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SHEAR DEFORMABLE BEAMS ON NONLINEAR VISCOELASTIC FOUNDATION UNDER MOVING LOADING

EVANGELOS J. SAPOUNTZAKIS[*] AND ANDREAS E. KAMPITSIS [†]

[*] School of Civil Engineering, National Technical University of Athens (NTUA)
Zografou Campus, GR-157 80, Athens, Greece
e-mail: cvsapoun@central.ntua.gr

[†] School of Civil Engineering, National Technical University of Athens (NTUA)
Zografou Campus, GR-157 80, Athens, Greece
e-mail: cvakamb@gmail.com

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Abstract. In this paper, a boundary element method is developed for the nonlinear response of shear deformable beams of simply or multiply connected constant cross section, traversed by moving loads, resting on tensionless nonlinear viscoelastic foundation, undergoing moderate large deflections under general boundary conditions. The beam is subjected to the combined action of arbitrarily distributed or concentrated transverse moving loading as well as to axial loading. To account for shear deformations, the concept of shear deformation coefficients is used. Three boundary value problems are formulated with respect to the transverse displacement, to the axial displacement and to a stress functions and solved using the Analog Equation Method, a BEM based method. Application of the boundary element technique yields a system of nonlinear differential – algebraic equations (DAE), which is solved using an efficient time discretization scheme, from which the transverse and axial displacements are computed. The evaluation of the shear deformation coefficient is accomplished from the aforementioned stress function using only boundary integration. Analyses are performed to investigate the effects of various parameters, such as the load velocity, load frequency, shear rigidity, foundation nonlinearity, damping, on the beam displacements and stress resultants and to examine how the consideration of shear and axial compression affect the response of the system.

1 INTRODUCTION

Vibration analysis of beams traversed by moving load is of great interest in the area of high-speed transportation or rocket-sledge technology. This problem can be modelled as a beam on elastic foundation subjected to loading moving at a constant speed.

When the beam deforms the conventional elastic foundation models can sustain both compression as well as tension. In order to address this issue tensionless foundation models were proposed, in which regions of no contact develop beneath the beam. These regions are unknown and the change of the transverse displacement sign provides the condition for the determination of the contact region. Besides, having in mind the magnitude of the arising compressive forces due to environmental loads such as changes in temperature or moisture or due to the train wheel and the importance of weight saving in engineering structures, the study of nonlinear effects on the analysis of supporting structural elements becomes essential. This non-linearity results from retaining the square of the slope in the strain–displacement relations (intermediate non-linear theory), avoiding in this way the inaccuracies arising from a linearized second – order analysis. Moreover, due to the intensive use of materials having relatively high transverse shear modulus, the error incurred from the ignorance of the effect of shear deformation may be substantial, particularly in the case of heavy lateral loading. All of the aforementioned concepts constitute the motive for a rigorous nonlinear dynamic analysis of shear deformable beams subjected to moving loads and resting on a tensionless nonlinear viscoelastic foundation.

When the beam deflections are small, a wide range of linear analysis tools, such as modal analysis, can be used, and some analytical results are possible. Analytical solutions of problems involving beam vibrations of simple geometry and boundary conditions under moving loads have received a good amount of attention in the literature, with pioneer the work of Krylov [1] and later the one of Timoshenko [2] who determined dynamic stresses in the beam structure. Linear transverse vibrations of a simply supported beam traversed by a constant force moving at a constant velocity were presented by Inglis [3], Lowan [4] and later on by Kceloušek [5] and Fryba [6].

Since then, important development has been achieved regarding also *linear* more rigorous dynamic analyses of beams under moving loads employing either analytical or numerical methods. Kargarnovin and Younesian [7-9] studied the response of infinite beams supported by nonlinear or Pasternak-type viscoelastic foundations subjected to harmonic moving loads employing a perturbation methods. Zehsaz et. al. [10] studied the dynamics of railway, as a Timoshenko beam of limited length, lying on a Pasternak viscoelastic foundation, subjected to moving load employing the modal superposition method.

