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TWO DIMENSIONAL SOLUTION OF THE ADVECTION-DIFFUSION EQUATION USING TWO COLLOCATION METHODS WITH LOCAL UPWINDING RBF

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Abstract. The two-dimensional advection-diffusion equation is solved using two local collocation methods with Multiquadric (MQ)Radial Basis Functions (RBFs). Although both methods use upwinding, the first one, similar to the method of Kansa, approximates the dependent variable with a linear combination of MQs. The nodes are grouped into two types of stencil: cross-shaped stencil to approximate the Laplacian of the variable and circular sector shape stencil to approximate the gradient components. The circular sector opens in opposite to the flow direction and therefore the maximum number of nodes and the shape parameter value are selected conveniently. The second method is based on the Hermitian interpolation where the approximation function is a linear combination of MQs and the resulting functions of applying partial differential equation (PDE) and boundary operators to MQs, all of them centred at different points. The performance of these methods is analysed by solving several test problems whose analytical solutions are known. Solutions are obtained for different Peclet numbers, Pe, and several values of the shape parameter. For high Peclet numbers the accuracy of the second method is affected by the ill-conditioning of the interpolation matrix while the first interpolation

method requires the introduction of additional nodes in the cross stencil. For low Pe both methods yield accurate results. Moreover, the first method is employed to solve the two-dimensional Navier-Stokes equations in velocity-vorticity formulation for the lid-driven cavity problem moderate Pe.

1 INTRODUCTION

Recently, the RBFs have been used as the base of meshless collocation approaches for solving PDEs. The use of RBF interpolation technique has become the foundation of the RBF meshless collocation methods for the solution of PDEs, since the pioneer work on the Unsymmetric method by Kansa [1]. Kansa used the MQ function to obtain an accurate meshless solution to the advection-diffusion and Poisson equations without employing any special treatment for the advective term (upwinding), due to the high order of the resultant scheme and the intrinsic relationship between governing equations and the interpolation.

With the aim of improving the Kansa's Method, Fasshauser [2] used Hermite interpolation to construct an RBF interpolating function which gives a non-singular symmetric collocation matrix. He concluded that the Hermitian (Symmetric) method performs slightly better than the Kansa (Unsymmetric) method. Jumarhon et al. [3] obtained a similar improvement using the Symmetric method and, more recently, Power and Barraco [4] attained better results by employing the Symmetric method for a variety of problems including advection-diffusion equation.

Although full-domain RBF methods are highly flexible and exhibit high order convergence rates, the fully-populated matrix systems they produce lead to the problem described by Shaback [5] as the uncertainty relation; better conditioning is associated with worse accuracy, and worse conditioning is associated with improved accuracy. As the system size is increased, this problem becomes more pronounced. Many techniques have been developed to reduce the effect of the uncertainty relation, such as RBF-specific preconditioners and adaptive selection of data centres. However, at present the only reliable method of controlling numerical ill-conditioning and computational cost as problem size increases is through domain decomposition. One of the first attempts in this direction was made by Lee et al. [6] who proposed the local MQ approximation in which only the nodes inside the influence subdomain of one central node are used in the Unsymmetric method for solving the Poisson equation. Divo and Kassab [7] solved the non-isothermal flow problem by implementing a localized Radial Basis method based on the formulation proposed by Sarler and Vertnik [8] and using a sequential algorithm. Afterwards, Stevens et al. [9] implemented the Local Hermitian Interpolation (LHI) method to solve transient and non-linear diffusion problems. Unlike the method proposed in [7], Stevens et al. solved accurately the two- and three-dimensional advection-diffusion equation by LHI method including the PDE operator and PDE centres in the approximation of the solution field for each local domain.

