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Numerical analysis of a type III thermo-porous-elastic problem with microtemperatures

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Abstract In this work, we consider, from the numerical point of view, a poro-thermoelastic problem. The thermal law is the so-called of type III and the microtemperatures are also included into the model. The variational formulation of the problem is written as a linear system of coupled first-order variational equations. Then, fully discrete approximations are introduced by using the classical finite element method and the implicit Euler scheme. A discrete stability property and an a priori error estimates result are proved, from which the linear convergence of the algorithm is derived under suitable additional regularity conditions. Finally, some one- and two-dimensional numerical simulations are presented to show the accuracy of the approximation and the behavior of the solution.

Keywords Type III thermoelasticity with voids · Microtemperatures · Numerical approximation · Error estimates · numerical solutions

1 Introduction

The most useful model to describe the heat conduction is based on the Fourier law that proposes a linear relation between the heat flux vector and the gra-

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dient of temperature. If we combine this equation with the usual energy equation, we obtain the existence of thermal waves propagating with an unbounded speed. That is, a thermal perturbation at one point is instantaneously felt at any other point of the space for every distance. It is clear that this effect contradicts the *causality principle*. For this reason, a big deal trying to overcome this paradox has been developed in the last and current centuries. It seems that the first works in this aspect correspond to Cattaneo and Maxwell [7]. They proposed the introduction of a relaxation time in the Fourier law. Recently, in the 1990's decade Green and Naghdi proposed several alternative models [13,20]. In fact, they proposed these new theories in the context of the thermoelasticity and the main difference concerning the classical theory corresponds to the thermal effects. The most general is the so-called type III and it contains the Fourier model as a limit case. It is also worth recalling the type II which is also called *without energy dissipation*. It also corresponds to another limit case of the type III theory.

A big interest has also been developed to understand models with microstructure. In fact, Eringen [14] contributed in an important way in this sense in the last century. An interesting case for these models corresponds to those where microtemperatures are taken into account. That is, among the microstructure effects we can consider the microtemperatures. First contribution on this kind of materials came back to the one by Grot [21] and some people used them to study several problems [43–45]. We recall the contribution [24] as a new reborn of the interest for this kind of questions, because many works have studied this kind of problems recently (see [5,8,10,22,23,27,35,40,41] among others). Last twenty years there has been a big deal of people interested in the study of elastic materials with microtemperatures.

Cowin and Nunziato [11,12,39] proposed a mathematical theory to model elastic materials with voids. Since these contributions many people have been interested in the study of thermoelastic materials with voids and the quantity of contributions involving this model is huge [3,6,15–19,26,29–31,33,34,42]. It is worth noting that the model has become useful to understand the behavior of elastic materials with small distributed porous and we can find them in the study of biological materials as bones or in the study of soils, woods, ceramics or rocks. It is also worth noting the structural similarity (from the mathematical point of view) of the system of equations for the poro-elasticity with the equations of the Timoshenko beam (see, for instance, [1]).

In the present paper, we want to joint these three basic ideas. On one side, we consider the type III theory, on the second aspect we consider microtemperatures and, on the third side, we consider porous aspects. First contribution concerning the three aspects at the same time can be seen at [36]. There, the authors consider the system of equations that we can obtain from the studies [2,22,25]. Here, we continue the research started in [36], introducing a fully discrete approximation based on the finite element method and the implicit Euler scheme, proving a discrete stability property and a priori error estimates, and performing some one- and two-dimensional numerical simulations to demonstrate the accuracy and the behavior of the discrete solutions.

We think that it is relevant to point out that the behavior of the thermoelastic materials in the context of the type III theory has been reveal different from the classical theory based in the Fourier law. We can cite several contributions [32, 35, 38, 37, 36] where we have detected relevant differences in the behavior of the solutions corresponding to this kind of materials. The main reason is that, when we consider type III theory, new coupling terms appear which are not present when we consider the theory based on the classical Fourier law. At the same time, when we consider microtemperatures there are also more new coupling terms which are not present in the case of the Fourier theory with microtemperatures. Therefore, our system is more complex from the mathematical point of view and then new and strong difficulties could appear when we consider the new theory. Furthermore, what we will develop here cannot be a direct extension of the classical theory, but we could consider new aspects in our study.

2 Mathematical and variational formulations

First, we describe the problem (see [36] for further details). Let Ω be a bounded domain in $\mathbb{R}^d$ (for $d = 1, 2, 3$) with boundary smooth enough to allow the application of the divergence theorem. We will use the standard notation where "$, i$" means the partial derivative with respect to the variable x_i, a superposed dot represents time derivative and summation on repeated indices is assumed. Moreover, let $[0, T]$, $T > 0$, be the time interval of interest.

Let us denote $\boldsymbol{u} = (u_i)_{i=1}^d$, φ, θ and $\boldsymbol{M} = (M_i)_{i=1}^d$ the displacement, the volume fraction, the temperature and the microtemperatures, respectively.

Since we are interested in the thermoelastic theory of type III with voids and microtemperatures, the corresponding thermo-mechanical problem is the following (see [2, 24, 25, 36]):

Problem P. *Find the displacement* $\boldsymbol{u} : \overline{\Omega} \times [0, T] \to \mathbb{R}^d$, *the volume fraction* $\varphi : \overline{\Omega} \times [0, T] \to \mathbb{R}$, *the thermal displacement* $\tau : \overline{\Omega} \times [0, T] \to \mathbb{R}$ *and the microthermal displacement* $\boldsymbol{R} : \overline{\Omega} \times [0, T] \to \mathbb{R}^d$ *such that,*

$$\rho\ddot{u}_i = (A_{ijkl}u_{k,l} - a_{ij}\theta + \zeta_{ij}\varphi + B_{ijkl}R_{k,l})_{,j} \quad \text{in} \quad \Omega \times (0, T) \quad \text{for } i = 1, \dots, d, \tag{1}$$

$$J\ddot{\varphi} = (A_{ij}\varphi_{,j} - \alpha_{ij}\dot{R}_i + H_{ij}\tau_{,i})_{,j} - \zeta_{ij}u_{i,j} + \kappa\dot{\tau} - F_{ij}R_{i,j} - \xi\varphi \quad \text{in} \quad \Omega \times (0, T), \tag{2}$$

$$c\ddot{\tau} = -a_{ij}\dot{u}_{i,j} + (H_{ij}\varphi_{,i})_{,j} - (d_{ij}\dot{R}_i)_{,j} + (K_{ij}\tau_{,i} + K^*_{ij}\dot{\tau}_{,i})_{,j} - b_{ij}\dot{R}_{i,j} - \kappa\dot{\varphi} + (A^1_{ij}M_i)_{,j} \quad \text{in} \quad \Omega \times (0, T), \tag{3}$$

$$c_{ij}\ddot{R}_j = (B_{klij}u_{k,l} - b_{ij}\dot{\tau} + F_{ij}\varphi + C_{ijkl}R_{k,l})_{,j} - d_{ij}\dot{\tau}_{,j} + (C^*_{ijkl}\dot{R}_{k,l})_{,j} - \alpha_{ij}\dot{\varphi}_{,j} - A^2_{ij}\theta_{,j} - A^3_{ij}M_j \quad \text{in} \quad \Omega \times (0, T) \quad \text{for } i = 1, \dots, d, \tag{4}$$

$$u_i(\boldsymbol{x},0)=u_i^0(\boldsymbol{x}),\quad \dot{u}_i(\boldsymbol{x},0)=v_i^0(\boldsymbol{x}),\quad \varphi(\boldsymbol{x},0)=\varphi^0(\boldsymbol{x}) \text{ for } \boldsymbol{x}\in\Omega, \quad (5)$$

$$\dot{\varphi}(\boldsymbol{x},0)=e^0(\boldsymbol{x}),\quad \tau(\boldsymbol{x},0)=\tau^0(\boldsymbol{x}),\quad \dot{\tau}(\boldsymbol{x},0)=\theta^0(\boldsymbol{x}) \text{ for } \boldsymbol{x}\in\Omega, \quad (6)$$

$$R_i(\boldsymbol{x},0)=R_i^0(\boldsymbol{x}),\quad \dot{R}_i(\boldsymbol{x},0)=M_i^0(\boldsymbol{x}) \text{ for } \boldsymbol{x}\in\Omega, \quad (7)$$

$$u_i(\boldsymbol{x},t)=\varphi(\boldsymbol{x},t)=\tau(\boldsymbol{x},t)=R_i(\boldsymbol{x},t)=0 \text{ for } \boldsymbol{x}\in\partial\Omega,\ t\in[0,T]. \quad (8)$$

Here, τ is the thermal displacement introduced by Green and Naghdi and $\boldsymbol{R}=(R_i)_{i=1}^d$ are the microthermal displacements, defined respectively by

$$\tau(\boldsymbol{x},t)=\tau^0(\boldsymbol{x})+\int_0^t \theta(\boldsymbol{x},s)\,ds,\quad R_i(\boldsymbol{x},t)=R_i^0(\boldsymbol{x})+\int_0^t M_i(\boldsymbol{x},s)\,ds.$$

As usual, ρ denotes the mass density, J the product of the mass density by the equilibrated inertia and c the thermal capacity. A_{ijkl} is the elastic tensor, a_{ij}, ζ_{ij} and B_{ijkl} are, respectively, the coupling tensors between the displacement and the temperature, the displacement and the volume fraction, and the displacement and the microtemperatures. A_{ij}, $A_{ij}^{(1)}$, $A_{ij}^{(2)}$, $A_{ij}^{(3)}$, α_{ij}, H_{ij}, F_{ij}, d_{ij} and b_{ij} are other coupling tensors between the variables. K_{ij} is the tensor introduced by Green and Naghdi and it is usually called rate conductivity, K_{ij}^* is the thermal conductivity tensor, c_{ij} is a typical tensor of the theories with microtemperatures, and, finally, C_{ijkl} and C_{ijkl}^* are the specific type III tensors with microtemperatures.

The following symmetries are assumed (see [2,36]):

$$\left.\begin{array}{l} A_{ijkl}=A_{klij},\quad A_{ij}=A_{ji},\quad K_{ij}=K_{ji},\quad K_{ij}^*=K_{ji}^*,\quad C_{ijkl}=C_{klij},\\ C_{ijkl}^*=C_{klij}^*,\quad c_{ij}=c_{ji},\quad A_{ij}^3=A_{ji}^3,\quad H_{ij}=H_{ji},\quad B_{ijkl}=B_{klij}. \end{array}\right\} \quad (9)$$

From the second law of thermodynamics the following inequality must be satisfied (see [24]):

$$K_{i,j}^*\xi_i\xi_j+(A_{ij}^1+A_{ij}^2)\eta_i\xi_j+A_{ij}^3\eta_i\eta_j+C_{ijkl}^*\eta_{ij}\eta_{kl}\geq K_0(\xi_i\xi_i+\eta_i\eta_i+\eta_{ij}\eta_{ij}), \quad (10)$$

for a positive constant K_0 and for each pair of vectors ξ_i and η_i and for each tensor η_{ij}.

We will also impose some assumptions over the constitutive coefficients. For each vector ξ_i, each pair of tensors ξ_{ij} and η_{ij} and each real number l, the following inequalities are assumed:

$$\left.\begin{array}{l} A_{ijkl}\xi_{ij}\xi_{kl}+2B_{ijkl}\xi_{ij}\eta_{kl}+C_{ijkl}\eta_{ij}\eta_{kl}+2\zeta_{ij}\xi_{ij}l+\xi l^2\\ \qquad\geq C_0(\xi_{ij}\xi_{ij}+\eta_{ij}\eta_{ij}+l^2),\\ A_{ij}\xi_i\xi_j+2H_{ij}\xi_i\eta_j+K_{ij}\eta_i\eta_j\geq C_1(\xi_i\xi_i+\eta_i\eta_j),\\ c_{ij}\xi_i\xi_j\geq C_2\xi_i\xi_i,\quad \rho\geq\rho_0>0,\quad J\geq J_0>0,\quad c\geq c_0>0, \end{array}\right\} \quad (11)$$

for positive constants J_0, c_0, C_0, C_1, C_2 and ρ_0. The first two conditions proposed here can be interpreted with the help of the stability theory for thermoelastic materials. The physical meaning of the assumptions in the third line of (11) is clear.

First, we show that the energy of the system is dissipative.

Proposition 1 *Let us define the energy of the system $\mathcal{E}(t)$ as follows:*

$$\mathcal{E}(t) = \frac{1}{2}\Big\{\rho(v_i(t), v_i(t))_Y + J\|e(t)\|_Y^2 + c\|\theta(t)\|_Y^2 + (c_{ij}M_j(t), M_i(t))_Y \\ +(A_{ijkl}u_{i,j}(t), u_{k,l}(t))_Y + (B_{ijkl}u_{i,j}(t), R_{k,l}(t))_Y + \xi\|\varphi(t)\|_Y^2 \\ +(C_{ijkl}R_{i,j}(t), R_{k,l}(t))_Y + (\zeta_{ij}u_{i,j}(t), \varphi(t))_Y + (A_{ij}\varphi_{,i}(t), \varphi_{,j}(t))_Y \\ +(H_{ij}\varphi_{,i}(t), \tau_{,j}(t))_Y + (K_{ij}\tau_{,i}(t), \tau_{,j}(t))_Y\Big\}, \quad (12)$$

where we have used the notation $Y = L^2(\Omega)$ and $(\cdot,\cdot)_Y$ for the usual scalar product in this space. Then, this energy is dissipative.

Proof We note that, from the previous definition, after a direct calculation we find that

$$\mathcal{E}'(t) = -\int_\Omega \Big(K^*_{ij}\theta_{,i}\theta_{,j} + (A^1_{ij} + A^2_{ij})M_i\theta_{,j} + A^3_{ij}M_iM_j\Big)\,dv \\ -\int_\Omega C^*_{ijkl}M_{i,j}M_{k,l}\,dv,$$

and using assumption (10) we then conclude that the energy is always dissipative.

Now, we recall the following existence and uniqueness result [36].

Theorem 1 *Under assumptions (9)-(11), if the following regularity on the initial conditions hold:*

$$\boldsymbol{u}^0, \boldsymbol{v}^0,\, \boldsymbol{R}^0,\, \boldsymbol{M}^0 \in [H^2(\Omega)]^d, \quad \varphi^0,\, e^0,\, \tau^0,\, \theta^0 \in H^2(\Omega),$$

then there exists a unique solution to Problem P with the regularity:

$$\boldsymbol{u},\, \boldsymbol{R} \in C^1([0,T];V) \cap C^2([0,T];H), \quad \varphi,\, \tau \in C^1([0,T];E) \cap C^2([0,T];Y).$$

In order to obtain the exponential decay of the solutions to Problem P, we will assume that, for every tensor ξ_{ij} and every vector ζ_i,

$$B_{klij}\xi_{kl}\xi_{ij} \geq C\xi_{ij}\xi_{ij}, \quad H_{ij}\zeta_i\zeta_j \geq C^*\zeta_i\zeta_i, \quad (13)$$

for two positive constants C and C^*.

Even if the above assumptions are quite natural, we need to impose also two more technical conditions on some of the tensors. Let us suppose that there exist two constants, m_1 and m_2, such that

$$a_{ij} = m_1\zeta_{ij} \quad \text{and} \quad \alpha_{ij} = m_2\zeta_{ij}. \quad (14)$$

Notice that, for *isotropic and homogeneous materials*, assumptions (14) *are satisfied* whenever the corresponding constitutive parameter is different from zero, because, in this case, $\zeta_{ij} = \zeta\delta_{ij}$ for a constant $\zeta \neq 0$ (δ_{ij} denotes the Kronecker delta).

Therefore, we have the following (see [36]).

Theorem 2 *Under the assumptions of Theorem 1 and (13)-(14), the solution to Problem P is asymptotically stable; that is, there exist two positive constants M and α such that*

$$\|\mathcal{E}(t)\| \leq M\|\mathcal{E}(0)\|e^{-\alpha t},$$

where the energy of the system $\mathcal{E}$ was defined in (12).

Finally, in order to provide the numerical approximation of Problem P in the next section, we will obtain the variational formulation of this problem. Thus, let $H = [L^2(\Omega)]^d$ and $Q = [L^2(\Omega)]^{d\times d}$, and denote by $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_Q$ the respective scalar products in these spaces, with corresponding norms $\|\cdot\|_H$ and $\|\cdot\|_Q$. Moreover, let us define the variational spaces $E = H_0^1(\Omega)$ and $V = [H_0^1(\Omega)]^d$.

Then, applying Green's formula to equations (1)-(4) and using boundary conditions (8) we have the following weak problem.