As the beam deflections become larger, the induced *geometric nonlinearities* result in effects that are not observed in linear systems. Chen et. al. [11] performing a geometrically nonlinear analysis with constant axial force presented the dynamic stiffness matrix of an infinite Timoshenko beam on viscoelastic foundation subjected to a harmonic moving load and determined the critical velocities and the resonant frequencies. Kim and Cho [12] presented the vibration and buckling of an infinite beam-column under constant axial force, resting on an elastic foundation and subjected to moving loads of either constant or harmonically varying amplitude with a constant advance velocity, taking into account shear deformation effect.

In this paper, a boundary element method is developed for the nonlinear response of shear deformable beams of simply or multiply connected constant cross section, traversed by moving loads, resting on tensionless nonlinear viscoelastic foundation, undergoing moderate large deflections under general boundary conditions. The beam is subjected to the combined action of arbitrarily distributed or concentrated transverse moving loading as well as to axial loading. To account for shear deformations, the concept of shear deformation coefficients is used. Three boundary value problems are formulated with respect to the transverse displacement, to the axial displacement and to a stress functions and solved using the Analog Equation Method [13], a BEM based method. Application of the boundary element technique yields a system of nonlinear differential–algebraic equations (DAE), which is solved using an efficient time discretization scheme. The evaluation of the shear deformation coefficient is accomplished from the aforementioned stress function using only boundary integration. The essential features and novel aspects of the present formulation compared with previous ones are summarized as follows.

i. Shear deformation effect and rotary inertia are taken into account in the nonlinear dynamic analysis of beams subjected to arbitrary (distributed or concentrated) transverse moving, as well as to axial loading.
ii. The homogeneous linear half-space is approximated by a tensionless three-parameter viscoelastic foundation.
iii. The beam is supported by the most general nonlinear boundary conditions.
iv. The proposed model takes into account the coupling effects of bending and shear deformations along the member as well as shear forces along the span induced by the applied axial loading.
v. The shear deformation coefficients are evaluated using an energy approach, instead of Timoshenko's [14] and Cowper's [15] definitions.
vi. The effect of the material's Poisson ratio v is taken into account.
vii. The proposed method employs a BEM approach (requiring boundary discretization) resulting in line or parabolic elements instead of area elements of the FEM solutions (requiring the whole cross section to be discretized into triangular or quadrilateral area elements), while a small number of line elements are required to achieve high accuracy.

Analyses are performed to investigate the effects of various parameters, such as the load velocity, load frequency, shear rigidity, foundation nonlinearity, damping, on the beam displacements and stress resultants and to examine how the consideration of shear and axial compression affect the response of the system.

2 STATEMENT OF THE PROBLEM

Let us consider a prismatic beam of length l (Fig.1a), of arbitrary constant cross-section of area A (Fig.1b), having at least one axis of symmetry (z-axis). The homogeneous isotropic and linearly elastic material of the beam cross-section, with modulus of elasticity E, shear modulus G and Poisson's ratio v occupies the two dimensional multiply connected region Ω of the y,z plane and is bounded by the $\Gamma_j\left(j=1,2,...,K\right)$ boundary curves, which are piecewise smooth, i.e. they may have a finite number of corners. The beam is supported on a homogeneous tensionless nonlinear three-parameter viscoelastic soil. The foundation model is characterized by the linear Winkler modulus k_L, the nonlinear Winkler modulus k_{NL}, the Pasternak (shear) foundation modulus k_P and the damping coefficient c_z. Taking into account the unbonded contact between beam and subgrade, the interaction pressure at the interface can be only compressive and is represented for the transverse direction by the following relations

$$p_{sz}\left(x,t\right)=U\left(x,t\right)p_{react}\left(x,t\right) \tag{1a}$$

$$p_{react}\left(x,t\right)=k_L w\left(x,t\right)+k_{NL}w^3\left(x,t\right)-k_P\frac{\partial^2 w\left(x,t\right)}{\partial x^2}+c\frac{\partial w\left(x,t\right)}{\partial t} \tag{1b}$$

where $U\left(x,t\right)$ is a unit step function defined as

$$U(x,t)=\begin{cases}1 & if \;\; p_{react}(x,t)>0 \\ 0 & if \;\; p_{react}(x,t)\leq 0\end{cases} \tag{2}$$