On the other hand, many authors have implemented upwinding schemes to avoid oscillations in the solution when dealing with the advective term. As the most common strategy, the stencil or subdomain employed to approximate the value of the variable at a given point contains points that are selected based on the flow direction. In the solution of advection-diffusion problems with Finite Volume Method (FVM) different upwinding schemes such as Upwind Differencing (UD) and QUICK have been widely used [10]. Several authors have reported the use of that kind of upwinding applied to mesh-based methods. Lin and Atluri [11] developed two upwinding schemes, USI and USII, for the Meshless Local Petrov Galerkin (MLPG) method applied to stationary advection-diffusion problems in one- and two-dimensions. In USI scheme, Lin and Atluri propose a circular stencil in which the central node moves a certain distance in the opposite way to the flow direction while in USII the complete stencil is moved. In both cases, the distances are previously defined and they depend on the local Peclet numbers. The authors report best results for USII scheme in most of the cases, mainly at high Reynolds numbers.

The RBF interpolation method has been used to solution different formulations of the Navier-Stokes equations. Several numerical techniques have been reported in the literature to solve viscous flow problems in terms of their velocity-vorticity formulation (Skerget and Rek [12] use a Boundary Element Method (BEM), Huang and Li [13] a Finite Difference Method (FDM) and Young, Liu and Eldho [14] a Finite Element Method (FEM)-BEM coupled scheme). More recently, Hribersek and Skerget [15] deal with complex geometry situations by the Boundary Domain Integral Method (BDIM) for high Reynolds numbers. Zunic et al. [16] use the scheme implemented by Young et al. in [14] for three-dimensional domains. With a similar formulation Pascazio and Napolitano [17] solve the Navier-Stokes equations for transient flow in staggered grids, where velocities are known at the volume faces and the vorticities at the nodes. Qian and Vezza [18] apply the Control Volume Method (CVM) to solve the kinetics equation and the Bio-Savart Law to compute velocities in an iterative time marching algorithm. They also used an additional scheme to compute vorticity values at boundaries. Among others, these are some examples of previously works published in the literature using the velocity-vorticity formulation.

In this paper, two RBF collocation methods with local upwinding to solve advection-dominated problems are implemented. The first method uses circular sector shape stencil for the advective term approximation. The second method is based on the Hermitian interpolation in wich the approximation functions are enforced at different locations and the PDE and boundary operators are employed at local level. Both methods are detailed in section 2 and are tested with problems in one- and two-dimensions in section 3.

2 LOCAL COLLOCATION METHOD FOR A SYSTEM OF EQUATIONS

With the vorticity vector, $\vec{\omega}$, understood as the curl of the velocity field,

$$\overrightarrow{\omega} = \overrightarrow{\nabla} \times \overrightarrow{v},$$

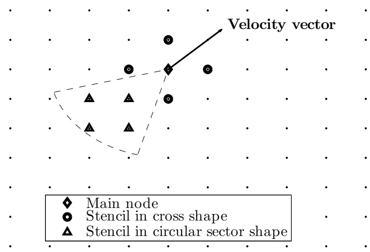


Figure 1: Stencil used in MSRBF method

the dimensionless kinematics and kinetics equations for an incompressible fluid in a two-dimensional domain, Ω, are given by

$$\nabla^2 v_1 = -\frac{\partial \omega}{\partial x_2}, \qquad \nabla^2 v_2 = \frac{\partial \omega}{\partial x_1} \tag{1}$$

and

$$\nabla^2 \omega - Re\overrightarrow{v}\cdot\overrightarrow{\nabla}\omega = 0, \tag{2}$$

where $\vec{v} = [v_1, v_2]$. The Reynolds number is defined, in terms of the density, ρ, dynamic viscosity, μ, relative velocity, U, and relative length, L, as $Re = \frac{\rho U L}{\mu}$ and this is equivalent to Pe in problems of section 3. Equation (1) is obtained by applying the curl operator to the vorticity definition and by considering the mass conservation equation, $\vec{\nabla}\cdot\vec{v} = 0$. Similarly, the curl operator applied to the Navier-Stokes equations in terms of the primitive variables leads to the vorticity transport equation (2). The equations (1)-(2) form the velocity-vorticity formulation of the Navier-Stokes system of equations for two-dimensional steady state flows. On the boundary, $\partial\Omega$, the variables v_1, v_2 and ω satisfy Dirichlet conditions. PDEs linear operators

$$\mathcal{L}_1(\cdot) : \nabla^2(\cdot) \quad \text{and} \quad \mathcal{L}_2(\cdot) : \nabla^2(\cdot) - Re\vec{v}\cdot\overrightarrow{\nabla}(\cdot),$$

