IV International Conference on Computational Methods for Coupled Problems in Science and Engineering
COUPLED PROBLEMS 2011
M. Papadrakakis, E. Oñate and B. Schrefler (Eds)

NUMERICAL MODELLING OF COUPLED ELECTRO-MECHANICAL PROBLEMS FOR THE STATE SPACE CONTROLLER DESIGN

TAMARA N. NESTOROVIĆ[*] AND MIROSLAV M. TRAJKOV[†]

[*] Mechanics of Adaptive Systems, Institute of Computational Engineering
Ruhr-Universität Bochum
Universitätsstraße 150, Geb. IA-01/128, D-44801 Bochum, Germany
e-mail: [*]tamara.nestorovic@rub.de, [†]miroslav.trajkov@rub.de

Key words: Coupled electro-mechanical behavior, State space models, Model reduction for the controller design.

Abstract. Model development for coupled electro-mechanical problems in light-weight smart structure design is the subject of this paper. The paper addresses development of reliable models for the controller design of piezoelectric smart structures and systems, within an overall design procedure. Model development is based on the finite element (FE) approach, with application of modal reduction techniques for obtaining the state space models convenient for the controller design. Modal truncation and balanced modal reduction are considered as modal reduction techniques, with regard to controllability and observability issues. From the model optimization and verification point of view the experimental modal analysis and identification issues are addressed as well. Examples of model application to controller design document the feasibility of the technique.

1 INTRODUCTION

Modelling of light-weight smart structures has been a research challenge over the past years. As a part of the overall design of smart structures it represents an important phase in development procedure, which supports all subsequent steps including simulation, controller design, testing and implementation. Smart structures and systems in general comprise integrity (structural and functional) of a structure or a system, multifunctional materials, actuators/sensors and appropriate control in order to achieve desired performances under varying environmental conditions. This paper considers a special type of active multifunctional materials – piezoelectric materials, which belong to the group of ferroelectrics. Due to special properties of the piezoelectric materials, which qualify them for the implementation as actuators and sensors in smart structures, their modelling deserves a special attention. Numerical modelling of smart light-weight structures, which include piezoelectric actuators and sensors, will be treated as a coupled electro-mechanical problem, which integrates the primary mechanical structural behaviour and piezoelectric influence of the active material.

Development of reliable coupled electro-mechanical models for the controller design of piezoelectric smart structures and systems, as a part of an overall design procedure, is the subject of this paper. Special attention is paid to development of state space models, since they represent the basis for many controller applications [1]. Model development is based on the finite element (FE) approach, and appropriate formulation of the coupled electro-mechanical problem [2], with application of modal reduction techniques for obtaining the state space models convenient for the controller design. Development of reduced state space models based on the FE approach has mostly been treated in literature by applying pure model truncation techniques, which assume retaining only the eigenmodes of interest, assessed by the designer without considering or adopting criteria for the mode selection. This approach does not consider controllability and observability issues within the model reduction algorithm, which represents its lack. The model reduction technique proposed in this paper overcomes the mentioned drawback of the pure model truncation, by introducing weighting of the eigenmodes of interest and consideration of the controllability and observability issues, via appropriate controllability and observability Gramians. In this way a consistent conclusion about the influence of the modes with respect to highest controllability and observability Gramians can be adopted as a criterion for the mode selection. An algorithm for such balanced modal reduction is developed and implemented in this paper for development of reduced order state space models, which are reliable and appropriate for the controller design. As an alternative approach, the subspace-based state space model identification is also presented. The implementation of the techniques is documented by an example of state space model development for coupled electro-mechanical behavior applied to vibration control of a piezoelectric light-weight structure.

2 FINITE ELEMENT BASED REDUCED MODELS

The finite element (FE) based modeling of piezoelectric smart systems and structures represents a suitable basis for the overall simulation and design procedure. This approach results in models which are convenient for the controller design [3-5] and for the analysis of appropriate actuator/sensor placement [6].

The FE analysis is based on the finite element semi-discrete form of the equations of motion of a piezoelectric smart system, which describe its electro-mechanical behavior. These equations can be derived using the established approximation method of displacements and electric potential and the standard finite element procedure [6]. Here the coupled electro-mechanical behavior of smart structures will be considered.