The foundation reaction p_{react} of eqn.(1b) takes into account the nonlinear behavior of the subsoil (e.g. ballast and rail-bed) as proposed by Dahlberg [16], demonstrating that the differences between the non-linear and linear models are considerable and a non-linear track model simulates the rail deflection quite well whereas the equivalent linear one cannot. Later, Wu and Thompson [17] presented a similar non-linear model and studied the problem of wheel/track impact employing the finite element method. Moreover, for real sample of the hardening behavior of the foundation one can refer to [18] where detailed field measurement results are presented.

The beam is subjected to the combined action of the arbitrarily distributed or concentrated transverse along the axis of symmetry moving loading $p_z = p_z(x,t)$ with constant velocity V as well as to axial loading $p_x = p_x(x,t)$, as shown in Fig.1a. Under the action of this loading, the displacement field of the beam taking into account shear deformation effect is given as

$$\bar{u}(x,y,z,t)=u(x,t)+z\theta_y(x,t) \qquad \bar{w}(x,t)=w(x,t) \tag{3a,b}$$

where $\bar{u}$, $\bar{w}$ are the axial and transverse beam displacement components with respect to the Cyz system of axes (Fig.1b); $u(x,t)$, $w(x,t)$ are the corresponding components of the centroid C and $\theta_y(x,t)$ is the angle of rotation due to bending of the cross-section with respect to the same point.

Employing the strain-displacement relations of the three - dimensional elasticity for moderate displacements the following strain components can be easily obtained

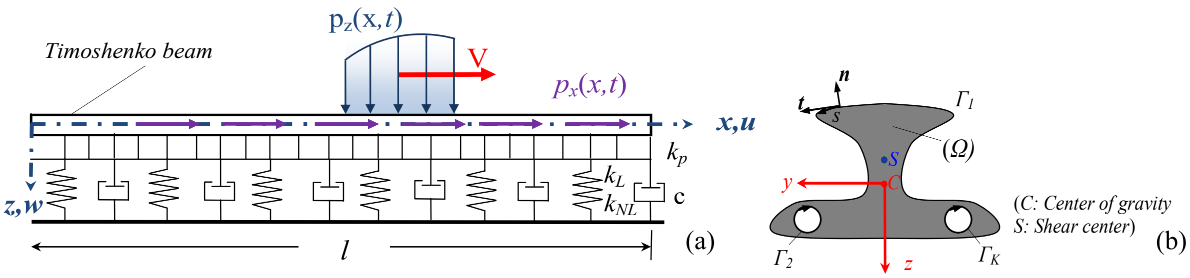

Figure 1: Prismatic beam resting on a nonlinear viscoelastic foundation (a) of an arbitrary mono symmetric cross-section occupying the two dimensional region Ω (b)

$$\varepsilon_{xx}=\frac{\partial\bar{u}}{\partial x}+\frac{1}{2}\left(\frac{\partial\bar{w}}{\partial x}\right)^2 \qquad \gamma_{xz}=\frac{\partial\bar{w}}{\partial x}+\frac{\partial\bar{u}}{\partial z}+\frac{\partial\bar{w}}{\partial x}\frac{\partial\bar{w}}{\partial z} \qquad \varepsilon_{zz}=\gamma_{yz}=0 \tag{4a,b,c}$$

where it has been assumed that for moderate displacements $\left(\partial\bar{u}/\partial x\right)^2 << \partial\bar{u}/\partial x$, $\left(\partial\bar{u}/\partial x\right)\left(\partial\bar{u}/\partial z\right) << \left(\partial\bar{u}/\partial x\right)+\left(\partial\bar{u}/\partial z\right)$. Substituting the displacement components (3) to the strain-displacement relations (4), the strain components can be written as

$$\varepsilon_{xx}\left(x,z,t\right)=u'+z\theta_y'+\frac{1}{2}w'^2 \qquad \gamma_{xz}=w'+\theta_y \tag{5a,b}$$

where γ_{xz} is the additional angle of rotation of the cross-section due to shear deformation.

