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Stability and convergence properties of Bergman kernel methods for numerical conformal mapping

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ABSTRACT

In this paper we study the stability and convergence properties of Bergman kernel methods, for the numerical conformal mapping of simply and doubly-connected domains. In particular, by using certain well-known results of Carleman, we establish a characterization of the level of instability in the methods, in terms of the geometry of the domain under consideration. We also explain how certain known convergence results can provide some theoretical justification of the observed improvement in accuracy which is achieved by the methods, when the basis set used contains functions that reflect the main singular behaviour of the conformal map.

1. <u>Introduction</u>

Let $\partial\Omega$ be a closed piecewise analytic Jordan curve in the complex z-plane, assume that 0 is in $\Omega = \mathrm{Int}(\partial\Omega)$, and let f be the function which maps conformally Ω onto the unit disc $\{w : |w| < 1\}$ so that $f(0) = 0$ and $f'(0) > 0$. Also, let $L_2(\Omega)$ be the Hilbert space of all square integrable analytic functions in Ω, denote by $<.,.>$ the inner product of $L_2(\Omega)$, i.e.

$$< u, v) = \iint_{\Omega} u(z)\overline{v(z)} ds_z , \tag{1.1}$$

and let K(.,0) be the Bergman kernel function of Ω. Then, the kernel K(.,0) is uniquely characterized by the reproducing property

$$< g, K(.,0) > = g(0) \quad , \qquad \forall \;\; g \in L_2(\Omega) \tag{1.2}$$

and is related to the mapping function f by means of

$$f'(z) = \left\{\frac{\pi}{k(0,0)}\right\}^{\frac{1}{2}} k(z,0) \;\; ; \tag{1.3}$$

see e.g. [1,7,8,12].

Let n_j ; = 1,2,3,..., be a complete set of functions of $L_2(\Omega)$. Then the reproducing property (1.2) and the relation (1.3) suggest the following procedure for approximating the mapping function f. The set $\{\eta j\}_{j=1}^{n}$ is orthonormalized by means of the Gram-Schmidt process to give the orthonormal set $\left\{\eta_j^*\right\}_{j=1}^{n}$. The kernel k(z,0) is then approximated by the finite Fourier

$$\begin{aligned} k_n(z,0) &= \sum_{j=1}^{n} < k(.,0) \quad , \eta_j^* > \eta_j^*(z) \\ &= \sum_{j=1}^{n} \overline{\eta_j^*(0)} \eta_j^*(z) \quad , \end{aligned} \tag{1.4}$$

and finally equation (1.3) is used to give the approximation

$$f_n(z) = \left\{\frac{\pi}{k_n(0,0)}\right\}^{\frac{1}{2}} \int_0^z k_n(\zeta,0) d\zeta \quad , \tag{1.5}$$

to the function f. In other words the approximation f_n is obtained after first determining the least squares approximation, in

$$\wedge_n = \mathrm{span}\{\eta 1, \eta 2, ...\eta_n\} \;\; , \tag{1.6}$$

to the Bergman kernel function $K(.,0)$. This method of approximating f is the well-known Bergman kernel method (BKM); see e.g. [1,2,4,7,8,10,12-14].

Let now $\partial\Omega_1$ and $\partial\Omega_2$ be two closed piecewise analytic Jordan curves such that $\partial\Omega_1 \subset \mathrm{Int}(\partial\Omega_2)$ and $0 \; \mathrm{Int}(\partial\Omega_1)$, denote by Ω the doubly-connected domain

$$\Omega = \mathrm{Ext}(\partial\Omega_1) \cap \mathrm{int}(\partial\Omega_2), \tag{1.7}$$

and let f be the function which maps conformally Ω onto a circular annulus $\{w: 1 < |w| < M\}$ so that $f(\zeta_1) = 1$, where ζ_1 is some fixed point on $\partial\Omega_1$. Also let

$$H(z) = f'(z)/f(z) - 1/z, \tag{1.8}$$

and denote by $L_2^S(\Omega)$ the Hilbert space of all functions in $L_2(\Omega)$ which also possess a single-valued indefinite integral in Ω. Then, it can be shown that for $\eta \in L_2^S(\Omega)$

$$<\eta, H> = i\int_{\partial\Omega_1 \cup \partial\Omega_2} \eta(z)\log|z|\,dz \tag{1.19}$$

provided that the function η satisfies certain boundary continuity requirements; see [7,p.249] and the remark in [15,§2,p.686]. In other words, the determination of $<\eta,H>$ does not require the explicit knowledge of H and, because of this, an approximation f_n to f can be determined by means of (1.8), in a manner similar to the BKM. That is, the approximation f_n to the conformal map of the doubly-connected domain (1.7) is determined from the least squares approximation of the function H in Λ_n, where now Λ_n is an n-dimensional subspace of $L^S_2(\Omega)$. Also, an approximation to the outer radius M of the annulus, i.e. to the conformal modulus of Ω, may be determined, from the least squares approximation of H by means of

$$\log M = \left\{\frac{1}{i}\int_{\partial\Omega}\frac{1}{2}\log|z|\,dz - \|H\|^2\right\}/2\pi, \tag{1.10}$$

where $\|\cdot\|^2 = <.,.>$ The above method for approximating f and M is the orthonormalization method ONM considered recently in [15,17]; see also [7,§53,p.249].

The purpose of this paper is to consider the stability and convergence properties of the BKM and the ONM, in relation to the basis set $\{\eta_j\}$ used and to the geometry of the domain Ω under consideration. In particular, we consider the effect that the geometry of Ω has on the stability and convergence of the methods when the "monomial" basis sets

$$\eta_j = z^{j-1}; \qquad j = 1,2,\ldots\ldots\ldots\ldots, \tag{1.11}$$

and

$$\eta_{2j-1} = z^{j-1}, \eta_{2j} = 1/z^{j+1}; \quad j = 1,2,\ldots\ldots., \tag{1.12}$$

are used respectively in the BKM and ONM. We also consider how the use of "augmented" basis sets, of the type considered in [10] and [13-18], affect the stability and convergence properties of the two methods. These augmented sets are formed by introducing into the monomial sets (1.11) and (1.12) "singular" functions that reflect the main singular behaviour of the conformal maps on $\partial\Omega$ and in compl($\Omega \cup \partial\Omega$).

The details of the presentation are as follows:

In Section 2 we consider various ways of measuring the level of instability in the Gram-Schmidt process and, in particular, we define an instability indicator which can be computed easily during the orthonormalization.

In Section 3 we consider the stability properties of the BKM and the ONM. In particular, we establish a geometrical characterization of the degree of instability in the orthonormalization of the monomial basis sets (1.11) and (1.12), by using certain well-known results of Carleman [5]. (It is, of course, well-known that the Gram-Schmidt process is numerically unstable. However, for the applications considered here, we are not aware of any detailed study regarding the dependence of the level of instability on the basis set used and on the geometry of Ω.)

Section 4 concerns the convergence properties of the BKM and ONM. Here, we discuss the significance of certain known convergence results contained in [7,8,11,20], In particular, we indicate how the results in [7,8] can be used to provide some theoretical explanation of the observed improvement in accuracy which is achieved when the monomial sets (1.11) and (1.12) are augmented by introducing appropriate singular functions.

Finally, in Section 5 we present several numerical examples, illustrating the stability and convergence results of the previous sections.

2. Instability Indicators

In what follows the function g and the Hilbert space Λ have the following meanings, depending on whether the domain Ω under consideration is simply or doubly-connected.

(i) When Ω is simply-connected then g is the Bergman kernel function $K(.,0)$ of Ω, and Λ is the space $L_2(\Omega)$ •

(ii) When Ω is doubly-connected then g is the function H of (1.8), and A is the space $L_2^S(\Omega)$

Let η_j; $j = 1,2,\ldots,$ be a complete set of Δ and let

$$A_n = \text{span}\{\eta_1,\eta_2\ldots,\eta_n\} \tag{2.1}$$

Then, with the notation introduced above, in both the BKM and ONM the approximation f_n to the mapping function f is determined after first computing the least squares approximation $g_n \in \wedge_n$ and $g \in \wedge$ That is,

$$gn = \sum_{j=1}^{n} < g, \quad \eta_j^* > \eta_j^* \ , \tag{2.2}$$

where $\{\eta_j^*\}_{j=1}^n$ is the orthonormal set obtained from $\{\eta_j\}_{j=1}^n$, by means of the Gram-Schmidt process. Of course, the approximation (2.2) can also be expressed as

$$g_n = \sum_{j=1}^{n} c_{n,j} \ \eta_j \ , \tag{2.3}$$

where the coefficients $c_{n,}$; $j = 1(1)n$, satisfy the Gram linear system

$$\sum_{j=1}^{n} < \eta_j,\eta_i > c_{n,j} =< g,\eta_i >; \qquad i = 1(1)n \ . \tag{2.4}$$

Let

$$G_n = \{<\eta_j,\eta_i>\} \ . \tag{2.5}$$

denote the coefficient matrix of (2.4) and let $C(G_n)$ be the spectral condition number of G_n, i.e.

$$C(G_n) = \rho(G_n)\rho(G_n^{-1}) \ , \tag{2.6}$$

where we use $\rho(\cdot)$ to denote the spectral radius of a matrix. Then, a small $C(G_n)$ implies that the Gram linear system (2.4) is well-conditioned, and suggests that there is no-excessive build-up of rounding errors in the corresponding orthonormalization process. Conversely, a large $C(G_n)$ suggests ill-conditioning and a rapid build-up of errors. However, a large $C(G_n)$ may be simply due to a badly scaled Gram linear system. For this reason, it is more appropriate to use, as a measure of instability, the condition number $C(\hat{G}_n)$ corresponding to the normalized Gram matrix

$$\left.\begin{aligned} &\hat{G}n = \{<\hat{\eta}_j, \hat{\eta}_i>\}, \\ \text{where} \qquad & \\ &\hat{\eta}_i = \eta_i / \|\eta_i\|. \end{aligned}\right\} \tag{2.7}$$

Of course, the condition number

$$C(\hat{G}_n) = \rho(\hat{G}_n)\rho(\hat{G}_n^{-1}) \ , \tag{2.8}$$

depends on the basis set $\{n_j.\}$ used and on the geometry of the domain Ω under consideration. However, it is very difficult to obtain, directly from definition (2.8), any information regarding the dependence of $C(\hat{G}_n)$ on $\{\eta_j.\}$ and on Ω. Furthermore, the determination of $C(\hat{G}_n)$ involves considerable computational effort. Ideally, we require an easily computable instability indicator, which can also be used to provide a characterization of the degree of instability in terms of the basis set $\{\eta_j.\}$ and the geometry of Ω. Such an indicator emerges from the result of the following theorem, which is due to Taylor [24].

