the fully nonlinear range. The theory leads to a conceptually clean, methodologically definite model and to drastic simplifications of both theoretical and computational aspects of geometrically non-linearized elasto-visco-plasticity. The geometric approach to nonlinear continuum mechanics developed in this paper, with explicit application to the theory of elasto-plastic constitutive models, is a major step in a geometrization program of continuum mechanics carried out under several points of view in [26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 1, 37, 2, 4, 38].

REFERENCES

[1] Romano, G. and Barretta, R. Covariant hypo-elasticity. *Eur. J. Mech. A-Solids* (2011) **30**:1012–1023.

[2] Romano, G. and Barretta, R. On Euler's Stretching Formula in Continuum Mechanics. *Acta Mech.* (2013) **224**:211–230.

[3] Simò, J.C. A framework for finite strain elastoplasticity based on maximum plastic dissipation and the multiplicative decomposition: Continuum formulation. *Comp. Meth. Appl. Mec. Eng.* (1988) **66**:199–219.

[4] Romano, G. and Barretta, R. Geometric constitutive theory and frame invariance. *Int. J. Non-Linear Mech.* (2013) **51**:75–86.

[5] Truesdell, C. Hypo-elasticity. *J. Rational Mech. Anal.* (1955) **4**:83–133, 1019–1020.

[6] Halphen, B. and Nguyen, Q.-S. Sur les matériaux standard généralisés. *J. Mécanique* (1975) **14**:1–37.

[7] Saunders, D.J. *The Geometry of Jet Bundles.* London Mathematical Society Lecture Note Series 142, Cambridge University Press, (1989).

[8] Truesdell, C. and Noll, W. *The non-linear field theories of mechanics*, Handbuch der Physik. Springer 1–591 (1965).

[9] Marsden, J.E. and Hughes, T.J.R. *Mathematical Foundations of Elasticity.* Prentice-Hall, Redwood City, Cal., (1983).

[10] Simó, J.C. and Pister, K.S. Remarks on rate constitutive equations for finite deformation problems: computational implications. *Comp. Meth. Appl. Mech. Eng.* (1984) **46**:201–215.

[11] Johnson, G.C. and Bammann, D.J. A discussion of stress rates in finite deformation problems. *Int. J. Solids Struct.* (1984) **20**(8):725–737.

[12] Sansour, C. and Bednarczyk, H. A study on rate-type constitutive equations and the existence of free energy function. *Acta Mech.* (1993) **100**:205–221.

[13] Šilhavý, M. *Mechanics and thermodynamics of continuous media.* Springer, Berlin, (1997).

[14] Lu, J. and Papadopoulos, P. A covariant formulation of anisotropic finite plasticity: theoretical developments. *Comp. Meth. Appl. Mech. Eng.* (2004) **193**:5339–5358.

[15] Bruhns, O.T., Xiao, H. and Meyers, A. Elastoplasticity beyond small deformations. *Acta Mech.* (2006) **182**:31–111.

[16] Marotti de Sciarra, F. On non-local and non-homogeneous elastic continua. *Int. J. Solids Structures* (2009) **46**:651–676.

[17] Marotti de Sciarra, F. *A finite element for nonlocal elastic analysis*, IV International Conference on Computational Methods for Coupled Problems in Science and Engineering, Greek, CD-Rom (2011).

[18] Romano, G. *Continuum Mechanics on Manifolds.* Lecture notes, University of Naples Federico II, Naples, Italy (2007) 1–695, http://wpage.unina.it/romano.

[19] Clerk-Maxwell, J. On the dynamical theory of gases. *Phil. Trans. Roy. Soc.* (1867) **157**:49–88.

[20] Marotti de Sciarra, F. Novel variational formulations for nonlocal plasticity. *Int. J. Plasticity* (2009) **25**:302–331.

[21] Marotti de Sciarra, F. Variational formulations, convergence and stability properties in nonlocal elastoplasticity. *Int. J. Solids Structures* (2008) **45**:2322–2354.

[22] Marotti de Sciarra, F. A nonlocal model with strain-based damage. *Int. J. Solids Structures* (2009) **46**:4107–41224.

[23] Marotti de Sciarra, F. Hardening plasticity with nonlocal strain damage. *Int. J. Plasticity* (2012) **34**:114–138.

[24] Simó, J.C. and Ortiz, M. A unified approach to finite deformation elastoplastic analysis based on the use of hyperelastic constitutive equations. *Comp. Meth. Appl. Mech. Eng.* (1985) **49**:221–245.

[25] Lubliner, J. *Plasticity theory.* Macmillan Publishing C., New York, (1990).

[26] Romano, G., Barretta, R. and Sellitto, C. *On the evaluation of the elastoplastic tangent stiffness*. Computational Plasticity: Fundamentals and Applications - Proceedings of the 8th International Conference on Computational Plasticity, COMPLAS VIII (PART 2), 1118–1121 (2005).

[27] Romano, G., Barretta, R. and Diaco, M. *Conservation laws for multiphase fracturing materials*. Proceedings of the 6th International Conference on Fracture Mechanics of Concrete and Concrete Structures - Fracture Mechanics of Concrete and Concrete Structures 1, 411–418 (2007).

[28] Romano, G., Diaco, M. and Barretta, R. On the theory of material inhomogeneities. *Mech. Res. Comm.* (2006) **33**(6):758–763.

[29] Romano G., Sellitto C. and Barretta R. Nonlinear shell theory: a duality approach. *J. Mech. Materials and Structures* (2007) **2**(7):1207–1230.

[30] Romano, G., Barretta, R. and Diaco, M. On the general form of the law of dynamics. *Int. J. Non-Linear Mech.* (2009) **44**(6):689–695.

[31] Romano, G., Barretta, R. and Barretta, A. On Maupertuis principle in dynamics. *Rep. Math. Phys.* (2009) **63**(3):331–346.

[32] Romano, G., Barretta, R. and Diaco, M. On continuum dynamics. *J. Math. Phys.* (2009) **50**:102903–1–26.

[33] Romano, G., Barretta, R. and Diaco, M. Algorithmic tangent stiffness in elastoplasticity and elastoviscoplasticity: a geometric insight. *Mech. Res. Comm.* (2010) **37**(3):289–292.

[34] Romano, G., Diaco, M. and Barretta, R. Variational Formulation of the First Principle of Continuum Thermodynamics. *Continuum Mech. Thermodyn.* (2010) **22**(3):177–187.

[35] Barretta, R. and Barretta, A. Shear stresses in elastic beams: an intrinsic approach. *Eur. J. Mech. A/Solids* (2010) **29**:400–409.

[36] Romano, G., Barretta, A. and Barretta, R. On Torsion and Shear of Saint-Venant Beams. *Eur. J. Mech. A/Solids* (2012) **35**:47–60.

[37] Barretta, R. On the relative position of twist and shear centres in the orthotropic and fiberwise homogeneous Saint-Venant beam theory. *Int. J. Solids Structures* (2012) **49**(21):3038–3046.

[38] Barretta, R. On Cesàro-Volterra method in orthotropic Saint-Venant beam. *J. Elasticity* (2013). Published online. doi: 10.1007/s10659-013-9432-7.