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APPLICATION OF THE PROPER GENERALIZED DECOMPOSITION METHOD TO A VISCOELASTIC MECHANICAL PROBLEM WITH A LARGE NUMBER OF INTERNAL VARIABLES AND A LARGE SPECTRUM OF RELAXATION TIMES

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Abstract. We here extend the use of the PGD to the case of a viscoelastic mechanical problem with a large number of internal variables and with a large spectrum of relaxation times. Such a number of internal variables leads to solving a system of non linear differential equations which correspond to the return to the equilibrium state. The feasibility and the robustness of the method are discussed in a simple case; a future application is the simulation of a polymer reaction under cyclic loading.

1 INTRODUCTION

To solve a problem with a large number of degrees of freedom (dofs), numerical techniques, as parallel computing and domain decomposition, can be used. In the case of a multiphysical problem or of a problem with a large number of internal variables, it leads to solving a large number of differential equations.

The PGD method, based on the radial approximation [4], has proved to be efficient for solving problems with a large number of dofs [2, 3], and particularly in the case of a coupled thermo mechanical problem [1].

We here extend the use of the PGD [1] to the case of a viscoelastic mechanical problem with a large number of internal variables and with a large spectrum of relaxation times (50 to 100 times [5]). Such a number of internal variables leads to solving a system of non linear differential equations which correspond to the return to the equilibrium state. The feasibility and the robustness of the method are discussed in a simple case; a future application is the simulation of a polymer reaction under cyclic loading.

Section 2 introduces the equations of the viscoelastic mechanical problem. In section 3, the PGD is used to solve the problem with internal variables. While in section 4, we present the numerical results of a problem with one internal variable but with different relaxation times.

2 RESOLUTION OF VISCOELASTIC PROBLEM WITH INTERNAL VARIABLES

2.1 Equations of problem

Let us consider a one-dimensional problem in space (noted x). The generic form of the mechanical problem with internal variables is written as:

$$\frac{\partial \sigma}{\partial x} + f = 0 \tag{1}$$

$$\frac{dz_j}{dt} = -\frac{1}{\tau_j}\left(z_j - z_j^{\infty}\right) \tag{2}$$

Where

$$\sigma = \mathrm{E}_V \frac{\partial u}{\partial x} - \sum_{j=1}^{m} z_j \tag{3}$$

$$z_j^{\infty} = \mathrm{E}_{\mathrm{rj}}^{\infty} \frac{\partial u}{\partial x} \tag{4}$$

Where z_j^{∞} represents the relaxed status of the internal variable z_j; τ_j is the relaxation time; $\mathrm{E}_{\mathrm{rj}}^{\infty}$ is the relaxed Young's modulus and E_v is the vitreous Young's modulus of the material.

The vitreous Young's modulus depends on the E_{∞} modulus and the relaxed one E_{r} and its form is: $\mathrm{E}_V = \mathrm{E}_{\infty} + \sum_{j}^{m} \mathrm{E}_{\mathrm{rj}}^{\infty}$ where $\mathrm{E}_{\mathrm{r}} = \sum_{j}^{m} \mathrm{E}_{\mathrm{rj}}^{\infty}$.

Problems (1)-(2) are assumed to be defined on the domain: $\Omega = \Omega_X \times \Omega_t$, where $\Omega_X = [0, \mathrm{L}_{\mathrm{X}}]$ and $\Omega_t = [0, \mathrm{L}_{\mathrm{t}}]$. The initial conditions are equal to zero and the boundary conditions are written as: $\underline{\underline{\sigma}} \cdot \underline{\mathrm{n}} = \underline{\mathrm{F}}$ on $\partial\Omega_{\mathrm{f}}$ and $\underline{\mathrm{u}} = \underline{0}$ on $\partial\Omega_{\mathrm{u}}$.

Remark: The subscript j concerns the internal variables, and it variates from 1 to m, where m is the number of internal variables. If we consider a problem with m internal variables, it means that the equation (2) is reported m times. The specificity of each equation is related to the relaxation time of this internal variable. The displacement field and the internal variable are then coupled.