<u>Theorem 2.1</u> (Taylor [24,p.p.46-47])

Let N_i; $i = 1(1)n$, denote the (n-l)-dimensional subspaces

$$N_i = \text{span}\{\eta_1, \eta_2, \ldots\ldots\eta_{i-1}, \eta_{i+1}, \ldots\ldots\eta_n\} \tag{2.9}$$

of Λ_n , and let $\underline{e}_i$ be the ith column of the $n \times n$ identity matrix. Then, for any $i = 1,2,\ldots,n$,

$$\{C(G_n)\}^{-1} \le \tilde{I}_{n,i} \le I_{n,i} \ , \tag{2.10}$$

where

$$\tilde{I}_{n,i} = e_i^H G_n^{-1} e_i, \tag{2.11}$$

and

$$I_{n,i} = \min_{u \in N_i} \ \| \hat{\eta}_i - u \|^2 . \tag{2.12}$$

As was remarked by Taylor [24], the deviation of the quantity

$$IS_n = \min_{1 \le i \le n} In, i \tag{2.13}$$

from zero measures the deviation of the set $\{\eta_i\}$ from linear dependence. (If the $\hat{\eta}_i$'s are linearly dependent, i.e. if is $\{\hat{\eta}_i\}_{i=1}^n$ not a basis, then $I_{n,i} = 0$, whilst if $\{\hat{\eta}_i\}_{i=1}^n$ is an orthonormal set then $I_{n-i} = 1$; $i = 1(1)n$.) This means that the deviation of the numbers IS_n from unity gives a measure of the level of instability in the orthonormalization process. However, a more easily computable instability indicator can be defined, by using (2.9)-(2.10), as follows.

We recall that the Gram-Schmidt process generates a triangular array a_{ij}; $i = 1(1)n$, $j \le i$, with diagonal elements $a_{ii} > 0$, so that each orthonormal function η_i^* is of the form

$$\eta_i^* = \sum_{j=1}^{i} a_{ij} \eta_j \ ; \qquad i = 1,2,\ldots\ldots n. \tag{2.14}$$

Let A be the n x n lower triangular matrix formed by the coefficients a_{ij}, in (2.14), Then, the orthonormality property $<\eta_i^*, \eta_j^*> = \delta_{ij}$ implies that

$$\overline{G}_n^{-1} = A_n^H A_n \ , \tag{2.15}$$

where G_n is the Gram matrix (2.5). Therefore, from (2.7), (2.11) and (2.15) we have that

$$\begin{aligned} \tilde{I}_{n,i} &= (A\, e_{n-i})^H (A\, e_{n-i}) / \| \eta_i \|^2 \\ &= \left\{ \| \eta_i \|^2 \sum_{j=1}^{n} | a_{ij} |^2 \right\}^{-1}, \end{aligned} \tag{2.16}$$

i.e.

$$\tilde{I}_{n,i} = \{\| \eta_i \|^2 | a_{ni} |^2 + 1/\tilde{I}_{n-1,i})^{-1}; \qquad i = 1(1)n-1 \ , \tag{2.17a}$$

and

$$\tilde{I}n,n = I_{n,n}$$

$$= 1/\| n_n \|^2 | a_{nn|}^2 . \tag{2.17b}$$

Let

$$\tilde{I}S_n = \min_{1 \le i \le n} \tilde{I}_{n,i} \quad . \tag{2.18}$$

Then, from (2,10),

$$\{C(G_n)\}^{-1} \le \tilde{I}s_n \le Is_n , \tag{2.19}$$

and this shows that, like IS_n , the number $\tilde{I}S_n$ may also be regarded as an instability indicator. However, unlike IS_n , the indicator (2.18) can be computed easily during the orthonormalization, by means of (2.17).

We end this section by considering briefly another instability indicator. This is the so-called Bauer's condition number of the Gram matrix G_n, which is defined by

$$\beta(G_n) = \inf_{\gamma_i > 0} C(\mathrm{diag}(\gamma_i) G_n \mathrm{diag}(\gamma_i)). \tag{2.20}$$

i.e. $\beta(G_n)$ is the spectral condition number corresponding to the best possible re-scaling of the matrix G_n. Clearly, it is very difficult to determine $\beta(G_n)$ and, for this reason, the measure (2.20) is of mainly theoretical value. However, we must state here an important result due to Švecova [23], concerning the condition number $\beta(G_n)$ of the matrix G_n corresponding to the monomials z^{j-1} ; j=1,2,.,.,n. Švecova has studied the asymptotic behaviour of this $\beta(G_n)$ and has shown that, unless Ω is a disc with its centre at 0,

$$\lim_{n \to \infty} \beta(G_n) = \infty. \tag{2.21}$$

3. Stability Properties

We examine first the stability properties of the BKM with monomial basis (1.11), i.e. with

$$\eta_j(z) = z^{j-1} ; \quad j = 1,2,3,\dots \ , \tag{3.1}$$

for the mapping of a simply-connected domain Ω. More specifically, we examine the rate of decrease of the sequence $\{I_{n,n}\}$ where, from (2,12) and (2.17)-(2.19),

$$\{C(\hat{G}_n)\}^{-1} \leq \hat{I}S_n \leq IS_n \leq I_{n,n}, \tag{3.2}$$

and where, for the set (3.1),

$$I_{n+1,n+1} = \min_{u \in \wedge_n} \{\| z^n - u \|^2 / \| z^n \|^2\} \tag{3.3}$$

with

$$\wedge_n = \text{span}\{1, z, z^2, \ldots, z^{n-1}\} \quad . \tag{3.4}$$

Our main result is Theorem 3.2, which gives a geometrical characterization of the rate of decrease of $\{I_{n,n}\}$ and hence, because of (3.2), of the level of instability in the orthonormalization of the set (3.1). The theorem is established by using two preliminary lemmas, concerning the sizes of $\| z^n \|^2$ and $\min_{u \in \wedge_n} \| z^n - u \|^2$.

Lemma 3.1 let $\partial\Omega$ be a closed piecewise analytic Jordan curve without cusps, and let

$$d = \max\{ |z| : z \in \partial\Omega\} \ . \tag{3.5}$$

Then, there exists a constant $\alpha > 0$ so that, for all $n > 0$,

$$\frac{\alpha d^{2n+2}}{n(n+1)} \leq \| z^n \|^2 \leq \frac{\pi d^2 n + 2}{(n+1)}. \tag{3.6}$$

Proof Let $D = \{z : |z[< d\}$. Then,$\Omega \subseteq D$ and the upper bound follows since

$$\iint_D | z^n |^2 \, dS_z = \pi d^{2n+2} / (n+1).$$

To establish the lower bound, let $z_0 \ \varepsilon \ \partial\Omega$ be such that $d = |z_0|$ and assume, without loss of generality, that Ω is orientated so that $z_0 = (d,0)$. Then the assumptions concerning the geometry of $\partial\Omega$ imply that there exist numbers $r_1 > 0$ and θ_1, θ_2, with $\theta_1 + \theta_2 > 0$, so that the sector

$$A = \{z : | z - d | < r_1, (\pi - \theta_1) < \arg(z - d) < (\pi + \theta_2)\}$$

is contained in Ω. This means that

$$\| z^n \|^2 \geq \iint_A | z^n |^2 \, dS_z \geq \int_{\pi-\theta_1}^{\pi+\theta_2} \int_0^{r_1} (d-r)^{2n} \, r dr d\theta \quad ,$$

and the lower bound follows.□

The next lemma contains essentially one of the results of Carleman [5], on asymptotic properties of orthonormal polynomials; see [7,p.136], [8,p.20] and [21,p.288], The result of the lemma is given in terms of the so-called capacity of the curve $\partial\Omega$, which is defined as follows. Let f_E be the function which maps conformally Ext$(\partial\Omega)$ onto $\{w : |w| > 1\}$, so that $f_E(\infty) = \infty$ and $\lim_{z\to\infty} f_E'(z) > 0$. Then,

$$\text{cap}\,(\partial\Omega) = \lim_{z\to\infty} \{f_E'(z)\}^{-1} \quad . \qquad (3.7)$$

Lemma 3.2 Let $\Omega \cup \partial\Omega$ be as in Lemma 3.1 and let

$$c = \text{cap}\,(\partial\Omega)$$

be the capacity of $\partial\Omega$ as defined by (3.7). Then

$$\min_{u\in\Lambda_n} \| z^n - u \|^2 \leq \pi c^{2n+2} /(n+1) \quad , \qquad (3.8)$$

where Λn is the polynomial space (3.4).□

The lemma can be established by modifying trivially the proof of Theorem 2 of Gaier [8,p.20-22]. More precisely in [8] Gaier establishes the sharper result

$$\min_{u\in\Lambda_n} \| z^n - u \|^2 = \pi c^{2n+2} /(n+1) + 0(r_0^{2n}), \; r_0 < c \quad , \qquad (3.9)$$

of Carleman [5], under the assumption that $\partial\Omega$ is an analytic curve. (To recognize the connection between (3.9) and the result proved in [8], recall that

$$\min_{u\in\Lambda_n} \| z^n - u \|^2 = 1 / a^2_{n+1,n+1} \; ,$$

where $a_{n+1,\,n+1}$ is the coefficient of z^n in the (n+1)th orthonormalized polynomial, and observe that Gaier denotes this coefficient by k_n)

Theorem 3.1 Let $\Omega \cup \partial\Omega$ be as in Lemma 3.1, let $I_{n+1,n+1}$ be defined by (3.3), and let

$$\delta = \{c/d\}^2 \qquad (3.10)$$

where, as before $d = \max\{|z| : z \in \partial\Omega\}$ and $c = \text{cap}\,(\partial\Omega)$. Then, there exists a constant $\beta > 0$ so that, for all $n \geq 1$,

$$I_{n+1,n+1} \le \beta n \delta^{n} . \tag{3.11}$$

Furthermore, if $\partial\Omega$ is analytic then there exist constants $\beta > 0, \gamma > 0$ and $0 < r_0 < c$ so that, for all $n \ge 1$,

$$\delta^{n+1}\left\{1-\gamma\left(\frac{r_0}{c}\right)^{2n}\right\} \le I_{n+1,n+1} \le \beta\sqrt{n}\,\delta^{n} . \tag{3.12}$$

(Observe that $\delta < 1$, unless Ω is a disc $\{z : |z| < R\}$ in which case $\delta = 1$.)

Proof The result (3.11) is a direct consequence of (3.8) and the lower bound in (3.6). The lower bound in (3.12) is established by using (3.9) and the upper bound in (3.6).

Finally, the upper bound in (3.12) is obtained from (3.8), by observing that if $\partial\Omega$ is analytic then the lower bound in (3.6) can be replaced by $ad^{2n+Z}/\{\sqrt{n}(n+1)\}$. □

It follows from (3.2) and the results (3.11)-(3.12) that the quantity δ may be regarded as a "geometrical" indicator, whose deviation from unity measures the level of instability in the orthonormalization of the monomial set (3.1). Because of this, the theorem provides theoretical justification for some intuitively apparent results, concerning the relation between the stability properties of the BKM with monomial basis and the geometry of Ω. For example, we have the following.

(i) For the purposes of stability, the origin 0 should be positioned so that its maximum distance from $\partial\Omega$ is as small as possible.