Problem VP. *Find the velocity $\boldsymbol{v} : [0,T] \to V$, the volume fraction speed $e : [0,T] \to E$, the temperature $\theta : [0,T] \to E$ and the microtemperatures $\boldsymbol{M} : [0,T] \to V$ such that $\boldsymbol{v}(0) = \boldsymbol{v}^0$, $e(0) = e^0$, $\theta(0) = \theta^0$, $\boldsymbol{M}(0) = \boldsymbol{M}^0$ and, for a.e. $t \in (0,T)$,*

$$\rho(\dot{v}_i(t), w_i)_Y + (A_{ijkl}u_{k,l}(t), w_{i,j})_Y = (a_{ij}\theta(t), w_{i,j})_Y - (B_{ijkl}R_{k,l}(t), w_{i,j})_Y \\ -(\zeta_{ij}\varphi(t), w_{i,j})_Y \quad \forall \boldsymbol{w} = (w_i)_{i=1}^d \in V, \tag{15}$$

$$J(\dot{e}(t), r)_Y + (A_{ij}\varphi_{,j}(t), r_{,i})_Y + \xi(\varphi(t), r)_Y = (\alpha_{ij}M_i(t), r_{,j})_Y + \kappa(\theta(t), r)_Y \\ -(H_{ij}\tau_{,i}(t), r_{,j})_Y - (\zeta_{ij}u_{i,j}(t), r)_Y - (F_{ij}R_{i,j}(t), r)_Y \quad \forall r \in E, \tag{16}$$

$$c(\dot{\theta}(t), z)_Y + (K^*_{ij}\theta_{,i}(t), z_{,j})_Y + (K_{ij}\tau_{,i}(t), z_{,j})_Y = (d_{ij}M_i(t), z_{,j})_Y \\ -(A^1_{ij}M_i(t), z_{,j})_Y - (b_{ij}M_{i,j}(t), z)_Y - (a_{ij}v_{i,j}(t), z)_Y - \kappa(e(t), z)_Y \\ -(H_{ij}\varphi_{,i}(t), z_{,j})_Y \quad \forall z \in E, \tag{17}$$

$$(c_{ij}\dot{M}_j(t), \xi_i)_Y + (C_{ijkl}R_{k,l}(t), \xi_{i,j}) + (C^*_{ijkl}M_{k,l}(t), \xi_{i,j})_Y = (b_{ij}\theta(t), \xi_{i,j})_Y \\ -(B_{klij}u_{k,l}(t), \xi_{i,j})_Y - (F_{ij}\varphi(t), \xi_{i,j})_Y - (d_{ij}\theta_{,j}(t), \xi_i)_Y - (\alpha_{ij}e_{,j}(t), \xi_i)_Y \\ -(A^2_{ij}\theta_{,j}(t), \xi_i)_Y - (A^3_{ij}M_j(t), \xi_i)_Y \quad \forall \boldsymbol{\xi} = (\xi_i)_{i=1}^d \in V, \tag{18}$$

where we recall that the displacement, the volume fraction, the thermal displacement and the microthermal displacements are then recovered from relations:

$$\boldsymbol{u}(t) = \int_0^t \boldsymbol{v}(s)\,ds + \boldsymbol{u}^0, \quad \varphi(t) = \int_0^t e(s)\,ds + \varphi^0, \tag{19}$$

$$\tau(t) = \int_0^t \theta(s)\,ds + \tau^0, \quad \boldsymbol{R}(t) = \int_0^t \boldsymbol{M}(s)\,ds + \boldsymbol{R}^0. \tag{20}$$

3 Fully discrete approximations: an a priori error analysis

In this section, we now consider a fully discrete approximation of Problem VP. This is done in two steps. First, we assume that the domain $\overline{\Omega}$ is polyhedral and we denote by $\mathcal{T}^h$ a regular triangulation in the sense of [9]. Thus, we construct the finite dimensional spaces $V^h \subset V$ and $E^h \subset E$ given by

$$V^h = \{\boldsymbol{z}^h \in [C(\overline{\Omega})]^d \; ; \; \boldsymbol{z}^h_{|Tr} \in [P_1(Tr)]^d \quad \forall Tr \in \mathcal{T}^h, \quad \boldsymbol{z}^h = \boldsymbol{0} \quad \text{on} \quad \partial\Omega\}, \tag{21}$$

$$E^h = \{\eta^h \in C(\overline{\Omega}) \; ; \; \eta^h_{|Tr} \in P_1(Tr) \quad \forall Tr \in \mathcal{T}^h, \quad \eta^h = 0 \quad \text{on} \quad \partial\Omega\}, \tag{22}$$

where $P_1(Tr)$ represents the space of polynomials of degree less or equal to one in the element Tr, i.e. the finite element spaces V^h and E^h are composed of continuous and piecewise affine functions. Here, $h > 0$ denotes the spatial discretization parameter. Moreover, we assume that the discrete initial conditions, denoted by $\boldsymbol{u}^{0h}$, $\boldsymbol{v}^{0h}$, φ^{0h}, e^{0h}, τ^{0h}, θ^{0h}, $\boldsymbol{R}^{0h}$ and $\boldsymbol{M}^{0h}$, are given by

$$\begin{array}{llll} \boldsymbol{u}^{0h} = \mathcal{P}_1^h \boldsymbol{u}^0, & \boldsymbol{v}^{0h} = \mathcal{P}_1^h \boldsymbol{v}^0, & \varphi^{0h} = \mathcal{P}_2^h \varphi^0, & e^{0h} = \mathcal{P}_2^h e^0, \\ \tau^{0h} = \mathcal{P}_2^h \tau^0, & \theta^{0h} = \mathcal{P}_2^h \theta^0, & \boldsymbol{R}^{0h} = \mathcal{P}_1^h \boldsymbol{R}^0, & \boldsymbol{M}^{0h} = \mathcal{P}_1^h \boldsymbol{M}^0, \end{array} \tag{23}$$

where $\mathcal{P}_1^h$ and $\mathcal{P}_2^h$ are the classical finite element interpolation operators over V^h and E^h, respectively (see, e.g., [9]).

Secondly, we consider a partition of the time interval $[0, T]$, denoted by $0 = t_0 < t_1 < \cdots < t_N = T$. In this case, we use a uniform partition with step size $k = T/N$ and nodes $t_n = n\,k$ for $n = 0, 1, \ldots, N$. For a continuous function $z(t)$, we use the notation $z_n = z(t_n)$ and, for the sequence $\{z_n\}_{n=0}^N$, we denote by $\delta z_n = (z_n - z_{n-1})/k$ its corresponding divided differences.

Therefore, using the backward Euler scheme, the fully discrete approximations are considered as follows.

Problem VP^{hk}. *Find the discrete velocity $\boldsymbol{v}^{hk} = \{\boldsymbol{v}^{hk,n}\}_{n=0}^N \subset V^h$, the discrete volume fraction speed $e^{hk} = \{e^{hk,n}\}_{n=0}^N \subset E^h$, the temperature $\theta^{hk} = \{\theta^{hk,n}\}_{n=0}^N \subset E^h$ and the microtemperatures $\boldsymbol{M}^{hk} = \{\boldsymbol{M}^{hk,n}\}_{n=0}^N \subset V^h$ such that $\boldsymbol{v}^{hk,0} = \boldsymbol{v}^{0h}$, $e^{hk,0} = e^{0h}$, $\theta^{hk,0} = \theta^{0h}$, $\boldsymbol{M}^{hk,0} = \boldsymbol{M}^{0h}$, and, for $n =$*

$1,\ldots,N,$

$$\rho(\delta v_i^{hk,n},w_i^h)_Y+(A_{ijkl}u_{k,l}^{hk,n},w_{i,j}^h)_Y=(a_{ij}\theta^{hk,n},w_{i,j}^h)_Y-(B_{ijkl}R_{k,l}^{hk,n},w_{i,j}^h)_Y$$
$$-(\zeta_{ij}\varphi^{hk,n},w_{i,j}^h)_Y\quad\forall\boldsymbol{w}^h=(w_i^h)_{i=1}^d\in V^h,\tag{24}$$
$$J(\delta e^{hk,n},r^h)_Y+(A_{ij}\varphi_{,j}^{hk,n},r_{,i}^h)_Y+\xi(\varphi^{hk,n},r^h)_Y=(\alpha_{ij}M_i^{hk,n},r_{,j}^h)_Y$$
$$-(H_{ij}\tau_{,i}^{hk,n},r_{,j}^h)_Y-(\zeta_{ij}u_{i,j}^{hk,n},r^h)_Y+\kappa(\theta^{hk,n},r^h)_Y$$
$$-(F_{ij}R_{i,j}^{hk,n},r^h)_Y\quad\forall r^h\in E^h,\tag{25}$$
$$c(\delta\theta^{hk,n},z^h)_Y+(K_{ij}^*\theta_{,i}^{hk,n},z_{,j}^h)_Y+(K_{ij}\tau_{,i}^{hk,n},z_{,j}^h)_Y=(d_{ij}M_i^{hk,n},z_{,j}^h)_Y$$
$$-(A_{ij}^1M_i^{hk,n},z_{,j}^h)_Y-(b_{ij}M_{i,j}^{hk,n},z^h)_Y-(a_{ij}v_{i,j}^{hk,n},z^h)_Y-\kappa(e^{hk,n},z^h)_Y$$
$$-(H_{ij}\varphi_{,i}^{hk,n},z_{,j}^h)_Y\quad\forall z^h\in E^h,\tag{26}$$
$$(c_{ij}\delta M_j^{hk,n},\xi_i^h)_Y+(C_{ijkl}R_{k,l}^{hk,n},\xi_{i,j}^h)+(C_{ijkl}^*M_{k,l}^{hk,n},\xi_{i,j}^h)_Y+(A_{ij}^3M_j^{hk,n},\xi_i^h)_Y$$
$$=(b_{ij}\theta^{hk,n},\xi_{i,j}^h)_Y-(F_{ij}\varphi^{hk,n},\xi_{i,j}^h)_Y-(d_{ij}\theta_{,j}^{hk,n},\xi_i^h)_Y-(\alpha_{ij}e_{,j}^{hk,n},\xi_i^h)_Y$$
$$-(A_{ij}^2\theta_{,j}^{hk,n},\xi_i^h)_Y-(B_{klij}u_{k,l}^{hk,n},\xi_{i,j}^h)_Y\quad\forall\boldsymbol{\xi}^h=(\xi_i^h)_{i=1}^d\in V^h,\tag{27}$$

where the discrete displacement, the discrete volume fraction, the discrete thermal displacement and the discrete microthermal displacement are then recovered from relations:

$$\boldsymbol{u}^{hk,n}=k\sum_{j=1}^n\boldsymbol{v}^{hk,j}+\boldsymbol{u}^{0h},\quad\varphi^{hk,n}=k\sum_{j=1}^ne^{hk,j}+\varphi^{0h},\tag{28}$$
$$\tau^{hk,n}=k\sum_{j=1}^n\theta^{hk,j}+\tau^{0h},\quad\boldsymbol{R}^{hk,n}=k\sum_{j=1}^n\boldsymbol{M}^{hk,j}+\boldsymbol{R}^{0h}.\tag{29}$$

The existence of a unique solution to Problem VP^{hk} can be easily proved using Lax-Milgram lemma and taking into account assumptions (11)-(14).

The aim of this section is to provide the numerical analysis of Problem VP. First, we have the following discrete stability result.

Lemma 1 *Under the assumptions of Theorem 2, it follows that the sequences* $\{\boldsymbol{u}^{hk},\boldsymbol{v}^{hk},\varphi^{hk},e^{hk},\tau^{hk},\theta^{hk},\boldsymbol{R}^{hk},\boldsymbol{M}^{hk}\}$ *generated by Problem* VP^{hk} *satisfy the stability estimate:*

$$\|\boldsymbol{v}^{hk,n}\|_H^2+\|\nabla\boldsymbol{u}^{hk,n}\|_Q^2+\|e^{hk,n}\|_Y^2+\|\nabla\varphi^{hk,n}\|_H^2+\|\varphi^{hk,n}\|_Y^2+\|\theta^{hk,n}\|_Y^2$$
$$+\|\nabla\tau^{hk,n}\|_H^2+\|\boldsymbol{M}^{hk,n}\|_H^2+\|\nabla\boldsymbol{R}^{hk,n}\|_Q^2\le C,$$

where C *is a positive constant assumed to be independent of the discretization parameters* h *and* k*.*

Proof First, if we take as a test function $w_i^h=v_i^{hk,n}$ in discrete variational equation (24) we find that

$$\rho(\delta v_i^{hk,n},v_i^{hk,n})_Y+(A_{ijkl}u_{k,l}^{hk,n},v_{i,j}^{hk,n})_Y=(a_{ij}\theta^{hk,n},v_{i,j}^{hk,n})_Y$$
$$-(B_{ijkl}R_{k,l}^{hk,n},v_{i,j}^{hk,n})_Y-(\zeta_{ij}\varphi^{hk,n},v_{i,j}^{hk,n})_Y.$$

Thus, taking into account that

$$(\delta v_i^{hk,n}, v_i^{hk,n})_Y \geq \frac{1}{2k}\left\{\|\boldsymbol{v}^{hk,n}\|_H^2 - \|\boldsymbol{v}^{hk,n-1}\|_H^2\right\},$$

we find that

$$\begin{aligned}
&\frac{\rho}{2k}\left\{\|\boldsymbol{v}^{hk,n}\|_H^2 - \|\boldsymbol{v}^{hk,n-1}\|_H^2\right\} + \frac{1}{2k}\Big\{(A_{ijkl}u_{k,l}^{hk,n}, u_{i,j}^{hk,n})_Y - (A_{ijkl}u_{k,l}^{hk,n-1}, u_{i,j}^{hk,n-1})_Y \\
&\qquad +(A_{ijkl}(u_{k,l}^{hk,n} - u_{k,l}^{hk,n-1}), u_{i,j}^{hk,n} - u_{i,j}^{hk,n-1})_Y\Big\} \\
&\quad = (a_{ij}\theta^{hk,n}, v_{i,j}^{hk,n})_Y - (B_{ijkl}R_{k,l}^{hk,n}, v_{i,j}^{hk,n})_Y - (\zeta_{ij}\varphi^{hk,n}, v_{i,j}^{hk,n})_Y. \qquad (30)
\end{aligned}$$

Secondly, taking $r^h = e^{hk,n}$ as a test function in (25) we have

$$\begin{aligned}
&J(\delta e^{hk,n}, e^{hk,n})_Y + (A_{ij}\varphi_{,j}^{hk,n}, e_{,i}^{hk,n})_Y + \xi(\varphi^{hk,n}, e^{hk,n})_Y = (\alpha_{ij}M_i^{hk,n}, e_{,j}^{hk,n})_Y \\
&\quad -(H_{ij}\tau_{,i}^{hk,n}, e_{,j}^{hk,n})_Y - (\zeta_{ij}u_{i,j}^{hk,n}, e^{hk,n})_Y + \kappa(\theta^{hk,n}, e^{hk,n})_Y - (F_{ij}R_{i,j}^{hk,n}, e^{hk,n})_Y,
\end{aligned}$$

and using the estimates

$$J(\delta e^{hk,n}, e^{hk,n})_Y \geq \frac{J}{2k}\left\{\|e^{hk,n}\|_Y^2 - \|e^{hk,n-1}\|_Y^2\right\},$$

we obtain

$$\begin{aligned}
&\frac{J}{2k}\left\{\|e^{hk,n}\|_Y^2 - \|e^{hk,n-1}\|_Y^2\right\} + \frac{1}{2k}\Big\{(A_{ij}\varphi_{,j}^{hk,n}, \varphi_{,i}^{hk,n})_Y - (A_{ij}\varphi_{,j}^{hk,n-1}, \varphi_{,i}^{hk,n-1})_Y \\
&\qquad +(A_{ij}(\varphi_{,j}^{hk,n} - \varphi_{,j}^{hk,n-1}), \varphi_{,i}^{hk,n} - \varphi_{,i}^{hk,n-1})_Y\Big\} \\
&\qquad +\frac{\xi}{2k}\Big\{\|\varphi^{hk,n}\|_Y^2 - \|\varphi^{hk,n-1}\|_Y^2 + \|\varphi^{hk,n} - \varphi^{hk,n-1}\|_Y^2\Big\} \\
&\quad = (\alpha_{ij}M_i^{hk,n}, e_{,j}^{hk,n})_Y - (H_{ij}\tau_{,i}^{hk,n}, e_{,j}^{hk,n})_Y - (\zeta_{ij}u_{i,j}^{hk,n}, e^{hk,n})_Y \\
&\qquad +\kappa(\theta^{hk,n}, e^{hk,n})_Y - (F_{ij}R_{i,j}^{hk,n}, e^{hk,n})_Y. \qquad (31)
\end{aligned}$$

Third, choosing $z^h = \theta^{hk,n}$ as a test function in (26) it follows that

$$\begin{aligned}
&c(\delta\theta^{hk,n}, \theta^{hk,n})_Y + (K_{ij}^*\theta_{,i}^{hk,n}, \theta_{,j}^{hk,n})_Y + (K_{ij}\tau_{,i}^{hk,n}, \theta_{,j}^{hk,n})_Y = (d_{ij}M_i^{hk,n}, \theta_{,j}^{hk,n})_Y \\
&\quad -(A_{ij}^1M_i^{hk,n}, \theta_{,j}^{hk,n})_Y - (b_{ij}M_{i,j}^{hk,n}, \theta^{hk,n})_Y - (a_{ij}v_{i,j}^{hk,n}, \theta^{hk,n})_Y \\
&\quad -(H_{ij}\varphi_{,i}^{hk,n}, \theta_{,j}^{hk,n})_Y - \kappa(e^{hk,n}, \theta^{hk,n})_Y.
\end{aligned}$$

Keeping in mind that

$$c(\delta\theta^{hk,n}, \theta^{hk,n})_Y \geq \frac{c}{2k}\left\{\|\theta^{hk,n}\|_Y^2 - \|\theta^{hk,n-1}\|_Y^2\right\},$$

we find that

$$\begin{aligned}
&\frac{c}{2k}\left\{\|\theta^{hk,n}\|_Y^2 - \|\theta^{hk,n-1}\|_Y^2\right\} + (K_{ij}^*\theta_{,i}^{hk,n}, \theta_{,j}^{hk,n})_Y + \frac{1}{2k}\Big\{(K_{ij}\tau_{,i}^{hk,n}, \tau_{,j}^{hk,n})_Y \\
&\qquad -(K_{ij}\tau_{,i}^{hk,n-1}, \tau_{,j}^{hk,n-1})_Y + (K_{ij}(\tau_{,i}^{hk,n} - \tau_{,i}^{hk,n-1}), \tau_{,j}^{hk,n} - \tau_{,j}^{hk,n-1})_Y\Big\} \\
&\quad = (d_{ij}M_i^{hk,n}, \theta_{,j}^{hk,n})_Y - (A_{ij}^1M_i^{hk,n}, \theta_{,j}^{hk,n})_Y - (b_{ij}M_{i,j}^{hk,n}, \theta^{hk,n})_Y - (a_{ij}v_{i,j}^{hk,n}, \theta^{hk,n})_Y \\
&\qquad -(H_{ij}\varphi_{,i}^{hk,n}, \theta_{,j}^{hk,n})_Y - \kappa(e^{hk,n}, \theta^{hk,n})_Y. \qquad (32)
\end{aligned}$$