2.2 Use of the PGD to solve the viscoelastic problem

The aim of the separated representation method is to compute N couples of functions $\{(\mathrm{A_i}, \mathrm{B_i}), (\mathrm{C_i}, \mathrm{D_i}), \mathrm{i} = 1,\ldots\ldots, \mathrm{N}\}$ such that $\{\mathrm{A_i}, \mathrm{C_i}, \mathrm{i} = 1,\ldots\ldots, \mathrm{N}\}$ and $\{\mathrm{B_i}, \mathrm{D_i}, \mathrm{i} = 1,\ldots\ldots, \mathrm{N}\}$ are defined respectively in Ω_x and Ω_t , and the solutions u and z of the coupled problem can be written in the following separated form:

$$\begin{cases} u(x,t) \approx \sum_{i=1}^{N} \mathrm{A_i}(\mathrm{x})\mathrm{B_i}(\mathrm{t}) \\ z_j(x,t) \approx \sum_{k=1}^{N} \mathrm{C_{kj}}(\mathrm{x})\mathrm{D_{kj}}(\mathrm{t}) \end{cases} \tag{5}$$

Once N separated variables functions are computed, the next separated variables functions to be computed are called $R(x)$, $S(t)$, $V_j(x)$ and $W_j(t)$ in this step. They are solutions of the Galerkin variational formulation related to equations (1) and (2):

$$\int_{\Omega} (div\sigma + f)u^* d\Omega + \int_{\partial\Omega_f} (\sigma(l) - F)u^* d\Omega = 0 \qquad \forall u^* \tag{6}$$

$$\int_{\Omega} \left(\frac{dz_j}{dt} + \frac{1}{\tau_j}\left(z_j - z_j^{\infty}\right) \right) z_j^* d\Omega = 0 \qquad \forall z_j^* \tag{7}$$

With the trial and test fields written as follows:

$$u(x,t) = \sum_{i=1}^{n} A_i(x)B_i(t) + R(x)S(t), \quad z_j(x,t) = \sum_{i=1}^{n} C_{ij}(x)D_{ij}(t) + V_j(x)W_j(t) \tag{8}$$

$$u^*(x,t) = R^*(x)S(t) + R(x)S^*(t), \quad z_j^*(x,t) = V_j^*(x)W_j(t) + V_j(x)W_j^*(t) \tag{9}$$

2.3 Specificities related to the internal variables

The displacement field and the internal variables are completely integral. We choose to solve equation (6) and then equation (7) at each step of enrichment in order to decouple the problem. The process is initialized with the solution of an elastic problem without internal variable. Once the displacement field is computed (6), the solution is introduced in equation (4) in order to compute the internal variable in equation (2). This internal variable is placed in equation (3) to compute the solution of the viscoelastic problem. This process is iterated until convergence.

In a step of enrichment, the method of fixed point is used to compute the functions $R(x)$, $S(t)$ and then $V_j(x)$ and $W_j(t)$. In what follows, the equations to compute these functions are:

For $R(x)$:

$$E_v\beta_t \frac{\partial^2 R}{\partial x^2} = -E_v \sum_{i=1}^{n} \alpha_t^i \frac{\partial^2 A_i}{\partial x^2} + \sum_{i=1}^{n}\sum_{j=1}^{m} \gamma_t^j \frac{\partial C_{ij}}{\partial x} + \sum_{j=1}^{m} \delta_t^j \frac{\partial V_j}{\partial x} - \theta_t \tag{10}$$

For $S(t)$:

$$E_v\beta_x S(t) = -E_v \sum_{i=1}^{n} \alpha_x^i B_i(t) + \sum_{i=1}^{n}\sum_{j=1}^{m} \gamma_x^j D_{ij}(t) + \sum_{j=1}^{m} \delta_x^j W_j(t) - \theta_x \tag{11}$$

