DYNAMICALLY COUPLED MODELS OF THE SLIDING AND SPINNING FRICTION BASED ON PADÉ EXPANSIONS

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Abstract. It is presented a new approach for dry friction modeling under conditions of combined kinematics. The main distinguish feature of this approach is building of friction models which are suitable for using in differential equations of motion. Under the proposed models of friction are understudied the interrelations between friction force components, torques and velocities which are represented be the analytical functions. The procedure of the models constructing consists of the two parts. In the first part, the exact integral expressions for the net vector and torque are formed with the assumption that Coulomb's friction law in classical forms or generalized differential forms is valid at each point of the contact area. In addition, in process of the exact integral models construction there are is used well known results from the theory of elasticity that tangent stresses lead to shift in the symmetric diagram of the normal contact stresses in the direction of the instantaneous sliding velocity. To use the theory of elasticity results in the dynamics problems, it is proposed the simple asymptotic representations for the contact stresses distributions based on their general properties known from the theoretical results of the theory of elasticity. In the second part the exact integral models are replaced by appropriate Pade expansions. The approximate models preserve all properties of the models based on the exact integral expressions and correctly describe the behaviour of the net vector and torque of the friction forces and their first derivatives at zero and infinity. Moreover, one does not have even to calculate the integrals to determine the coefficients of the Pade approximations. The corresponded coefficients can be identified from experiments. Consequently, the models based on Pade expansions may be considered as phenomenological models of combined dry friction.

1 INTRODUCTION

One of the first models describing the relation between the sliding friction and the whirling friction in the case of nonpoint contact between the moving bodies was proposed by in [1]. A principally new development of the theory was given by in [2], where exact analytic expressions for the resultant vector and the frictional moment for circular contact sites were obtained under the assumption that the distribution of contact stresses in the contact spot obeys the Hertz law. In [2], to apply the obtained dependencies to problems of dynamics, the linear-fractional Pade approximations of these dependencies were constructed. The developed in [2] theory was used in [3] to study the dynamics of a homogeneous circular disk sliding with rotation on a plane. Under the assumption that the distribution of contact stresses obeys the Galin law, exact analytic expressions for the resultant vector and the frictional moment were obtained and their linear-fractional Pade′ approximations were constructed.

The convenience in the use of the Pade approximations, which permit describing the effects of combined dry frictions for the entire range of angular and linear velocities, allowed one to construct principally new the two-dimensional coupled models of the sliding and spinning friction on the basis of these approximations [4].

The two-dimensional friction model was constructed under supposition that the classical Coulomb law in differential form is validated for an infinitesimal area inside of contact spot. Its generalizing for the case of more realistic dry friction characteristic (validity of Coulomb law in generalized differential form) was given in [5]. It was shown that in the case of combined kinematics using of the Coulomb law in generalized differential form leads to new qualitative properties of the friction force dependence on the sliding and spinning velocities, but does not change the model dimension. All these models of the sliding and spinning friction were constructed in the assumption that, in the case of circular contact sites, the distributions of normal contact stresses depend only on the position vector with origin at the contact spot center. But, it is known [6] that in the case of the rigid solids sliding it is appears tangent stresses that leads to shifting in the symmetric diagram of the normal contact stresses in the direction of the instantaneous sliding velocity. Investigations carried out in [7] shown this shifting even for uniform distribution of the normal contact stresses cause, in the case of combined kinematics, the dynamics coupling between components defining the force state of rubbed solids.

Proposed below the dry friction models generalizing permits to take into account, simultaneously, both the dynamics coupling of the components defining force state and the more realistic representations about dry friction characteristics and the normal contact stresses distributions in the case of combined kinematics.

2 COUPLED MODELS OF THE SLIDING AND SPINNING FRICTION

2.1 Basic relationships

The combined model of sliding and rolling friction is constructed for circular contact sites under the assumption that the Coulomb law in differential form holds for the small surface element dS in the interior of the contact spot, according to which the differentials of the resultant vector $d\mathbf{F}$ and the moment of friction dM_C with respect to the disk center are determined by the formulas:

$$d\mathbf{F} = -f\sigma\frac{\mathbf{V}}{|\mathbf{V}|}\left(1+\mu_1|\mathbf{V}|^3-\mu_2|\mathbf{V}|\right)dS,\ dM_C = -f\sigma\frac{\mathbf{r}\times\mathbf{V}}{|\mathbf{V}|}\left(1+\mu_1|\mathbf{V}|^3-\mu_2|\mathbf{V}|\right)dS, \tag{1}$$
$$\mathbf{V} = (v-\omega y, \omega x),\ \mathbf{r} = (x, y)$$

where f is the coefficient of friction, $\mathbf{r} = (x, y)$ is the position vector of an elemental area in the interior of the contact spot with respect to its center (Fig. 1), ω is the angular velocity of rotation of the contact spot center, but μ_1 and μ_2 are the coefficients which can be defined in practice from experiments.