Constitutive equations in the stress-charge form (1) are used for the development of the equations of motion for a smart structure:

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon} - \mathbf{e}\mathbf{E}, \qquad \mathbf{D} = \mathbf{e}^{\mathrm{T}}\boldsymbol{\varepsilon} + \boldsymbol{\kappa}\mathbf{E} \tag{1}$$

with following notations:

$\boldsymbol{\sigma}^{\mathrm{T}} = [\sigma_{11} \quad \sigma_{22} \quad \sigma_{33} \quad \sigma_{12} \quad \sigma_{23} \quad \sigma_{31}]$ mechanical stress vector, $\mathbf{C}_{(6\times6)}$ symmetric elasticity matrix, $\boldsymbol{\varepsilon}^{\mathrm{T}} = [\varepsilon_{11} \quad \varepsilon_{22} \quad \varepsilon_{33} \quad 2\varepsilon_{12} \quad 2\varepsilon_{23} \quad 2\varepsilon_{31}]$ strain vector, $\mathbf{E}^{\mathrm{T}} = [E_1 \quad E_2 \quad E_3]$ electric field vector, $\mathbf{e}_{(6\times3)}$ piezoelectric matrix, $\mathbf{D}^{\mathrm{T}} = [D_1 \quad D_2 \quad D_3]$ vector of electrical displacement and $\boldsymbol{\kappa}_{(3\times3)}$ symmetric dielectric matrix.

The system of equations which describe electro-mechanical behavior consists of the constitutive equations (1) together with the mechanical equilibrium and electric equilibrium (charge equation of electrostatics resulting from the 4th Maxwell equation):

$$\mathbf{D}_u^{\mathrm{T}}\boldsymbol{\sigma} + \mathbf{P} - \rho\,\mathbf{v} = \mathbf{0}, \quad \mathbf{D}_\phi^{\mathrm{T}}\mathbf{D} = \mathbf{0} \tag{2}$$

where $\mathbf{P}^{\mathrm{T}} = [P_1 \;\; P_2 \;\; P_3]$ represents the body force vector, $\mathbf{v}^{\mathrm{T}} = [v_1 \;\; v_2 \;\; v_3]$ is the vector of mechanical displacements, ρ is the mass density and $\mathbf{D}_u$ and $\mathbf{D}_\phi$ are differentiation matrices:

$$\mathbf{D}_u^{\mathrm{T}} = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_3} \\ 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_3} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \end{bmatrix}, \quad \mathbf{D}_\phi^{\mathrm{T}} = \begin{bmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_2} \end{bmatrix}. \tag{3}$$

Variational statement of the governing equations for the coupled electro-mechanical problem derived from the Hamilton's principle represents the basis for development of the FE model [2]. It is obtained in the form:

$$\begin{aligned} &-\int_V (\rho\,\delta\mathbf{v}^{\mathrm{T}}\ddot{\mathbf{v}} - \delta\boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{C}\boldsymbol{\varepsilon} + \delta\boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{e}^{\mathrm{T}}\mathbf{E})dV + \int_V (\delta\mathbf{E}^{\mathrm{T}}\mathbf{e}\boldsymbol{\varepsilon} + \delta\mathbf{E}^{\mathrm{T}}\boldsymbol{\kappa}\,\mathbf{E} + \delta\mathbf{v}^{\mathrm{T}}\mathbf{F}_{\mathrm{V}})dV \\ &+ \int_{\Omega_1} \delta\mathbf{v}^{\mathrm{T}} F_\Omega d\Omega + \delta\mathbf{v}^{\mathrm{T}}\mathbf{F}_{\mathrm{P}} - \int_{\Omega_2} \delta\phi\, q\, d\Omega - \delta\phi Q = 0 \end{aligned} \tag{4}$$