For $V_j(x)$:

$$\left(\gamma_{tj}^{j}+\frac{1}{\tau_j}\theta_t^{j}\right)V_j=-\sum_{i=1}^{n}\beta_t^{i}C_{ij}-\frac{1}{\tau_j}\sum_{i=1}^{n}\delta_{tj}^{i}C_{ij}+\frac{B}{\sqrt{\tau_j}}\sum_{i=1}^{n}\beta_{tj}^{i}\frac{\partial A_i}{\partial x}+\frac{B}{\sqrt{\tau_j}}\frac{\partial R}{\partial x}\Gamma_t^{i} \tag{12}$$

For $W_j(t)$:

$$\frac{\partial W_j}{\partial t}\gamma_{xj}^{j}+\frac{1}{\tau_j}W_j\gamma_{xj}^{j}=-\sum_{i=1}^{n}\beta_{xj}^{i}\frac{\partial D_{ij}}{\partial t}-\frac{1}{\tau_j}\sum_{i=1}^{n}\beta_{xj}^{i}D_{ij}+\frac{B}{\sqrt{\tau_j}}\sum_{i=1}^{n}\alpha_{xj}^{i}B_i+\frac{B}{\sqrt{\tau_j}}S\delta_{xj}^{i} \tag{13}$$

Where :

$$\beta_t=\int_{\Omega_t}S^2(t)dt,\ \alpha_t^i=\int_{\Omega_t}B_i(t)S(t)dt,\ \gamma_t^j=\int_{\Omega_t}D_{ij}(t)S(t)dt,\ \delta_t^j=\int_{\Omega_t}W_j(t)S(t)dt,\ \theta_t=\int_{\Omega_t}(\sigma(l)-F)S(t)dt.$$

$$\beta_x=\int_{\Omega_x}R(x)\frac{\partial^2R(x)}{\partial^2x}dx,\ \alpha_x^i=\int_{\Omega_x}R(x)\frac{\partial^2A_i(x)}{\partial^2x}dx,\ \gamma_x^j=\int_{\Omega_x}R(x)\frac{\partial C_{ij}(x)}{\partial x}dx,\ \delta_x^j=\int_{\Omega_x}R(x)\frac{\partial V_j(x)}{\partial x}dx$$

$$\theta_x=\int_{\Omega_x}(\sigma(l)-F)R(x)dx.\ \theta_t^j=\int_{\Omega_t}W_j^2(t)dt,\ \beta_t^j=\int_{\Omega_t}\frac{\partial D_{ij}(t)}{\partial x}W_j(t)dt,\ \delta_{tj}^j=\int_{\Omega_t}D_{ij}(t)W_j(t)dt,\ \alpha_{tj}^i=\int_{\Omega_t}B_i(t)W_j(t)dt,$$

$$\Gamma_t^j=\int_{\Omega_t}S(t)W_j(t)dt,\ \gamma_{tj}^j=\int_{\Omega_t}\frac{\partial W_j(t)}{\partial x}W_j(t)dt.\ \gamma_{xj}^j=\int_{\Omega_x}V_j^2(x)dx,\ \beta_{xj}^j=\int_{\Omega_x}C_{ij}(x)V_j(x)dx,$$

$$\alpha_{xj}^i=\int_{\Omega_x}\frac{\partial A_i(x)}{\partial x}V_j(x)dx,\ \delta_{xj}^i=\int_{\Omega_x}\frac{\partial R(x)}{\partial x}V_j(x)dx.$$

After each iteration *l*, two residuals are computed; R_u related to equation (1) and R_{z_j} related to equation (2).

$$R_u=\frac{\|(u_{l+1}-u_l)\|}{\|u_l\|}\le\varepsilon \tag{14}$$

$$R_{z_j}=\max\left(\frac{\|(z_{j\,l+1}-z_{j\,l})\|}{\|z_{j\,l}\|}\right)\le\varepsilon \tag{15}$$

Where $\|\ \|$ stands for the L^2 norms. The iterative procedure is stopped when $\max(R_u,R_{z_j})$ is small enough. The solution of the problem is then given by equation (5).