Finally, taking $\xi_i^h = M_i^{hk,n}$ as a test function in (27) we obtain

$$
\begin{aligned}
&(c_{ij}\delta M_j^{hk,n}, M_i^{hk,n})_Y + (C_{ijkl}R_{k,l}^{hk,n}, M_{i,j}^{hk,n})_Y + (C^*_{ijkl}M_{k,l}^{hk,n}, M_{i,j}^{hk,n})_Y \\
&\quad +(A^3_{ij}M_j^{hk,n}, M_i^{hk,n})_Y - (B_{klij}u_{k,l}^{hk,n}, M_{i,j}^{hk,n})_Y - (b_{ij}\theta^{hk,n}, M_{i,j}^{hk,n})_Y + (F_{ij}\varphi^{hk,n}, M_{i,j}^{hk,n})_Y \\
&\qquad +(d_{ij}\theta_{,j}^{hk,n}, M_i^{hk,n})_Y + (\alpha_{ij}e_{,j}^{hk,n}, M_i^{hk,n})_Y + (A^2_{ij}\theta_{,j}^{hk,n}, M_i^{hk,n})_Y = 0,
\end{aligned}
$$

and, since using (11) it follows that

$$
(c_{ij}\delta M_j^{hk,n}, M_i^{hk,n})_Y \geq \frac{C_2}{2k}\left\{\|\boldsymbol{M}^{hk,n}\|_H^2 - \|\boldsymbol{M}^{hk,n-1}\|_H^2\right\},
$$

we have

$$
\begin{aligned}
&\frac{C_2}{2k}\left\{\|\boldsymbol{M}^{hk,n}\|_H^2 - \|\boldsymbol{M}^{hk,n-1}\|_H^2\right\} + (C^*_{ijkl}M_{k,l}^{hk,n}, M_{i,j}^{hk,n})_Y + \frac{1}{2k}\Big\{(C_{ijkl}R_{k,l}^{hk,n}, R_{i,j}^{hk,n})_Y \\
&\quad -(C_{ijkl}R_{k,l}^{hk,n-1}, R_{i,j}^{hk,n-1})_Y + (C_{ijkl}(R_{k,l}^{hk,n} - R_{k,l}^{hk,n-1}), R_{i,j}^{hk,n} - R_{i,j}^{hk,n-1})_Y\Big\} \\
&= (B_{klij}u_{k,l}^{hk,n}, M_{i,j}^{hk,n})_Y + (b_{ij}\theta^{hk,n}, M_{i,j}^{hk,n})_Y - (F_{ij}\varphi^{hk,n}, M_{i,j}^{hk,n})_Y - (d_{ij}\theta_{,j}^{hk,n}, M_i^{hk,n})_Y \\
&\quad -(\alpha_{ij}e_{,j}^{hk,n}, M_i^{hk,n})_Y - (A^2_{ij}\theta_{,j}^{hk,n}, M_i^{hk,n})_Y. \qquad (33)
\end{aligned}
$$

Combining now estimates (30)-(33), after easy algebraic manipulations we find that

$$
\begin{aligned}
&\frac{\rho}{2k}\left\{\|\boldsymbol{v}^{hk,n}\|_H^2 - \|\boldsymbol{v}^{hk,n-1}\|_H^2\right\} + \frac{1}{2k}\Big\{(A_{ijkl}u_{k,l}^{hk,n}, u_{i,j}^{hk,n})_Y - (A_{ijkl}u_{k,l}^{hk,n-1}, u_{i,j}^{hk,n-1})_Y \\
&\quad +(A_{ijkl}(u_{k,l}^{hk,n} - u_{k,l}^{hk,n-1}), u_{i,j}^{hk,n} - u_{i,j}^{hk,n-1})_Y\Big\} \\
&\quad +\frac{J}{2k}\left\{\|e^{hk,n}\|_Y^2 - \|e^{hk,n-1}\|_Y^2\right\} + \frac{1}{2k}\Big\{(A_{ij}\varphi_{,j}^{hk,n}, \varphi_{,i}^{hk,n})_Y - (A_{ij}\varphi_{,j}^{hk,n-1}, \varphi_{,i}^{hk,n-1})_Y \\
&\quad +(A_{ij}(\varphi_{,j}^{hk,n} - \varphi_{,j}^{hk,n-1}), \varphi_{,i}^{hk,n} - \varphi_{,i}^{hk,n-1})_Y\Big\} \\
&\quad +\frac{\xi}{2k}\Big\{\|\varphi^{hk,n}\|_Y^2 - \|\varphi^{hk,n-1}\|_Y^2 + \|\varphi^{hk,n} - \varphi^{hk,n-1}\|_Y^2\Big\} \\
&\quad +\frac{c}{2k}\left\{\|\theta^{hk,n}\|_Y^2 - \|\theta^{hk,n-1}\|_Y^2\right\} + (K^*_{ij}\theta_{,i}^{hk,n}, \theta_{,j}^{hk,n})_Y + \frac{1}{2k}\Big\{(K_{ij}\tau_{,i}^{hk,n}, \tau_{,j}^{hk,n})_Y \\
&\quad -(K_{ij}\tau_{,i}^{hk,n-1}, \tau_{,j}^{hk,n-1})_Y + (K_{ij}(\tau_{,i}^{hk,n} - \tau_{,i}^{hk,n-1}), \tau_{,j}^{hk,n} - \tau_{,j}^{hk,n-1})_Y\Big\} \\
&\quad +\frac{C_2}{2k}\left\{\|\boldsymbol{M}^{hk,n}\|_H^2 - \|\boldsymbol{M}^{hk,n-1}\|_H^2\right\} + (C^*_{ijkl}M_{k,l}^{hk,n}, M_{i,j}^{hk,n})_Y \\
&\quad +(\zeta_{ij}u_{i,j}^{hk,n}, e^{hk,n})_Y + (A^3_{ij}M_j^{hk,n}, M_i^{hk,n})_Y + \frac{1}{2k}\Big\{(C_{ijkl}R_{k,l}^{hk,n}, R_{i,j}^{hk,n})_Y \\
&\quad -(C_{ijkl}R_{k,l}^{hk,n-1}, R_{i,j}^{hk,n-1})_Y + (C_{ijkl}(R_{k,l}^{hk,n} - R_{k,l}^{hk,n-1}), R_{i,j}^{hk,n} - R_{i,j}^{hk,n-1})_Y\Big\} \\
&\quad +(B_{ijkl}R_{k,l}^{hk,n}, v_{i,j}^{hk,n})_Y + (B_{klij}u_{k,l}^{hk,n}, M_{i,j}^{hk,n})_Y + (\zeta_{ij}\varphi^{hk,n}, v_{i,j}^{hk,n})_Y \\
&\quad +(H_{ij}\tau_{,i}^{hk,n}, e_{,j}^{hk,n})_Y + (H_{ij}\varphi_{,i}^{hk,n}, \theta_{,j}^{hk,n})_Y + ((A^1_{ij} + A^2_{ij})\theta_{,j}^{hk,n}, M_i^{hk,n})_Y \\
&\leq C\Big(\|e^{hk,n}\|_Y^2 + \|\varphi^{hk,n}\|_Y^2 + \|\nabla\boldsymbol{R}^{hk,n}\|_Q^2\Big) + \epsilon\|\nabla\boldsymbol{M}^{hk,n}\|_Q^2,
\end{aligned}
$$

where $\epsilon > 0$ is a positive constant assumed small enough, and C is a generic constant, whose value may change from line to line, and it is independent of the discretization parameters h and k.

Keeping in mind assumptions (10), we find that

$$
\begin{aligned}
&(C^*_{ijkl}M^{hk,n}_{k,l},M^{hk,n}_{i,j})_Y+((A^1_{ij}+A^2_{ij})\theta^{hk,n}_{,j},M^{hk,n}_i)_Y+(K^*_{ij}\theta^{hk,n}_{,i},\theta^{hk,n}_{,j})_Y\\
&\quad+(A^3_{ij}M^{hk,n}_j,M^{hk,n}_i)_Y\geq C(\|\nabla\theta^{hk,n}\|^2_H+\|\nabla\boldsymbol{M}^{hk,n}\|^2_Q+\|\boldsymbol{M}^{hk,n}\|^2_H).
\end{aligned}
$$

Observing that

$$
\begin{aligned}
&(B_{ijkl}R^{hk,n}_{k,l},v^{hk,n}_{i,j})_Y+(B_{klij}u^{hk,n}_{k,l},M^{hk,n}_{i,j})_Y=\frac{1}{k}\Big\{(B_{ijkl}R^{hk,n}_{k,l},u^{hk,n}_{i,j})_Y\\
&\qquad-(B_{ijkl}R^{hk,n-1}_{k,l},u^{hk,n-1}_{i,j})_Y+(B_{ijkl}(R^{hk,n}_{k,l}-R^{hk,n-1}_{k,l}),u^{hk,n}_{i,j}-u^{hk,n-1}_{i,j})_Y\Big\},\\
&(\zeta_{ij}\varphi^{hk,n},v^{hk,n}_{i,j})_Y+(\zeta_{ij}u^{hk,n}_{i,j},e^{hk,n})_Y=\frac{1}{k}\Big\{(\zeta_{ij}\varphi^{hk,n},u^{hk,n}_{i,j})_Y-(\zeta_{ij}\varphi^{hk,n-1},u^{hk,n-1}_{i,j})_Y\\
&\qquad+(\zeta_{ij}(\varphi^{hk,n}-\varphi^{hk,n-1}),u^{hk,n}_{i,j}-u^{hk,n-1}_{i,j})_Y\Big\},\\
&(H_{ij}\tau^{hk,n}_{,i},e^{hk,n}_{,j})_Y+(H_{ij}\varphi^{hk,n}_{,i},\theta^{hk,n}_{,j})_Y=\frac{1}{k}\Big\{(H_{ij}\tau^{hk,n}_{,i},\varphi^{hk,n}_{,j})_Y-(H_{ij}\tau^{hk,n-1}_{,i},\varphi^{hk,n-1}_{,j})_Y\\
&\qquad+(H_{ij}(\tau^{hk,n}_{,i}-\tau^{hk,n-1}_{,i}),\varphi^{hk,n}_{,j}-\varphi^{hk,n-1}_{,j})_Y\Big\},
\end{aligned}
$$

using assumptions (11) it follows that

$$
\begin{aligned}
&(A_{ijkl}(u^{hk,n}_{k,l}-u^{hk,n-1}_{k,l}),u^{hk,n}_{i,j}-u^{hk,n-1}_{i,j})_Y+2(B_{ijkl}(R^{hk,n}_{k,l}-R^{hk,n-1}_{k,l}),u^{hk,n}_{i,j}-u^{hk,n-1}_{i,j})_Y\\
&\quad+(C_{ijkl}(R^{hk,n}_{k,l}-R^{hk,n-1}_{k,l}),R^{hk,n}_{i,j}-R^{hk,n-1}_{i,j})_Y+2(\zeta_{ij}(\varphi^{hk,n}-\varphi^{hk,n-1}),u^{hk,n}_{i,j}-u^{hk,n-1}_{i,j})_Y\\
&\quad+\xi\|\varphi^{hk,n}-\varphi^{hk,n-1}\|^2_Y\geq 0,\\
&(A_{ij}(\varphi^{hk,n}_{,j}-\varphi^{hk,n-1}_{,j}),\varphi^{hk,n}_{,i}-\varphi^{hk,n-1}_{,i})_Y+2(H_{ij}(\tau^{hk,n}_{,i}-\tau^{hk,n-1}_{,i}),\varphi^{hk,n}_{,j}-\varphi^{hk,n-1}_{,j})_Y\\
&\quad+(K_{ij}(\tau^{hk,n}_{,i}-\tau^{hk,n-1}_{,i}),\tau^{hk,n}_{,j}-\tau^{hk,n-1}_{,j})_Y\geq 0.
\end{aligned}
$$

Therefore, multiplying the above estimates by k and summing up to n we have

$$
\begin{aligned}
&\rho\|\boldsymbol{v}^{hk,n}\|^2_H+(A_{ijkl}u^{hk,n}_{k,l},u^{hk,n}_{i,j})_Y+J\|e^{hk,n}\|^2_Y+(A_{ij}\varphi^{hk,n}_{,j},\varphi^{hk,n}_{,i})_Y+\xi\|\varphi^{hk,n}\|^2_Y+c\|\theta^{hk,n}\|^2_Y\\
&\quad+(K_{ij}\tau^{hk,n}_{,i},\tau^{hk,n}_{,j})_Y+C_2\|\boldsymbol{M}^{hk,n}\|^2_H+(C_{ijkl}R^{hk,n}_{k,l},R^{hk,n}_{i,j})_Y+2(B_{ijkl}R^{hk,n}_{k,l},u^{hk,n}_{i,j})_Y\\
&\quad+2(\zeta_{ij}\varphi^{hk,n},u^{hk,n}_{i,j})_Y+2(H_{ij}\tau^{hk,n}_{,i},\varphi^{hk,n}_{,j})_Y\\
&\\
&\leq Ck\sum_{j=1}^n\Big(\|e^{hk,j}\|^2_Y+\|\varphi^{hk,j}\|^2_Y+\|\nabla\boldsymbol{R}^{hk,j}\|^2_Q\Big)+C\Big(\|\boldsymbol{v}^{0h}\|^2_H+\|\nabla\boldsymbol{u}^{0h}\|^2_Q+\|e^{0h}\|^2_Y\\
&\quad+\|\nabla\varphi^{0h}\|^2_H+\|\varphi^{0h}\|^2_Y+\|\theta^{0h}\|^2_Y+\|\nabla\tau^{0h}\|^2_H+\|\boldsymbol{M}^{0h}\|^2_H+\|\nabla\boldsymbol{R}^{0h}\|^2_Q\Big).
\end{aligned}
$$

Finally, using again assumptions (11) we obtain

$$
\begin{aligned}
&(A_{ijkl}u^{hk,n}_{k,l},u^{hk,n}_{i,j})_Y+2(B_{ijkl}R^{hk,n}_{k,l},u^{hk,n}_{i,j})_Y+(C_{ijkl}R^{hk,n}_{k,l},R^{hk,n}_{i,j})_Y+2(\zeta_{ij}\varphi^{hk,n},u^{hk,n}_{i,j})_Y\\
&\quad+\xi\|\varphi^{hk,n}\|^2_Y\geq C\Big(\|\nabla\boldsymbol{u}^{hk,n}\|^2_Q+\|\nabla\boldsymbol{R}^{hk,n}\|^2_Q+\|\varphi^{hk,n}\|^2_Y\Big),\\
&(A_{ij}\varphi^{hk,n}_{,j},\varphi^{hk,n}_{,i})_Y+2(H_{ij}\tau^{hk,n}_{,i},\varphi^{hk,n}_{,j})_Y+(K_{ij}\tau^{hk,n}_{,i},\tau^{hk,n}_{,j})_Y\geq C\Big(\|\nabla\varphi^{hk,n}\|^2_H+\|\nabla\tau^{hk,n}\|^2_H\Big),
\end{aligned}
$$

and so,

$$
\begin{aligned}
&\|\boldsymbol{v}^{hk,n}\|_H^2+\|\nabla\boldsymbol{u}^{hk,n}\|_Q^2+\|e^{hk,n}\|_Y^2+\|\nabla\varphi^{hk,n}\|_H^2+\|\varphi^{hk,n}\|_Y^2+\|\theta^{hk,n}\|_Y^2+\|\nabla\tau^{hk,n}\|_H^2\\
&\quad+\|\boldsymbol{M}^{hk,n}\|_H^2+\|\nabla\boldsymbol{R}^{hk,n}\|_Q^2\\
&\le Ck\sum_{j=1}^{n}\Big(\|e^{hk,j}\|_Y^2+\|\varphi^{hk,j}\|_Y^2+\|\nabla\boldsymbol{R}^{hk,j}\|_Q^2\Big)+C\Big(\|\boldsymbol{v}^{0h}\|_H^2+\|\nabla\boldsymbol{u}^{0h}\|_Q^2+\|e^{0h}\|_Y^2\\
&\quad+\|\nabla\varphi^{0h}\|_H^2+\|\varphi^{0h}\|_Y^2+\|\theta^{0h}\|_Y^2+\|\nabla\tau^{0h}\|_H^2+\|\boldsymbol{M}^{0h}\|_H^2+\|\nabla\boldsymbol{R}^{0h}\|_Q^2\Big).
\end{aligned}
$$

Therefore, applying a discrete version of Gronwall's inequality (see, e.g., [4]) we deduce the desired stability property.

Now, our aim will be to obtain a priori error estimates on the numerical errors from the approximations given in Problem VP^{hk}. We have the following.