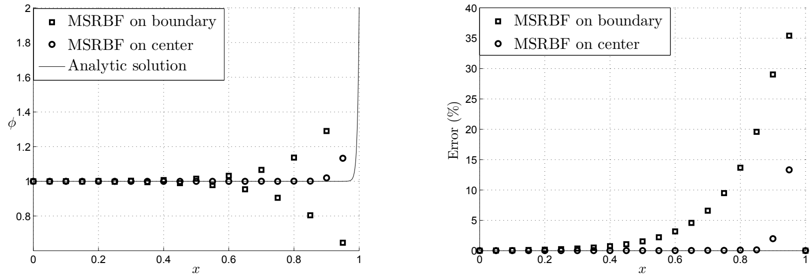
respectively, are associated to (1) and (2) and, if ϕ is used instead of v_1, v_2 and ω, they are represented by $\mathcal{J}$. Namely,

$$\mathcal{J}(\phi(\vec{x})) = \begin{cases} \mathcal{L}(\phi(\vec{x})) & \text{if} \quad \vec{x} \in \Omega, \\ \mathcal{B}(\phi(\vec{x})) & \text{if} \quad \vec{x} \in \partial\Omega, \end{cases} \quad \text{and} \quad \mathcal{L}(\phi) = \begin{cases} \mathcal{L}_1(\phi) & \text{if} \quad \phi = v_i, i = 1, 2, \\ \mathcal{L}_2(\phi) & \text{if} \quad \phi = \omega, \end{cases}$$

where $\mathcal{B}$ represents the boundary operator, which in this case is the identity operator.

An RBF, ψ, is a symmetric function respect to a source point, $\vec{\xi}$, and it is defined in terms of the Euclidean distance, r, between the source point and an evaluation point, $\vec{x}$. There are different types of RBF widely referenced in the literature, but in this case only the multiquadric, $\psi(r) = (r^2 + c^2)^{\frac{m}{2}}$, with $m = 1$ is considered. The real constant c is the shape parameter.

The Kansa's method builds the solution approximation, $\widehat{\phi}(\vec{x})$, as the linear combination (3), consisting of two kinds of continuous functions: a polynomial of degree $m - 1$, with



a) b)

Figure 2: **a)** Solutions and **b)** Relative errors for MSRBF without fictitious node

NT terms that depend on the degree of the polynomial and the spatial dimensions, and RBFs centered at N points located in Ω and $\partial\Omega$, namely,

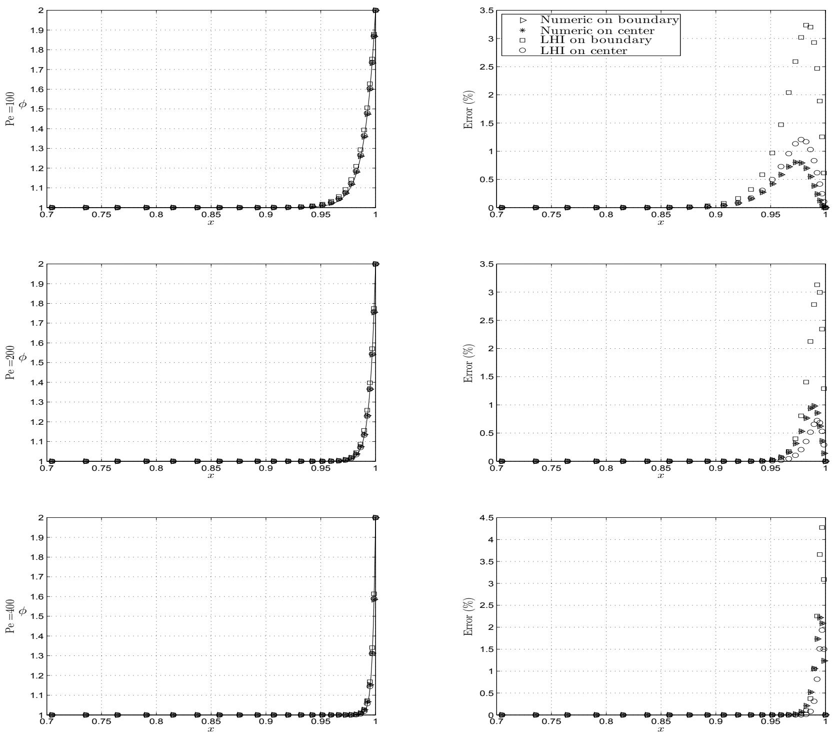
$$\widehat{\phi}\,(\vec{x}) = \sum_{i=1}^{N} \alpha_i \psi_i(r) + \sum_{k=1}^{NT} \beta_k P_{m-1}^{k}(\vec{x}). \tag{3}$$