3 APPLICATION TO A PROBLEM WITH ONE INTERNAL VARIABLE

The simulation test is a 10 mm long one-dimensional bar subjected to a load $F(x,t)=G(x=10)\times H(t)$ at x = 10 mm as shown in figure 1 and with null boundary conditions at x = 0 mm. The time L_t equals 10 s. The parameters of material are given in the table 1. The time step equals 0,1 sec, and the space one equals 0,25 mm. We here consider a problem with one internal variable with different relaxation times and study its influence on the displacement field.

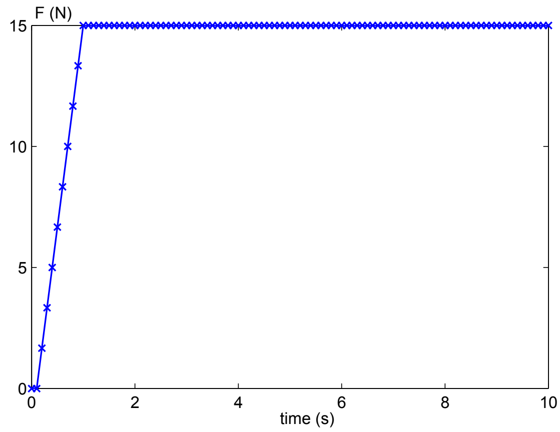


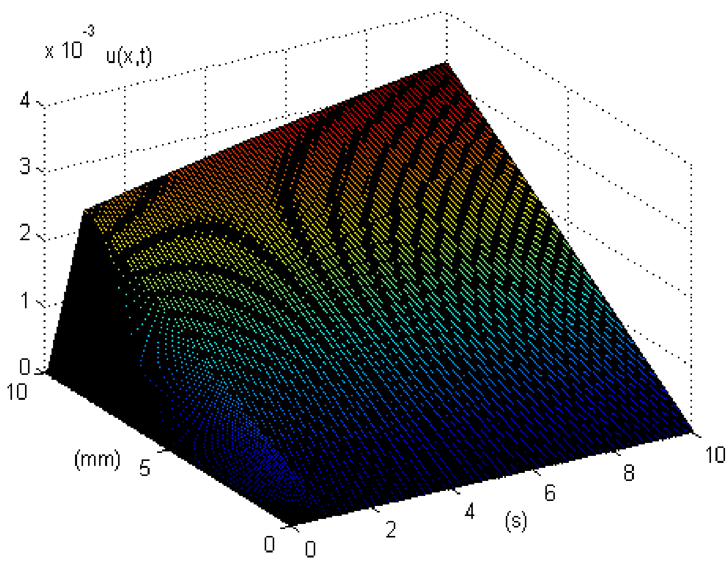
Figure 1: Creep load at the extremity of the bar