Theorem 3 *Let the assumptions of Theorem 2 still hold. If we also assume that*

$$
a_{ij}(\boldsymbol{x})=a_{ij},\quad \alpha_{ij}(\boldsymbol{x})=\alpha_{ij}\quad \textit{for all}\quad \boldsymbol{x}\in\Omega, \tag{34}
$$

and if we denote by $(\boldsymbol{v},e,\theta,\boldsymbol{M})$ and $(\boldsymbol{v}^{hk},e^{hk},\theta^{hk},\boldsymbol{M}^{hk})$ the respective solutions to problems VP and VP^{hk}, then, we have the following a priori error estimates for all $\boldsymbol{w}^h=\{\boldsymbol{w}^{h,n}\}_{n=0}^N$, $\boldsymbol{\xi}^h=\{\boldsymbol{\xi}^{h,n}\}_{n=0}^N\subset V^h$ and $r^h=\{r^{h,n}\}_{n=0}^N$, $z^h=\{z^{h,n}\}_{n=0}^N\subset E^h$,

$$
\begin{aligned}
&\max_{0\le n\le N}\Big\{\|\boldsymbol{v}^n-\boldsymbol{v}^{hk,n}\|_H^2+\|\nabla(\boldsymbol{u}^n-\boldsymbol{u}^{hk,n})\|_Q^2+\|e^n-e^{hk,n}\|_Y^2+\|\nabla(\varphi^n-\varphi^{hk,n})\|_H^2\\
&\quad+\|\theta^n-\theta^{hk,n}\|_Y^2+\|\varphi^n-\varphi^{hk,n}\|_Y^2+\|\nabla(\tau^n-\tau^{hk,n})\|_H^2+\|\boldsymbol{M}^n-\boldsymbol{M}^{hk,n}\|_H^2\\
&\quad+\|\nabla(\boldsymbol{R}^n-\boldsymbol{R}^{hk,n})\|_Q^2\Big\}\\
&\le Ck\sum_{j=1}^{n}\Big(\|\dot{\boldsymbol{v}}^j-\delta\boldsymbol{v}^j\|_H^2+\|\nabla(\dot{\boldsymbol{u}}^j-\delta\boldsymbol{u}^j)\|_Q^2+\|\nabla(\boldsymbol{v}^j-\boldsymbol{w}^{h,j})\|_Q^2+\|\boldsymbol{v}^j-\boldsymbol{w}^{h,j}\|_H^2\\
&\quad+\|\dot{e}^j-\delta e^j\|_Y^2+\|\nabla(\dot{\varphi}^j-\delta\varphi^j)\|_H^2+\|\nabla(e^j-r^{h,j})\|_H^2+\|e^j-r^{h,j}\|_Y^2\\
&\quad+\|\dot{\theta}^j-\delta\theta^j\|_Y^2+\|\nabla(\dot{\tau}^j-\delta\tau^j)\|_H^2+\|\nabla(\theta^j-z^{h,j})\|_H^2+\|\theta^j-z^{h,j}\|_Y^2+\|\dot{\boldsymbol{M}}^j-\delta\boldsymbol{M}^j\|_H^2\\
&\quad+\|\nabla(\dot{\boldsymbol{R}}^j-\delta\boldsymbol{R}^j)\|_Q^2+\|\nabla(\boldsymbol{M}^j-\boldsymbol{\xi}^{h,j})\|_Q^2+\|\boldsymbol{M}^j-\boldsymbol{\xi}^{h,j}\|_H^2\Big)\\
&\quad+\frac{C}{k}\sum_{j=1}^{N-1}\Big\{\|\boldsymbol{v}^j-\boldsymbol{w}^{h,j}-(\boldsymbol{v}^{j+1}-\boldsymbol{w}^{h,j+1})\|_H^2+\|e^j-r^{h,j}-(e^{j+1}-r^{h,j+1})\|_Y^2\\
&\qquad+\|\boldsymbol{M}^j-\boldsymbol{\xi}^{h,j}-(\boldsymbol{M}^{j+1}-\boldsymbol{\xi}^{h,j+1})\|_H^2+\|\theta^j-z^{h,j}-(\theta^{j+1}-z^{h,j+1})\|_Y^2\Big\}\\
&\quad+C\max_{0\le n\le N}\Big\{\|\boldsymbol{v}^n-\boldsymbol{w}^{h,n}\|_H^2+\|e^n-r^{h,n}\|_Y^2+\|\theta^n-z^{h,n}\|_Y^2+\|\boldsymbol{M}^n-\boldsymbol{\xi}^{h,n}\|_H^2\Big\}\\
&\quad+C\Big(\|\boldsymbol{v}^0-\boldsymbol{v}^{0h}\|_H^2+\|\nabla(\boldsymbol{u}^0-\boldsymbol{u}^{0h})\|_Q^2+\|e^0-e^{0h}\|_Y^2+\|\nabla(\varphi^0-\varphi^{0h})\|_H^2+\|\xi^0-\xi^{0h}\|_Y^2\\
&\quad+\|\theta^0-\theta^{0h}\|_Y^2+\|\nabla(\tau^0-\tau^{0h})\|_H^2+\|\boldsymbol{M}^0-\boldsymbol{M}^{0h}\|_H^2+\|\nabla(\boldsymbol{R}^0-\boldsymbol{R}^{0h})\|_Q^2\Big),
\end{aligned} \tag{35}
$$

where $C > 0$ is a positive constant assumed to be independent of the discretization parameters but depending on the continuous solution.

Remark 1 We note that assumption (34) implies that these terms are homogeneous. Such conditions are found, for instance, in the case that the material is homogeneous and isotropic, that is

$$a_{ij}(\boldsymbol{x}) = a\delta_{ij}, \quad \alpha_{ij}(\boldsymbol{x}) = \alpha\delta_{ij} \quad \text{for all} \quad \boldsymbol{x} \in \Omega,$$

where a and α are constants and δ_{ij} represents the Kronecker symbol.

Proof First, we will derive the estimates for the velocity. Thus, subtracting variational equation (15) at time t_n for a test function $\boldsymbol{w} = \boldsymbol{w}^h \in V^h$ and discrete variational equation (24) it follows that, for all $\boldsymbol{w}^h = (w_i^h)_{i=1}^d \in V^h$,

$$\begin{aligned}&\rho(\dot{v}_i^n - \delta v_i^{hk,n}, w_i^h)_Y + (A_{ijkl}(u_{k,l}^n - u_{k,l}^{hk,n}), w_{i,j}^h)_Y - (a_{ij}(\theta^n - \theta^{hk,n}), w_{i,j}^h)_Y \\ &\quad +(B_{ijkl}(R_{k,l}^n - R_{k,l}^{hk,n}), w_{i,j}^h)_Y + (\zeta_{ij}(\varphi^n - \varphi^{hk,n}), w_{i,j}^h)_Y = 0,\end{aligned}$$

and so, for all $\boldsymbol{w}^h = (w_i^h)_{i=1}^d \in V^h$,

$$\begin{aligned}&\rho(\dot{v}_i^n - \delta v_i^{hk,n}, v_i^n - v_i^{hk,n})_Y + (A_{ijkl}(u_{k,l}^n - u_{k,l}^{hk,n}), v_{i,j}^n - v_{i,j}^{hk,n})_Y - (a_{ij}(\theta^n - \theta^{hk,n}), v_{i,j}^n - v_{i,j}^{hk,n})_Y \\ &\quad +(B_{ijkl}(R_{k,l}^n - R_{k,l}^{hk,n}), v_{i,j}^n - v_{i,j}^{hk,n})_Y + (\zeta_{ij}(\varphi^n - \varphi^{hk,n}), v_{i,j}^n - v_{i,j}^{hk,n})_Y \\ &= \rho(\dot{v}_i^n - \delta v_i^{hk,n}, v_i^n - w_i^h)_Y + (A_{ijkl}(u_{k,l}^n - u_{k,l}^{hk,n}), v_{i,j}^n - w_{i,j}^h)_Y - (a_{ij}(\theta^n - \theta^{hk,n}), v_{i,j}^n - w_{i,j}^h)_Y \\ &\quad +(B_{ijkl}(R_{k,l}^n - R_{k,l}^{hk,n}), v_{i,j}^n - w_{i,j}^h)_Y + (\zeta_{ij}(\varphi^n - \varphi^{hk,n}), v_{i,j}^n - w_{i,j}^h)_Y.\end{aligned}$$

Keeping in mind that

$$\begin{aligned}&(\dot{v}_i^n - \delta v_i^{hk,n}, v_i^n - v_i^{hk,n})_Y \geq (\dot{v}_i^n - \delta v_i^n, v_i^n - v_i^{hk,n})_Y + \frac{1}{2k}\left\{\|\boldsymbol{v}^n - \boldsymbol{v}^{hk,n}\|_H^2 - \|\boldsymbol{v}^{n-1} - \boldsymbol{v}^{hk,n-1}\|_H^2\right\}, \\ &(A_{ijkl}(u_{k,l}^n - u_{k,l}^{hk,n}), v_{i,j}^n - v_{i,j}^{hk,n})_Y = (A_{ijkl}(u_{k,l}^n - u_{k,l}^{hk,n}), \dot{u}_{i,j}^n - \delta u_{i,j}^n)_Y \\ &\qquad +(A_{ijkl}(u_{k,l}^n - u_{k,l}^{hk,n}), \delta u_{i,j}^n - \delta u_{i,j}^{hk,n})_Y, \\ &(A_{ijkl}(u_{k,l}^n - u_{k,l}^{hk,n}), \delta u_{i,j}^n - \delta u_{i,j}^{hk,n})_Y = \frac{1}{2k}\Big\{(A_{ijkl}(u_{k,l}^n - u_{k,l}^{hk,n}), u_{i,j}^n - u_{i,j}^{hk,n})_Y \\ &\quad -(A_{ijkl}(u_{k,l}^{n-1} - u_{k,l}^{hk,n-1}), u_{i,j}^{n-1} - u_{i,j}^{hk,n-1})_Y \\ &\quad +(A_{ijkl}(u_{k,l}^n - u_{k,l}^{hk,n} - (u_{k,l}^{n-1} - u_{k,l}^{hk,n-1})), u_{i,j}^n - u_{i,j}^{hk,n} - (u_{i,j}^{n-1} - u_{i,j}^{hk,n-1}))_Y\Big\},\end{aligned}$$

where we used the notations $\delta \boldsymbol{v}^n = (\boldsymbol{v}^n - \boldsymbol{v}^{n-1})/k$ and $\delta \boldsymbol{u}^n = (\boldsymbol{u}^n - \boldsymbol{u}^{n-1})/k$, we obtain the following estimates for the velocity:

$$\begin{aligned}
&\frac{\rho}{2k}\left\{\|\boldsymbol{v}^n - \boldsymbol{v}^{hk,n}\|_H^2 - \|\boldsymbol{v}^{n-1} - \boldsymbol{v}^{hk,n-1}\|_H^2\right\} + \frac{1}{2k}\Big\{(A_{ijkl}(u^n_{k,l} - u^{hk,n}_{k,l}), u^n_{i,j} - u^{hk,n}_{i,j})_Y \\
&\quad -(A_{ijkl}(u^{n-1}_{k,l} - u^{hk,n-1}_{k,l}), u^{n-1}_{i,j} - u^{hk,n-1}_{i,j})_Y \\
&\quad +(A_{ijkl}(u^n_{k,l} - u^{hk,n}_{k,l} - (u^{n-1}_{k,l} - u^{hk,n-1}_{k,l})), u^n_{i,j} - u^{hk,n}_{i,j} - (u^{n-1}_{i,j} - u^{hk,n-1}_{i,j}))_Y\Big\} \\
&\quad +(B_{ijkl}(R^n_{k,l} - R^{hk,n}_{k,l}), v^n_{i,j} - v^{hk,n}_{i,j})_Y + (\zeta_{ij}(\varphi^n - \varphi^{hk,n}), v^n_{i,j} - v^{hk,n}_{i,j})_Y \\
&\quad -(a_{ij}(\theta^n - \theta^{hk,n}), v^n_{i,j} - v^{hk,n}_{i,j})_Y \\
&\leq C\Big(\|\dot{\boldsymbol{v}}^n - \delta\boldsymbol{v}^n\|_H^2 + \|\nabla(\dot{\boldsymbol{u}}^n - \delta\boldsymbol{u}^n)\|_Q^2 + \|\nabla(\boldsymbol{v}^h - \boldsymbol{w}^h)\|_Q^2 + \|\nabla(\boldsymbol{u}^n - \boldsymbol{u}^{hk,n})\|_Q^2 \\
&\quad +\|\boldsymbol{v}^n - \boldsymbol{w}^h\|_H^2 + \|\theta^n - \theta^{hk,n}\|_Y^2 + \|\varphi^n - \varphi^{hk,n}\|_Y^2 + \|\nabla(\boldsymbol{R}^n - \boldsymbol{R}^{hk,n})\|_Q^2 \\
&\quad +(\delta v^n_i - \delta v^{hk,n}_i, v^n_i - w^h_i)_Y\Big) \quad \forall \boldsymbol{w}^h = (w^h_i)^d_{i=1} \in V^h.
\end{aligned} \tag{36}$$

Now, we will derive the estimates on the volume fraction speed. Therefore, subtracting variational equation (16) at time t_n for all $r = r^h \in E^h$ and discrete variational equation (25) we get

$$\begin{aligned}
&J(\dot{e}^n - \delta e^{hk,n}, r^h)_Y + (A_{ij}(\varphi^n_{,j} - \varphi^{hk,n}_{,j}), r^h_{,i})_Y + \xi(\varphi^n - \varphi^{hk,n}, r^h)_Y \\
&\quad -(\alpha_{ij}(M^n_i - M^{hk,n}_i), r^h_{,j})_Y + (H_{ij}(\tau^n_{,i} - \tau^{hk,n}_{,i}), r^h_{,j})_Y + (\zeta_{ij}(u^n_{i,j} - u^{hk,n}_{i,j}), r^h)_Y \\
&\quad -\kappa(\theta^n - \theta^{hk,n}, r^h)_Y + (F_{ij}(R^n_{i,j} - R^{hk,n}_{i,j}), r^h)_Y = 0 \quad \forall r^h \in E^h,
\end{aligned}$$

and so, for all $r^h \in E^h$

$$\begin{aligned}
&J(\dot{e}^n - \delta e^{hk,n}, e^n - e^{hk,n})_Y + (A_{ij}(\varphi^n_{,j} - \varphi^{hk,n}_{,j}), e^n_{,i} - e^{hk,n}_{,i})_Y + \xi(\varphi^n - \varphi^{hk,n}, e^n - e^{hk,n})_Y \\
&\qquad -(\alpha_{ij}(M^n_i - M^{hk,n}_i), e^n_{,j} - e^{hk,n}_{,j})_Y + (H_{ij}(\tau^n_{,i} - \tau^{hk,n}_{,i}), e^n_{,j} - e^{hk,n}_{,j})_Y \\
&\qquad +(\zeta_{ij}(u^n_{i,j} - u^{hk,n}_{i,j}), e^n - e^{hk,n})_Y - \kappa(\theta^n - \theta^{hk,n}, e^n - e^{hk,n})_Y \\
&\qquad +(F_{ij}(R^n_{i,j} - R^{hk,n}_{i,j}), e^n - e^{hk,n})_Y \\
&\quad = J(\dot{e}^n - \delta e^{hk,n}, e^n - r^h)_Y + (A_{ij}(\varphi^n_{,j} - \varphi^{hk,n}_{,j}), e^n_{,i} - r^h_{,i})_Y + \xi(\varphi^n - \varphi^{hk,n}, e^n - r^h)_Y \\
&\qquad -(\alpha_{ij}(M^n_i - M^{hk,n}_i), e^n_{,j} - r^h_{,j})_Y + (H_{ij}(\tau^n_{,i} - \tau^{hk,n}_{,i}), e^n_{,j} - r^h_{,j})_Y \\
&\qquad +(\zeta_{ij}(u^n_{i,j} - u^{hk,n}_{i,j}), e^n - r^h)_Y - \kappa(\theta^n - \theta^{hk,n}, e^n - r^h)_Y \\
&\qquad +(F_{ij}(R^n_{i,j} - R^{hk,n}_{i,j}), e^n - r^h)_Y.
\end{aligned}$$