(ii) Best stability occurs when $\partial\Omega$ is nearly circular and 0 is positioned properly so that δ is close to unity. Conversely, when Ω is a "thin" domain then δ is small and the orthonormalization process is very unstable.

We consider now the use of the ONM with "monomial" basis (1.12), for the mapping of a doubly-connected domain . Ω As before, we define the quantities $I_{n,n}$ by means of

$$I_{n,n} = \min_{u \in \wedge_{n-1}} \{\| \eta_n - u \|^2 / \| \eta_n \|^2 , \tag{3.13}$$

where

$$\wedge_{n-1} = \text{span}\{\eta_1, \eta_2, \ldots\ldots, \eta_{n-1}\}, \tag{3.14}$$

and assume that the basis functions (1.12) are introduced in the order

$$\eta_{2j-1}(z) = z^{j-1}, \eta_{2j} = 1/z^{j+1}; \qquad j = 1,2,....... \tag{3.15}$$

Then, corresponding to Theorem 3.1 we have the following.

Theorem 3.2 Let Ω be a doubly-connected domain whose inner and outer boundary components $\partial\Omega_1$ and $\partial\Omega_2$ are closed piecewise analytic Jordan curves without cusps. Assume that $0 \in \mathrm{Int}(\partial\Omega_1)$, and let

$$d_1 = \min\{|z| : z \in \partial\Omega_1\}, \; d_2 = \max\{|z| : z \,\varepsilon\, \partial\Omega_2\}. \tag{3.16}$$

and

$$c_2 = \mathrm{cap}(\partial\Omega_2). \tag{3.17}$$

Also, let R_1, be the conformal radius of $\mathrm{Int}(\partial\Omega_1)$ at 0, i.e.

$$R_1 = 1/f_1'(0), \tag{3.18}$$

where f_1 is the function that maps conformally $\mathrm{Int}(\partial\Omega_1)$ onto $\{w : |w| < 1\}$, so that $f_1(0) = 0$ and $f_1'(0) > 0$. Then, there exist constants $\alpha > 0$ and $\beta > 0$ so that, for all $n \geq 1$,

$$I_{2n+1,n+1} \leq \alpha n \left\{\frac{C_2}{d_2}\right\}^{2n} \tag{3.19}$$

and

$$I_{2n+2,\,2n+2} \leq \beta n \left\{\frac{d_2}{R_2}\right\}^{2n} . \tag{3.20}$$

Proof Let

$$A_n = \mathrm{span}\ \{1, z, z^2, ..., z^{n-1}\},$$

$$B_n = \mathrm{span}\ \{1/z^2, 1/z^3, ..., 1/z^{n+1}\},$$

and observe that $A_n \subset \wedge_{2n}$ and $ß_n \subset \Delta_{2n+1}$. where, because of the ordering (3.15)

$$\wedge_{2n} = \mathrm{span}\ \{1, 1/z^2, z, \ldots, z^{n-1}, 1/z^{n+1}\}$$

and

$$\wedge_{2n+1} = \mathrm{span}\ \{1, 1/z^2, z, ...z^{n-1}, 1/z^{n+1}, z^n\} .$$

Also, let Ω_1^*. be the image of $\mathrm{Ext}(\partial\Omega_1)$ under the inversion $w = {}^1/_z$,and observe that

$$\operatorname{cap}(\partial\Omega_1^*) = 1/R_1 \tag{3.21}$$

and

$$\max\{|w| : w \in \partial\Omega_1^* = 1/d . \tag{3.22}$$

Then, by using Lemma 3.2, we have that

$$\min_{u\in\wedge_{2n}} \| zn - u \|^2 \le \min_{u\in A_n}\left\{\iint_{\operatorname{Int}(\partial\Omega_2)} |z^n - u|^2 \, dS_z\right\} \le \pi c_2^{2n+2}/(n+1) , \tag{3.23}$$

and

$$\min_{u\in\wedge_{2n+1}} \| 1/z^{n+2} - u \|^2 \le \min_{u\in B_n} \left\{\iint_\Omega |1/z^{n+2} - u|^2 \, dS_z\right\}$$

$$\le \min_{u\in A_n}\left\{\iint_{\Omega_1^*} |w^n - u|^2 dS_w\right\}$$

$$\le \pi R_1^{-(2n+2)}/(n+1). \tag{3.24}$$

The results (3.19) and (3.20). follow at once from (3.23) and (3.24), by observing that the lower bound of Lemma 3.1 also holds when the domain is doubly-connected and d is the distance of 0 from the outer boundary.□

We consider next the use of the BKM or ONM with augmented basis, for the mapping of simply or doubly-connected domains respectively. That is, we consider the case where the basis set is formed by introducing into one of the monomial sets (3.1) or (3.15) a fixed number m of "singular" functions of the type used in [10,13-18]. As before, we denote the basis set by $\{\eta_j\}$ and assume that, corresponding to the ordering $\eta_1,\eta_2\ldots,$ the m singular functions are

$$\eta_{s_1}, \eta_{s_2}, \ldots\ldots\ldots, \eta_{s_m} . \tag{3.25}$$

Then, it follows immediately that the level of instability in the orthonormalization of the set $\{\eta_j\}_{j=1}^n$, $n \ge s_i$, is at least as serious as in the orthonormalization of the n-i monomials in (3.1) or (3.15).For this reason, we cannot expect to improve significantly the stability of the BKM or ONM by introducing singular functions into the monomial basis sets. In fact, the use of an augmented basis may lead to a substantial deterioration of the stability, when one or more of the singular basis functions η_{s_i} are "nearly" linearly dependent on the other basis functions. The situation is characterized by a rapid decrease of the sequence $\{I_{n,s_i}\}$ where, as in (2.12),

$$I_{n,s_i} = \min_{u \in N_{s_i}} \| \hat{\eta}_{s_i} - u \|^2 . \tag{3.26}$$

For this reason, when an augmented basis is used it is essential to measure the level of instability by means of the indicators IS_n or $\hat{IS}_n$, defined by (2.12) and (2.18), rather than by the size of the quantities $I_{n,n}$.

To illustrate the deterioration in stability that the introduction of singular functions may cause, we consider the use of the BKM and assume that, due to the presence of a pole of f at a point $p \in \text{compl}\,\Omega \cup \partial\Omega$, the basis set used is

$$\eta_1(z) = p/(z-p)^2, \quad \eta_j(z) = z^{j-1} \quad ; \; j = 1, 2, \ldots \; ; \tag{3.27}$$

see [10,§2.1], [13,§4.1] and [18,§5]. In this case, if p is "far" from $\partial\Omega$ then the singular function η_1 has the series expansion

$$\eta_1(z) = -\frac{1}{p} \sum_{j=0}^{\infty} j(z/p)^j \quad , \tag{3.28}$$

which converges rapidly in Ω. That is, if p is far from $\partial\Omega$ then there is "near" linear dependence between η_1, and the first few monomials $1, z, z^2, \ldots$ and, because of this, the sequence of indicators $\{I_{n,1}\}$ tends rapidly to zero. More generally, the above situation arises when singular functions are used to reflect pole type singularities, of the form described in [18,§5], at points which are far from the boundary. In general, however, such weak singularities do not affect seriously the rate of convergence of the numerical methods, and do not require special treatment.

4. Convergence Properties

Let Ω be either a simply or doubly-connected domain, and let f denote the associated mapping function. Also, let f_n be the nth approximation to f, obtained by applying to an appropriate basis set either the BKM or the ONM.

Then, it is well-known that for each of the two methods the sequence $\{f_n\}$ converges uniformly to f on any compact subset of Ω. Furthermore, the two books of Gaier [7,8] and the papers by Simonenko [20] and Kulikov [11] contain a number of results which establish the uniform convergence in $\overline{\Omega} = \Omega \cup \partial\Omega$ of the approximations $\{f_n\}$, obtained by using as basis one of the monomial sets (1.11) or (1.12). The purpose of this section is to discuss the significance of the convergence results of [7,8,11,20], and to indicate how they can be used to provide some theoretical explanation of the experimentally observed improvement in accuracy, which is achieved when the monomial basis sets are augmented by introducing appropriate singular functions.

We consider first the use of the BKM for the mapping of a simply-connected domain $\Omega = \mathrm{Int}(\partial\Omega)$ and, as before, we let f_E be the function which maps conformally $\mathrm{Ext}(\partial\Omega)$ onto $\{w : |w| > 1\}$, so that $f_E(\infty) = \infty$ and $\lim_{z\to\infty} f_E'(z) > 0$. Then, the level curves of the region $\mathrm{Ext}(\partial\Omega)$ are defined by

$$C_R = \{z : |f_E(z)| = R, \quad R > 1\}. \tag{4.1}$$

Assume that there exists an $R > 1$ so that f is analytic in $\mathrm{Int}(C_R)$, and observe that this assumption holds whenever $\partial\Omega$ is an analytic curve or more generally, whenever the mapping function f is analytic on $\partial\Omega$. Then the theory of maximal convergence of polynomial approximations of Walsh [25,pp.77-79] leads to the results contained in the following theorem.

<u>Theorem 4.1</u> Assume that the mapping function f is analytic on $\partial\Omega$, and let

$$\hat{R} = \sup\{R : f \text{ is analytic in } \mathrm{Int}(C_R)\} . \tag{4.2}$$

Also, let f_n denote the nth BKM approximation to f, corresponding to the monomial basis (1.11). Then, the following results hold:

(i) For each R, $1 < R < \hat{R}$, there exists a constant M(R), independent of n, so that

$$\max_{z\in\overline{\Omega}} | f(z) - f_n(z) | \le M(R)/R^n . \tag{4.3}$$

(ii) An inequality of the form (4.3) cannot hold with $R>\hat{R}$

(iii) $\overline{\lim_{n\to\infty}}\left\{\max_{z\in\Omega}|f(z)|^{1/n}\right\}=1/\hat{R}$. □

(4.4)

A detailed proof of Theorem 4.1 can be found in Gaier [8,pp,33-35]; see also [7,p.125] and Ellacott [6]. The theorem states that if f is analytic on $\partial\Omega$, and hence analytic in the interior of some level curve C_R, then the convergence of the sequence of polynomial approximations $\{f_n\}$ is maximal in $\overline{\Omega}$, in the sense that

$$\max_{z\in\Omega}|f(z)-f_n(z)|=0(1/R^n),\quad \forall R,\quad 1<R<\hat{R} \quad . \tag{4.5}$$

It follows that we have maximal convergence, of the form (4.5), whenever the boundary $\partial\Omega$ of Ω is an analytic curve. It also follows from the results of Lehman [9], concerning the asymptotic expansion of the mapping function in the neighbourhood of a corner, that we may have convergence of the form (4.5) in some other cases where $\partial\Omega$ is piecewise analytic and involves only corners with interior angles π/q, q = 2,3,4,,.,; see [18,§4], For more general piecewise analytic boundaries we have the following two theorems:

<u>Theorem 4.2</u> Let the boundary $\partial\Omega$ of Ω be a piecewise analytic Jordan curve with parametric equation

$$z=p(s)\quad,\qquad 0\le s\le L\,,\ p^{(j)}(0)=p^{(j)}(L)\quad;\quad j=0,1\quad, \tag{4.6}$$

where s denotes are length, and assume that, for some $k\geq 1$, $p^{(k)}(s)$ is of Lipschitz class 1 in the interval [0,L], Then, there exists a constant C, independent of n, so that

$$\max_{z\in\Omega}|f(z)-f_n(z)|\le C\ \log\ n/n^{k+1}\quad , \tag{4.7}$$

where, as in Theorem 4.1, f_n is the nth BKM approximation to f corresponding to the monomial basis (1.11), □

<u>Theorem 4.3</u> If the boundary $\partial\Omega$ of Ω is a piecewise analytic Jordan curve without cusps then there exist constants C>0 and $\gamma>0$, independent of n, so that

$$\max_{z\in\Omega} |f(z) - f_n(z)| \le C/n^{\gamma}, \tag{4.8}$$

where f_n is as in Theorem 4.1. □

Theorem 4.2 is a special case of a slightly more general result due to Suetin [22]; see Gaier [8,p.40] and compare with Walsh [25,Theor.1,p.371]. The theorem applies only to domains bounded by curves with continuously varying tangents. For example, let Ω be the domain whose boundary consists of the half circle

$$\Gamma_2 = \{z = x + iy : |z| = 1, x \le 0\} \tag{4.9}$$

and the half ellipse

$$\Gamma_2 = \{z = x + iy : x^2/a^2 + y^2 = 1, x > 0, \ a > 1\} . \tag{4.10}$$

In this case, Theorem 4.1 is not applicable, because f has branch point singularities at the points $\pm i$, where the two curves r_j; $j = 1,2$, meet each other. This follows from the results of Lehman [9], which show that the asymptotic expansions of f at the points $z_1 = i$ and $z_2 = -i$ involve respectively the singular functions

$$g_j(z) = (z_j - z)^2 \operatorname{Log}(z_j - z) \quad ; \quad j = 1, 2 . \tag{4.11}$$

However, the boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$ satisfies the smoothness condition of Theorem 4.2 with $k = 1$, and thus (4.7) gives

$$\max_{z\in\Omega} |f(z) - f_n(z)| \le \text{const} . \log n/n^2. \tag{4.12}$$

It is of interest to observe that the Maclaurin series expansion of the singular functions (4.11) satisfy

$$\max_{|z|\le 1} \left| g_j(z) - \sum_{r=0}^{n} \left\{ g_j^{(r)}(0)/r! \right\} z^r \right| \le 1/n/(n-1) \quad ; \quad j = 1, 2 \quad . \tag{4.13}$$

In other words, the Maclaurin expansions of (4.11) display a similar type of convergence as the sequence $\{f_n\}$.

Theorem 4.3 is a recent result due to Simonenko [20]. This important theorem establishes the uniform convergence in $\overline{\Omega} = \Omega \cup \partial\Omega$ of the BKM polynomial approximations to f, associated with any piecewise analytic boundary without cusps.

Unfortunately, however, the theorem does not provide any information about the magnitudes of the constants C and γ in (4.8). A more recent result of Kulikov [11] gives a domain dependent constant $\gamma_1 > 0$ such that (4.8) holds for all $\gamma \varepsilon (0,\gamma_1)$. However, this constant γ_1 of [11] cannot exceed 1/16 for any domain. For this reason, the estimate of the rate of convergence implied by the result of [11] can be very pessimistic; see §5,Ex.5.3.

As was remarked by Ellacott [6,p.189], in some special cases, Theorem 4.1 can be used to explain the improved convergence which is achieved when rational functions, that reflect the dominant pole singularities of f in Ext($\partial\Omega$), are introduced into the basis set (1.11). To see this, we let f be analytic on $\partial\Omega$, and assume that its analytic extension, across $\partial\Omega$, has simple poles at the points

$$p_j \in \mathrm{Ext}(\partial\Omega) \quad ; \qquad j = 1(1)k \ , \tag{4.14}$$

where

$$|f_E(p_1)| = |f_E(p_2)| = \dots = |f_E(p_k)|. \tag{4.15}$$

We also assume that the other singularities of the analytic extension of f occur at the points $p_{k+1}, p_{k+2}, \ldots\ldots$, where

$$|f_E(p_1) < |f_E(p_{k+1})| \leq |f_E(p_{k+2})| \leq \ldots\ldots\ldots \tag{4.16}$$

Then, from Theorem 4.1 we have that

$$\max_{z\in\Omega} |f(z)| = 0(1/R^n) \ , \quad \forall R, \ \ 1 < R < \hat{R}, \tag{4.17a}$$

where

$$\hat{R} = |f_E(p_1)|. \tag{4.17b}$$

We now let $f_n^{(A)}$ be the nth BKM approximation corresponding to the augmented basis

$$\begin{aligned} \eta_j(z) &= -p_j/(z-p_j)^2 \quad ; \quad j = 1(1)k \ , \\ \eta_{k+j}(z) &= 2^{j-1} \quad ; \qquad\qquad j = 1,2,\ldots \ , \end{aligned} \tag{4.18}$$

which reflects the dominant singularities of f; see [10,§2.1] and [18,§5]. Then the k functions η_j; j = 1(1)k, "cancel out" the nearest singularities of f at the k points (4,14) in the sense that the approximations $f_n^{(A)}$ satisfy

$$\max_{z\in G} |f(z) - f_n^{(a)}(z)| = 0(1/R^n)\ ,\ \forall R,\ \ 1 < R < \hat{R}_A \qquad (4.19a)$$

where G is any compact subset of Ω and

$$\hat{R}_A = |f_E(p_{k+1})| > \hat{R}. \qquad (4.19b)$$

In fact our numerical results in Section 5 suggest that, in some cases, (4.19) might also hold with G replaced by $\overline{\Omega}$, but we have not been able to prove this.

The above discussion explains the improvement in convergence which is achieved by using an appropriate augmented basis, in cases where f is analytic on $\partial\Omega$ and the singular functions in the basis set reflect exactly the dominant singularities of f in compl$(\overline{\Omega})$, In general however the situation regarding the convergence of approximations obtained by using an augmented basis is not clear. For example, the numerical experiments of [10] and [13,14] indicate clearly that substantial improvement in accuracy is achieved when, in the presence of a "singular" corner at $z_0 \in \partial\Omega$, the basis set contains functions of the form

$$(z - z_0)^{\beta}\{\mathrm{Log}(z - z_0)\}^{m}\ ,$$

where the real number $\beta \geq 0$ and the integer $m \geq 0$ depend on the size of the angle at z_0. However, such singular functions do not reflect exactly the corner singularities of f on $\partial\Omega$. For this reason, when $\partial\Omega$ involves singular corners we do not expect the BKM approximations to f to satisfy convergence results of the form (4.5), even when an augmented basis is used. We can only speculate that in the presence of corner singularities the convergence is always of the type described in (4.8), and that the use of an appropriate augmented basis leads to a larger exponent γ.

We consider next the convergence of the ONM approximations to the function f, which maps conformally a finite doubly-connected domain Ω onto a circular annulus $\{w : 1 < |w| < M\}$. As before, we let $\partial\Omega_1$ and $\partial\Omega_2$ be respectively the inner and outer components of the boundary $\partial\Omega$, and assume that $0 \in \mathrm{Int}\,(\partial\Omega_1)$. We also recall that in the ONM the approximations to f are obtained after first determining a least squares approximation to the function

$$H(z) = f'(z)/f(z) - 1/z . \tag{4.20}$$

Then, in the case where both $\partial\Omega_1$ and $\partial\Omega_2$ are analytic Jordan curves we have a result due to Gaier [7,p.250], which establishes the uniform convergence in $\bar{\Omega} = \Omega \cup \partial\Omega$ of the ONM approximations corresponding to the use of the monomial basis (1.12). More precisely, the result of [7] states that if $\partial\Omega_1$ and $\partial\Omega_2$ are analytic Jordan curves then, for some $\rho<1$, there exists a constant A independent of n so that

$$\max_{z\in\bar{\Omega}} |H(z) - H_n(z)| \le A\rho^n , \tag{4.21}$$

where H_n denotes the nth ONM approximation to H, corresponding to the monomial basis (1.12). (We point out that Gaier in his book does not consider the ONM, but an equivalent variational method.)

Our purpose here is to express the above convergence result of [7] in a slightly more detailed form, analogous to that of Theorem 4.1. In order to do this we need to make the following four observations:

(i) The function H of (4.20) can be expressed as

$$H = H_I + H_E , \tag{4.22}$$

where H_I is analytic in $\mathrm{Int}(\partial\Omega_2)$, and $z^2 H_E(z)$ is analytic in $\mathrm{Ext}(\partial\Omega_1)$ including the point at infinity. This means that the function

$$H_E^*(z) = H_E(1/z)/z^2 \tag{4.23}$$

is analytic in $\mathrm{Int}(\partial\Omega_1^*)$,, where $\partial\Omega_1^*$ is the image of $\partial\Omega_1$ under the inversion $z \to 1/z$.

(ii) If the mapping function f is analytic on $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ then the function H is also analytic on $\partial\Omega$.

(iii) Because of (i), the least squares property of the ONM approximation H_{2n} implies that

$$\| H - H_{2n} \|_\Omega \le \inf_{u\in A_n} \| H_I - u \|_{\mathrm{Int}(\partial\Omega_2)} + \inf_{u\in A_n} \| H_{R}^* - u \|_{\mathrm{Int}(\partial\Omega_1^*)} , \tag{4.24}$$

where $A_n = \mathrm{span}\{1, z, z^2, \ldots, z^{n-1}\}$, and where we used $\|\bullet\|_G$ to denote the norm of the space $L_2(G)$.

(iv) If G is a bounded domain of finite connectivity then convergence in the norm of the space $L_2(G)$ implies uniform convergence in every compact subset of G.

We also need to define the two families of curves

$$C_{R1} = \{z : |f_{I1}(z)| = R,\ R < 1\} \tag{4.25}$$

and

$$C_{R1} = \{z : |f_{E2}(z)| = R, R > 1\} \tag{4.26}$$

where f_I is the interior mapping function associated with $\operatorname{Int}(\partial\Omega_1)$, and f_{E2} is the exterior mapping function associated with $\operatorname{Ext}(\partial\Omega_2)$. Then, the observations (i)-(iv) in conjunction with the theory of maximal convergence of polynomial approximations lead easily to the following theorem.