Considering strains to be small, employing the second Piola – Kirchhoff stress tensor and assuming an isotropic and homogeneous material, the stress components are defined in terms of the strain ones as

$$\begin{Bmatrix} S_{xx} \\ S_{xz} \end{Bmatrix}=\begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix}\begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{xz} \end{Bmatrix} \tag{6}$$

or employing eqns. (5) as

$$S_{xx}=E\left(u'+z\theta_y'+\frac{1}{2}w'^2\right) \qquad S_{xz}=G\cdot\left(w'+\theta_y\right) \tag{7a,b}$$

On the basis of Hamilton's principle, the variations of the Lagrangian equation defined as

$$\delta\int_{t_1}^{t_2}\left(U-K-W_{ext}\right)dt=0 \tag{8}$$

and expressed as a function of the stress resultants acting on the cross section of the beam in the deformed state provide the governing equations and the boundary conditions of the beam subjected to nonlinear vibrations. In eqn.(8), $\delta(\cdot)$ denotes variation of quantities, while U, K, W_{ext} are the strain energy, the kinetic energy and the external load work, respectively given as

$$\delta U=\int_V\left(S_{xx}\delta\varepsilon_{xx}+S_{xz}\delta\gamma_{xz}\right)dV \qquad \delta K=\frac{1}{2}\int_V\rho\left(\delta\dot{\bar{u}}^2+\delta\dot{\bar{w}}^2\right)dV \tag{9a,b}$$

$$\delta W_{ext}=\int_L\left(p_x\delta u+p_z\delta w-\delta\left(p_{sz}w\right)\right)dx \tag{9c}$$

Moreover, the stress resultants of the beam are given as

$$N=\int_\Omega S_{xx}d\Omega \qquad \mathrm{M}_y=\int_\Omega S_{xx}zd\Omega \qquad Q_z=\int_{A_z}S_{xz}d\Omega \tag{10a,b,c}$$

Substituting the expressions of the stress components (7) into equations (10), the stress resultants are obtained as

$$N=EA\left(u'+\frac{1}{2}w'^2\right) \qquad M_y=EI_y\theta_y' \qquad Q_z=GA_z\gamma_{xz} \tag{11a,b,c}$$

where A is the cross section area, I_y the moment of inertia with respect to z-axis given as

$$A=\int_\Omega d\Omega \tag{12}$$

$$I_y=\int_\Omega z^2 d\Omega \tag{13}$$

and GA_z is its shear rigidity of the Timoshenko's beam theory, where

$$A_z=\kappa_z A=\frac{1}{a_z}A \tag{14}$$

is the shear area, respectively with κ_z the shear correction factor and a_z the shear deformation coefficient. Substituting the stress components given in eqns. (7) and the strain resultants given in eqns. (5) to the strain energy variation δE_{int} (eqn.9a) and employing eqn. (8), the equilibrium equations of the beam are derived as

$$\rho A\ddot{u} - EA\left(u'' + w'w''\right) = p_x \tag{15a}$$

$$\rho A\ddot{w} - \left(Nw'\right)' - GA_z\left(w'' + \theta_y'\right) + p_{sz} = p_z \qquad EI_y\theta_y'' - GA_z\left(w' + \theta_y\right) = \rho I_y\ddot{\theta}_y \tag{15b,c}$$

where ($\dot{}$), ($'$) denote differentiation with respect to t, x, respectively. Combining equations (15b,c) the following differential equations with respect to u, w are derived as