Taking into account that

$$\begin{aligned}
&(\dot{e}^n - \delta e^{hk,n}, e^n - e^{hk,n})_Y \geq (\dot{e}^n - \delta e^n, e^n - e^{hk,n})_Y \\
&\qquad +\frac{1}{2k}\left\{\|e^n - e^{hk,n}\|_Y^2 - \|e^{n-1} - e^{hk,n-1}\|_Y^2\right\}, \\
&(A_{ij}(\varphi^n_{,j} - \varphi^{hk,n}_{,j}), e^n_{,i} - e^{hk,n}_{,i})_Y = (A_{ij}(\varphi^n_{,j} - \varphi^{hk,n}_{,j}), \dot{\varphi}^n_{,i} - \delta\varphi^n_{,i})_Y \\
&\qquad +(A_{ij}(\varphi^n_{,j} - \varphi^{hk,n}_{,j}), \delta\varphi^n_{,i} - \delta\varphi^{hk,n}_{,i})_Y, \\
&(A_{ij}(\varphi^n_{,j} - \varphi^{hk,n}_{,j}), \delta\varphi^n_{,i} - \delta\varphi^{hk,n}_{,i})_Y = \frac{1}{2k}\Big\{(A_{ij}(\varphi^n_{,j} - \varphi^{hk,n}_{,j}), \varphi^n_{,i} - \varphi^{hk,n}_{,i})_Y \\
&\qquad -(A_{ij}(\varphi^{n-1}_{,j} - \varphi^{hk,n-1}_{,j}), \varphi^{n-1}_{,i} - \varphi^{hk,n-1}_{,i})_Y \\
&\qquad +(A_{ij}(\varphi^n_{,j} - \varphi^{hk,n}_{,j} - (\varphi^{n-1}_{,j} - \varphi^{hk,n-1}_{,j})), \varphi^n_{,i} - \varphi^{hk,n}_{,i} - (\varphi^{n-1}_{,i} - \varphi^{hk,n-1}_{,i}))_Y\Big\},
\end{aligned}$$

$$(\varphi^n-\varphi^{hk,n},e^n-e^{hk,n})_Y=(\varphi^n-\varphi^{hk,n},\dot\varphi^n-\varphi^{hk,n})_Y+\frac{1}{2k}\Big\{\|\xi^n-\xi^{hk,n}\|_Y^2$$
$$-\|\xi^{n-1}-\xi^{hk,n-1}\|_Y^2+\|\xi^n-\xi^{hk,n}-(\xi^{n-1}-\xi^{hk,n-1})\|_Y^2\Big\},$$

where we used the notations $\delta e^n=(e^n-e^{n-1})/k$ and $\delta\varphi^n=(\varphi^n-\varphi^{n-1})/k$, we have the following estimates for the volume fraction speed, for all $r^h\in E^h$,

$$\begin{aligned}
&\frac{J}{2k}\left\{\|e^n-e^{hk,n}\|_Y^2-\|e^{n-1}-e^{hk,n-1}\|_Y^2\right\}+\frac{1}{2k}\Big\{(A_{ij}(\varphi^n_{,j}-\varphi^{hk,n}_{,j}),\varphi^n_{,i}-\varphi^{hk,n}_{,i})_Y\\
&\quad-(A_{ij}(\varphi^{n-1}_{,j}-\varphi^{hk,n-1}_{,j}),\varphi^{n-1}_{,i}-\varphi^{hk,n-1}_{,i})_Y\\
&\quad+(A_{ij}(\varphi^n_{,j}-\varphi^{hk,n}_{,j}-(\varphi^{n-1}_{,j}-\varphi^{hk,n-1}_{,j})),\varphi^n_{,i}-\varphi^{hk,n}_{,i}-(\varphi^{n-1}_{,i}-\varphi^{hk,n-1}_{,i}))_Y\Big\}\\
&\quad+\frac{\xi}{2k}\Big\{\|\xi^n-\xi^{hk,n}\|_Y^2-\|\xi^{n-1}-\xi^{hk,n-1}\|_Y^2+\|\xi^n-\xi^{hk,n}-(\xi^{n-1}-\xi^{hk,n-1})\|_Y^2\Big\}\\
&\quad-(\alpha_{ij}(M_i^n-M_i^{hk,n}),e^n_{,j}-e^{hk,n}_{,j})_Y+(H_{ij}(\tau^n_{,i}-\tau^{hk,n}_{,i}),e^n_{,j}-e^{hk,n}_{,j})_Y\\
&\quad+(\zeta_{ij}(u^n_{i,j}-u^{hk,n}_{i,j}),e^n-e^{hk,n})_Y-\kappa(\theta^n-\theta^{hk,n},e^n-e^{hk,n})_Y\\
&\quad+(F_{ij}(R^n_{i,j}-R^{hk,n}_{i,j}),e^n-e^{hk,n})_Y\\
&\le C\Big(\|\dot e^n-\delta e^n\|_Y^2+\|\nabla(\dot\varphi^n-\delta\varphi^n)\|_H^2+\|\nabla(e^h-r^h)\|_H^2+\|\nabla(\varphi^n-\varphi^{hk,n})\|_H^2\\
&\quad+\|e^n-r^h\|_Y^2+\|\theta^n-\theta^{hk,n}\|_Y^2+\|\varphi^n-\varphi^{hk,n}\|_Y^2+\|\nabla(\boldsymbol{R}^n-\boldsymbol{R}^{hk,n})\|_Q^2\\
&\quad+\|\nabla(\boldsymbol{u}^n-\boldsymbol{u}^{hk,n})\|_Q^2+\|\boldsymbol{M}^n-\boldsymbol{M}^{hk,n}\|_H^2+\|\nabla(\tau^n-\tau^{hk,n})\|_H^2\\
&\quad+(\delta e^n-\delta e^{hk,n},e^n-r^h)_Y\Big).
\end{aligned}\tag{37}$$

Now, we will obtain the error estimates on the temperature. Subtracting variational equation (17) at time $t=t_n$ for a test function $z=z^h\in E^h$ and discrete variational equation, we find that, for all $z^h\in E^h$,

$$\begin{aligned}
&c(\dot\theta^n-\delta\theta^{hk,n},z^h)_Y+(K^*_{ij}(\theta^n_{,i}-\theta^{hk,n}_{,i}),z^h_{,j})_Y+(K_{ij}(\tau^n_{,i}-\tau^{hk,n}_{,i}),z^h_{,j})_Y\\
&\quad-(d_{ij}(M_i^n-M_i^{hk,n}),z^h_{,j})_Y+(A^1_{ij}(M_i^n-M_i^{hk,n}),z^h_{,j})_Y+(b_{ij}(M^n_{i,j}-M^{hk,n}_{i,j}),z^h)_Y\\
&\quad+(a_{ij}(v^n_{i,j}-v^{hk,n}_{i,j}),z^h)_Y+(H_{ij}(\varphi^n_{,i}-\varphi^{hk,n}_{,i}),z^h_{,j})_Y-\kappa(e^n-e^{hk,n},z^h)_Y=0,
\end{aligned}$$

and so, for all $z^h\in V^h$, it follows that

$$\begin{aligned}
&c(\dot\theta^n-\delta\theta^{hk,n},\theta^n-\theta^{hk,n})_Y+(K^*_{ij}(\theta^n_{,i}-\theta^{hk,n}_{,i}),\theta^n_{,j}-\theta^{hk,n}_{,j})_Y+(K_{ij}(\tau^n_{,i}-\tau^{hk,n}_{,i}),\theta^n_{,j}-\theta^{hk,n}_{,j})_Y\\
&\qquad-(d_{ij}(M_i^n-M_i^{hk,n}),\theta^n_{,j}-\theta^{hk,n}_{,j})_Y+(A^1_{ij}(M_i^n-M_i^{hk,n}),\theta^n_{,j}-\theta^{hk,n}_{,j})_Y\\
&\qquad+(b_{ij}(M^n_{i,j}-M^{hk,n}_{i,j}),\theta^n-\theta^{hk,n})_Y+(a_{ij}(v^n_{i,j}-v^{hk,n}_{i,j}),\theta^n-\theta^{hk,n})_Y\\
&\qquad+(H_{ij}(\varphi^n_{,i}-\varphi^{hk,n}_{,i}),\theta^n_{,j}-\theta^{hk,n}_{,j})_Y-\kappa(e^n-e^{hk,n},\theta^n-\theta^{hk,n})_Y\\
&\quad=c(\dot\theta^n-\delta\theta^{hk,n},\theta^n-z^h)_Y+(K^*_{ij}(\theta^n_{,i}-\theta^{hk,n}_{,i}),\theta^n_{,j}-z^h_{,j})_Y+(K_{ij}(\tau^n_{,i}-\tau^{hk,n}_{,i}),\theta^n_{,j}-z^h_{,j})_Y\\
&\qquad-(d_{ij}(M_i^n-M_i^{hk,n}),\theta^n_{,j}-z^h_{,j})_Y+(A^1_{ij}(M_i^n-M_i^{hk,n}),\theta^n_{,j}-z^h_{,j})_Y\\
&\qquad+(b_{ij}(M^n_{i,j}-M^{hk,n}_{i,j}),\theta^n-z^h)_Y+(a_{ij}(v^n_{i,j}-v^{hk,n}_{i,j}),\theta^n-z^h)_Y\\
&\qquad+(H_{ij}(\varphi^n_{,i}-\varphi^{hk,n}_{,i}),\theta^n_{,j}-z^h_{,j})_Y-\kappa(e^n-e^{hk,n},\theta^n-z^h)_Y.
\end{aligned}$$

Keeping in mind that

$$
\begin{aligned}
&(\dot\theta^n-\delta\theta^{hk,n},\theta^n-\theta^{hk,n})_Y\geq(\dot\theta^n-\delta\theta^n,\theta^n-\theta^{hk,n})_Y+\frac{1}{2k}\left\{\|\theta^n-\theta^{hk,n}\|_Y^2-\|\theta^{n-1}-\theta^{hk,n-1}\|_Y^2\right\},\\
&(K_{ij}(\tau^n_{,j}-\tau^{hk,n}_{,j}),\theta^n_{,i}-\theta^{hk,n}_{,i})_Y=(K_{ij}(\tau^n_{,j}-\tau^{hk,n}_{,j}),\dot\tau^n_{,i}-\delta\tau^n_{,i})_Y\\
&\qquad+(K_{ij}(\tau^n_{,j}-\tau^{hk,n}_{,j}),\delta\tau^n_{,i}-\delta\tau^{hk,n}_{,i})_Y,\\
&(K_{ij}(\tau^n_{,j}-\tau^{hk,n}_{,j}),\delta\tau^n_{,i}-\delta\tau^{hk,n}_{,i})_Y=\frac{1}{2k}\Big\{(K_{ij}(\tau^n_{,j}-\tau^{hk,n}_{,j}),\tau^n_{,i}-\tau^{hk,n}_{,i})_Y\\
&\qquad-(K_{ij}(\tau^{n-1}_{,j}-\tau^{hk,n-1}_{,j}),\tau^{n-1}_{,i}-\tau^{hk,n-1}_{,i})_Y\\
&\qquad+(K_{ij}(\tau^n_{,j}-\tau^{hk,n}_{,j}-(\tau^{n-1}_{,j}-\tau^{hk,n-1}_{,j})),\tau^n_{,i}-\tau^{hk,n}_{,i}-(\tau^{n-1}_{,i}-\tau^{hk,n-1}_{,i}))_Y\Big\},\\
&(a_{ij}(v^n_{i,j}-v^{hk,n}_{i,j}),\theta^n-z^h)_Y=-(a_{ij}(v^n_i-v^{hk,n}_i),\theta^n_{,j}-z^h_{,j})_Y,
\end{aligned}
$$

where we used the notations $\delta\theta^n=(\theta^n-\theta^{n-1})/k$ and $\delta\tau^n=(\tau^n-\tau^{n-1})/k$ and assumption (34), we get the following estimates for the temperature, for all $z^h\in E^h$,

$$
\begin{aligned}
&\frac{1}{2k}\left\{\|\theta^n-\theta^{hk,n}\|_Y^2-\|\theta^{n-1}-\theta^{hk,n-1}\|_Y^2\right\}+(K^*_{ij}(\theta^n_{,i}-\theta^{hk,n}_{,i}),\theta^n_{,j}-\theta^{hk,n}_{,j})_Y\\
&\qquad+\frac{1}{2k}\Big\{(K_{ij}(\tau^n_{,j}-\tau^{hk,n}_{,j}),\tau^n_{,i}-\tau^{hk,n}_{,i})_Y-(K_{ij}(\tau^{n-1}_{,j}-\tau^{hk,n-1}_{,j}),\tau^{n-1}_{,i}-\tau^{hk,n-1}_{,i})_Y\\
&\qquad+(K_{ij}(\tau^n_{,j}-\tau^{hk,n}_{,j}-(\tau^{n-1}_{,j}-\tau^{hk,n-1}_{,j})),\tau^n_{,i}-\tau^{hk,n}_{,i}-(\tau^{n-1}_{,i}-\tau^{hk,n-1}_{,i}))_Y\Big\}\\
&\qquad-(d_{ij}(M^n_i-M^{hk,n}_i),\theta^n_{,j}-\theta^{hk,n}_{,j})_Y+(A^1_{ij}(M^n_i-M^{hk,n}_i),\theta^n_{,j}-\theta^{hk,n}_{,j})_Y\\
&\qquad+(b_{ij}(M^n_{i,j}-M^{hk,n}_{i,j}),\theta^n-\theta^{hk,n})_Y+(a_{ij}(v^n_{i,j}-v^{hk,n}_{i,j}),\theta^n-\theta^{hk,n})_Y\\
&\qquad+(H_{ij}(\varphi^n_{,i}-\varphi^{hk,n}_{,i}),\theta^n_{,j}-\theta^{hk,n}_{,j})_Y-\kappa(e_n-e^{hk,n},\theta^n-\theta^{hk,n})_Y\\
&\leq C\Big(\|\dot\theta^n-\delta\theta^n\|_Y^2+\|\nabla(\dot\tau^n-\delta\tau^n)\|_H^2+\|\nabla(\theta^n-z^h)\|_H^2+\|\nabla(\tau^n-\tau^{hk,n})\|_H^2\\
&\qquad+\|\theta^n-z^h\|_Y^2+\|e^n-e^{hk,n}\|_Y^2+\|\nabla(\varphi^n-\varphi^{hk,n})\|_H^2+\|\boldsymbol{M}^n-\boldsymbol{M}^{hk,n}\|_H^2\\
&\qquad+\|\boldsymbol{v}^n-\boldsymbol{v}^{hk,n}\|_H^2+(\delta\theta^n-\delta\theta^{hk,n},\theta^n-z^h)_Y+\|\theta^n-\theta^{hk,n}\|_Y^2\Big)\\
&\qquad+\epsilon\|\nabla(\theta^n-\theta^{hk,n})\|_H^2+\epsilon\|\nabla(\boldsymbol{M}^n-\boldsymbol{M}^{hk,n}\|_Q^2,
\end{aligned}\tag{38}
$$

where $\epsilon>0$ is assumed small enough.

Finally, we will obtain the error estimates on the microtemperatures. Therefore, we subtract variational equation (18) at time t_n for a test function $\boldsymbol{\xi}=\boldsymbol{\xi}^h\in V^h$ and discrete variational equation (27) to obtain the following estimates, for all $\boldsymbol{\xi}^h=(\xi^h_i)_{i=1}^d\in V^h$,

$$
\begin{aligned}
&(c_{ij}(\dot M^n_j-\delta M^{hk,n}_j),\xi^h_i)_Y+(C_{ijkl}(R^n_{k,l}-R^{hk,n}_{k,l}),\xi^h_{i,j})+(C^*_{ijkl}(M^n_{k,l}-M^{hk,n}_{k,l}),\xi^h_{i,j})_Y\\
&\quad+(A^3_{ij}(M^n_j-M^{hk,n}_j),\xi^h_i)_Y-(b_{ij}(\theta^n-\theta^{hk,n}),\xi^h_{i,j})_Y\\
&\quad+(F_{ij}(\varphi^n-\varphi^{hk,n}),\xi^h_{i,j})_Y+(d_{ij}(\theta^n_{,j}-\theta^{hk,n}_{,j}),\xi^h_i)_Y+(\alpha_{ij}(e^n_{,j}-e^{hk,n}_{,j}),\xi^h_i)_Y\\
&\quad+(A^2_{ij}(\theta^n_{,j}-\theta^{hk,n}_{,j}),\xi^h_i)_Y+(B_{klij}(u^n_{k,l}-u^{hk,n}_{k,l}),\xi^h_{i,j})_Y=0,
\end{aligned}
$$

and so, for all $\boldsymbol{\xi}^h=(\xi_i^h)_{i=1}^d\in V^h$ it follows that

$$
\begin{aligned}
&(c_{ij}(\dot M_j^n-\delta M_j^{hk,n}),M_i^n-M_i^{hk,n})_Y+(C_{ijkl}(R_{k,l}^n-R_{k,l}^{hk,n}),M_{i,j}^n-M_{i,j}^{hk,n})\\
&\quad+(C^*_{ijkl}(M_{k,l}^n-M_{k,l}^{hk,n}),M_{i,j}^n-M_{i,j}^{hk,n})_Y+(A^3_{ij}(M_j^n-M_j^{hk,n}),M_i^n-M_i^{hk,n})_Y\\
&\quad-(b_{ij}(\theta^n-\theta^{hk,n}),M_{i,j}^n-M_{i,j}^{hk,n})_Y+(F_{ij}(\varphi^n-\varphi^{hk,n}),M_{i,j}^n-M_{i,j}^{hk,n})_Y\\
&\quad+(d_{ij}(\theta_{,j}^n-\theta_{,j}^{hk,n}),M_i^n-M_i^{hk,n})_Y+(\alpha_{ij}(e_{,j}^n-e_{,j}^{hk,n}),M_i^n-M_i^{hk,n})_Y\\
&\quad+(A^2_{ij}(\theta_{,j}^n-\theta_{,j}^{hk,n}),M_i^n-M_i^{hk,n})_Y+(B_{klij}(u_{k,l}^n-u_{k,l}^{hk,n}),M_{i,j}^n-M_{i,j}^{hk,n})_Y\\
&=(c_{ij}(\dot M_j^n-\delta M_j^{hk,n}),M_i^n-\xi_i^h)_Y+(C_{ijkl}(R_{k,l}^n-R_{k,l}^{hk,n}),M_{i,j}^n-\xi_{i,j}^h)\\
&\quad+(C^*_{ijkl}(M_{k,l}^n-M_{k,l}^{hk,n}),M_{i,j}^n-\xi_{i,j}^h)_Y+(A^3_{ij}(M_j^n-M_j^{hk,n}),M_i^n-\xi_i^h)_Y\\
&\quad-(b_{ij}(\theta^n-\theta^{hk,n}),M_{i,j}^n-\xi_{i,j}^h)_Y+(F_{ij}(\varphi^n-\varphi^{hk,n}),M_{i,j}^n-\xi_{i,j}^h)_Y\\
&\quad+(d_{ij}(\theta_{,j}^n-\theta_{,j}^{hk,n}),M_i^n-\xi_i^h)_Y+(\alpha_{ij}(e_{,j}^n-e_{,j}^{hk,n}),M_i^n-\xi_i^h)_Y\\
&\quad+(A^2_{ij}(\theta_{,j}^n-\theta_{,j}^{hk,n}),M_i^n-\xi_i^h)_Y+(B_{klij}(u_{k,l}^n-u_{k,l}^{hk,n}),M_{i,j}^n-\xi_{i,j}^h)_Y.
\end{aligned}
$$