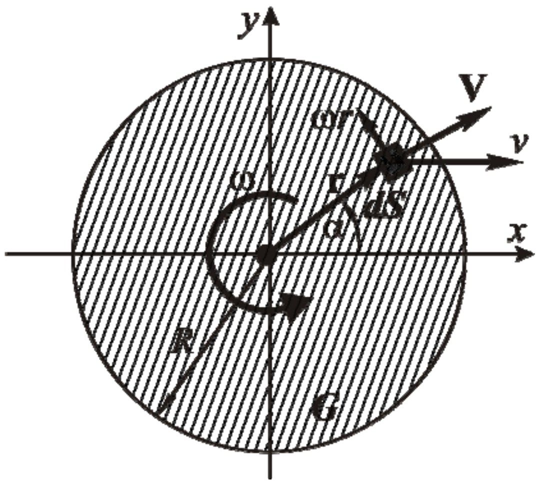

Figure 1. Kinematics inside the contact spot letters

To use the theory of elasticity results in the dynamics problems, a simple linear approximation of the normal contact stresses distribution is proposed:

$$\sigma(x,y)=\sigma_0\left(1+kx/R\right) \tag{2}$$

where $\sigma_0=\sigma_0(r)$ - distribution of normal contact stresses at absence of motion having the properties of central symmetry, R - radius of contact spot, x - axe of the restangular coordination systems with origin in the center of contact circle (Fig. 1) which is directed parallelly to vector of the instantenious sliding.

To calculate coefficient k in the formula (1) it is used the condition of equality of the external force F torque to the normal reaction force N torque which is appears from the shifting of the center of gravity of the contact spot in the direction of sliding on the value s:

$$Fh=Ns \tag{3}$$

where h - distance from the moving solid center mass to the plane of sliding. On the other hand the shifting s of the gravity center relatively of the contact spot center can be defined by the following formula:

$$s=\iint_G x\sigma(x,y)dxdy\bigg/\iint_G \sigma(x,y)dxdy,\ G=\{(x,y):x^2+y^2\le R^2\} \tag{4}$$

Substitution of the representation (2) to the (4) yields:

$$s=\frac{\pi k}{R}\int_0^R \sigma_0(r)r^3dr \tag{5}$$

Equalization values s calculated from the formulas (3) and (5) allows to calculate coefficient k which is characterized the dynamical coupling of the components defining the force state inside of contact spot.

If the distribution of normal contact stresses $\sigma_0(r)$ at the moving absence are obeyed by the Hertz $\sigma_0 = 3N\sqrt{1-r^2/R^2}\,/(2\pi R^2)$ or Galin $\sigma_0 = N\Big/\left(2\pi R^2\sqrt{1-r^2/R^2}\right)$ laws then $k = 5Fh/(NR)$ or $k = 3Fh/(NR)$, correspondently.

2.1 Integral model

To obtain the resultant vector and the moment of friction, it is necessary to integrate the expressions (1) over the contact spot. The obtained dependencies, where $F_{\|}$ and $F_{\perp}$ denote the respective components of the resultant vector directed along the tangent and the normal to the trajectory of motion, present an exact combined integral model of sliding and spinning friction

$$F_{\|}(\omega,v) = -f\iint_G\left(\frac{(v-\omega y)}{\sqrt{\omega^2(x^2+y^2)+v^2-2\omega v y}} + \mu_1 v^3 - \mu_2 v + 2\mu_1 v\omega^2(x^2+y^2)\right)\sigma_0 dxdy$$
$$F_{\perp}(\omega,v) = -\frac{kf}{R}\iint_G \frac{\omega x^2\sigma_0}{\sqrt{\omega^2(x^2+y^2)+v^2-2\omega v y}}dxdy,\ \ G=\{(x,y): x^2+y^2\le R^2\}$$
$$M_C(\omega,v) = -f\iint_G\left(\frac{(\omega(x^2+y^2)-vy)}{\sqrt{\omega^2(x^2+y^2)+v^2-2\omega v y}} + (2\mu_1 v^2-\mu_2)\omega(x^2+y^2) + \mu_1\omega^3(x^2+y^2)^2\right)\sigma_0 dxdy \tag{6}$$