<u>Theorem 4.4</u> Assume that the mapping function f is analytic on $\partial\Omega$, and let

$$\hat{R}_1 = \inf\{R : H \text{ is analytic in } \operatorname{Ext}(C_{R1}) \cap \operatorname{Int}(\partial\Omega_2)\} \tag{4.27}$$

and

$$\hat{R}_2 = \sup\{R : H \text{ is analytic in } \operatorname{Int}(CR_2) \cap \operatorname{Ext}(\partial\Omega_1)\}\,. \tag{4.28}$$

Also, let f_k denote the kth ONM approximation to f, corresponding to the monomial basis (1.12). Then, for each R, $1 < R < \min(1/\hat{R}_1, \hat{R}_2)$, there exists a constant $M(R)$, independent of n so that

$$\max_{z\in\Omega} |f(z) - f_{2n}(z) \le M(R)/R^n. \square \tag{4.29}$$

Theorem 4.4 applies only to a slightly wider class of domains than the class of doubly-connected domains whose boundary components $\partial\Omega_1$ and $\partial\Omega_2$ are both analytic Jordan curves. Unfortunately, we do not know of any results, similar to those of Theorems 4.2 and 4.3, which establish the uniform convergence of the ONM approximations in $\overline{\Omega}$, when $\partial\Omega_1$ and $\partial\Omega_2$ are more general piecewise analytic Jordan curves. Regarding the use of augmented basis sets, the remarks we made in connection with the BKM also apply to the ONM. In the case of the ONM however, it has been shown in [17] that it is not possible to reflect exactly the singularities of H, even when the singular points are in compl $(\Omega \cup \partial\Omega)$.

5. Numerical Examples

In this section we present four numerical examples, illustrating the stability and convergence results of Sections 3 and 4. The computational details of the BKM and ONM procedures used in these examples are as described in [10,§3] and [15,§5] respectively. In particular, when Ω is simply-connected then the estimate E_n, of the maximum error in the modulus of the nth BKM approximation f_n, is given by

$$E_n \max_j \left| 1 - | f_n(z_j) | \right| \quad , \tag{5.1}$$

where $\{z_j\}$ is a set of "boundary test points" on $\partial\Omega$. Similarly, when Ω is doubly-connected then the estimate E_n, corresponding to the nth ONM approximation f_n, is given by

$$E_n = \max \left\{ \max_j \left| 1 - | f_n(z_{1,j}) \right| , \max_j \left| M_n - | f_n(z_{2,j}) | \right| \right\} \quad , \tag{5.2}$$

where $\{z_{i,j}\}$ and $\{z_{2,j}\}$ are two sets of boundary test points on $\partial\Omega_1$ and $\partial\Omega_2$ respectively, and M_n is the nth ONM approximation to the conformal modulus M of Ω. As was remarked in Section 1, the approximation M_n is computed by using formula (1.10).

Each of the BKM and ONM algorithms computes recursively a sequence of approximations f_n. Also, each algorithm includes a termination criterion for terminating the process at some "optimum" value n = Nopt, which gives a "best" approximation f_{Nopt} in some pre-defined sense. In [10,13,15], the number Nopt is determined by using essentially the following procedure:

A minimum number n_{min} of basis functions to be used is defined and. for each $n \geq n_{min}$, the error estimate E_n is computed. If at the (n+1)th stage the inequality

$$E_{n+1} < E_n \tag{5.3}$$

is satisfied then the approximation f_{n+2} is computed. When for a certain value of n, due to numerical instability, (5.3) no longer holds then the process is terminated and n is taken to be the optimum number Nopt of basis functions. Naturally, the number Nopt depends critically on the precision of the computer arithmetic used, and on the stability and convergence properties of the numerical method. Regarding the choice of the parameter n_{min}, when a monomial basis is used then $n_{min}=2$ is appropriate. However, when an augmented basis is used then n_{min} should always be chosen sufficiently large so that the basis set $\{\eta_1, \eta_2, \ldots, \eta_{n_{min}}\}$ includes the "main" singular functions.

A drawback of the above procedure for determining Nopt is that it does not take into account the possibility of non-monotonic convergence. (Even with exact arithmetic, there is no guarantee that sequence $\{E_n\}$ will decrease monotonically.) In the present paper we attempt to remedy this shortcoming, by determining the numbers $\tilde{E}_n$ defined by

$$\tilde{E}_{n_{min}} = E_{n_{min}}, \tilde{E}_n = \min\{\tilde{E}_n, \tilde{E}_{n-1}\} \ ; \ n = n_{min}+1, \ n_{min}+2, \ldots\ldots, \tag{5.4}$$

and taking as Nopt the first $n \geq n_{min}$ for which

$$\tilde{E}_{n+j} = \tilde{E}_n \ ; \quad j = 1,2,3 \ . \tag{5.5}$$

In general, the procedure for determining Nopt also includes a termination criterion which safeguards against "slow" convergence. In the present paper however, because of the nature of our investigation, we introduce such a criterion only in Ex. 5.3, where we anticipate very slow convergence. More precisely, in this example we take Nopt to be the first $n \geq n_{min}$ for which either the equalities (5.5) or the inequality

$$\tilde{E}_{n+5} > 0.9 \quad \tilde{E}_n \tag{5.6}$$

are satisfied.

For comparison purposes, we present numerical results obtained by implementing our BKM and ONM Fortran algorithms on each of the following three computers, in the precision ε indicated.

COMP1 : <u>Honeywell level 68 computer.</u>

Single precision: $\varepsilon = 2^{-26} \simeq 1.5\text{x } 10^{-8}$. (5.7)

C0MP2 : <u>CDC 7600 computer.</u>

Single precision: $\varepsilon = 2^{-47} \simeq 7.1 \text{ x } 10^{-15}$ (5.8)

C0MP3 : <u>IBM Amdahl computer.</u>

Extended precision: $\varepsilon = 2^{-100} \simeq 3.1 \text{ x } 10^{-33}$. (5.9)

In presenting the results, we use the abbreviations BKM/MB and BKM/AB to denote respectively the BKM with monomial basis (1.11) and with augmented basis.Similarly, we use ONM/MB and ONM/AB to denote the ONM with monomial basis (1.12) and with augmented basis. Also, in the tables of results we use the abbreviation a(-M) to denote a 10^{-M} .

Example 5.1 BKM for ellipse

$$\Omega = \{(x,y) : x^2/a^2 + y^2 < 1, \quad a > 1\} . \tag{5.10}$$

Monomial basis. Because Ω has two-fold rotational symmetry about the origin, the monomial basis is taken to be

$$\eta_j(z) = z^{2(j,-1)} \quad ; \qquad j = 1,2,3 \tag{5.11}$$

Augmented basis. In this case, the exact mapping function f is given by an elliptic sine; see e.g. [12,Eq.51,p,296]. From this it follows that f has simple poles at the infinite array of points

$$z = i(a^2-1)^{\frac{1}{2}} \sinh(2k+1)\sinh^{-1}(2a/(a^2-1))\} \quad ; \quad k = 0,\pm1,\pm2, \ldots , \tag{5.12}$$

on the imaginary axis.

The augmented basis is formed so that it reflects the two dominant singularities of f, i.e. the two simple poles at the points $\pm ip_1$, where

$$p_1 = 2a/(a^2-1)^{\frac{1}{2}} . \tag{5.13}$$

Because of the symmetry, this is done by introducing into the monomial set (5.11) the single singular function $\{z/(z^2+p_1^2)\}'$. That is, the augmented basis is

$$(\eta_1(z) = \{z/(z^2+p_1^2)\}' \ , \eta_{j+1} = \eta^{2(j-1)} \quad ; \quad j = 1,2,3,\ldots\ldots \quad ; \tag{5.14}$$

see [10 , Ex.5].

Optimum results. The values of Nopt and E_{Nopt} obtained by applying the BKM/MB and the BKM/AB to the four ellipses corresponding to the values a = 1.2, 2.0, 4.0 and 8.0 are listed in Table 5.1. For each geometry, the table contains the results obtained by carrying out the computations on each of the three computers COMP1, COMP2 and COMP3, in the precision given respectively by (5.7), (5.8) and (5.9).

TABLE 5.1

Values of Nopt and E_{Nopt}

		COMP1 ε = 1.5(-8)		COMP2 ε = 7.1(-15)		COMP3 ε - 3.1(-33)	
a	BKM/	Nopt	E_{Nopt}	Nopt	E_{Nopt}	Nopt	E_{Nopt}
1.2	MB	8	6.7(-8)	15	5.0(-14)	32	8.3(-31)
	AB	2	3.4(-7)	3	5.8(-13)	6	2.9(-28)
2.0	MB	11	1.7(-5)	19	2.6 (-9)	42	5.1(-19)
	AB	4	1.0 (-7)	6	3.5(-13)	12	5.6(-28)
4.0	MB	8	1.7(-2)	14	1.2(-3)	32	5.9 (-7)
	AB	7	2.2(-7)	11	7.5(-12)	21	3.0(-23)
8.0	MA	7	1.7 (-1)	12	6.6(-2)	27	1.5(-3)
	BA	9	1.5(-5)	16	5.6(-8)	27	8.2(-15)

From the table, we observe that in the case a =1.2 the optimum BKM/MB approximations are accurate to almost machine precision. We also observe that, for all four values of a, the use of the augmented basis (5.14) improves considerably the rate of convergence of the method. However, when a =1.2 the optimum BKM/AB approximations are not as accurate as those obtained by using the monomial basis (5.11). For the other three values of a the use of (5.14) leads to a substantial improvement in accuracy. In what follows, we shall attempt to explain the above observations by examining the stability and convergence properties of the BKM/MB and BKM/AB.

Stability. The exterior mapping function f_E associated with Ext($\partial\Omega$) is

$$f_E(z) = \{z + (z^2 - a^2 + 1)^{\frac{1}{2}}\}/(a+1) \quad . \tag{5.15}$$

Thus, cap ($\partial\Omega$) =(1+a)/2 and, since max $\{|z| : z \in \partial\Omega\}$ = a, the quantity δ defined by (3.10) is given by $\delta = \{(1 + a)/2a\}^2$.

In this example, because of the symmetry of Ω, the monomial set (5.11) involves only the even powers of z. For this reason, it follows from Theorem 3.1

that the rate of decrease of the sequence of indicators $\{I_{n,n}\}$, associated with the set (5.11), is at least as rapid as that of the sequence

$$\{(2n-1)\Delta^n\} \ , \text{ where } \ \Delta = \ \delta^2 = \{(1+a)/2a\}^4 \quad . \tag{5.16}$$

Furthermore, since in this case $\partial\Omega$ is analytic and the bound (3.12) holds, we expect the rate of decrease of (5.16) to reflect closely that of $\{I_{n,n}\}$.

We note that for all $a > 1$,

$$1/16 < \Delta < 1 \ , \tag{5.17}$$

where the limiting values $\Delta = 1$ and $\Delta = 1/16$, for "perfect" and "worst" stability, correspond respectively to the cases where Ω is the unit disc and, by rescaling, the straight line slit joining the points ± 1. We also note the following in connection with the limiting value $\Delta = 1/16$. As $a\to\infty$ the matrix $\hat{G}_n$ of (2.7), corresponding to the set (5.11), tends to the Gram matrix $\hat{L}_n$ associated with the construction of even degree Legendre polynomials. As is well-known, the condition number of the matrix $\hat{L}_n$ increases with n at least as rapidly as the sequence $\{16^n\}$; see e.g. [24].