$$\rho A\ddot{u} - EA\left(u'' + w'w''\right) = p_x \tag{16a}$$

$$EI_y w'''' + \rho A\ddot{w} + p_{sz} + \frac{EI_y}{GA_z}\left(\left(Nw'\right)''' - \rho A\frac{\partial^2\ddot{w}}{\partial x^2} - p_{sz}'' + p_z''\right) - \left(Nw'\right)' - \rho I_y\frac{\partial^2\ddot{w}}{\partial x^2}$$
$$-\frac{\rho I_y}{GA_z}\left(\frac{\partial^2\left(Nw'\right)'}{\partial t^2} - \rho A\ddot{\ddot{w}} - \ddot{p}_{sz} + \ddot{p}_z\right) = p_z \tag{16b}$$

Eqns. (16) constitute the governing differential equations of a Timoshenko beam, supported on a tensionless nonlinear three-parameter viscoelastic foundation, subjected to nonlinear vibrations due to the combined action of arbitrarily distributed or concentrated transverse moving loading as well as to axial loading. These equations are also subjected to the pertinent boundary conditions of the problem at hand given as

$$a_1 u\left(x,t\right) + \alpha_2 N\left(x,t\right) = \alpha_3 \tag{17}$$

$$\beta_1 w\left(x,t\right) + \beta_2 V_z\left(x,t\right) = \beta_3 \qquad \gamma_1\theta_y\left(x,t\right) + \gamma_2 M_y\left(x,t\right) = \gamma_3 \tag{18a,b}$$

at the beam ends $x = 0, l$, together with the initial conditions

$$u\left(x,0\right) = \bar{u}_0\left(x\right) \qquad \dot{u}\left(x,0\right) = \dot{\bar{u}}_0\left(x\right) \tag{19a,b}$$

$$w\left(x,0\right) = \bar{w}_0\left(x\right) \qquad \dot{w}\left(x,0\right) = \dot{\bar{w}}_0\left(x\right) \tag{20a,b}$$

where $\bar{u}_0\left(x\right)$, $\bar{w}_0\left(x\right)$, $\dot{\bar{u}}_0\left(x\right)$ and $\dot{\bar{w}}_0\left(x\right)$ are prescribed functions. In eqns. (18) V_z and M_y are the reaction and bending moment, which together with the angle of rotation due to bending θ_y are given as

$$V_z = Nw' - EI_y w''' - \frac{EI_y}{GA_z}\left[\left(Nw'\right)'' + p_z' - p_{sz}' - \rho A\frac{\partial\ddot{w}}{\partial x}\right] - \rho I_y\ddot{\theta}_y \tag{21a}$$

$$M_y = -EI_y w'' - \frac{EI_y}{GA_z}\left[\left(Nw'\right)' + p_z - p_{sz} - \rho A\ddot{w}\right] \tag{21b}$$

$$\theta_y = \frac{EI_y}{G^2 A_z^2}\left(-p_z' + p_{sz}' - \left(Nw'\right)'' + \rho A\frac{\partial\ddot{w}}{\partial x}\right) - \frac{1}{GA_z}\left(EI_y w''' + \rho I_y\ddot{\theta}_y + GA_z w'\right) \tag{21c}$$

Finally, $\alpha_k, \beta_k, \gamma_k$ ($k = 1,2,3$) are functions specified at the beam ends $x = 0, l$. Eqs. (17)-(18) describe the most general nonlinear boundary conditions associated with the problem at hand

and can include elastic support or restraint. It is apparent that all types of the conventional boundary conditions (clamped, simply supported, free or guided edge) can be derived from these equations by specifying appropriately these functions (e.g. for a clamped edge it is $\alpha_1 = \beta_1 = 1$, $\gamma_1 = 1$, $\alpha_2 = \alpha_3 = \beta_2 = \beta_3 = \gamma_2 = \gamma_3 = 0$).