The constant coefficients α_i satisfy the following homogeneity conditions

$$\sum_{i=1}^{N} \alpha_i P_{m-1}^{k}(\vec{x_i}) = 0,\ 1 \leq k \leq NT. \tag{4}$$

For each equation, this approach leads to a system of linear equations with a matrix made of four blocks resulting from the application of the PDE and the boundary operators to equation (3), one block generated by conditions (4) and a last block of zeros. Regarding the fact that this matrix suffers ill-conditioning problems as N becomes very large and the calculations require a large computational cost, the local formulation of the methods is a valid alternative to avoid these problems.

The local collocation consist in selecting a suitable number of points that form a stencil whereby the value of the solution is approximated in one of the selected nodes, main node. In the first method, that we call Modified Stencil RBF (MSRBF) collocation method, $\widehat{\phi}$ is calculated with two kinds of stencil: cross shape and circular sector shape (Figure 1). The main node is the one who is at once cross center and sector vertex. The stencil in cross shape is used to approximate the diffusive term, stencil for diffusion, SD, while the advective term is approximated with the stencil in sector circular shape, stencil for advection, SA. The SA opens in upwind direction and the maximum number of nodes within is conveniently selected. If the SA contains only one point then both terms are approximated with SD. If N_i represents the number of nodes in the stencil i, $1 \leq i \leq N$, wherein $\vec{x}_i$ is the main node, the interpolation (3) is given by



a) b)

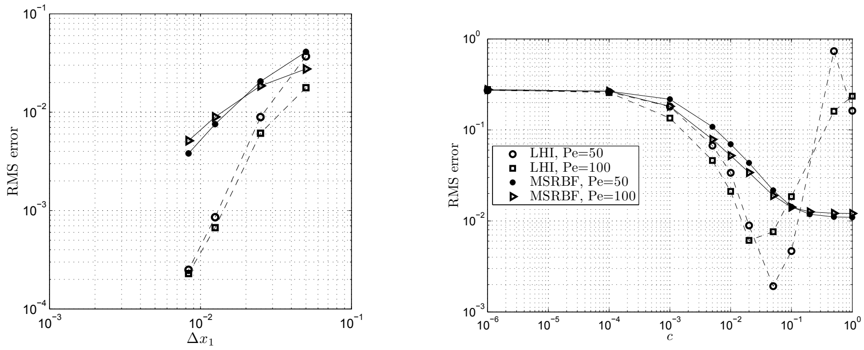
Figure 3: **a)** Solutions and **b)** Relative errors for problem 3.1

$$\widehat{\phi}\,(\vec{x}) = \sum_{j=1}^{N_i} \alpha_j \psi_j(r) + \beta_i. \tag{5}$$

Evaluation of the interpolation function at all nodes in the stencil allows to express the left side of (1), (2) and the boundary condition at $\vec{x}_i$ in terms of the solution at the remaining nodes of stencil i, as shown in the following expression