where F_Ω represents the surface applied forces (defined on surface Ω_1), F_P the point loads, ϕ the electric potential, q the surface charge brought on surface Ω_2 and Q the applied concentrated electric charges. Applying the approximation of displacements and electric potential with the shape functions over an element, representing the structure by a finite number of elements and adding up all elements contributions, the finite element semi-discrete form of the equations of motion is obtained:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{D}_d\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \overline{\mathbf{E}}\mathbf{f}(t) + \overline{\mathbf{B}}\mathbf{u}(t) \tag{5}$$

where vector **q** represents the vector of generalized displacements including mechanical displacements and electric potential and contains all degrees of freedom. Matrices **M**, $\mathbf{D}_d$ and **K** are the mass matrix, the damping matrix and the stiffness matrix, respectively. The total load vector is divided into the vector of the external forces $\mathbf{F}_E = \overline{\mathbf{E}}\mathbf{f}(t)$ and the vector of the control forces $\mathbf{F}_C = \overline{\mathbf{B}}\mathbf{u}(t)$, where the forces are generalized quantities including also electric charges. Vector $\mathbf{f}(t)$ represents the vector of external disturbances, and $\mathbf{u}(t)$ is the vector of the controller influence on the structure. Matrices $\overline{\mathbf{E}}$ and $\overline{\mathbf{B}}$ describe the positions of the forces and the control parameters in the finite element structure, respectively.

Model (5) represents a standard FE formulation of equations of motion for a piezoelectric structure. Since FE models in this form often may have more thousands of degrees of freedom (depending on the meshing, i.e. number of finite elements), they are often not suitable for subsequent investigation phases, especially not for the controller design, and therefore a model reduction is required. A technique often used for the model reduction of large flexible vibrating structures is modal truncation, which will be briefly represented in the following.

2.1 Modal truncation

This model reduction technique is based on the modal analysis, which enables determination of structural eigenmodes and eigenfrequencies. Eigenmodes, i.e. eigenvectors, form the modal matrix $\mathbf{\Phi}_\mathrm{m}$. Modal coordinates $\mathbf{z}$ can be introduced in the following way:

$$\mathbf{q}(t) = \mathbf{\Phi}_\mathrm{m}\mathbf{z}(t)\ . \tag{6}$$

If (6) is substituted into equation of motion (5), then the following ortho-normalization relations can be applied $\mathbf{\Phi}_\mathrm{m}^\mathrm{T}\mathbf{M}\mathbf{\Phi}_\mathrm{m} = \mathbf{I}$, $\mathbf{\Phi}_\mathrm{m}^\mathrm{T}\mathbf{K}\mathbf{\Phi}_\mathrm{m} = \mathbf{\Omega}$, $\mathbf{\Delta} = \mathbf{\Phi}_\mathrm{m}^\mathrm{T}\mathbf{D}_\mathrm{d}\mathbf{\Phi}_\mathrm{m}$, where $\mathbf{\Omega}$ represents the spectral matrix and $\mathbf{\Delta}$ the modal damping matrix. In modal truncation only a limited number of eigenmodes of interest is taken into account. The remaining modes are truncated, which enables model reduction. Introducing the state space vector:

$$\mathbf{x}(t) = \begin{bmatrix}\mathbf{z} & \dot{\mathbf{z}}\end{bmatrix}^\mathrm{T} \tag{7}$$

after performing appropriate transformations, the state equation of the modally reduced state space model can be obtained in the form:

$$\dot{\mathbf{x}} = \begin{bmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{\Omega} & -\mathbf{\Delta}\end{bmatrix}\mathbf{x} + \begin{bmatrix}\mathbf{0}\\ \mathbf{\Phi}_\mathrm{m}^\mathrm{T}\overline{\mathbf{B}}\end{bmatrix}\mathbf{u}(t) + \begin{bmatrix}\mathbf{0}\\ \mathbf{\Phi}_\mathrm{m}^\mathrm{T}\overline{\mathbf{E}}\end{bmatrix}\mathbf{f}(t)\ . \tag{8}$$

Equation (8) corresponds to the state equation in the state space form of a structural model (9), which is convenient for the controller design.