The solution of the initial boundary value problem given from eqns (16), subjected to the boundary conditions (17)-(18) and the initial conditions (19)-(20), which represents the nonlinear flexural dynamic analysis of a Timoshenko beam, supported on a tensionless nonlinear three-parameter viscoelastic foundation, presumes the evaluation of the shear deformation coefficient a_z corresponding to the principal centroidal system of axes Cyz. This coefficient is established equating the approximate formula of the shear strain energy per unit length [19]

$$U_{appr.} = \frac{a_z Q_z^2}{2AG} \tag{22}$$

with the exact one given from

$$U_{exact} = \int_\Omega \frac{(\tau_{xz})^2}{2G} d\Omega \tag{23}$$

and are obtained as [20]

$$a_z = \frac{1}{\kappa_z} = \frac{A}{\Delta^2} \int_\Omega \left[(\nabla\Phi) - \boldsymbol{d} \right] \cdot \left[(\nabla\Phi) - \boldsymbol{d} \right] d\Omega \tag{24}$$

where (τ_{xz}) is the transverse (direct) shear stress component, $(\nabla) \equiv \boldsymbol{i}_y (\partial/\partial y) + \boldsymbol{i}_z (\partial/\partial z)$ is a symbolic vector with $\boldsymbol{i}_y, \boldsymbol{i}_z$ the unit vectors along y and z axes, respectively, Δ is given as

$$\Delta = 2(1+\nu) I_y I_z \tag{25}$$

ν is the Poisson ratio of the cross section material, $\boldsymbol{d}$ is a vector defined as

$$\boldsymbol{d} = (\nu I_z yz) \boldsymbol{i}_y - \left(\nu I_z \frac{y^2 - z^2}{2} \right) \boldsymbol{i}_z \tag{26}$$

and $\Phi(y,z)$ is a stress function evaluated from the solution of the following Neumann type boundary value problem [20]

$$\nabla^2 \Phi = -2 I_z z \quad \text{in } \Omega \qquad \frac{\partial \Phi}{\partial n} = \boldsymbol{n} \cdot \boldsymbol{d} \quad \text{on } \Gamma = \bigcup_{j=1}^{K+1} \Gamma_j \tag{27a,b}$$

where $\boldsymbol{n}$ is the outward normal vector to the boundary Γ. In the case of negligible shear deformations $a_z = 0$. It is also worth here noting that the boundary conditions (27b) have been derived from the physical consideration that the traction vector in the direction of the normal vector $\boldsymbol{n}$ vanishes on the free surface of the beam.

3 INTEGRAL REPRESENTATIONS – NUMERICAL SOLUTION

According to the precedent analysis, the nonlinear flexural dynamic analysis of

Timoshenko beams, supported on a tensionless nonlinear three-parameter viscoelastic foundation, undergoing moderate large deflections reduces in establishing the displacement components $u\left(x,t\right)$ and $w\left(x,t\right)$ having continuous derivatives up to the second and up to the fourth order with respect to x, respectively, and also having derivatives up to the second order with respect to *t* (ignoring the inertia terms of the fourth order [21]). These displacement components must satisfy the coupled governing differential equations (16) inside the beam, the boundary conditions (17)-(18) at the beam ends $x=0,l$ and the initial conditions (19)-(20). Eqns (16) are solved using the Analog Equation Method [13] as this is developed for hyperbolic differential equations [22].

4 NUMERICAL EXAMPLES

On the basis of the analytical and numerical procedures presented, a computer program has been written and a representative example has been studied to demonstrate the efficiency of the developed method. In the example, the results have been obtained using $L=21$ nodal points (longitudinal discretization), *400* boundary elements (cross section discretization) and a time step of $\Delta t=1.0\,\mu\sec$.

4.1 Example

In order to illustrate the importance of the nonlinear analysis, a simply supported UIC60 rail track, resting on a nonlinear viscoelastic bilateral foundation is examined. The geometric constants of the track and the foundation are given in Table 1. The track is subjected to a concentrated moving harmonic load $p_z\left(x,t\right)=P\delta\left(x-Vt\right)\sin\left(\Omega t\right)$, where P,Ω are the amplitude and the frequency of the harmonic load, respectively and δ is the Dirac's delta function. Moreover, the track is subjected to an either tensile or compressive distributed axial load $p_x\left(x,t\right)=\pm 2500\left(kN/m\right)$.