Keeping in mind that

$$
\begin{aligned}
&(c_{ij}(\dot M_j^n-\delta M_j^{hk,n}),M_i^n-M_i^{hk,n})_Y\geq(c_{ij}(\dot M_j^n-\delta M_j^n),M_i^n-M_i^{hk,n})_Y\\
&\quad+\frac{C_2}{2k}\Big\{\|\boldsymbol{M}^n-\boldsymbol{M}^{hk,n}\|_H^2-\|\boldsymbol{M}^{n-1}-\boldsymbol{M}^{hk,n-1}\|_H^2\Big\},\\
&(C_{ijkl}(R_{k,l}^n-R_{k,l}^{hk,n}),M_{i,j}^n-M_{i,j}^{hk,n})_Y=(C_{ijkl}(R_{k,l}^n-R_{k,l}^{hk,n}),\dot R_{i,j}^n-\delta R_{i,j}^n)_Y\\
&\quad+(C_{ijkl}(R_{k,l}^n-R_{k,l}^{hk,n}),\delta R_{i,j}^n-\delta R_{i,j}^{hk,n})_Y,\\
&(C_{ijkl}(R_{k,l}^n-R_{k,l}^{hk,n}),\delta R_{i,j}^n-\delta R_{i,j}^{hk,n})_Y=\frac{1}{2k}\Big\{(C_{ijkl}(R_{k,l}^n-R_{k,l}^{hk,n}),R_{i,j}^n-R_{i,j}^{hk,n})_Y\\
&\quad-(C_{ijkl}(R_{k,l}^{n-1}-R_{k,l}^{hk,n-1}),R_{i,j}^{n-1}-R_{i,j}^{hk,n-1})_Y\\
&\quad+(C_{ijkl}(R_{k,l}^n-R_{k,l}^{hk,n}-(R_{k,l}^{n-1}-R_{k,l}^{hk,n-1})),R_{i,j}^n-R_{i,j}^{hk,n}-(R_{i,j}^{n-1}-R_{i,j}^{hk,n-1}))_Y\Big\},\\
&(\alpha_{ij}(e_{,j}^n-e_{,j}^{hk,n}),M_i^n-\xi_i^h)_Y=-(\alpha_{ij}(e^n-e^{hk,n}),M_{i,j}^n-\xi_{i,j}^h)_Y,
\end{aligned}
$$

where we used the notations $\delta\boldsymbol{M}^n=(\boldsymbol{M}^n-\boldsymbol{M}^{n-1})/k$ and $\delta\boldsymbol{R}^n=(\boldsymbol{R}^n-\boldsymbol{R}^{n-1})/k$, and assumption (34), we obtain the following estimates for the microtemperatures, for all $\boldsymbol{\xi}^h=(\xi_i^h)_{i=1}^d\in V^h$,

$$
\begin{aligned}
&\frac{C_2}{2k}\Big\{\|\boldsymbol{M}^n-\boldsymbol{M}^{hk,n}\|_H^2-\|\boldsymbol{M}^{n-1}-\boldsymbol{M}^{hk,n-1}\|_H^2\Big\}+(C^*_{ijkl}(M_{k,l}^n-M_{k,l}^{hk,n}),M_{i,j}^n-M_{i,j}^{hk,n})_Y\\
&\quad+\frac{1}{2k}\Big\{(C_{ijkl}(R_{k,l}^n-R_{k,l}^{hk,n}),R_{i,j}^n-R_{i,j}^{hk,n})_Y-(C_{ijkl}(R_{k,l}^{n-1}-R_{k,l}^{hk,n-1}),R_{i,j}^{n-1}-R_{i,j}^{hk,n-1})_Y\\
&\quad+(C_{ijkl}(R_{k,l}^n-R_{k,l}^{hk,n}-(R_{k,l}^{n-1}-R_{k,l}^{hk,n-1})),R_{i,j}^n-R_{i,j}^{hk,n}-(R_{i,j}^{n-1}-R_{i,j}^{hk,n-1}))_Y\Big\}\\
&\quad+(A^3_{ij}(M_j^n-M_j^{hk,n}),M_i^n-M_i^{hk,n})_Y\\
&\quad-(b_{ij}(\theta^n-\theta^{hk,n}),M_{i,j}^n-M_{i,j}^{hk,n})_Y+(F_{ij}(\varphi^n-\varphi^{hk,n}),M_{i,j}^n-M_{i,j}^{hk,n})_Y\\
&\quad+(d_{ij}(\theta_{,j}^n-\theta_{,j}^{hk,n}),M_i^n-M_i^{hk,n})_Y-(\alpha_{ij}(e^n-e^{hk,n}),M_{i,j}^n-M_{i,j}^{hk,n})_Y\\
&\quad+(A^2_{ij}(\theta_{,j}^n-\theta_{,j}^{hk,n}),M_i^n-M_i^{hk,n})_Y+(B_{klij}(u_{k,l}^n-u_{k,l}^{hk,n}),M_{i,j}^n-M_{i,j}^{hk,n})_Y\\
&\leq C\Big(\|\dot{\boldsymbol{M}}^n-\delta\boldsymbol{M}^n\|_H^2+\|\nabla(\dot{\boldsymbol{R}}^n-\delta\boldsymbol{R}^n)\|_Q^2+\|\nabla(\boldsymbol{M}^h-\boldsymbol{\xi}^h)\|_Q^2+\|\nabla(\boldsymbol{u}^n-\boldsymbol{u}^{hk,n})\|_Q^2
\end{aligned}
$$

$$+\|\boldsymbol{M}^n-\boldsymbol{\xi}^h\|_H^2+\|\theta^n-\theta^{hk,n}\|_Y^2+\|\varphi^n-\varphi^{hk,n}\|_Y^2+\|\nabla(\boldsymbol{R}^n-\boldsymbol{R}^{hk,n})\|_Q^2$$
$$+\|e^n-e^{hk,n}\|_Y^2+(\delta M_i^n-\delta M_i^{hk,n},M_i^n-\xi_i^h)_Y+\|\boldsymbol{M}^n-\boldsymbol{M}^{hk,n}\|_H^2\Big)$$
$$+\epsilon\|\nabla(\theta^n-\theta^{hk,n})\|_H^2+\epsilon\|\nabla(\boldsymbol{M}^n-\boldsymbol{M}^{hk,n})\|_Q^2, \quad (39)$$

where $\epsilon>0$ is assumed small enough.

Using assumption (10) we find that

$$(C^*_{ijkl}(M^n_{k,l}-M^{hk,n}_{k,l}),M^n_{i,j}-M^{hk,n}_{i,j})_Y+((A^1_{ij}+A^2_{ij})(\theta^n_{,j}-\theta^{hk,n}_{,j}),M^n_i-M^{hk,n}_i)_Y$$
$$+(K^*_{ij}(\theta^n_{,i}-\theta^{hk,n}_{,i}),\theta^n_{,j}-\theta^{hk,n}_{,j})_Y+(A^3_{ij}(M^n_j-M^{hk,n}_j),M^n_i-M^{hk,n}_i)_Y$$
$$\geq K_0(\|\nabla(\theta^n-\theta^{hk,n})\|_H^2+\|\nabla(\boldsymbol{M}^n-\boldsymbol{M}^{hk,n})\|_Q^2+\|\boldsymbol{M}^n-\boldsymbol{M}^{hk,n}\|_H^2).$$

Therefore, combining estimates (36)-(39) it follows that

$$\frac{\rho}{2k}\left\{\|\boldsymbol{v}^n-\boldsymbol{v}^{hk,n}\|_H^2-\|\boldsymbol{v}^{n-1}-\boldsymbol{v}^{hk,n-1}\|_H^2\right\}+\frac{1}{2k}\Big\{(A_{ijkl}(u^n_{k,l}-u^{hk,n}_{k,l}),u^n_{i,j}-u^{hk,n}_{i,j})_Y$$
$$-(A_{ijkl}(u^{n-1}_{k,l}-u^{hk,n-1}_{k,l}),u^{n-1}_{i,j}-u^{hk,n-1}_{i,j})_Y$$
$$+(A_{ijkl}(u^n_{k,l}-u^{hk,n}_{k,l}-(u^{n-1}_{k,l}-u^{hk,n-1}_{k,l})),u^n_{i,j}-u^{hk,n}_{i,j}-(u^{n-1}_{i,j}-u^{hk,n-1}_{i,j}))_Y\Big\}$$
$$+(B_{ijkl}(R^n_{k,l}-R^{hk,n}_{k,l}),v^n_{i,j}-v^{hk,n}_{i,j})_Y+(B_{klij}(u^n_{k,l}-u^{hk,n}_{k,l}),M^n_{i,j}-M^{hk,n}_{i,j})_Y$$
$$+\frac{J}{2k}\left\{\|e^n-e^{hk,n}\|_Y^2-\|e^{n-1}-e^{hk,n-1}\|_Y^2\right\}+\frac{1}{2k}\Big\{(A_{ij}(\varphi^n_{,j}-\varphi^{hk,n}_{,j}),\varphi^n_{,i}-\varphi^{hk,n}_{,i})_Y$$
$$-(A_{ij}(\varphi^{n-1}_{,j}-\varphi^{hk,n-1}_{,j}),\varphi^{n-1}_{,i}-\varphi^{hk,n-1}_{,i})_Y$$
$$+(A_{ij}(\varphi^n_{,j}-\varphi^{hk,n}_{,j}-(\varphi^{n-1}_{,j}-\varphi^{hk,n-1}_{,j})),\varphi^n_{,i}-\varphi^{hk,n}_{,i}-(\varphi^{n-1}_{,i}-\varphi^{hk,n-1}_{,i}))_Y\Big\}$$
$$+\frac{\xi}{2k}\Big\{\|\xi^n-\xi^{hk,n}\|_Y^2-\|\xi^{n-1}-\xi^{hk,n-1}\|_Y^2+\|\xi^n-\xi^{hk,n}-(\xi^{n-1}-\xi^{hk,n-1})\|_Y^2\Big\}$$
$$+(H_{ij}(\tau^n_{,i}-\tau^{hk,n}_{,i}),e^n_{,j}-e^{hk,n}_{,j})_Y+(H_{ij}(\varphi^n_{,i}-\varphi^{hk,n}_{,i}),\theta^n_{,j}-\theta^{hk,n}_{,j})_Y$$
$$+(\zeta_{ij}(u^n_{i,j}-u^{hk,n}_{i,j}),e^n-e^{hk,n})_Y+(\zeta_{ij}(\varphi^n-\varphi^{hk,n}),v^n_{i,j}-v^{hk,n}_{i,j})_Y$$
$$+(F_{ij}(R^n_{i,j}-R^{hk,n}_{i,j}),e^n-e^{hk,n})_Y+(F_{ij}(\varphi^n-\varphi^{hk,n}),M^n_{i,j}-M^{hk,n}_{i,j})_Y$$
$$+\frac{1}{2k}\left\{\|\theta^n-\theta^{hk,n}\|_Y^2-\|\theta^{n-1}-\theta^{hk,n-1}\|_Y^2\right\}+\frac{1}{2k}\Big\{(K_{ij}(\tau^n_{,j}-\tau^{hk,n}_{,j}),\tau^n_{,i}-\tau^{hk,n}_{,i})_Y$$
$$-(K_{ij}(\tau^{n-1}_{,j}-\tau^{hk,n-1}_{,j}),\tau^{n-1}_{,i}-\tau^{hk,n-1}_{,i})_Y$$
$$+(K_{ij}(\tau^n_{,j}-\tau^{hk,n}_{,j}-(\tau^{n-1}_{,j}-\tau^{hk,n-1}_{,j})),\tau^n_{,i}-\tau^{hk,n}_{,i}-(\tau^{n-1}_{,i}-\tau^{hk,n-1}_{,i}))_Y\Big\}$$
$$+\frac{C_2}{2k}\left\{\|\boldsymbol{M}^n-\boldsymbol{M}^{hk,n}\|_H^2-\|\boldsymbol{M}^{n-1}-\boldsymbol{M}^{hk,n-1}\|_H^2\right\}$$
$$+\frac{1}{2k}\Big\{(C_{ijkl}(R^n_{k,l}-R^{hk,n}_{k,l}),R^n_{i,j}-R^{hk,n}_{i,j})_Y-(C_{ijkl}(R^{n-1}_{k,l}-R^{hk,n-1}_{k,l}),R^{n-1}_{i,j}-R^{hk,n-1}_{i,j})_Y$$
$$+(C_{ijkl}(R^n_{k,l}-R^{hk,n}_{k,l}-(R^{n-1}_{k,l}-R^{hk,n-1}_{k,l})),R^n_{i,j}-R^{hk,n}_{i,j}-(R^{n-1}_{i,j}-R^{hk,n-1}_{i,j}))_Y\Big\}$$

$$
\begin{aligned}
&\leq C\Big(\|\dot{\boldsymbol{v}}^n-\delta\boldsymbol{v}^n\|_H^2+\|\nabla(\dot{\boldsymbol{u}}^n-\delta\boldsymbol{u}^n)\|_Q^2+\|\nabla(\boldsymbol{v}^h-\boldsymbol{w}^h)\|_Q^2+\|\nabla(\boldsymbol{u}^n-\boldsymbol{u}^{hk,n})\|_Q^2\\
&\quad+\|\boldsymbol{v}^n-\boldsymbol{w}^h\|_H^2+\|\theta^n-\theta^{hk,n}\|_Y^2+\|\varphi^n-\varphi^{hk,n}\|_Y^2+\|\nabla(\boldsymbol{R}^n-\boldsymbol{R}^{hk,n})\|_Q^2\\
&\quad+(\delta v_i^n-\delta v_i^{hk,n},v_i^n-w_i^h)_Y+\|\dot{e}^n-\delta e^n\|_Y^2+\|\nabla(\dot{\varphi}^n-\delta\varphi^n)\|_H^2+\|\nabla(e^h-r^h)\|_H^2\\
&\quad+\|\nabla(\varphi^n-\varphi^{hk,n})\|_H^2+\|e^n-r^h\|_Y^2+\|\boldsymbol{M}^n-\boldsymbol{M}^{hk,n}\|_H^2+\|\nabla(\tau^n-\tau^{hk,n})\|_H^2\\
&\quad+(\delta e^n-\delta e^{hk,n},e^n-r^h)_Y+\|\dot{\theta}^n-\delta\theta^n\|_Y^2+\|\nabla(\dot{\tau}^n-\delta\tau^n)\|_H^2+\|\nabla(\theta^n-z^h)\|_H^2\\
&\quad+\|\theta^n-z^h\|_Y^2+\|e^n-e^{hk,n}\|_Y^2+\|\boldsymbol{v}^n-\boldsymbol{v}^{hk,n}\|_H^2+(\delta\theta^n-\delta\theta^{hk,n},\theta^n-z^h)_Y\\
&\quad+\|\dot{\boldsymbol{M}}^n-\delta\boldsymbol{M}^n\|_H^2+\|\nabla(\dot{\boldsymbol{R}}^n-\delta\boldsymbol{R}^n)\|_Q^2+\|\nabla(\boldsymbol{M}^h-\boldsymbol{\xi}^h)\|_Q^2+\|\boldsymbol{M}^n-\boldsymbol{\xi}^h\|_H^2\\
&\quad+(\delta M_i^n-\delta M_i^{hk,n},M_i^n-\xi_i^h)_Y\Big)\quad\forall\boldsymbol{w}^h,\,\boldsymbol{\xi}^h\in V^h,\,r^h,\,z^h\in E^h. \qquad (40)
\end{aligned}
$$