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{E}\mathbf{f}(t)\\ \mathbf{y} &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) + \mathbf{F}\mathbf{f}(t)\end{aligned} \tag{9}$$

Controllability and observability issues play an important role in controller design and its application. For state space models of a smart structure, controllability and observability can be proven using appropriate numeric criteria. In literature different criteria can be found, see e.g. [7]. Controllability/observability of single selected modes of interest cannot be strictly assessed for truncated models using standard controllability/observability criteria based on the rank of the controllability and observability matrices. Balanced modal reduction overcomes this drawback.

2.2 Balanced modal reduction

Ranks of the controllability and observability matrices, although relatively simple criteria, provide only an answer to the controllability/observability question in terms “yes” or “no”. As very well known, if the rank of the controllability/observability matrices is equal to the number of states, the model i.e. the realization is controllable/observable. This approach gives good results only for lower system orders, otherwise numerical difficulties may be encountered. Controllability and observability properties of the state space systems can be qualitatively expressed in terms of controllability (**P**) and observability (**Q**) Gramians, defined in the following way:

$$\mathbf{P} = \int_0^\infty e^{\mathbf{A}t}\mathbf{B}\mathbf{B}^\mathrm{T}e^{\mathbf{A}^\mathrm{T}t}dt,\quad \mathbf{Q} = \int_0^\infty e^{\mathbf{A}t}\mathbf{C}^\mathrm{T}\mathbf{C}\,e^{\mathbf{A}^\mathrm{T}t}dt\ . \tag{10}$$

P and **Q** satisfy algebraic Lyapunov linear matrix equations:

$$\mathbf{AP} + \mathbf{PA}^{\mathrm{T}} = -\mathbf{BB}^{\mathrm{T}}, \quad \mathbf{A}^{\mathrm{T}}\mathbf{Q} + \mathbf{QA} = -\mathbf{C}^{\mathrm{T}}\mathbf{C}\,. \tag{11}$$

For an arbitrary transformation of the states by some transformation matrix, appropriate Gramians are obtained, with the property that the eigenvalues of the controllability and observability Gramians products remain invariant. This invariants are the Hankel singular values of the system, and they represent the basis of the balanced model reduction. In balanced realization each state (mode) is equally controllable and observable and the reduced order model is obtained by truncating the least controllable and observable modes. The task of the balanced reduction is actually to find such state transformation, which provides equal controllability and observability of retained modes. In other words, the controllability and observability Gramians of the retained modes are diagonal and equal, and based on this criteria balanced model reduction can be performed.

3 MODAL REDUCTION FOR A SMART PIEZOELECTRIC BEAM

For a piezoelectric beam represented in Figure 1 the finite element model was obtained using the standard FE procedure for modeling piezoelectric materials and mechanical structures.

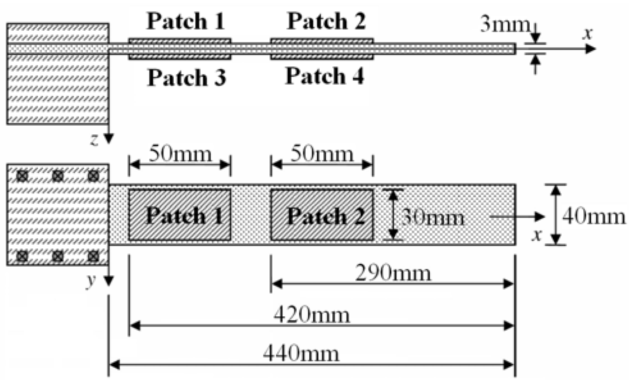

Figure 1: Geometry and layout of the smart structure

The smart structure consists of a cantilever aluminum beam (Young's modulus 70 GPa and density 2.7 g/cm³) and four piezoelectric patches (DuraActTM P-876.A15), which are attached to the beam, two on each side of the beam. For control purposes these four patches are used as actuators to enable active vibration control of the beam. A scanning digital laser Doppler vibrometer (VH-1000-D), which acts as a sensor, is used to measure the velocity of the bending vibration at the tip of the beam. The sensor provides the feedback signal in the active control algorithm. Among investigations of different MIMO and SISO models, some selected results for the model with one input (piezoelectric actuator excitation) and one output (measured velocity at the tip of the beam) are presented here. The model was obtained using the balanced reduction procedure previously described.