After introducing dimensionless variables: $x=\hat{x}R,\ y=\hat{y}R$ and $\sigma(\hat{x},\hat{y}) = \hat{\sigma}(\hat{x},\hat{y})\,N/R^2$ it is convenient to calculate the modulus of integrals (6) in the polar coordinates: $x = r\cos\alpha,\ y = r\sin\alpha,\ r\in[0,1],\ \alpha\in[0,2\pi]$ (Fig. 1) in which the functions (6) take the form

$$F_{\|} = fN\int_0^{2\pi}\int_0^1 \frac{(v-ur\sin\alpha)r\sigma_0(r)}{\sqrt{u^2r^2+v^2-2uvr\sin\alpha}}drd\alpha + 2\pi f\left((\mu_1 v^3-\mu_2 v)\int_0^1 r\sigma_0(r)dr + 2\mu_1 v u^2\int_0^1 r^3\sigma_0(r)dr\right)$$
$$F_{\perp} = kfN\int_0^{2\pi}\int_0^1 \frac{ur^3\sigma_0(r)\cos^2\alpha}{\sqrt{u^2r^2+v^2-2uvr\sin\alpha}}drd\alpha,\ \ u=\omega R$$
$$M_C = fRN\int_0^{2\pi}\int_0^1 \frac{(ur^2-vr\sin\alpha)r\sigma_0(r)}{\sqrt{u^2r^2+v^2-2uvr\sin\alpha}}drd\alpha + 2\pi f\left((2\mu_1 v^2-\mu_2)u\int_0^1 r^3\sigma_0(r)dr + \mu_1 u^3\int_0^1 r^5\sigma_0(r)dr\right) \tag{7}$$

where the “hat” symbol is omitted for brevity

If $k=0$, then model (7) is fully agree to the model, investigated in [3] and can be considered as the first approximation, but presented in this investigation as the second approximation. Thus, we have substantial approximation to the real situation in dependence on the general properties of the normal contact stresses distribution. At the supposition that external forces are absence, the coefficient k in formula (1), (5), (6), (7) is defined by the friction force component $F_{\|}$ from the first expressions in the relations (6-8) and, consequently, the dynamically coupled integral friction model is

$$F_{\|} = fN\int_0^{2\pi}\int_0^1 \frac{(v-ur\sin\alpha)r\sigma_0(r)}{\sqrt{u^2r^2+v^2-2uvr\sin\alpha}}drd\alpha + 2\pi f\left((\mu_1 v^3-\mu_2 v)I_1 + 2\mu_1 vu^2 I_3\right) \quad (8)$$

$$F_{\perp} = kfN\int_0^{2\pi}\int_0^1 \frac{ur^3\sigma_0(r)\cos^2\alpha}{\sqrt{u^2r^2+v^2-2uvr\sin\alpha}}drd\alpha,\ k=\frac{fhR}{\pi I_3}\int_0^{2\pi}\int_0^1 \frac{(v-ur\sin\alpha)r\sigma_0(r)}{\sqrt{u^2r^2+v^2-2uvr\sin\alpha}}drd\alpha$$

$$M_C = fRN\int_0^{2\pi}\int_0^1 \frac{(ur^2-vr\sin\alpha)r\sigma_0(r)}{\sqrt{u^2r^2+v^2-2uvr\sin\alpha}}drd\varphi + 2\pi f\left((2\mu_1 v^2-\mu_2)uI_3 + \mu_1 u^3 I_5\right)$$

where coefficients of polynomials terms in formulas (3) are the first moments of the normal contact stresses distribution: $I_1 = \int_0^1 r\sigma_0(r)dr$ - moment of the first order, $I_3 = \int_0^1 r^3\sigma_0(r)dr$ - moment of the third order and $I_5 = \int_0^1 r^5\sigma_0(r)dr$ - moment of the fifth order.

They can be calculated in elementary functions for the most used functions of the normal contact stresses distributions [5].

If the distribution of normal contact stresses is obeyed to the Hertz law: $\sigma(r) = 3\sqrt{1-r^2}/(2\pi)$ then: $I_1 = 1/2\pi, I_3 = 1/5\pi, I_5 = 4/35\pi$.

If the distribution of normal contact stresses is obeyed to the Galin law: $\sigma(r) = (2\pi\sqrt{1-r^2})^{-1}$ then: $I_1 = 1/2\pi, I_3 = 1/3\pi, I_5 = 4/15\pi$.