Now, we observe that

$$
\begin{aligned}
&(B_{ijkl}(R_{k,l}^n-R_{k,l}^{hk,n}),\delta u_{i,j}^n-\delta u_{i,j}^{hk,n})_Y+(B_{klij}(u_{k,l}^n-u_{k,l}^{hk,n}),\delta R_{i,j}^n-\delta R_{i,j}^{hk,n})_Y\\
&\quad=\frac{1}{k}\Big\{(B_{ijkl}(R_{k,l}^n-R_{k,l}^{hk,n}),u_{i,j}^n-u_{i,j}^{hk,n})_Y-(B_{ijkl}(R_{k,l}^{n-1}-R_{k,l}^{hk,n-1}),u_{i,j}^{n-1}-u_{i,j}^{hk,n-1})_Y\\
&\qquad+(B_{ijkl}(R_{k,l}^n-R_{k,l}^{hk,n}-(R_{k,l}^{n-1}-R_{k,l}^{hk,n-1})),u_{i,j}^n-u_{i,j}^{hk,n}-(u_{i,j}^{n-1}-u_{i,j}^{hk,n-1}))_Y\Big\},\\
&(\zeta_{ij}(u_{i,j}^n-u_{i,j}^{hk,n}),\delta\varphi^n-\delta\varphi^{hk,n})_Y+(\zeta_{ij}(\varphi^n-\varphi^{hk,n}),\delta u_{i,j}^n-\delta u_{i,j}^{hk,n})_Y\\
&\quad=\frac{1}{k}\Big\{(\zeta_{ij}(u_{i,j}^n-u_{i,j}^{hk,n}),\varphi^n-\varphi^{hk,n})_Y-(\zeta_{ij}(u_{i,j}^{n-1}-u_{i,j}^{hk,n-1}),\varphi^{n-1}-\varphi^{hk,n-1})_Y\\
&\qquad+(\zeta_{ij}(u_{i,j}^n-u_{i,j}^{hk,n}-(u_{i,j}^{n-1}-u_{i,j}^{hk,n-1})),\varphi^n-\varphi^{hk,n}-(\varphi^{n-1}-\varphi^{hk,n-1}))_Y\Big\},\\
&(H_{ij}(\tau_{,i}^n-\tau_{,i}^{hk,n}),\delta\varphi_{,j}^n-\delta\varphi_{,j}^{hk,n})_Y+(H_{ij}(\varphi_{,i}^n-\varphi_{,i}^{hk,n}),\delta\tau_{,j}^n-\delta\tau_{,j}^{hk,n})_Y\\
&\quad=\frac{1}{k}\Big\{(H_{ij}(\tau_{,i}^n-\tau_{,i}^{hk,n}),\varphi_{,j}^n-\varphi_{,j}^{hk,n})_Y-(H_{ij}(\tau_{,i}^{n-1}-\tau_{,i}^{hk,n-1}),\varphi_{,j}^{n-1}-\varphi_{,j}^{hk,n-1})_Y\\
&\qquad+(H_{ij}(\tau_{,i}^n-\tau_{,i}^{hk,n}-(\tau_{,i}^{n-1}-\tau_{,i}^{hk,n-1})),\varphi_{,j}^n-\varphi_{,j}^{hk,n}-(\varphi_{,j}^{n-1}-\varphi_{,j}^{hk,n-1}))_Y\Big\},
\end{aligned}
$$

so using again assumptions (11) it follows that

$$
\begin{aligned}
&(A_{ijkl}(u_{k,l}^n-u_{k,l}^{hk,n}-(u_{k,l}^{n-1}-u_{k,l}^{hk,n-1})),u_{i,j}^n-u_{i,j}^{hk,n}-(u_{i,j}^{n-1}-u_{i,j}^{hk,n-1}))_Y\\
&\quad+2(B_{ijkl}(R_{k,l}^n-R_{k,l}^{hk,n}-(R_{k,l}^{n-1}-R_{k,l}^{hk,n-1})),u_{i,j}^n-u_{i,j}^{hk,n}-(u_{i,j}^{n-1}-u_{i,j}^{hk,n-1}))_Y\\
&\quad+(C_{ijkl}(R_{k,l}^n-R_{k,l}^{hk,n}-(R_{k,l}^{n-1}-R_{k,l}^{hk,n-1})),R_{i,j}^n-R_{i,j}^{hk,n}-(R_{i,j}^{n-1}-R_{i,j}^{hk,n-1}))_Y\\
&\quad+2(\zeta_{ij}(\varphi^n-\varphi^{hk,n}-(\varphi^{n-1}-\varphi^{hk,n-1})),u_{i,j}^n-u_{i,j}^{hk,n}-(u_{i,j}^{n-1}-u_{i,j}^{hk,n-1}))_Y\\
&\quad+\xi\|\varphi^n-\varphi^{hk,n}-(\varphi^{n-1}-\varphi^{hk,n-1})\|_Y^2\geq 0,\\
&(A_{ij}(\varphi_{,j}^n-\varphi_{,j}^{hk,n}-(\varphi_{,j}^{n-1}-\varphi_{,j}^{hk,n-1})),\varphi_{,i}^n-\varphi_{,i}^{hk,n}-(\varphi_{,i}^{n-1}-\varphi_{,i}^{hk,n-1}))_Y\\
&\quad+2(H_{ij}(\tau_{,i}^n-\tau_{,i}^{hk,n}-(\tau_{,i}^{n-1}-\tau_{,i}^{hk,n-1})),\varphi_{,j}^n-\varphi_{,j}^{hk,n}-(\varphi_{,j}^{n-1}-\varphi_{,j}^{hk,n-1}))_Y\\
&\quad+(K_{ij}(\tau_{,i}^n-\tau_{,i}^{hk,n}-(\tau_{,i}^{n-1}-\tau_{,i}^{hk,n-1})),\tau_{,j}^n-\tau_{,j}^{hk,n}-(\tau_{,j}^{n-1}-\tau_{,j}^{hk,n-1}))_Y\geq 0.
\end{aligned}
$$

Multiplying estimates (40) by k and summing up to n we have

$$
\begin{aligned}
&\|\boldsymbol{v}^n-\boldsymbol{v}^{hk,n}\|_H^2+(A_{ijkl}(u_{k,l}^n-u_{k,l}^{hk,n}),u_{i,j}^n-u_{i,j}^{hk,n})_Y+2(B_{ijkl}(R_{k,l}^n-R_{k,l}^{hk,n}),u_{i,j}^n-u_{i,j}^{hk,n})_Y\\
&\quad+\|e^n-e^{hk,n}\|_Y^2+(A_{ij}(\varphi_{,j}^n-\varphi_{,j}^{hk,n}),\varphi_{,i}^n-\varphi_{,i}^{hk,n})_Y+\|\xi^n-\xi^{hk,n}\|_Y^2\\
&\quad+2(H_{ij}(\tau_{,i}^n-\tau_{,i}^{hk,n}),\varphi_{,j}^n-\varphi_{,j}^{hk,n})_Y+2(\zeta_{ij}(u_{i,j}^n-u_{i,j}^{hk,n}),e^n-e^{hk,n})_Y\\
&\quad+(K_{ij}(\tau_{,j}^n-\tau_{,j}^{hk,n}),\tau_{,i}^n-\tau_{,i}^{hk,n})_Y+\|\theta^n-\theta^{hk,n}\|_Y^2+\|\boldsymbol{M}^n-\boldsymbol{M}^{hk,n}\|_H^2\\
&\quad+(C_{ijkl}(R_{k,l}^n-R_{k,l}^{hk,n}),R_{i,j}^n-R_{i,j}^{hk,n})_Y
\end{aligned}
$$

$$
\begin{aligned}
&\leq Ck\sum_{j=1}^{n}\Big(\|\dot{\boldsymbol{v}}^j-\delta\boldsymbol{v}^j\|_H^2+\|\nabla(\dot{\boldsymbol{u}}^j-\delta\boldsymbol{u}^j)\|_Q^2+\|\nabla(\boldsymbol{v}^j-\boldsymbol{w}^{h,j})\|_Q^2+\|\nabla(\boldsymbol{u}^j-\boldsymbol{u}^{hk,j})\|_Q^2\\
&\quad+\|\boldsymbol{v}^j-\boldsymbol{w}^{h,j}\|_H^2+\|\theta^j-\theta^{hk,j}\|_Y^2+\|\varphi^j-\varphi^{hk,j}\|_Y^2+\|\nabla(\boldsymbol{R}^j-\boldsymbol{R}^{hk,j})\|_Q^2\\
&\quad+(\delta v_i^j-\delta v_i^{hk,j},v_i^j-w_i^{h,j})_Y+\|\dot{e}^j-\delta e^j\|_Y^2+\|\nabla(\dot{\varphi}^j-\delta\varphi^j)\|_H^2+\|\nabla(e^j-r^{h,j})\|_H^2\\
&\quad+\|\nabla(\varphi^j-\varphi^{hk,j})\|_H^2+\|e^j-r^{h,j}\|_Y^2+\|\boldsymbol{M}^j-\boldsymbol{M}^{hk,j}\|_H^2+\|\nabla(\tau^j-\tau^{hk,j})\|_H^2\\
&\quad+(\delta e^j-\delta e^{hk,j},e^j-r^{h,j})_Y+\|\dot{\theta}^j-\delta\theta^j\|_Y^2+\|\nabla(\dot{\tau}^j-\delta\tau^j)\|_H^2+\|\nabla(\theta^j-z^{h,j})\|_H^2\\
&\quad+\|\theta^j-z^{h,j}\|_Y^2+\|e^j-e^{hk,j}\|_Y^2+\|\boldsymbol{v}^j-\boldsymbol{v}^{hk,j}\|_H^2+(\delta\theta^j-\delta\theta^{hk,j},\theta^j-z^{h,j})_Y\\
&\quad+\|\dot{\boldsymbol{M}}^j-\delta\boldsymbol{M}^j\|_H^2+\|\nabla(\dot{\boldsymbol{R}}^j-\delta\boldsymbol{R}^j)\|_Q^2+\|\nabla(\boldsymbol{M}^j-\boldsymbol{\xi}^{h,j})\|_Q^2+\|\boldsymbol{M}^j-\boldsymbol{\xi}^{h,j}\|_H^2\\
&\quad+(\delta M_i^j-\delta M_i^{hk,j},M_i^j-\xi_i^{h,j})_Y\Big)+C\Big(\|\boldsymbol{v}^0-\boldsymbol{v}^{0h}\|_H^2+\|\nabla(\boldsymbol{u}^0-\boldsymbol{u}^{0h})\|_Q^2+\|e^0-e^{0h}\|_Y^2\\
&\quad+\|\nabla(\varphi^0-\varphi^{0h})\|_H^2+\|\xi^0-\xi^{0h}\|_Y^2+\|\theta^0-\theta^{0h}\|_Y^2+\|\nabla(\tau^0-\tau^{0h})\|_H^2+\|\boldsymbol{M}^0-\boldsymbol{M}^{0h}\|_H^2\\
&\quad+\|\nabla(\boldsymbol{R}^0-\boldsymbol{R}^{0h})\|_Q^2\Big)\quad\forall\boldsymbol{w}^h,\,\boldsymbol{\xi}^h\in V^h,\,r^h,\,z^h\in E^h.
\end{aligned}
$$

Finally, using assumptions (11) we find that

$$
\begin{aligned}
&(A_{ijkl}(u_{k,l}^n-u_{k,l}^{hk,n}),u_{i,j}^n-u_{i,j}^{hk,n})_Y+2(B_{ijkl}(R_{k,l}^n-R_{k,l}^{hk,n}),u_{i,j}^n-u_{i,j}^{hk,n})_Y\\
&\quad+(C_{ijkl}(R_{k,l}^n-R_{k,l}^{hk,n}),R_{i,j}^n-R_{i,j}^{hk,n})_Y+2(\zeta_{ij}(\varphi^n-\varphi^{hk,n}),u_{i,j}^n-u_{i,j}^{hk,n})_Y\\
&\quad+\xi\|\varphi^n-\varphi^{hk,n}\|_Y^2\geq C\Big(\|\nabla(\boldsymbol{u}^n-\boldsymbol{u}^{hk,n})\|_Q^2+\|\nabla(\boldsymbol{R}^n-\boldsymbol{R}^{hk,n})\|_H^2+\|\varphi^n-\varphi^{hk,n}\|_Y^2\Big),\\
&(A_{ij}(\varphi_{,j}^n-\varphi_{,j}^{hk,n}),\varphi_{,i}^n-\varphi_{,i}^{hk,n})_Y+2(H_{ij}(\tau_{,i}^n-\tau_{,i}^{hk,n}),\varphi_{,j}^n-\varphi_{,j}^{hk,n})_Y\\
&\quad+(K_{ij}(\tau_{,i}^n-\tau_{,i}^{hk,n}),\tau_{,j}^n-\tau_{,j}^{hk,n})_Y\geq C\Big(\|\nabla(\varphi^n-\varphi^{hk,n})\|_H^2+\|\nabla(\tau^n-\tau^{hk,n})\|_H^2\Big).
\end{aligned}
$$

Therefore, taking into account that

$$
\begin{aligned}
&k\sum_{j=1}^{n}(\delta\boldsymbol{v}^j-\delta\boldsymbol{v}^{hk,j},\boldsymbol{v}^j-\boldsymbol{w}^{h,j})_H=\sum_{j=1}^{n}(\boldsymbol{v}^j-\boldsymbol{v}^{hk,j}-(\boldsymbol{v}^{j-1}-\boldsymbol{v}^{hk,j-1}),\boldsymbol{v}^j-\boldsymbol{w}^{h,j})_H\\
&\quad=(\boldsymbol{v}^n-\boldsymbol{v}^{hk,n},\boldsymbol{v}^n-\boldsymbol{w}^{h,n})_H+(\boldsymbol{v}^{0h}-\boldsymbol{v}^0,\boldsymbol{v}^1-\boldsymbol{w}^{h,1})_H\\
&\qquad+\sum_{j=1}^{n-1}(\boldsymbol{v}^j-\boldsymbol{v}^{hk,j},\boldsymbol{v}^j-\boldsymbol{w}^{h,j}-(\boldsymbol{v}^{j+1}-\boldsymbol{w}^{h,j+1}))_H,\\
&k\sum_{j=1}^{n}(\delta e^j-\delta e^{hk,j},e^j-r^{h,j})_Y=\sum_{j=1}^{n}(e^j-e^{hk,j}-(e^{j-1}-e^{hk,j-1}),e^j-r^{h,j})_Y\\
&\quad=(e^n-e^{hk,n},e^n-r^{h,n})_Y+(e^{0h}-e^0,e^1-r^{h,1})_Y\\
&\qquad+\sum_{j=1}^{n-1}(e^j-e^{hk,j},e^j-r^{h,j}-(e^{j+1}-r^{h,j+1}))_Y,
\end{aligned}
$$

$$
\begin{aligned}
k\sum_{j=1}^{n}(\delta\theta^j-\delta\theta^{hk,j},\theta^j-z^{h,j})_Y &= \sum_{j=1}^{n}(\theta^j-\theta^{hk,j}-(\theta^{j-1}-\theta^{hk,j-1}),\theta^j-r^{h,j})_Y \\
&= (\theta^n-\theta^{hk,n},\theta^n-r^{h,n})_Y+(\theta^{0h}-\theta^0,\theta^1-r^{h,1})_Y \\
&\quad+\sum_{j=1}^{n-1}(\theta^j-\theta^{hk,j},\theta^j-r^{h,j}-(\theta^{j+1}-r^{h,j+1}))_Y,
\end{aligned}
$$

$$
\begin{aligned}
k\sum_{j=1}^{n}(\delta\boldsymbol{M}^j-\delta\boldsymbol{M}^{hk,j},\boldsymbol{M}^j-\boldsymbol{\xi}^{h,j})_H &= \sum_{j=1}^{n}(\boldsymbol{M}^j-\boldsymbol{M}^{hk,j}-(\boldsymbol{M}^{j-1}-\boldsymbol{M}^{hk,j-1}),\boldsymbol{M}^j-\boldsymbol{\xi}^{h,j})_H \\
&= (\boldsymbol{M}^n-\boldsymbol{M}^{hk,n},\boldsymbol{M}^n-\boldsymbol{\xi}^{h,n})_H+(\boldsymbol{M}^{0h}-\boldsymbol{M}^0,\boldsymbol{M}^1-\boldsymbol{\xi}^{h,1})_H \\
&\quad+\sum_{j=1}^{n-1}(\boldsymbol{M}^j-\boldsymbol{M}^{hk,j},\boldsymbol{M}^j-\boldsymbol{\xi}^{h,j}-(\boldsymbol{M}^{j+1}-\boldsymbol{\xi}^{h,j+1}))_H,
\end{aligned}
$$

applying again a discrete version of Gronwall's inequality (see [4]) we derive error estimates (35).

We remark that these a priori error estimates can be used to obtain the convergence order of the approximations given by Problem VP^{hk}. Thus, as an example, we have the following result which states the linear convergence of the algorithm under suitable additional regularity conditions.

Corollary 1 *Let the assumptions of Theorem 2 hold. Therefore, if we assume the following additional regularity:*

$$
\begin{aligned}
&\boldsymbol{u},\,\boldsymbol{R}\in H^3(0,T;H)\cap W^{1,\infty}(0,T;[H^2(\Omega)]^d)\cap H^2(0,T;V),\\
&\theta,\,\tau\in H^2(0,T;Y)\cap L^\infty(0,T;H^2(\Omega))\cap H^1(0,T;E),
\end{aligned}
$$

it follows that the approximations obtained by Problem VP^{hk} *are linearly convergent; that is, there exists a positive constant* C*, independent of the discretization parameters* h *and* k*, such that*

$$
\begin{aligned}
\max_{0\le n\le N}\Big\{&\|\boldsymbol{v}^n-\boldsymbol{v}^{hk,n}\|_H+\|\nabla(\boldsymbol{u}^n-\boldsymbol{u}^{hk,n})\|_Q+\|e^n-e^{hk,n}\|_Y+\|\nabla(\varphi^n-\varphi^{hk,n})\|_H \\
&+\|\theta^n-\theta^{hk,n}\|_Y+\|\varphi^n-\varphi^{hk,n}\|_Y+\|\nabla(\tau^n-\tau^{hk,n})\|_H+\|\boldsymbol{M}^n-\boldsymbol{M}^{hk,n}\|_H \\
&+\|\nabla(\boldsymbol{R}^n-\boldsymbol{R}^{hk,n})\|_Q\Big\}\le C(h+k).
\end{aligned}
$$

4 Numerical results

In this final section, we will present some numerical results obtained in one- and two-dimensional examples.

In the numerical resolution we assume that the material is homogeneous and isotropic and therefore, the tensors defined in Problem P can be simplified.