For the values $a = 1.2$, 2.0, 4.0 and 8,0, considered in this example, the corresponding values of Δ are respectively

$$0.706\ 07 \ , \quad 0.316\ 41 \ , \quad 0.152\ 59 \quad \text{and} \quad 0.100\ \ 11 \ . \tag{5.18}$$

Thus, we expect the level of instability in the case $a = 2.0$ to be substantially higher than in the case $a = 1.2$. Similarly, we expect the levels of instability in the two cases $a = 4.0$ and $a = 8.0$ to be higher than in the case $a = 2.0$.

In Table 5.2 we list COMP3 values of the instability indicators $I_{n,n}$, $\widetilde{IS}_n$ and $\{C(G_n)\}^{-1}$, associated with the use of both the monomial and augmented basis sets (5.11) and (5.14). The values $I_{n,n}$ and $\widetilde{IS}_n$ are determined during the orthonormalization process from (2.17b) and (2.17)-(2.18), by allowing when necessary the process to continue after the value Nopt is reached. The values of $\{C(\hat{G}_n)\}^{-1}$ are determined from (2.8), by computing the largest and smallest eigenvalues of $\hat{G}_n$ by means of the NAG Library Subroutine F02AAF. In the table we also compare the observed and theoretical rates of decrease of the sequence $\{I_{n,n}\}$, associated with the monomial set (5.11). We do this by comparing the computed values

$$\Delta_n = \{(2n-3)I_{n,n} /(2n-1)I_{n-1,n-1}\} \quad , \tag{5.19}$$

with the exact values of Δ given in (5.18).

TABLE 5.2

Instability -indicators

		$I_{n,n}$		$\tilde{I}S_n$		$\{C(\hat{G}_n)\}^{-1}$			
a	n	MB	AB	MB	AB	MB	AB	Δ_n	Δ
1.2	6 12 18	9(-1) 1(-1) 4(-2)	5(-3) 1(-3) 2(-2)	4(-1) 2(-2) 1(-3)	4(-10) 3(-22) 1(-31)	1(-1) 3(-3) 5(-4)	8(-5) 4(-23) 2(-32)	0.639 0.676 0.687	0.706
2.0	6 12 18	2(-2) 3(-5) 3(-8)	7(-3) 9(-6) 1(-8)	4(-3) 3(-7) 2(-11)	1(-4) 5(-10) 1(-15)	1(-5) 2(-8) 5(-13)	1(-5) 2(-11) 4 (-17)	0.292 0.303 0.308	0.316
4.0	6 12 18	7(-4) 1(-8) 2(-13)	2(-3) 3(-8) 5(-13)	1(-4) 5(-11) 2(-17)	3(-4) 1(-10) 4(-17)	1(-5) 2(-12) 4(-19)	4(-5) 6(-12) 9(-19)	0.139 0.146 0.147	0.153
8.0	6 12 18	1(-4) 2(-10) 3(-16)	9(-4) 1(-9) 2(-15)	2(-5) 6(-13) 1(-20)	2(-4) 6(-12) 1(-19)	2(-6) 2(-14) *	2(-5) 3(-13) *	0.092 0.096 0.097	0.100

* *The eigenvalues of* $\hat{G}_n$ *cannot be computed to sufficient accuracy.*

The results of Table 5.2 confirm completely the theoretical predictions made in Section 3. In particular, we observe that the use of the augmented basis (5.14) causes the level of instability to increase substantially in the case a =1.2. The reason for this is that when a =1.2 the two points ± ip are "far " from ∂Ω and, because of this, there is "near" linear dependence between the singular function $\{z/(z^2+p_1^2)\}'$ and the first few terms of (5.11). The same remark, but to a much lesser extent, also applies to the case a = 2.0. By contrast, in the two cases a =4.0 and a =8.0, when the points $\pm i\,p_1$ are close to $\partial\Omega$ and the singularities are much more serious, the introduction of the singular function into the set (5.11) does not lead to a deterioration of the stability.

Convergence, As before, we let $\pm$ ip be the singular points of f nearest to $\partial\Omega$, and observe that the next nearest singular points are $\pm$ ip where

$$p_2 = p_1\{3 + 4p_1^2/(a^2 - 1)\} \ ; \tag{5.20}$$

see Eq. (5.12). We also let f_E be the exterior mapping function (5.15). Then, it follows from Section 4 that the BKM/MB and BKM/AB approximations of f satisfy respectively

$$\max_{z\in\overline{\Omega}} | f(z) - f_n(z) | = 0(1/R^n) \ , \forall \ R \ , \quad 1 < R < \hat{R}_1 , \tag{5.21}$$

and

$$\max_{z\in G} | f(z) - f_n(z) | = 0(1/R^n) , \ \forall \ R \ , \ 1 < R < \hat{R}_2 , \tag{5.22}$$

where G is any compact subset of Ω and where, because the basis sets (5.11) and (5.14) reflect the symmetry of Ω,

$$\hat{R}_1 = | f_E(\pm ip_1)^2 = (a+1)/(a-1) \ , \tag{5.23a}$$

and

$$\hat{R}_2 = | f_E(\pm ip_2)|^2 = \{p_2^2 + a^2 - 1)^{\frac{1}{2}}\}^2/(a+1)^2 \ . \tag{5.24b}$$

For the four cases a = 1.2, 2.0, 4.0 and 8.0, the values of the constants (5.22) and (5.23) are as follows:

$$\left.\begin{array}{lll} (i) & a = 1.2 : \hat{R}_1 = 11.0 , & \tilde{R}_2 = 161\ 051.0 , \\ (ii) & a = 2.0 : \hat{R}_1 = 3.0 , & \hat{R}_2 = 243.0 , \\ (iii) & a = 4.0 : \hat{R}_1 = 1.66\ 667 , & \hat{R}_2 = 12.8\ 601 , \\ (iv) & a = 8.0 : \hat{R}_1 = 1.28\ 571 , & \hat{R}_2 = 3.51\ 336 , \end{array}\right\} \tag{5.24}$$

Therefore, for the geometry under consideration, the theory indicates clearly the serious effect that singularities close to $\partial\Omega$ have on the rate of convergence of the BKM/MB. Also, (5.22) provides some explanation for the observed improvement in convergence which is achieved when the monomial basis set (5.11) is replaced by the augmented set (5.14)

In Table 5.3 we examine further the rates of convergence of the BKM/MB and BKM/AB by listing values of the ratios

$$r_n = E_n / E_{n-1} \ , n \simeq Nopt / 2+2 \ , \tag{5.25}$$

TABLE 5.3

Convergence ratios

a	BKM/MB	BKM/AB
1.2	r_{18} =11.000 000 000 3 $\hat{R}_1$ =11.0	r_5 = 1.619(5) $\hat{R}_2$ = 1.6 11(5)
2.0	r_{24} =2.999 999 999 7 $\hat{R}_1$ = 3.0	r_9 = 243.9 $\hat{R}_2$ = 243.0
4.0	r_{18} =1.666 1 $\hat{R}_1$ = 1.666 6 7	r_{12} =12.94 $\hat{R}_2$ =12.86
8.0	r_{18} = 1.276 $\hat{R}_1$ = 1	r_{18} = 3.29 $\hat{R}_2$ = 3.51

where the E_j are COMP3 error estimates, and comparing them with the corresponding values $\hat{R}_1$ and $\hat{R}_2$ given in (5.24). The results of the table suggest that, for the ellipse under consideration, Theorem 4.1 gives a sharp estimate of the rate of convergence of the BKM/MB. The results also suggest that, in this case, the BKM/AB approximations satisfy (5.22) with G replaced by $\overline{\Omega}$.

Example 5.2 BKM for rectangle

$$\Omega = \{(x,y) : |x| < a , |y| < 1 , a \geq 1 \} \tag{5.26}$$

Monomial basis. Because of the rotational symmetry the monomial basis set is taken to be

$$\eta_j(z) = z^{4(j-1)} ; \qquad j = 1, 2, 3, ... , \tag{5.27a}$$

when $a = 1$, i.e. when Ω is a square, and

$$\eta^j(z) = z^{2(j-1)} ; \qquad j = 1, 2, 3, ... , \tag{5.27b}$$

when $a > 1$.

Augmented basis. In this case, it follows at once from the Schwarz reflection principle that the mapping function f has simple poles at the mirror images of the origin with respect to each of the four sides of Ω, i.e. at the four points

$$z = \pm 2i \quad \text{and} \quad z = \pm 2a . \tag{5.28}$$

More precisely, the repeated application of the reflection principle shows that f has simple poles at all points

$$z = 2ma + i2n \; ; \quad m,n = 0,\pm 1,\pm 2,\ldots, \quad m+n = \text{odd} . \tag{5.29}$$

The augmented basis is formed, as in Levin et al [10,Ex.1], by introducing into the appropriate monomial set (5.27) the functions that reflect the singularities of f at the four points (5.28). However, when $a > 1$ we also consider the use of the augmented basis that reflects the singularities of f only at the two points $\pm 2i$ nearest to $\partial\Omega$ Thus, because of the symmetry, the two augmented basis sets considered are as follows:

AB: (i) When $a = 1$,

$$\eta_1(z) = \{z / z^2 + 16)\}', \; \eta_{j+1}(z) = z^{4(j-1)} ; \; j = 1,2,\ldots\ldots\ldots \tag{5.30a}$$

(ii) When a> 1,

$$\eta_1(z) = \{z/(z^2+4)\}', \; \eta_2(z) = \{z / z^2 - 4a^2)\}', \; \eta_{j+2}(z) = z^{2(j-1)} ;$$

$$j = 1,2,\ldots \quad . \tag{5.30b}$$

<u>AB'</u>: When $a > 1$,

$$\eta_1(z) = \{z / z^2 + 4)\}', \; \eta_{j+1}(z) = z^{2(j-1)} ; \quad j = 1,2,\ldots.. \quad . \tag{5.31}$$

Optimum results. The COMP1, COMP2, and COMP3 values of Nopt and E_{Nopt} obtained by applying the BKM/MB, BKM/AB and BKM/AB' to the four domains corresponding to a = 1,2,4 and 8 are given in Table 5.4. (We observe that when $a = 1$ the value of E_{Nopt}(COMP2) given in Table 5.4 is considerably less than the corresponding value obtained by Levin et al [10,Ex.1], also on COMP2. This discrepancy must be due to the slightly different methods used by Levin et al for performing the orthonormalization and for determining Nopt.)