Table 1: Mechanical parameters of the material

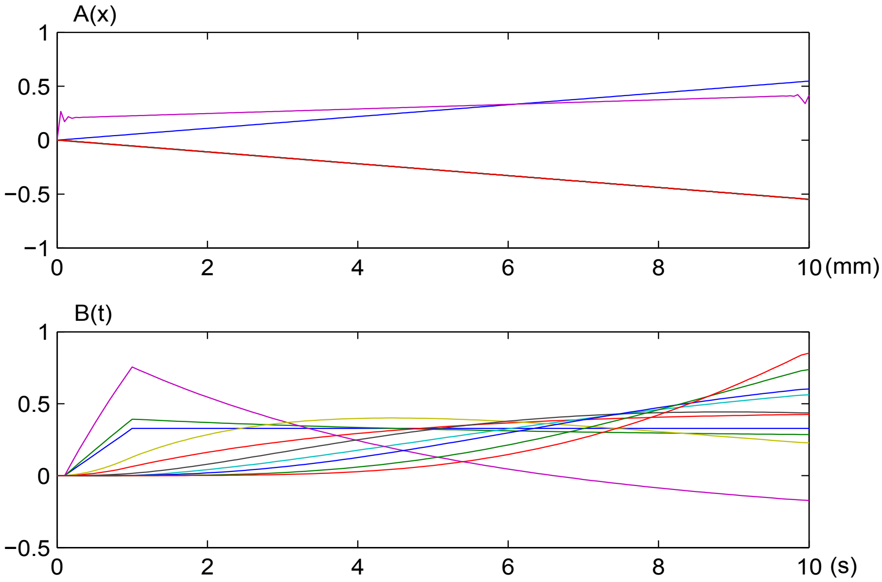
E_v (MPa)	E_∞ (MPa)
1140	140

3.1 Results and Discussion

For τ equals 1 second, the solution is reached with 10 iterations with $(\|A_i\|\times\|B_i\|)$ = (0.01 – 0.07 – 0.009 – 0.0025 – 0.0001 – 0.002 – 0.001 – 0.0004 – 0.0001 – 0.00003) for the displacement field; and $(\|C_i\|\times\|D_i\|)$ = (10.41 – 7.53 – 0.85 – 0.19 – 0.03 – 0.21 – 0.09 – 0.03 – 0.009 – 0.002) for the internal variable.



a)



b)

Figure 2: (a) Displacement field and (b) Spatial and Temporal modes.

Figure 2a shows the displacement field computed. Temporal and spatial modes for the displacement field are represented in Figure 2b. Figure 3 shows the convergence of the solution via the residuals (equations (14) and (15)) compared to the iteration number.

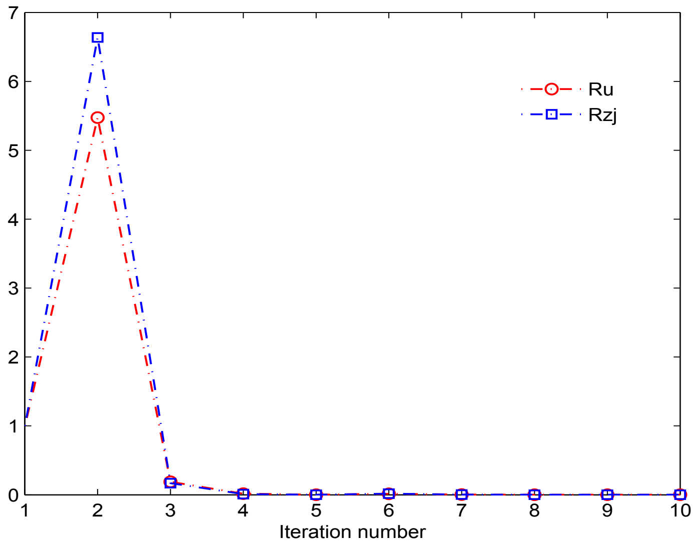
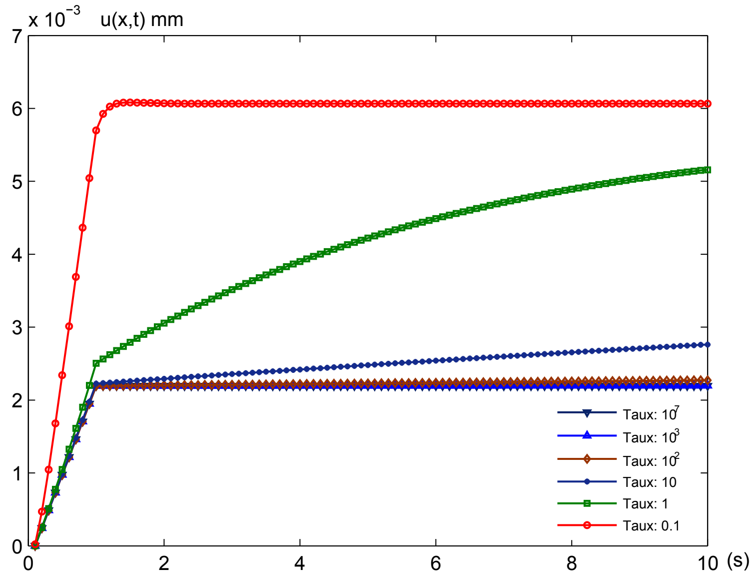
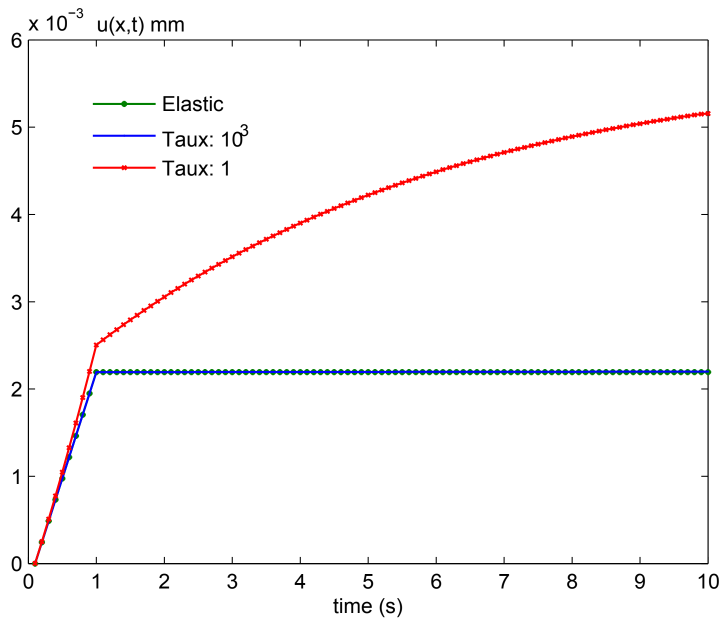