Balanced realization with 20 states was obtained from the original FE model which was reduced by selecting 10 eigenmodes of the beam. A diagram of controllability and observability Gramians for the balanced realization is represented in Figure 2, showing diagonal elements with highest magnitudes for the states which are most controllable /observable.

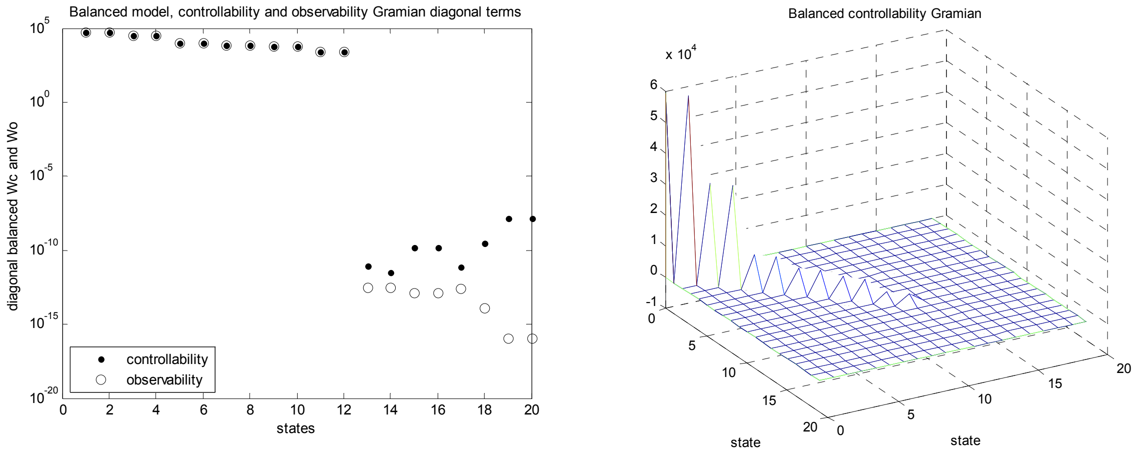

Figure 2: Balanced controllability/observability Gramians

Based on this diagram, selection of the states, which are most controllable and observable at the same time, can be performed. In this case those are obviously first twelve states. Furthermore, the number of the states which should be retained can be selected as well. In this case four eigenmodes are selected. Balanced reduction determines, which of the modes should be retained based on the previously stated controllability and observability criteria for the states. The order of the states in the diagram of grammians does not necessarily correspond to the order of the structural eigenmodes.

Analysis of the balanced reduced order model and its comparison with the original unreduced FE model shows an excellent agreement of frequency responses for the retained modes (Figure 3). It can be clearly seen that the fourth and the sixth modes or the original model are truncated, which was determined based on the balanced reduction and maximal controllability/observability influence of the states corresponding to appropriate structural eigenmodes.

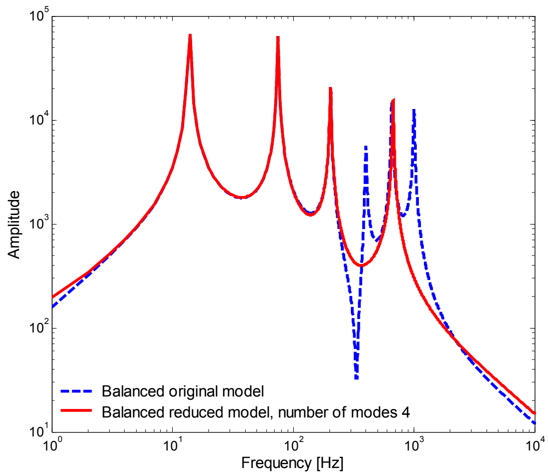

Figure 3: Frequency responses of the original and reduced model

4 MODEL DEVELOPMENT USING SYSTEM IDENTIFICATION

Another possibility for development of reliable smart structural models is system identification. The premise for this approach is the existence of a real structure, which should undergo experimental identification procedure. *Subspace based system identification* was proven to be a very efficient means for the identification of smart structural models, [3]. This procedure is suitable for the identification of discrete-time state space models in the form:

$$\mathbf{x}[k+1] = \mathbf{\Phi x}[k] + \mathbf{\Gamma u}[k], \quad \mathbf{y}[k] = \mathbf{Cx}[k] + \mathbf{Du}[k]\,. \tag{12}$$