In Fig. 2 the time history and the extreme values of the central deflection $w\left(l/2,t\right)$ of the track resting on the viscoelastic Winkler foundation and subjected to a concentrated harmonic load at its midpoint ($V=0m/s$,$\Omega=100rad/s$) is presented, performing either a linear or a nonlinear analysis and taking into account both rotary inertia and shear deformation effect. To illustrate the significant effect of the load frequency, in Table 2 the maximum values of the deflection $w\left(x,t\right)$ of the track resting on the nonlinear viscoelastic foundation, subjected to a moving with constant velocity $V=100m/s$ harmonic load are presented for various values of the excitation frequency Ω, performing either linear or nonlinear (for both cases of tensile or compressive axial load) analysis. Moreover, in Table 3 the maximum deflections and bending moments of the track are presented for different types of foundation reaction, for $\Omega=400rad/s$, $V=100m/s$, while in Fig. 3 the deflection curves $w(x,0.055)$ along the track axis at the time instant $t=0.055s$ as well as their maximum values are also presented for linear and nonlinear analysis for Winkler and Nonlinear viscoelastic foundation. From the obtained results, it is concluded that the discrepancy between the linear and the nonlinear analysis is not negligible and should not be ignored, while the influence of the shear deformation effect (increasing the transverse displacements and decreasing the bending moments) in both linear and nonlinear analysis is observed. This latter influence is more pronounced as the length of the track becomes smaller.

Table 1: Geometric constants of the UIC60 rail track [8,16] and the foundation of the beam

$l\,(m)$	10	a_z	2.68
$E\,(GPa)$	210	$k_L\,(MPa)$	35
$G\,(GPa)$	77	$k_{NL}\,(MN/m^2)$	4×10^8
$I_y\,(m^4)$	30.55×10^{-6}	$k_P\,(kN)$	200
$A\,(m^2)$	76.86×10^{-4}	$c\,(kNs/m^2)$	145
$\rho\,(kg/m^3)$	7850	$P\,(kN)$	100

Table 2: Maximum values of the deflection $w\cdot10^{-1}\,(mm)$ of the track, for various values of the excitation frequency Ω

$\Omega\,(rad/s)$	Linear	Nonlinear – Tensile Load	Nonlinear –Compressive Load
0.1	0.1377	0.1278	0.1511
0.5	0.6738	0.6278	0.7358
1.0	1.2708	1.1970	1.3770
5.0	3.8139	3.7529	3.9737
10	5.2252	5.1603	5.3181
50	5.7224	5.7052	6.7736
100	5.8639	5.8475	7.0310
200	5.7268	5.6447	6.5544
400	5.7828	5.6259	6.8505

Table 3: Maximum deflections w_{max} and bending moments $M_{y\max}$ of the track, for different types of foundation reaction

$w_{max}\cdot10^{-1}\,(mm)$ $M_{y\max}\,(kNm)$	Without Shear Deformation		With Shear Deformation	
	Linear Analysis	Nonlinear Analysis	Linear Analysis	Nonlinear Analysis
Linear Winkler	9.879 15.449	14.336 37.154	9.973 15.353	14.436 33.345
Linear and Nonlinear Winkler	5.788 13.468	6.859 22.196	5.923 13.142	6.992 20.608
Pasternak	9.861 15.422	14.235 36.917	9.937 15.312	14.452 33.123
3-Parameter	5.783 13.439	6.851 22.132	5.820 13.127	6.893 20.557