In the case of thin circle, the distribution of normal contact stresses can be described by the following function: $\sigma(r) = \delta(r-1)/(2\pi)$, where $\delta(r-1)$ - Dirac delta function in the point $r=1$ and $I_1 = I_3 = I_5 = 1/4\pi$.

Plots of the tangent $F_{\|}$ (left figure) and normal $F_{\perp}$ (right figure) friction force components normalized on the their maximum values as function of velocity of sliding v at the constant velocity of whirling $u=1$ are presented on the Fig. 2. As concerned friction torque then, qualitatively, its behavior is the same as case of using classical form Coulomb law: there are only small quantitative distinctions.

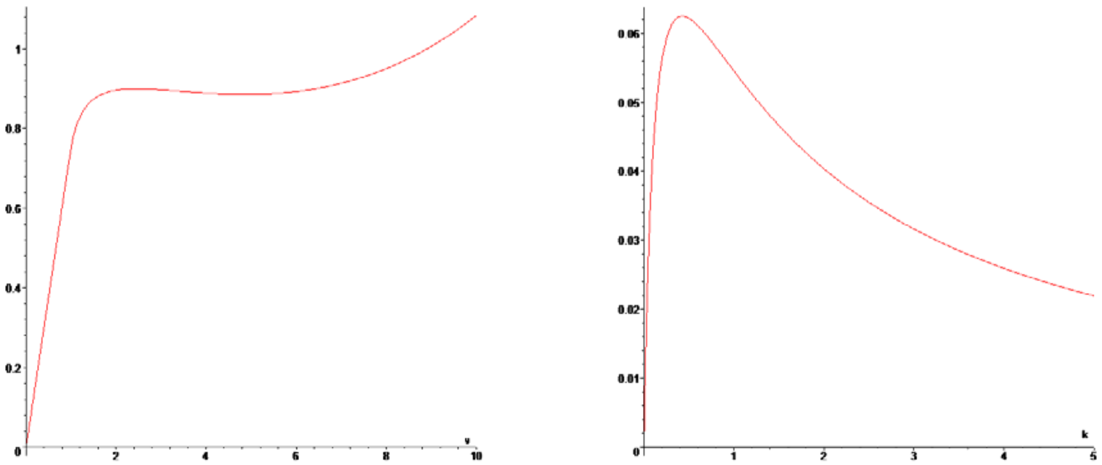

Figure 2. Tangent and normal friction force components

The expressions for the components of the resultant vector and the moment of friction in relations (8) have several important properties as functions of u and v.

Property 1. The distortion in symmetric diagram of the normal contact stresses distribution results in the appearance of the resultant vector component $F_{\perp}$ directed along the normal to the trajectory of motion. The resultant vector is not directed opposite to the velocity of sliding.

Property 2. The distortion in the symmetric diagram of distribution of normal stresses does not affect to the moment M_C and the resultant vector component $F_{\|}$ directed along the tangent to the trajectory.

Property 3. The first terms of the tangent $F_{\|}$ force component and torque M_C, just as normal $F_{\perp}$ force component, are homogeneous functions of the variables u and v of zero order of homogeneity and hence are invariant under the similarity group:

Property 4. The expressions (9), for the moment and both components of the friction force as functions of u and v have a singularity at the point $(u,v)=(0,0)$, because they do not have any limit at this point with respect to both of the variables u and v.

Property 5. In the case of pure sliding $u=0$ or spinning $v=0$, the moment M_C and the tangential component $F_{\|}$ are homogeneous models corresponding to the usual Coulomb law:

$$F_{\|}(0,v)=F_0 \equiv fN,\ M_C(u,0)=M_0, M_0=2\pi fNRI_2,\ I_2=\int_0^1 \sigma_0(r)r^2dr$$

Property 6. In the case of pure sliding, the normal component vanishes: $F_{\perp}(0,v)=0$, and hence the friction force is directed opposite to the velocity vector; in the case of pure spinning, it is equal to $F_{\perp}(u,0)=\mu F_0,\ \mu=fhR/(\pi I_3)$.

