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DECOMPOSITIONS OF THE STRESS AND THE RATE OF DEFORMATION TENSORS FOR MATERIALS UNDERGOING PHASE TRANSFORMATIONS

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Summary: An extension of the "Duhamel-Neumann hypothesis" for materials undergoing phase transformations and for arbitrary magnitudes of strains and rotations is provided.

1 INTRODUCTION

Generalized plasticity theory has been successfully used by the author and his co-workers in order to model phase transformations of shape memory alloy materials. Our work has been initially carried out within small strains (Panoskaltsis et al. [1], Ramanathan et al. [2]) and very recently within finite deformations and rotations and under both isothermal and non-isothermal conditions (Panoskaltsis [3]). In this paper we will develop, for the first time, two important decompositions applied to materials undergoing phase transformations. In the first decomposition (Theorem 1) the rate of the Kirchhoff stress tensor is given in terms of the rate of deformation tensor and the rate of the temperature.

In the second decomposition (Theorem 2), which can be thought of as a conjugate to the first one, the rate of deformation tensor is expressed as a sum of the (objective) rate of the stress tensor and the rate of the temperature. In the next section we will review the formulation of generalized plasticity theory for modeling phase transformations in finite deformations and in both reference and current configurations of the body (for an exhaustive development, see Panoskaltsis [3]). Finally, in the last section, we will prove Theorems 1 and 2.

2 GENERALIZED PLASTICITY FOR PHASE TRANSFORMATIONS

2.1 Formulation of the governing equations in the reference configuration

Generalized plasticity is a local internal variable theory of rate – independent behavior which is based primarily on loading – unloading irreversibility. In the theory it is assumed that the local thermomechanical state in a body is determined uniquely by the couple (**G**, **Q**) where **G** stands for the vector of the controllable state variables and **Q** stands for the vector of the internal variables, which are related to phase transformations. In this work, we follow a *material (referential) approach* within a *strain – space formulation.* Accordingly, **G** may be identified by (**E**, T) where **E** is the referential (Green – St. Venant) strain tensor and T is the (absolute) temperature. Depending on the nature of the (material) internal variable vector **Q**, the theory may, in principle, be formulated equivalently with respect to the *macro –, meso –, or micro – scale structure* of the material.

The central concept of generalized plasticity is that of the *elastic range* (e.g., see Lubliner [4]) which is defined at any material state, as the region in the strain – temperature space comprising the strains which can be attained elastically (i.e., with no change in the internal variables) from the current strain – temperature point. It is assumed that the elastic range is a regular set in the sense that it is the closure of an open set. The boundary of this set is defined as a *loading surface* at **Q** (see Eisenberg and Phillips [5], Lubliner [4]). In turn a *material state* may be defined as *elastic* if it is an interior point of its elastic range and *inelastic* if it is a boundary point of its elastic range; in the latter case the material state lies on a loading surface. It should be added that the notion of process is introduced implicitly here. By assuming that the loading surface is smooth at the current strain - temperature point and by invoking some basic axioms and results from set theory and topology, Lubliner [4] showed that the rate equations for the evolution of the internal variable vector may be written in the form:

$$\dot{\mathbf{Q}} = H\mathbf{L}(\mathbf{G},\mathbf{Q})\left\langle \mathbf{N}:\dot{\mathbf{G}} \right\rangle, \tag{1}$$

where <·> stands for the Macauley bracket which is defined as:

$$\left\langle x \right\rangle = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$$

and H denotes a scalar function of the state variables enforcing the defining property of an inelastic state. Accordingly, the value of H must be positive at any inelastic state and zero at any elastic one. Finally, **L** denotes a non - vanishing (tensorial) function of the state variables, which is associated with the properties of the phase transformation and **N** is the outward normal to the loading surface at the current state, while the symbol **:** between two second order tensors denotes their double contraction. Furthermore, the set of the material states defined as $H = H(\mathbf{G},\mathbf{Q}) = 0$, which comprises all the elastic states, is called the *elastic domain* and its projection on the set defined by **Q** = constant is defined at the elastic domain at **Q**. In general, the elastic domain at **Q** is a subset of the elastic range (Lubliner [4]). The particular case where the two sets coincide corresponds to classical plasticity and the boundary of the elastic domain, that is the *initial loading surface,* constitutes the *yield surface* (see Eisenberg and Phillips [5], Lubliner [4], Panoskaltsis et al. [6]).

It is emphasized that Equation (1) has been derived under the assumption of a smooth loading surface at the current strain – temperature point, which implies that only one loading mechanism can be considered. On the other hand, the phase transformations include multiple and sometimes interacting loading mechanisms, which may result in the appearance of a vertex or a corner at the current strain – temperature point. This fact calls for an appropriate modification of the rate Equation (1).