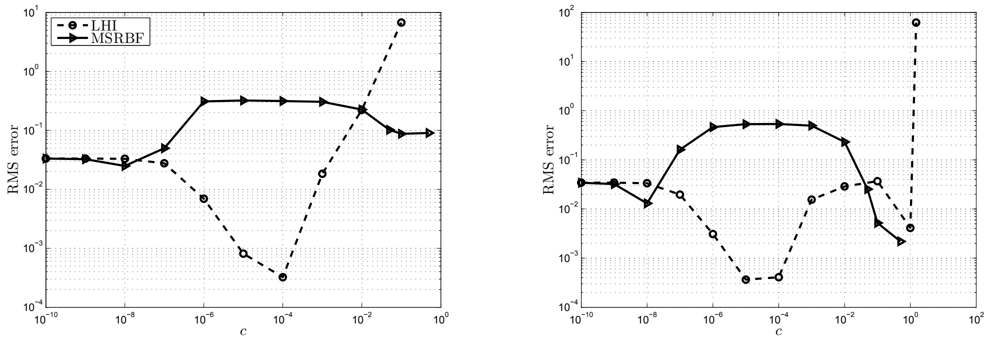
$$\mathcal{J}(\widehat{\phi}\,(\vec{x}_i)) = \mathcal{J}(\psi^i)\,[\Psi_{ij}]^{-1}\,\widehat{\phi}_i, \tag{6}$$

where $\widehat{\phi}_i$, Ψ_{ij} and $\mathcal{J}(\psi^i)$ represent, respectively, a vector formed by unknowns variables at stencil nodes, the interpolation matrix on the stencil i and a vector whose components are obtained by applying the operator $\mathcal{J}$ to the RBFs associated to the node i evaluated at all nodes in the stencil i. By applying equation (6) to every stencil, $1 \leq i \leq N$, it is obtain a sparse linear system that is solved to find the solutions $\widehat{\phi}\,(\vec{x}_i)$.



a) b)

Figure 4: RMS errors in terms of **a)** nodal distance and **b)** shape parameter in problem 3.1



a) b)

Figure 5: RMS errors for **a)** 21×21 and **b)** 51×51 nodal distributions in problem 3.2

3 Numerical Results

In this section we use MSRBF and LHI methods to solve one- and two-dimensional advection-dominated problems. The equation (2) with unidirectional velocity is solved in a rectangle in section 3.1 and the equation with a source term and skew velocity is solved in a unit square in section 3.2. The results obtained with MSRBF method in the mentioned sections suggest the parameter values employed to solve the lid-driven cavity flow problem at $Re = 100$ and 200 in section 3.3. The Picard iteration is used in order to solve the coupled equations (1)-(2). From an initial vorticity on Ω, a velocity field is obtained from equation (1). Then, this velocity field is used in equation (2) to get a vorticity solution on Ω with which the next iteration starts. The Root Mean Square (RMS) error [20], given by $\epsilon_{rms} = (\phi_{max}\sqrt{N})^{-1} \left(\sum_{i=1}^{N} [\phi(\vec{x}_i) - \widehat{\phi}(\vec{x}_i)]^2\right)^{1/2}$ is selected to assess the solution error.