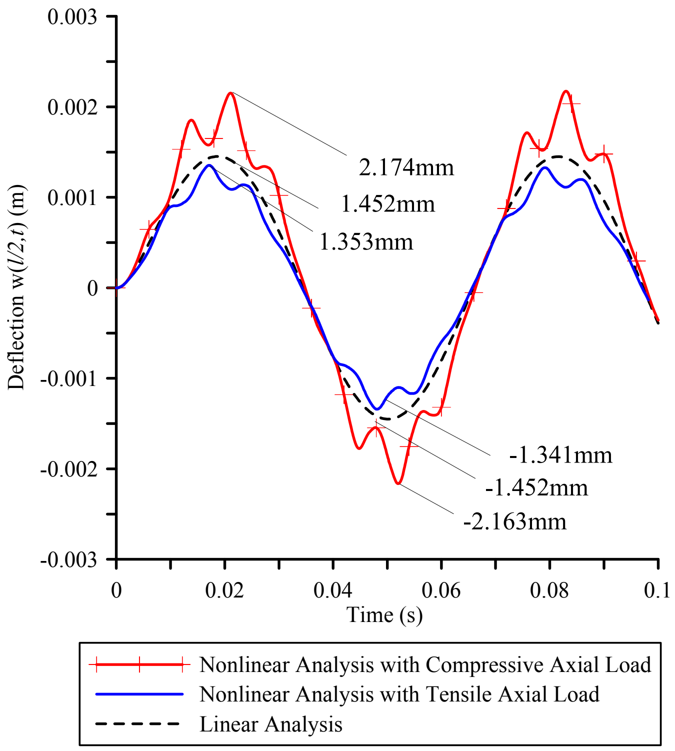

Figure 2: Time history and extreme values of the midpoint deflection $w(l/2,t)$ of the track

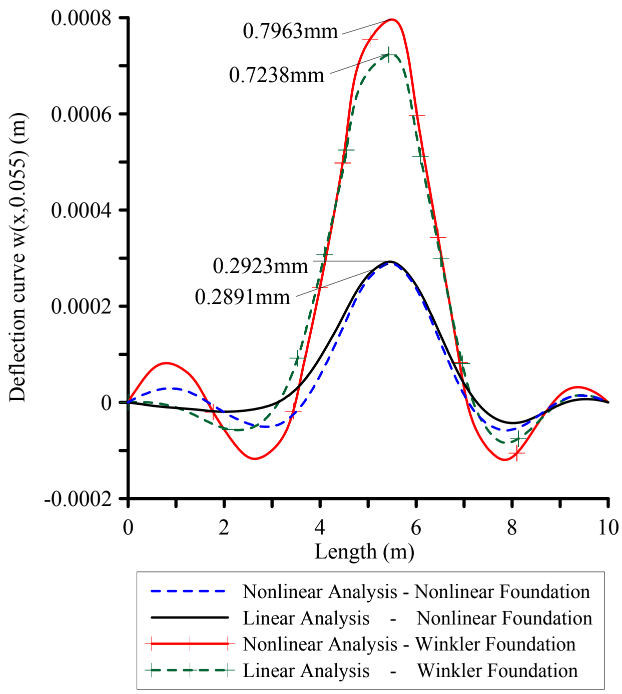

Figure 3: Deflection curves $w(x,0.055)$, at the time instant $t = 0.055s$ and their maximum values

5 CONCLUDING REMARKS

The main conclusions that can be drawn from this investigation are

a. The numerical technique presented in this investigation is well suited for computer aided analysis for beams of arbitrary simply or multiply connected cross section having at least one axis of symmetry.
b. The proposed method is developed for general dynamic moving loading, while the beam is subjected to the most general boundary conditions and rests on a linear or nonlinear viscoelastic foundation.
c. The lift up of the beam caused by the tensionless character of the foundation is observed, leading to magnification of the consequences of the dynamic response.
d. In some cases, the effect of shear deformation is significant, especially for low beam slenderness values, increasing the transverse displacements and decreasing the bending moments in both linear and nonlinear analysis.
e. The discrepancy between the results of the linear and the nonlinear analysis is remarkable.
f. The response of the beam is strongly influenced by the linear and nonlinear parameters of the foundation reaction.
g. The damping coefficient is of paramount importance for beams on viscoelastic foundations, as it reduces the vibration amplitude and the consequences of the dynamic response.

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