Property 7. The moment M_C and both components of the friction force $F_{\|}$ and $F_{\perp}$ have only one nonzero first partial derivative (the others are zero):

$$\left.\frac{\partial M_C}{\partial u}\right|_{u=0} \neq 0,\ \left.\frac{\partial F_{\|}}{\partial v}\right|_{v=0} \neq 0,\ \left.\frac{\partial F_{\perp}}{\partial u}\right|_{u=0} \neq 0$$

2.2 Models based on Pade expansions

The integral models (8) give a good description of the combined sliding and spinning friction, but are inconvenient to be used in problems of dynamics, because it is required to calculate multiple integrals in the right-hand sides of the equations of motion. This difficult procedure can be eliminated by replacing the exact integral expressions by the corresponding Pade approximations. The simplest of them is the linear-fractional approximation preserving the value at zero and at infinity of both for the torque M_C and for the tangent force component $F_{\|}$. But, for the normal friction force component, corresponded Pade approximation, naturally, became of the second order.

$$M_C = M_0\left(\frac{u}{u+mv} + 2\pi\left((2\mu_1 v^2 - \mu_2)uI_3 + \mu_1 u^3 I_5\right)\right),\ \frac{1}{m} = \frac{v}{M_0}\frac{\partial M_C}{\partial u}\bigg|_{u=0} \tag{9}$$
$$F_{\|} = F_0\left(\frac{v}{v+au} + 2\pi\left((\mu_1 v^3 - \mu_2 v)I_1 + 2\mu_1 v u^2 I_3\right)\right),\ \frac{1}{a} = \frac{u}{F_0}\frac{\partial F_{\|}}{\partial v}\bigg|_{v=0}$$
$$F_{\perp} = \frac{\mu F_0 uv}{(u+bv)(v+au)},\ \frac{1}{b} = \frac{v}{\mu F_0}\frac{\partial F_{\perp}}{\partial u}\bigg|_{u=0}$$

The linear-fractional Pade´ approximations (9) preserve the values of the functions $F_{\|}(u,v)$, $F_{\perp}(u,v)$ and $M_C(u,v)$ at zero, as well as their behavior and the behavior of their first derivatives at infinity. But model of this type cannot completely preserve the values of all first partial derivatives of these functions at zero. To obtain a correct description of the behavior of the first derivatives at zero, it is required to use the second-order Pade´ approximations, and then the coupled model of sliding and spinning friction takes the form

$$M_C = M_0\left(\frac{u^2+muv}{v^2+muv+u^2} + 2\pi\left((2\mu_1 v^2 - \mu_2)uI_3 + \mu_1 u^3 I_5\right)\right),\ m = \frac{v}{M_0}\frac{\partial M_C}{\partial u}\bigg|_{u=0} \tag{10}$$
$$F_{\|} = F_0\left(\frac{v^2+auv}{v^2+auv+u^2} + 2\pi\left((\mu_1 v^3 - \mu_2 v)I_1 + 2\mu_1 v u^2 I_3\right)\right),\ a = \frac{u}{F_0}\frac{\partial F_{\|}}{\partial v}\bigg|_{v=0}$$
$$F_{\perp} = \frac{\mu F_0 uv}{(u+bu)(v+au)},\ \frac{1}{b} = \frac{v}{\mu F_0}\frac{\partial F_{\perp}}{\partial u}\bigg|_{u=0}$$

The second-order model (10) completely satisfies all properties 1–7 of the exact integral models (8). But, for the majority of the problems of dynamics, it is sufficient to use the first order model (9). The second-order model (10) is required for a more precise qualitative analysis, for example, for determining the boundaries of the stagnant region and the motion stopping time.

The approximations (9) and (10) hold for positive values of u and v. They can be easily generalized to the case of arbitrary (in sign) velocities u and v by a formal change by absolute values in the denominators of the corresponding expressions.

The use of the friction models based on the Pade´ expansions allows one to avoid calculations of multiple integrals over the contact spot, which significantly simplifies their use in problems of dynamics.

The approximate models preserve all properties of the models based on the exact integral expressions and correctly describe the behaviour of the net vector and torque of the friction forces and their first derivatives at zero and infinity. Moreover, the models coefficients can be identified from experiments [8]. Consequently, the models based on Pade expansions may be considered as phenomenological models of the combined dry friction.

CONCLUSIONS

It is developed a dynamically coupled integral dry friction model. It is shown that the distortion in the symmetry of the normal contact stresses distribution in the case of circular contact sites results to the appearance of the friction force component directed along the normal to the trajectory of the mass center of the rubbed solids and, consequently, the mass center trajectory is inclined from the stright line.

To escape the double integrals calculation in the motion equations, the exact integral expressions are replaced by appropriate Pade expansions. Models based on Pade expansions may be considered as phenomenological models of the combined dry friction because their coefficients can be defined from the experiments.

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