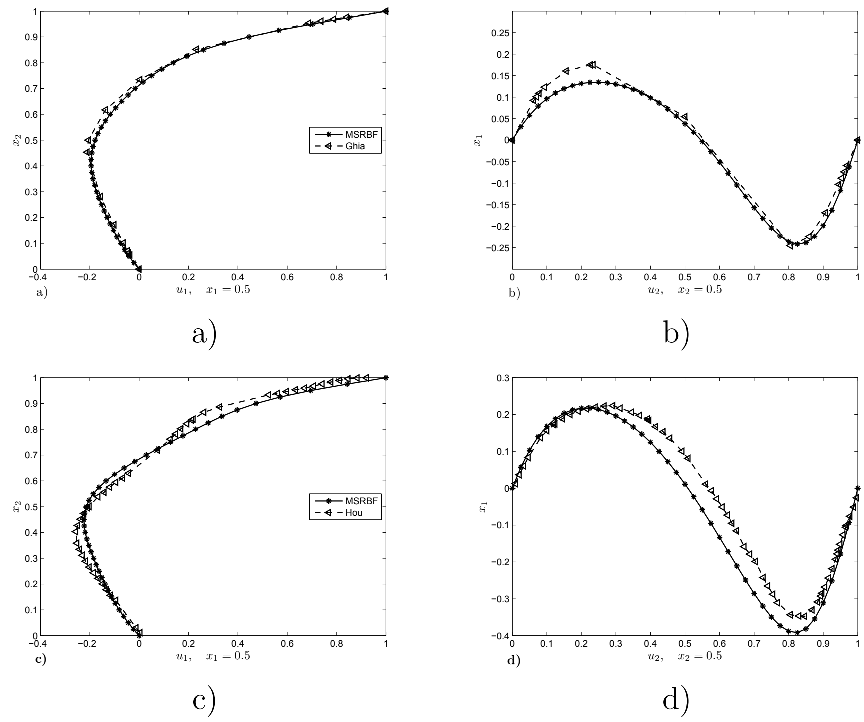


Figure 6: **a)** u_1 velocity at line $x_1 = 0.5$, $Re = 100$; **b)** u_2 velocity at line $x_2 = 0.5$, $Re = 100$; **c)** u_1 velocity at line $x_1 = 0.5$, $Re = 200$, and **d)** u_2 velocity at line $x_2 = 0.5$, $Re = 200$

3.1 One-dimensional advection-dominated problem

Let's consider the equation (2) with $\vec{u} = [u_1, 0]$ where $u_1 = 50$, 100, 200 and 400 in $\Omega = (0,1) \times (0, 0.2)$. The following boundary conditions are imposed

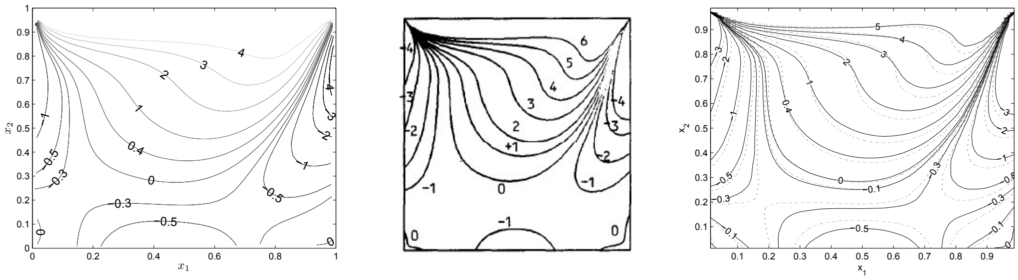
$$\phi = 1, \quad x_1 = 0, \quad 0 < x_2 < 0.2,$$
$$\phi = 2, \quad x_1 = 1, \quad 0 < x_2 < 0.2,$$

and $\dfrac{\partial \phi}{\partial n} = 0$ at the remaining walls. The analytic solution is given as

$$\phi(x_1, x_2) = 2 - \frac{1 - e^{u_1(x_1 - 1)}}{1 - e^{-u_1}}.$$

Solutions obtained with the MSRBF method, with a uniform node distribution of 21×5, $u_1 = 200$ and $c = 0.05$ on the edge of the rectangle ($x_1 \in [0,1]$ and $x_2 = 0$) and on the central line ($x_1 \in [0,1]$ and $x_2 = 0.1$) oscillate as shown in Figure 2 where solutions and relative errors at each node are presented.