Figure 3: Convergence of the PGD



a)



b)

Figure 4: (a) Influence of relaxation time on the displacement field, (b) and comparison with elastic solution

The same simulation is done with different value of relaxation time. Based on the value of τ, the solution was not reached with the same number of modes which increases when τ decreases. Figure 4a shows the influence of the relaxation time on the displacement field.

We can observe that the influence of the internal variable is important when the value of the relaxation time is weak. Figure 4b shows that when the relaxation time is high, the reached solution is that of the elastic problem. In that case, fewer modes are needed to represent the solution. In table 2, we show the number of iterations with respect to the variation of the relaxation time τ .

Table 2: Influence τ of on the number of modes

τ (s)	0.1	1	10	100	1000
iteration	15	10	9	5	4

4 CONCLUSION

In this work, the PGD was validated in the case of a viscoelastic problem with one internal variable. The solution was computed with different relaxation time, and we observed that the number of modes needed to describe the displacement field depends on the relaxation time.

These results are encouraging and will be extended to the case with many internal variables and relaxation time in order to simulate the behavior of the *PE-HD* polymer under cyclic loading. It will be our future application.

REFERENCES

[1] M. Béringhier, M. Gueguen, J.C. Grandidier. Solution of strongly coupled multiphysics problems using space-time separated representations – Application to thermoviscoelasticity, *Archives of Computational Methods in Engineering*, (2010), doi: 10.1007/s11831-010-9050-5.

[2] F. Chinesta, A. Ammar, E. Cueto. Proper generalized decomposition of multiscale models, *International Journal for Numerical Methods in Engineering*, (2010), Volume 83, 1114-1132,

[3] A. Ammar, B. Mokdad, F. Chinesta, R. Keunings. A new family of solvers for some classes of multidimensional partial differential equations encountered in kinetic theory modeling of complex fluids - Part II: Transient simulation using space-time separated representations, *Journal of Non-Newtonian Fluid Mechanics*, (2007), 144, 98-121.

[4] P. Ladevèze. Nonlinear computational structural mechanics - New approaches and non-incremental methods of calculation, (1999), *Springer Verlag*.

[5] Ch. Cunat. A thermodynamic theory of relaxation based on a distribution of non-linear processes, *Journal of Non-Crystalline Solids*, Volumes 131-133, Part 1, (1991), Pages 196-199.