The task of the subspace identification is to determine the order n of the unknown system as well as the system matrices $\mathbf{\Phi} \in \mathrm{R}^{n\times n}$, $\mathbf{\Gamma} \in \mathrm{R}^{n\times m}$, $\mathbf{C} \in \mathrm{R}^{l\times n}$, $\mathbf{D} \in \mathrm{R}^{l\times m}$. Input and output measurement data are organized in the form of the following input-output relation:

$$\mathbf{Y}[k] = \mathbf{\Gamma}_\alpha \mathbf{x}[k] + \mathbf{\Phi}_\alpha \mathbf{U}[k]\,, \tag{13}$$

where $\mathbf{\Gamma}_\alpha$ represents the observability matrix for the system (13), and $\mathbf{\Phi}_\alpha$ is the Toeplitz matrix of impulse responses from **u** to **y.** A detailed subspace identification procedure is described in [1,3]. Comparison of the frequency responses obtained for identified models using subspace identification method with different model orders is represented in Figure 4.

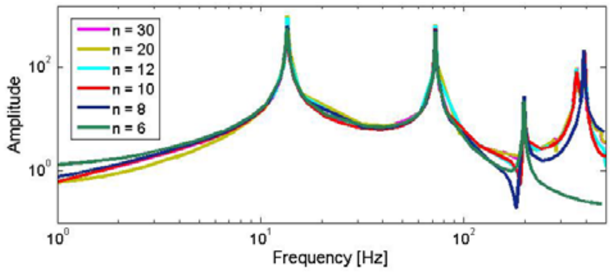

Figure 4: Frequency responses of identified models with different orders

5 CONTROLLER IMPLEMENTATION RESULTS

State space models obtained by previously described procedures can be successfully implemented in the controller design for vibration suppression. Development of reliable models, which enable an overall design of smart structures including controller design, plays a very important role for further application steps.

Active control of the piezoelectric beam represented in Figure 1 was performed by applying a discrete-time negative feedback control in the form:

$$\mathbf{u}[k] = -\mathbf{Lx}[k] \tag{14}$$

where **u** represents control voltages applied to the piezoelectric patches, and the state variables **x** are estimated using an observer (Kalman filter), [8]. The feedback gain matrix **L** is determined for an optimal linear quadratic regulator (LQR). The controller design task is to determine the control law which minimizes the performance index:

$$J = \sum_{k=0}^{\infty} \left(\mathbf{x}[k]^{\mathrm{T}} \mathbf{Qx}[k] + \mathbf{u}[k]^{\mathrm{T}} \mathbf{Ru}[k] \right) \tag{15}$$

where the matrices **Q** and **R** are the designer specified symmetric positive definite weighting matrices.

Implementation of the controllers, designed based on developed state space structural models with the purpose of vibration suppression is illustrated in the following figures. Figure 5 represents the sensor signal (velocity at the tip of the beam) in the presence of a random noise; the controller is switched on after 2 seconds. Active control of the piezoelectric beam results in significant reduction of the vibration magnitudes in comparison with uncontrolled case.

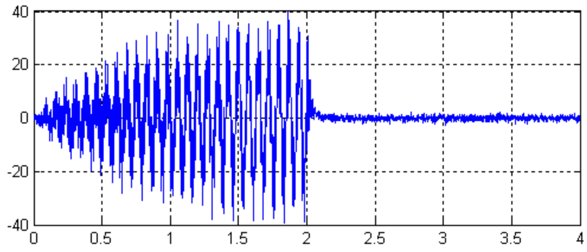

Figure 5: Velocity of a point at the tip of the beam (uncontrolled and controlled using piezoelectric patches)

Successful performance of the controlled system is also demonstrated for the case of the initial displacement disturbance. Free vibrations of the beam caused by an initial displacement applied to the tip of the beam are comparable with impulse disturbance vibrations.