The oscillations increase by increasing Pe or aspect ratio, that is, the ratio between the horizontal and vertical distances from one node to its neighbouring nodes. These difficulties are avoided when using a fictitious node strategy. Figure 3 shows the solutions



a) b) c)

Figure 7: Vorticity contours at $Re = 100$ obtained by: **a)** MSRBF; **b)** Ghia; and **c)** Bustamante

and relative errors on the edge of the rectangle and on the central line with a nodal distribution of 41×9, $c = 0.05$ and $u_1 = 100$, 200, 400. The RMS errors obtained by the two methods are shown in Figure 4. The absence of fictitious node in the LHI method leads to greater error on the side of the rectangle. Because of the ease of deployment of the RBFs and the smaller number of nodes required in the interpolations of the MSRBF method, the similarity in the solutions on the center line (Figure 3) shows that the MSRBF is a good option in the one-dimensional case. According to Figure 4a, the LHI method has a higher order convergence with a chosen value of c, but simultaneously it is very sensitive to the shape parameter, as shown in Figure 4b. A much more stable behaviour is observed in the MSRBF in comparison to the LHI in a double-precision arithmetic.

3.2 Two-dimensional advection-dominated problem

Regarding equation (2), in this case we take $\vec{u} = [10^6, -10^6]$, $Re = 1$, $\Omega = (0,1)^2$ and a source term and the Dirichlet boundary conditions in agreement with the exact solution

$$\phi(x_1, x_2) = x_1 \cos(0.5\pi x_2).$$

Figure 5 shows the RMS errors as a function of the shape parameter for $Pe = 10^6$. Both methods report similar accuracy for $c < 10^{-7}$. The LHI results are more accurate within the range $10^{-7} < c < 10^{-2}$. However, ill-conditioning problems in the local interpolation matrices produce an unstable behaviour of the solution in terms of the shape parameter, making difficult the selection of a suitable value. On the other hand, the MSRBF method presents better behaviour as the shape parameter is increased in agreement with the typical trend of the RBF direct collocation schemes. Moreover, the MSRBF method shows convergence when comparing results presented in Figures 5a and 5b, where the minimum RMS errors are of order 10^{-1} and 10^{-3}, respectively. For relatively high shape parameter values, the LHI method does not produce accurate results according to the mentioned issues.

3.3 Two-dimensional driven cavity flow problem

In this problem, we consider the system of equations (1)-(2) with $Re = 100$ and 200 in $\Omega = (0,1)^2$, that describes the movement of an incompressible, isothermal and Newtonian fluid that fills a square cavity. The flow field is due to the motion of the upper wall located at $x_2 = 1$, with a dimensionless prescribed velocity $u_1 = 1$ and $u_2 = 0$. On the remaining walls the velocities are zero, i.e. $u_i = 0$ for $i = 1, 2$. The values of the boundary vorticity are computed from the definition. The initial guess value in the algorithm for $Re = 200$ is set to be equal to the obtained solution at $Re = 100$.
Results obtained with MSRBF method are compared with Ghia et al. [21] at $Re = 100$ and Hou et al. [22] at $Re = 200$. Figure 6 shows the velocities on the cavity central lines where the good agreement with the reference can be observed. A 41×41 nodal distribution is used in all cases.

The achieved vorticity contours for $Re = 100$, Figure 7a), are qualitatively similar to the results presented by Ghia, Figure 7b), and Bustamante et al [23], Figure 7c). In the solution of this problem, every SA has an angle 120°, the selected radius guarantees no more than seven nodes per SA and the shape parameter has a value of 1.0. The parameter value is in the stability range shown in figure 5b by 51×51 nodal distribution.

4 CONCLUSIONS

- The two-dimensional advection-diffusion equation has been solved by means of two different upwinding strategies. By using the MSRBF method, accurate results were found with stencils for advection including 3 to 7 nodes and covering an angle between 90° and 100°. The LHI method presents higher order convergence than the MSRBF when solving a one-dimensional problem. Nevertheless, the ill-conditioning problems makes the LHI method more shape parameter sensitive. The MSRBF method is a suitable option regarding a straightforward implementation and a good behaviour in terms of the shape parameter.

- The MSRBF method has been used to solve the two-dimensional driven cavity flow at Reynolds numbers of 100 and 200. The parameter value used in the RBF interpolations was 1.0 and the maximum number of points in every SA was 7 as in the advection-dominated problems. The numerical results are in good agreement with the reference solution. However, the authors are currently working on the improvement of the solution algorithm with the aim of having better results.

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