Stability. Let

$$\delta = \{\text{cap}(\partial\Omega)\}^2 / (1 + a^2) \tag{5.32}$$

and observe that for any value of a, $\text{cap}(\partial\Omega)$ can be determined from the exact formula of Bickley [3]. Then, it follows from Theorem 3.1 that the sequence

TABLE 5.4

Values of Nopt and E_{Nopt}

		COMP1 $e \simeq 1.5(-8)$		COMP2 $e \simeq 7.(-15)$		COMP3 $\varepsilon \simeq 3.1(-33)$	
a	BKM/	Nopt	E_{Nopt}	Nopt	E_{Nopt}	Nopt	E_{Nopt}
1	MB	7	8.8(-7)	13	6.3(-12)	26	1.0(-24)
	AB	3	1-2(-6)	5	2.1(-11)	11	1.9(-24)
2	MB	11	7.6(-4)	22	2.5(-6)	45	3.3(-13)
	AB	6	2.1(-6)	10	1.0(-10)	21	2.1(-20)
	AB'	9	3.0(-6)	15	2.3(-10)	19	6.3(-22)
4	MB	8	4.9(-2)	18	2.4(-3)	37	3.5(-6)
	AB	7	1.6(-5)	11	2.3(-8)	24	3.7(-17)
	AB'	8	2.0(-6)	13	2.0(-10)	28	2.4(-21)
8	MB	8	2.0(-1)	12	9.0(-2)	31	2.4(-3)
	AB	8	2.1(-4)	14	7.6(-6)	27	3.4(-12)
	AB'	8	1.2(-4)	13	7.9(-7)	31	1.2(-13)

of indicators $\{I_{n,n,}\}$, associated with the BKM/MB, decreases at least as rapidly as the sequence

$$\{N\Delta^n\} \tag{5.33}$$

where, because of the form of the sets (5.27),N = 4n-1 and $\Delta = \delta^4$ when a-1, and N = 2n-1 and $\Delta = \delta^2$, when a >1. The values of Δ corresponding to a =1,2,4 and 8 are respectively

$$0.235\ 47\ ,\quad 0.374\ 74\ ,\quad 0.222\ 22\quad \text{and}\quad 0.138\ 91\ . \tag{5.34}$$

In Table 5.5 we list COMP3 values of the instability indicators $\tilde{I}S_n$ associated respectively with the use of the basis sets (5.27), (5.30) and (5.31). In the table we also compare the observed and theoretical rates of decrease of the sequence $\{I_{n,n}\}$, associated with (5.27), by comparing the exact values of Δ, given in (5.34), with the computed values Δ where, when a-1,

$$\tilde{\Delta} = \{59 I_{25,25} / 99 I_{15,15}\}^{1/10} \tag{5.35a}$$

and when $a > 1$,

$$\widetilde{\Delta} = \{29\, I_{25,25} \,/\, 49\, I_{15,15}\}^{1/10} \tag{5.35b}$$

TABLE 5.5

Instability -indicators $\widetilde{I}Sn$

a	n	MB	AB	AB'	$\widetilde{\Delta}$	Δ
1	5 15 25	2.4(-2) 8.4(-10) 8.3(-18)	4.0(-7) 3.4(-25) *	- - -	0.236	0.235
2	5 15 25	3.2(-2) 1.6(-8) 2.5(-15)	3.5(-4) 5.1(-17) 3.7(-30)	3.0 (-2) 2.2(-9) 2.8 (-16)	0.377	0.375
4	5 15 25	1.8(-3) 7.2(-13) 3.4(-22)	6.0(-5) 4.1(-21) *	5.4(-3) 1.8(-12) 3.1 (-22)	0.222	0.222
8	5 15 25	3.4 (-4) 8.5(-16) 1.0(-29)	2.1(-5) 5.8(-24) *	2.8(-3) 6.8(-15) 3.4(-26)	0.134	0.139

* *The value is less than* $\varepsilon \underset{\sim}{~} 3.1(-33)$, *the precision of COMP3.*

Convergence. Let f_E be the exterior mapping function associated with Ext $(\partial\Omega)$, and recall that the singularities of f in comp (Ω) occur at the points (5.29). Then, since f is analytic on $\partial\Omega$, the BKM/MB and BKM/AB approximations to f satisfy respectively (5.21) and (5.22) where, because of the form of the sets (5.27), (5.30) and (5.31), the values of $\hat{R}_1$ and $\hat{R}_2$ are as follows:

<u>BKM/MB</u>: $\hat{R}_1 = |f_E(2i)|^4$ when a= 1 and $\hat{R}_1 = |f_E(2i)|^2$ when $a = 2,4,8$.

BKM/AB : $\hat{R}_2 = |f_E(2a+4i)|^4$ when a= 1, $\hat{R}_2 = |f_E(2a+4i)|^2$ when $a = 2$, and $\hat{R}_2 = |f_E(6i)|^2$ when $a = 4,8$.

<u>BKM/AB'</u>: $\hat{R}_1$ $|f_E(2a)|^2$ when $a = 2$, and $\hat{R}_2 = |f_E(6i)|^2$ when $a = 4,8$.

In Table 5.6 we compare the values of $\hat{R}_1$ and $\hat{R}_2$, determined from the expressions given above, with the observed rates of convergence given by the ratios

TABLE 5.6

Convergence ratios

a	BKM/MB	BKM/AB	BKM/AB'
1	$r_{16} = 8.8820$ $\hat{R}_1 = 8.8839$	$r_{10} = 257.5$ $\hat{R}_2 = 205.9$	–
2	$r_{25} = 1.9866$ $\hat{R}_1 = 1.989$	$r_{13} = 11.31$ $\hat{R}_2 = 10.45$	$r_{17} = 4.623$ $\hat{R}_2 = 4.664$
4	$r_{21} = 1.4839$ $\hat{R}_1 = 1.4862$	$r_{15} = 5.600$ $\hat{R}_2 = 5.596$	$r_{17} = 5.422$ $\hat{R}_2 = 5.596$
8	$r_{18} = 1.2713$ $\hat{R}_1 = 1.2422$	$r_{16} = 2.785$ $\hat{R}_2 = 2.813$	$r_{18} = 2.758$ $\hat{R}_2 = 2.813$

$$r_n = \{E_n/E_{n-6}\}^{1/6} \quad , \quad n \simeq \ \text{Nopt}/2+3 \ , \tag{5.36}$$

where the E_j are COMP3 error estimates. (In this case, the mapping function f_E is not known in closed form. For this reason, the values of R listed in the table are only estimates, obtained by computing BKM approximations to f_E in the manner described in [14,Ex.3.2].)

All the remarks made in connection with the results of Ex.5.1 also apply to the results of the present example. In particular, we observe that in all three cases where $a > 1$ the level of instability in the BKM/AB is substantially higher than in the BKM/AB', This is of course due to the fact that when $a = 2,4$ and 8 the points $\pm 2a$ are "far" from $\partial\Omega$, and there is near linear dependence between the singular function η_2 in (5.30b) and the first few monomials in (5.27). Furthermore, the convergence of the BKM/AB is noticeably faster than that of the BKM/AB' only in the case $a = 2$. However, even when $a = 2$ the improvement in convergence is not sufficient to overcome the increased instability and, with the exception of the COMP2 results for a=2, all the BKM/AB' approximations are more accurate than those obtained by the BKM/AB.

Example 5.3 BKM for the L-shaped domain

$$\Omega = \{(x,y) :- 1 < x < 3, |y| < 1\} \cup \{x,y) \mid x| <1 ,-1< y < 3\} . \tag{5.37}$$

Monomial basis. The monomial basis set used is

$$\eta_i(z) = z^{J-1}; \qquad j = 1,2,3,... \tag{5.38}$$

Augmented basis. In this case, the mapping function f has a serious branch point singularity at the re-entrant corner of the L-shape, i.e. at the point $z = 1 + i$. This follows from the results of Lehman [9], which show that in the neighbourhood of 2_0 f has an asymptotic expansion of the form

$$f(z) - f(z_0) = \sum_{\ell=1}^{\infty} a_\ell (z - z_0)^{2\ell/3}, \qquad a_1 \neq 0 ; \tag{5.39}$$

see [18,§4]. Furthermore, by the Schwarz reflection principle, f has simple pole singularities at the points

$$p_1 = -2i , \quad p_2 = -2 , \quad p_3 = 6 \quad \text{and } p_4 = 6i . \tag{5.40}$$

Because of the above observations we consider the use of the following two augmented basis sets:

AB(Sm): This basis set takes into account only the branch point singularity of f at z_0, and is constructed by introducing into the monomial set (5.38) the first m functions of the set

$$S_\ell(z) = (Z-Z_0)^{2\ell/3-1}; \qquad = 1,2,4,5,7,8,... ; \tag{5.41}$$

see Eq. (5.39). The basis functions are ordered so that when $k-1 < (2\ell/3-1) < k$ the singular function S_ℓ lies between the monomials z^{k-1} and z^k.

AB(PSm): This set is the same as AB(Sm), except that here we also introduce the singular functions

$$\eta_1(z) = \{z/(z-p_1)\}' \quad \text{and} \quad \eta_2(z) = \{z/(z-p_2)\}' , \tag{5.42}$$

which reflect the poles of f at the two points p_1 and p_2 .(The other pole singularities at the points p_3 and p_4 are "far" from $\partial\Omega$, and are not considered here.)

Optimum results. The values of Nopt and E_{Nopt} otained on COMP1, COMP2 and COMP3 by using respectively the monomial basis (5.38) and the two augmented sets AB(Sm) and AB(PSm), each with m= 1,3 and 6, are listed in Table 5.7.

TABLE 5.7

Values of Nopt and E_{Nopt}

	COMP1 $\varepsilon \simeq 1.5(-8)$		COMP2 $\varepsilon \simeq 7.1(-15)$		COMP3 $\varepsilon \simeq 3.1(-33)$	
Method	Nopt	E_{Nopt}	Nopt	E_{Nopt}	Nopt	E_{Nopt}
MB	13(S)	2.4(-1)	13(S)	2.4(-1)	13(S)	2.4(-1)
AB(S1)	13	2.3(-2)	16	5.7(-3)	27(S)	2.2(-3)
AB(PS1)	13	7.9(-3)	29	2.2(-3)	28(S)	2.1(-3)
AB(S3)	15	2.3(-2)	28	6.0(-4)	45(M)	5.9(-5)
AB(PS3)	15	1.3(-3)	26	2.9(-4)	45(M)	2.2(-5)
AB(S6)	10(B)	2.2(-1)	26	114(-3)	45(M)	2.8(-6)
AB(PS6)	16	6.8(-4)	26	2.5(-5)	45(M)	7.8(-8)

(S): *Slow convergence, i.e. the process terminates because criterion* (5.6) *is satisfied.*
(M): *This is the maximum number of basis functions used, i.e. the process is stopped at* n = 45, *without* (5.5) *or* (5.6) *being satisfied.*
(B): *The orthonormalization process breaks down before* n_{min} = 13 *is reached.*

Stability. Typical COMP3 values of the instability indicators $\widetilde{IS}_n$, associated respectively with the use of the monomial basis (5.38) and the six augmented sets AB(Sm) and AB(PSm); m = 1,3,6 are listed in Table 5.8.