In particular, we will assume the following form for all of them:

$$
\begin{aligned}
&A_{ijkl}u_{k,l}w_{i,j} = (\lambda+\mu)u_{i,i}w_{i,i} + \mu u_{i,j}w_{i,j} \quad \text{for all} \quad \boldsymbol{u}=(u_i)_{i=1}^d,\ \boldsymbol{w}=(w_i)_{i=1}^d \in V,\\
&a_{ij}\theta w_{i,j} = a\theta w_{i,i} \quad \text{for all} \quad \theta\in E,\ \boldsymbol{w}=(w_i)_{i=1}^d \in V,\\
&B_{ijkl}R_{k,l}w_{i,j} = BR_{i,j}w_{i,j} \quad \text{for all} \quad \boldsymbol{R}=(R_i)_{i=1}^d,\ \boldsymbol{w}=(w_i)_{i=1}^d \in V,\\
&\zeta_{ij}\varphi w_{i,j} = \zeta\varphi w_{i,i} \quad \text{for all} \quad \varphi\in E,\ \boldsymbol{w}=(w_i)_{i=1}^d \in V,\\
&A_{ij}\varphi_{,j}r_{,i} = A\varphi_{,i}r_{,i} \quad \text{for all} \quad \varphi,\ r\in E,\\
&\alpha_{ij}M_i r_{,j} = \alpha M_i r_{,i} \quad \text{for all} \quad \boldsymbol{M}=(M_i)_{i=1}^d \in V,\ r\in E,\\
&H_{ij}z_{,i}r_{,j} = Hz_{,i}r_{,i} \quad \text{for all} \quad z,\ r\in E,\\
&F_{ij}R_{i,j}r = FR_{i,i}r \quad \text{for all} \quad \boldsymbol{R}=(R_i)_{i=1}^d \in V,\ r\in E,\\
&K^*_{ij}\theta_{,j}z_{,i} = K^*\theta_{,i}z_{,i} \quad \text{for all} \quad \theta,\ z\in E,\\
&K_{ij}\tau_{,j}z_{,i} = K\tau_{,i}z_{,i} \quad \text{for all} \quad \tau,\ z\in E,\\
&d_{ij}M_i z_{,j} = DM_i z_{,i} \quad \text{for all} \quad \boldsymbol{M}=(M_i)_{i=1}^d \in V,\ z\in E,\\
&A^1_{ij}M_i z_{,j} = A^1 M_i z_{,i} \quad \text{for all} \quad \boldsymbol{M}=(M_i)_{i=1}^d \in V,\ z\in E,\\
&b_{ij}M_i z_{,j} = bM_i z_{,i} \quad \text{for all} \quad \boldsymbol{M}=(M_i)_{i=1}^d \in V,\ z\in E,\\
&C_{ijkl}R_{k,l}w_{i,j} = C_1R_{i,i}w_{i,i} + C_2R_{i,j}w_{i,j} \quad \text{for all} \quad \boldsymbol{R}=(R_i)_{i=1}^d,\ \boldsymbol{w}=(w_i)_{i=1}^d \in V,\\
&C^*_{ijkl}R_{k,l}w_{i,j} = C^*_1R_{i,i}w_{i,i} + C^*_2R_{i,j}w_{i,j} \quad \text{for all} \quad \boldsymbol{R}=(R_i)_{i=1}^d,\ \boldsymbol{w}=(w_i)_{i=1}^d \in V,\\
&A^2_{ij}\theta_{,j}w_i = A^2\theta_{,i}w_i \quad \text{for all} \quad \theta\in E,\ \boldsymbol{w}=(w_i)_{i=1}^d \in V,\\
&A^3_{ij}M_j w_i = A^3 M_i w_i \quad \text{for all} \quad \boldsymbol{M}=(M_i)_{i=1}^d,\ \boldsymbol{w}=(w_i)_{i=1}^d \in V,\\
&c_{ij}M_j\xi_i = c^*M_i\xi_i \quad \text{for all} \quad \boldsymbol{M}=(M_i)_{i=1}^d,\ \boldsymbol{\xi}=(\xi_i)_{i=1}^d \in V.
\end{aligned}
$$

Hence, using these tensors, discrete problem VP^{hk} leads to a linear system for a variable U in an adequate product space which is solved by using classical Cholesky's method. This numerical scheme was implemented on a 3.2 Ghz PC using MATLAB, and a typical 1D run ($h = k = 0.01$) took about 0.622 seconds of CPU time, meanwhile a typical 2D run took about 3.66 seconds of CPU time.

4.1 Numerical convergence and asymptotic behavior in a one-dimensional problem

As a simpler one-dimensional case, we will consider the following one-dimensional version of Problem P using the isotropic and homogeneous expressions given above. We note that, in some cases, the coefficients are collected together because they lead to the same term.

Problem $\mathbf{P}^{1D}$. *Find the displacement $u : [0,1]\times[0,1] \to \mathbb{R}$, the volume fraction $\varphi : [0,1]\times[0,1] \to \mathbb{R}$, the thermal displacement $\tau : [0,1]\times[0,1] \to \mathbb{R}$ and the microthermal displacement $R : [0,1]\times[0,1] \to \mathbb{R}$ such that*

$$
\begin{aligned}
&\rho\ddot{u} = \mu u_{xx} + \zeta\varphi_x - a\dot{\tau}_x + BR_{xx} + F_1 \quad \textit{in} \quad (0,1)\times(0,1),\\
&J\ddot{\varphi} = A\varphi_{xx} - \zeta u_x - \xi\varphi + \kappa\dot{\tau} + H\tau_{xx} - FR_x - \alpha\dot{R}_x + F_2 \quad \textit{in} \quad (0,1)\times(0,1),\\
&c\ddot{\tau} = K\tau_{xx} + H\varphi_{xx} - a\dot{u}_x - \kappa\dot{\varphi} + A^1\dot{R}_x + K^*\dot{\tau}_{xx} - b\dot{R}_x - D\dot{R}_x + F_3\\
&\qquad\qquad \textit{in} \quad (0,1)\times(0,1),\\
&c^*\ddot{R} = CR_{xx} + Bu_{xx} + F\varphi_x - \alpha\dot{\varphi}_x - D\dot{\tau}_x + C^*\dot{R}_{xx} - A^3\dot{R} - b\dot{\tau}_x - A^2\dot{\tau}_x + F_4\\
&\qquad\qquad \textit{in} \quad (0,1)\times(0,1),
\end{aligned}
$$

$$u(0,t) = u(1,t) = \varphi(0,t) = \varphi(1,t) = 0 \quad \textit{for a.e.} \quad t \in (0,1),$$
$$\tau(0,t) = \tau(1,t) = R(0,t) = R(1,t) = 0 \quad \textit{for a.e.} \quad t \in (0,1),$$
$$u(x,0) = \dot{u}(x,0) = \varphi(x,0) = \dot{\varphi}(x,0) = x(x-1) \quad \textit{for a.e. } x \in (0,1),$$
$$\tau(x,0) = \dot{\tau}(x,0) = R(x,0) = \dot{R}(x,0) = x(x-1) \quad \textit{for a.e. } x \in (0,1),$$

where the artificial volume forces F_i, $i = 1,2,3,4$, are given by

$$\begin{aligned} F_1(x,t) &= e^t\left(x(x-1) - 8\right), \\ F_2(x,t) &= e^t\left(8x - 8\right), \\ F_3(x,t) &= e^t\left(4x(x-1) - 14 + 4x\right), \\ F_4(x,t) &= e^t\left(3x(x-1) - 14 + 6x\right), \end{aligned}$$

and we used the following data in the simulations:

$$\begin{aligned} &\rho = 1, \quad \mu = 2, \quad \zeta = 1, \quad a = 1, \quad B = 2, \quad J = 1, \quad A = 1, \quad \xi = 2, \quad \kappa = 3, \\ &H = 1, \quad F = 2, \quad \alpha = 1, \quad c = 1, \quad K = 3, \quad K^* = 3, \quad D = 1, \quad b = 1, \\ &c^* = 2, \quad C = 3, \quad C^* = 2, \quad A^1 = 1, \quad A^3 = 1, \quad A^2 = 2. \end{aligned}$$

The exact solution to Problem P^{1D} can be easily calculated and it has the following form, for $(x,t) \in (0,1) \times (0,1)$,

$$u(x,t) = \varphi(x,t) = \tau(x,t) = R(x,t) = e^t x(x-1).$$

The numerical errors, given by

$$\begin{aligned} \max_{0\leq n\leq N} \Big\{ &\|v^n - v^{hk,n}\|_Y + \|(u^n - u^{hk,n})_x\|_Y + \|e^n - e^{hk,n}\|_Y + \|(\varphi^n - \varphi^{hk,n})_x\|_Y \\ &+ \|\theta^n - \theta^{hk,n}\|_Y + \|\varphi^n - \varphi^{hk,n}\|_Y + \|(\tau^n - \tau^{hk,n})_x\|_Y \\ &+ \|M^n - M^{hk,n}\|_Y + \|(R^n - R^{hk,n})_x\|_Y \Big\}, \end{aligned}$$

and obtained for different discretization parameters h and k, are depicted in Table 1. Moreover, the evolution of the error depending on the parameter $h+k$ is plotted in Fig. 1. We notice that the convergence of the algorithm is clearly observed, and the linear convergence, stated in Corollary 1, is achieved.

$h \downarrow k \rightarrow$	0.1	0.05	0.01	0.005	0.001	0.0005	
$1/2^3$	1.070021	1.068799	1.068409	1.068338	1.068314	1.068303	1.068300
$1/2^4$	0.537092	0.534805	0.534131	0.534026	0.533997	0.533986	0.533984
$1/2^5$	0.272919	0.268519	0.267229	0.267039	0.266989	0.266974	0.266972
$1/2^6$	0.144580	0.136480	0.133980	0.133609	0.133515	0.133488	0.133484
$1/2^7$	0.085436	0.072347	0.067714	0.066989	0.066804	0.066751	0.066744
$1/2^8$	0.060382	0.042813	0.035229	0.033857	0.033494	0.033390	0.033375
$1/2^9$	0.050677	0.030320	0.019950	0.017616	0.016929	0.016724	0.016695
$1/2^{10}$	0.047323	0.025500	0.013302	0.009978	0.008808	0.008420	0.008362
$1/2^{11}$	0.046323	0.023846	0.010663	0.006656	0.004989	0.004322	0.004209
$1/2^{12}$	0.046056	0.023357	0.009722	0.005338	0.003328	0.002360	0.002159
$1/2^{13}$	0.045988	0.023227	0.009432	0.004868	0.002668	0.001481	0.001176

Table 1 Example 1: Numerical errors for some h and k.

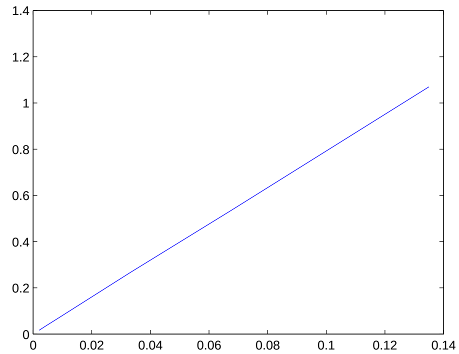

Fig. 1 Example 1: Asymptotic constant error.

If we assume now that there are not volume forces, and we use the following data:

$$T=20,\quad \rho=0.5,\quad \mu=7,\quad \zeta=1,\quad a=1,\quad B=2,\quad J=10,\quad A=1,$$
$$\xi=5,\quad \kappa=3,\quad H=1,\quad F=0.1,\quad \alpha=1,\quad c=1,\quad K=3,\quad K^*=3,$$
$$D=1,\quad b=1,\quad c^*=2,\quad C=3,\quad C^*=5,\quad A^1=2,\quad A^3=1,\quad A^2=2,$$

and the initial conditions:

$$u^0(x)=v^0(x)=R^0(x)=M^0(x)=x(x-1)\text{ for }x\in(0,1),\quad \varphi^0=e^0=\tau^0=\theta^0=0,$$

taking the discretization parameters $h=k=10^{-3}$ the evolution in time of the discrete energy $\mathcal{E}^{hk,n}$, defined as

$$\begin{aligned}\mathcal{E}^{hk,n}=\frac{1}{2}\Big\{&\rho\|v^{hk,n}\|_Y^2+J\|e^{hk,n}\|_Y^2+c\|\theta^{hk,n}\|_Y^2+c^*\|M^{hk,n}\|_Y^2+A^1\|u_x^{hk,n}\|_Y^2\\&+(Bu_x^{hk,n},R_x^{hk,n})_Y+C\|R_x^{hk,n}\|_Y^2+(\zeta u_x^{hk,n},\varphi^{hk,n})_Y+\xi\|\varphi^{hk,n}\|_Y^2\\&+A^2\|\varphi_x^{hk,n}\|_Y^2+(H\varphi_x^{hk,n},\tau_x^{hk,n})_Y+K\|\tau_x^{hk,n}\|_Y^2\Big\},\end{aligned}$$

is plotted in Fig. 2. As can be seen, it converges to zero and an exponential decay seems to be achieved.

4.2 Numerical results in a two-dimensional problem

For this second example, the square domain $[0,1]\times[0,1]$ is considered, assumed to be clamped on its vertical boundaries $\{0,1\}\times[0,1]$ and traction-free on the rest of the boundary.

The following data have been employed in this simulation:

$$\Omega=(0,1)\times(0,1),\quad T=1,\quad \rho=1,\quad \lambda=1,\quad \mu=1,\quad \zeta=1,\quad a=1,$$
$$B=2,\quad J=1,\quad A=1,\quad \xi=2,\quad \kappa=3,\quad H=1,\quad F=2,\quad \alpha=1,\quad c=1,$$
$$K=3,\quad K^*=3,\quad D=2,\quad b=2,\quad c^*=2,\quad C_1=2,\quad C_2=2,\quad C_1^*=1,$$
$$C_2^*=1,\quad A^1=2,\quad A^2=1,\quad A^3=1,$$

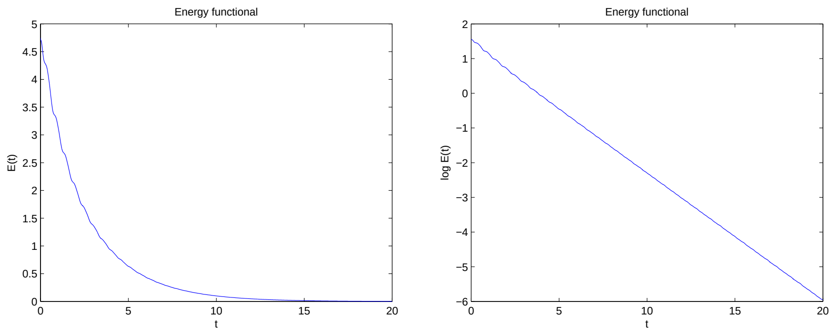

Fig. 2 Example 1: Evolution of the discrete energy in natural and semi-log scales.

and the initial conditions:

$$\begin{aligned}&\boldsymbol{u}^0 = \boldsymbol{v}^0 = \boldsymbol{R}^0 = \boldsymbol{M}^0 = \boldsymbol{0}, \quad \tau^0 = \theta^0 = 0,\\ &\varphi^0(x,y) = e^0(x,y) = x(x-1) \quad \text{for all} \quad (x,y) \in (0,1)\times(0,1).\end{aligned}$$

Taking the time discretization parameter $k = 0.01$, in Fig. 3 we plot the norm of both the displacement (left) and microtemperatures (right) at final time and over the deformed mesh. As expected, due to the clamping conditions, the displacement and the microtemperatures, which are generated by the volume fraction, have a similar behavior.

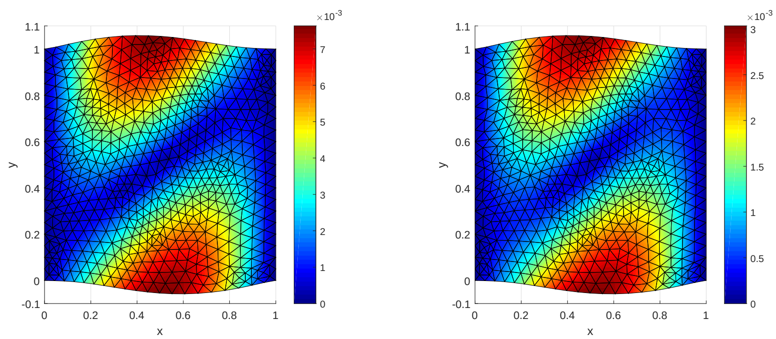

Fig. 3 Example 2: Norms of the displacement (left) and microtemperatures (right) at final time over the deformed mesh (multiplied by 10).

Moreover, in Fig. 4 we plot the microthermal displacement (left) and the volume fraction (right) at final time. We note that the volume fraction, even if it has a quadratic behavior, changes its sign, being now positive. Thus, in Fig. 5 the evolution in time of the volume fraction at middle point $\boldsymbol{x} = (0.5, 0.5)$ is shown. As we can see, it starts to increase after some time and it seems to converge to a steady state.

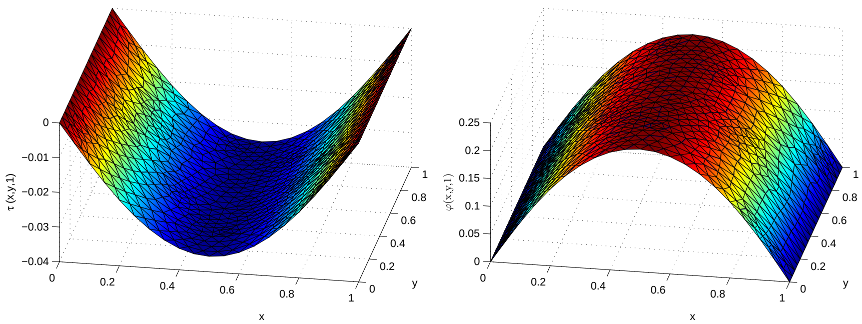

Fig. 4 Example 2: Microthermal displacement and volume fraction at final time.

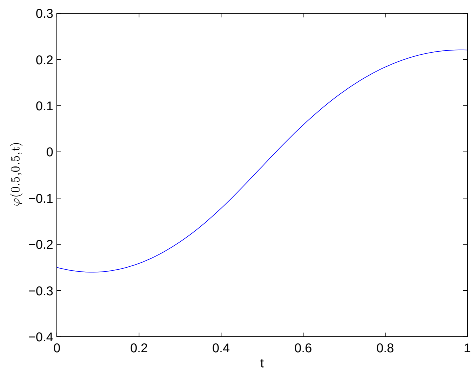

Fig. 5 Example 2: Evolution in time of the volume fraction at point $\boldsymbol{x} = (0.5, 0.5)$.

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