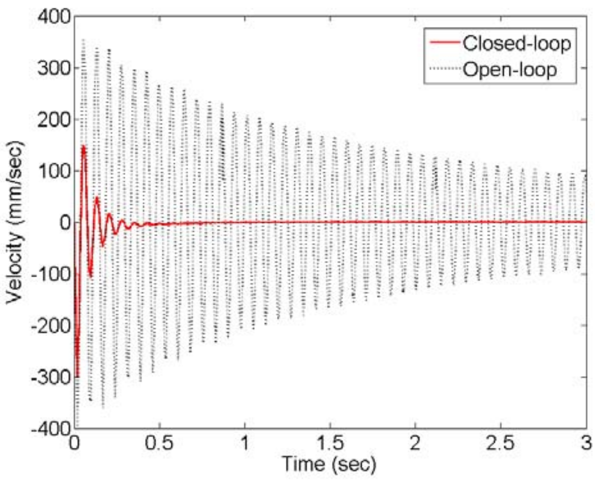

Figure 6: Free vibration response (velocity) of the controlled and uncontrolled system

The free vibration response (velocity) of the open-loop and closed-loop system subjected to an initial displacement of is represented in Figure 6. Designed controller attenuates significantly the magnitudes of the free end displacement [8].

In both cases controller development and application were performed owing to reliable reduced state space models.

6 CONCLUSIONS

In this paper the modeling of coupled electro-mechanical behavior of piezoelectric light-weight structures is considered. Numerical modeling based on finite element procedure is used to obtain the reduced state space models, which are convenient for the controller design. For the model reduction the balanced modal reduction is employed, which is superior in comparison with the pure modal truncation, since it provides necessary information on model controllability and observability issues. Besides, the numerical modeling using the subspace based model identification is also presented, as a technique for reliable modeling in case when a real structure or a prototype is available. State space models obtained applying proposed techniques provide a suitable basis for all subsequent steps in the overall design and implementation of actively controlled smart structures. The feasibility of the models for the controller design was demonstrated on example of a flexible cantilever beam with piezoelectric patches. Controllers designed based on the developed state space models perform significant vibration reduction in comparison with the uncontrolled cases.

ACKNOWLEDGEMENT

Parts of the research presented in the paper are supported by the German Research Foundation Project DFG-837. This support is highly appreciated.

REFERENCES

[1] Nestorović, T. *Controller Design for the Vibration Suppression of Smart Structures*. Fortschritt-Berichte VDI, Vol. 8, Nr. 1071, Düsseldorf: VDI Verlag (2005)

[2] Tiersten H.F. Hamilton's principle for linear piezoelectric media. *Proc. IEEE* (1967) 1523–1524.

[3] Nestorović-Trajkov, T. and Gabbert, U. Active control of a piezoelectric funnel-shaped structure based on subspace identification. *Structural Control and Health Monitoring* (2006) **13** (6): 1068–1079.

[4] Nestorović-Trajkov, T. Köppe, H. and Gabbert, U. Active Vibration Control Using Optimal LQ Tracking System with Additional Dynamics. *Int. J Control* (2005) **78** (15): 1182–1197.

[5] Nestorović, T. and Trajkov, M. Development of Reliable Models for the Controller Design of Smart Structures and Systems, *The International Conference "Mechanical Engineering in XXI Century" – Proceedings* University of Niš, Faculty of Mechanical Engineering, Niš, Serbia, (2010) pp. 263-266, ISBN: 978-86-6055-008-0

[6] Gabbert, U., Lefèvre, J., Laugwitz, F. and Nestorović, T. Modelling and analysis of piezoelectric smart structures for vibration and noise control, *International Journal of Applied Electromagnetics and Mechanics* (2009) **31** (1): 29-39

[7] Kailath, T. *Linear Systems*. Prentice Hall (1980)

[8] Durrani, N. and Nestorović, T. Active Vibration Control of Piezoelectric Structures. *DAGA 2011 (37th annual convention for acoustics)* 21-24 March 2011, Düsseldorf (submitted, Paper Nr. DAGA2011/14)