TABLE 5.8

Instability indicators $\widetilde{IS}_n$

n	MB	AB(S1)	AB(PS1)	AB(S3)	AB(PS3)	AB(S6)	AB(PS6)
5	4.3(-2)	1.0(-1)	1.1(-1)	1.2(-3)	1.4(-2)	1.2(-3)	1.4(-2)
15	2.1(-7)	7.4(-7)	2.8(-7)	3.8(-6)	5.4(-6)	1.2(-11)	8.4(-11)
25	6.7(-13)	2.4(-12)	4.8(-12)	2.8(-11)	3.0(-11)	1.1(-15)	1.1(-13)

From Theorem 3.1 we know that the sequence of indicators $\{I_{n,n}\}$, associated with the monomial set (5.38), decreases at least as rapidly as the sequence $\{(n-1)\delta^n\}$ where $\delta = \{\mathrm{cap}(\partial\Omega)\}^2/5$. Although $\mathrm{cap}(\partial\Omega)$ is not known exactly, the BKM/AB results of [14,Ex.3.4] give the estimate $\delta=0.4708$. By comparison, the C0MP3 ratio $\{14I_{25,25}/24I_{15,15}\}^{1/20}$ gives the value $\delta = 0.4690$. These two estimates of δ, in conjunction with the MB values of IS_n listed in Table 5.8, indicate that the level of instability in the BKM/MB is not particularly high, by comparison with the levels of instability associated with some of the geometries considered in Exs. 5.1 and 5.2. Furthermore, the results of Table 5.8 indicate that, in this example, the introduction of singular functions does not affect significantly the stability properties of the orthonormalization process.

Convergence, In this example, it is very difficult to draw precise conclusions, concerning the convergence of the BKM approximations, from the behaviour of the error estimates E_n. We can only make the following general remarks:

(i) The results of Table 5.7 indicate that the convergence of all BKM approximations is slow.

(ii) Theorem 4.3 of Simonenko [20] applies to the L-shaped geometry considered here, i.e. there exist constants $C>0$ and $\Gamma>0$ such that

$$\max_{z\in\bar{\Omega}} |f_n(z) - f(z)| \le C/n^{\gamma}, \tag{5.43}$$

where f_n is the nth BKM/MB approximation to f. Furthermore, the result of Kulikov [11] shows that (5.43) holds for all $\gamma\varepsilon(0,\gamma_1)$ where, in this case, $\gamma_1 = 1/480$. Here, we attempt to provide some further information about the rate of convergence by computing values of the quantity

$$\gamma_n = \{\log(\hat{E}_{n-10}/\hat{E}_n)\}/\{\log(n/(n-10)\}\ , \ \hat{E}_n = \min_{1\le k\le n} E_k\ , \tag{5.44}$$

where the E_k are the BKM/MB(COMP3) error estimates. (We use E_n, rather than E_n, to take into account the possibility of non-monotonic convergence.) Typical results are as follows:

$$Y_{25} = 0.23\ ,\quad Y_{35} = 0.26\ ,\quad \gamma_{45} = 0.20$$

These results suggest that the exponent γ in (5.43) is approximately 0.2 and, as we remarked in Section 4, the value γ_1 of Kulikov appears to be very pessimistic.

(iii) The results of Table 5.7 indicate clearly that the introduction of singular functions into the basis set improves considerably the accuracy of the BKM approximations. In this example, the dominant singularity is due to the re-entrant corner at $z_0 = 1+i$, and for improved approximations the augmented basis must contain functions that reflect this corner singularity; see also the examples in [10] and [13,14],

(iv) The results indicate that the AB(PSm) approximations are, in general, at least as accurate as those obtained by using the basis AB(Sm). However, the C0MP3 results suggest that the introduction of the "pole" singular functions (5.42) does not improve the asymptotic rate of convergence of the BKM/AB(Sm). (The reason for the improved approximations, which are sometimes obtained by the AB(PSm), appears to be that the introduction of the functions (5.42) causes a noticeable initial improvement.)

(v) In this case, we do not have any theoretical results concerning the rate of convergence of the AB(Sm) and AB(PSm) approximations. In Section 4 we speculated that the improved accuracy achieved by the BKM/AB is due to a larger exponent γ in (5.43). We performed several numerical experiments for testing this speculation, but our numerical results were not conclusive.

<u>Example 5.4</u> ONM for the mapping of an equilateral triangle with a circular hole. Here, Ω is the doubly-connected domain bounded internally by the circle

$$\partial\Omega_1 = \{z : |z| = a, \ a < 1\}, \tag{5.45}$$

and externally by the equilateral triangle $\partial\Omega_2$ with vertices at the points $1 \pm i\sqrt{3}$ and -2

Monomial basis. Because Ω has three-fold rotational symmetry about the origin the monomial basis is taken to be

$$\eta_{2j-1}(z) = z^{3j-1}, \ \eta_{2j}(z) = 1/z^{3j+1}; \quad j = 1,2,3,\ldots\ldots \tag{5.46}$$

Augmented basis. The domain Ω has no corner singularities and for this reason, an augmented basis need only reflect any singularities that the function $H(z) = f'(z)/f(z) - 1/z$ may have in comp $(\Omega U \partial\Omega)$ .As is shown in [17], the function H does in fact have such singularities at the common symmetric points associated with the circle $\partial\Omega_1$ and each of the three sides of the triangle $\partial\Omega_2$ i.e. at the points

$$\left.\begin{aligned} &\zeta_1^{(j)} = \zeta_1 w^{j-1} \text{ and } \zeta_1^{(j)} = \zeta_2 w^{j-1}; \quad j = 1,2,3, \\ &\zeta_1 = \zeta_1^{(1)} = 1 - (1-a^2)^{\frac{1}{2}}, \ \zeta_2 = \zeta_2^{(1)} = 1 + (1-a^2)^{\frac{1}{2}} \end{aligned}\right\} \qquad (5.47)$$

and $w = \exp(2\pi i/3)$ It is also shown in [17] that these singularities can be reflected, but only approximately, by introducing into the monomial set (5.46) functions corresponding to simple poles at the points (5.47). The use of augmented basis sets constructed in this manner leads to improved ONM approximations, especially when the radius a of $\partial\Omega_1$ is close to unity. However, since the singularities of H at the points (5.47) can only be reflected approximately, we do not have any theoretical results concerning the rate of convergence of the ONM/AB. For this reason, in this example we consider only the use of the ONM/MB and refer the reader to [15] and [17], where several examples involving the use of the ONM with augmented basis sets are considered.

Optimum results. The COMP1, COMP2 and COMP3 values of Nopt and EL_{Nopt} ,obtained by applying the ONM/MB to the three domains corresponding to a=0.3,0.5 and 0.8, are listed in Table 5.9. (When $a = 0.5$ the COMP2 values of Nopt and E_{Nopt} given in Table 5.9 differ somewhat from the corresponding values obtained in [17], also on COMP2. This discrepancy is due to slightly different implementation details regarding the calculation of inner products.)

Stability. From Theorem 3.2 we know that the subsequence of indicators $\{I_{2n+1,2n+1}\}$ decreases at least as rapidly as the sequence $\{N\Delta^n\}$ where, because the set (5.46) reflects the symmetry of Ω, N=3n+1 and $\Delta = \{\text{cap}(\partial\Omega_2)/2\}^6$. Here $\text{cap}(\partial\Omega_2)$ can be determined from the exact formula of Pólya and Szegö [19,p.256].

TABLE 5.9

Values of Nopt and E_{Nopt}

	COMP1 ε = 1.5(-8)		COMP2 ε = 7.1(-15)		COMP3 ε = 3.1(-33)	
a	Nopt	E_{Nopt}	Nopt	E_{Nopt}	Nopt	E_{Nopt}
0.3	15	3.1(-5)	26	2.2(-8)	45 (M)	6.1(-4)
0.5	15	2.5(-5)	28	3,9(-8)	45 (M)	5.5(-3)
0.8	18	1.8(-4)	25	2.1(-6)	45 (M)	5.5(-10)

(M): *This is the maximum number of basis functions used, i.e. the process is stopped at n =45, without (5.5) or (5.6) being satisfied.*

This gives A = 0.151 96. By comparison, the COMP3 ratios $\{(28I_{39,39})/(58I_{19,19})\}^{1/10}$, associated with the application of the ONM to the three domains with a =0.3, 0.5 and 0.8, are

0.151 76 , 0.151 76 and 0.151 77 ,

respectively.

Regarding the subsequence of indicators $\{I_{2n+2,2n+2}\}$, the ratio d_1/R_1 in (3.20) is, in this case, $d_1/R_1 = 1$ for all values of a. Because of this, we do not expect the size of the circular hole to affect the stability of the ortho-normalization process. This is confirmed by our numerical results. For example, the COMP3 values of the indicators $\tilde{I}S_{15}$ $\tilde{I}S_{25}$ $\tilde{I}S_{35}$ are respectively

1.5(-6) , 1.6(-11) and 1.2(-16) ,

in all three cases a =0.3, 0.5 and 0.8. Furthermore, our experiments show that for "large" n the introduction of basis functions of the form $1/z^{3n+1}$ does not affect the stability of the method. In fact, as n increases the indicators $I_{2n+2,2n+2}$ appear to aproach the value unity. For example, the COMP3 values of $I_{6,6}$ corresponding to a =0.3,0.5 and 0.8, are

1.000 00 , 1.000 00 and 0.999 91 ,

respectively.

Convergence. Since f is analytic on $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$, Theorem 4.4. applies. That is, because of the form of the set (5.46), the ONM approximations to f satisfy

$$|f(z) - f_{2n}(z)| = 0(1/R^{3n}) \ , \ \forall R \ , 1 < R < \min(1/\hat{R}_1, \hat{R}_2) \ , \tag{5.48}$$

where $\hat{R}_1$ and $\hat{R}_2$ are defined by (4.27) and(4.28). Therefore $\hat{R}_1 = a/\zeta_1$ and $\hat{R}_2 = f_E \ (\zeta_2)$ where the singular points ζ_1 and ζ_2 are given by (5.47), and f_{E2} is the exterior mapping function associated with the outer triangular boundary $\partial\Omega_2$

TABLE 5.10

Convergence ratios

	a = 0.3	a = 0.5	a = 0.8
r_{20}	1.902	1.760	1.517
r_{30}	1.867	1.751	1.503
$\hat{R}_2^{\ 3/2}$	1.811	1.720	1.465
$\hat{R}_1^{\ -3/2}$	16.622	7.210	2.828

In Table 5.10 we compare the observed rate of convergence with that pre-dieted by (5.48) by listing the C0MP3 values

$$r_n = \{ E_n / E_{n+10} \}^{1/10} \quad ; \quad n = 20, 30 \ ,$$

and comparing them with $\min\{\hat{R}_1^{-3/2}, \hat{R}_2^{3/2}\}$.(It should be observed that, as in Ex.5.2, the mapping function f_{E2} is not known exactly. For this reason, the values of $\hat{R}_2^{3/2}$ listed in the table are estimates computed by using BKM/AB approximations to f_{E2} ; see [14,Ex.3.3].)

The results of Table 5.10 reflect the theory, and indicate that rate of convergence of the ONM decreases as the radius a of the circular hole increases,

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