For this purpose we assume that the loading surfaces are defined in the state space by a number – say n – of smooth surfaces, which are defined by expressions of the form:

$$\Phi_{\mathrm{i}}(\mathbf{G},\mathbf{Q}) = 0. \quad \mathrm{i}=1, 2,..., \mathrm{n} \tag{2}$$

These surfaces can be either disjoint, or intersect in a possibly non – smooth fashion. Each of these surfaces is associated with a particular transformation mechanism which may be active at the current strain – temperature point. Then by assuming that each equation $\Phi_{\mathrm{i}}(\mathbf{G},\mathbf{Q}) = 0$ defines independent (non – redundant) active surfaces at the current stress temperature point the rate equations for the evolution of the internal variables in view of Equation (1) can be stated in the following general form:

$$\dot{\mathbf{Q}} = \sum_{\mathrm{i}=1}^{\mathrm{n}} H_{\mathrm{i}} \mathbf{L}_{\mathrm{i}}(\mathbf{G},\mathbf{Q}) \left\langle \mathbf{N}_{\mathrm{i}} : \dot{\mathbf{G}} \right\rangle, \tag{3}$$

where H_{i}, $\mathbf{L}_{\mathrm{i}}$ and $\mathbf{N}_{\mathrm{i}}$ are functions of the state variables defined as in Equation (1) and each set of them – defined by the index i – refers to the specific transformation associated with the part of the loading surface defined by $\Phi_{\mathrm{i}}(\mathbf{G},\mathbf{Q}) = 0$. From Equation (3) one can deduce directly the loading – unloading criteria for the proposed formulation as follows: Lets denote $n_{adm} \le n$ the number of loading surfaces that may be active at an inelastic state i.e., $H_i > 0$, and lets denote by J_{adm} the set of n_{adm} indices associated with those surfaces, i.e.,

$$J_{adm} = \{\alpha \in \{1,2,...,n\} / H_\alpha > 0\}.$$

Then Equation (3) implies the following loading – unloading conditions:

$$\begin{aligned}
&\text{If } J_{adm} = \varnothing, \text{ then } \dot{\mathbf{Q}}=0.\\
&\text{If } J_{adm} \neq \varnothing, \text{ then:}\\
&\text{i If } \mathbf{N}_\alpha : \dot{\mathbf{G}} \le 0 \text{ for all } \alpha \in J_{adm} \text{ then } \dot{\mathbf{Q}}=0,\\
&\text{ii If } \mathbf{N}_\alpha : \dot{\mathbf{G}} > 0 \text{ for at least one } \alpha \in J_{adm} \text{ then } \dot{\mathbf{Q}} \neq 0.
\end{aligned}$$

So, if we denote further by $n_{act} \le n_{adm}$ the number of parts for which (ii) holds, and we set:

$$J_{act} = \{\alpha \in J_{adm} / \mathbf{N}_\alpha : \dot{\mathbf{G}} > 0\},$$

the loading criteria in terms of the sets J_{adm} and J_{act} may be stated as:

$$\begin{cases}
\text{If } J_{adm} = \varnothing: & \text{elastic state.}\\
\text{If } J_{adm} \neq \varnothing \text{ and } J_{act} = \varnothing: & \\
\quad \text{i. If } \mathbf{N}_\alpha : \dot{\mathbf{G}} < 0 \text{ for all } \alpha \in J_{adm}: & \text{elastic unloading,}\\
\quad \text{ii. If } \mathbf{N}_\alpha : \dot{\mathbf{G}} = 0 \text{ for at least one } \alpha \in J_{adm}: & \text{neutral loading,}\\
\text{If } J_{adm} \neq \varnothing \text{ and } J_{act} \neq \varnothing: & \text{inelastic loading.}
\end{cases} \tag{4}$$

2.2 Equivalent spatial formulation

The equivalent assessment of the governing equations in the spatial configuration can be done on the basis of a push – forward operation (e.g., see Marsden and Hughes [5], pp. 67 – 68) to the basic equations. Hence, by performing a push – forward operation onto Equation (3) the latter is written in the form:

$$\mathrm{L_V}\mathbf{q} = \sum_{\mathrm{i}=1}^{\mathrm{n}} h_{\mathrm{i}} \mathbf{l}_{\mathrm{i}}(\mathbf{g},\mathbf{q},\mathbf{F})\langle r_{\mathrm{i}}\rangle, \tag{5}$$

where **F** stands for the deformation gradient and **g** denotes the vector of the controllable variables in the spatial configuration. These are, the Almansi strain tensor **e** – defined as the push – forward of the Green – St. Venant strain tensor – and the temperature T. Moreover, in Equation (5) **q** stands for the push forward of the internal variable vector **Q** and $\mathrm{L_V}(\cdot)$ denotes the Lie derivative of its argument (e.g., see Marsden and Hughes [5], pp. 93 – 104), defined as the convected derivative relative to the spatial configuration. The use of Lie derivatives guarantees the objectivity of rate equations in the current configuration (Marsden and Hughes [5], p. 99). Finally, h_{i} denotes the expression of the scalar invariant functions H_{i} in terms of the spatial variables (**e**, T, **q**) and the deformation gradient **F**, while $\mathbf{l}_{\mathrm{i}}$ denotes the push – forward of the tensorial functions $\mathbf{L}_{\mathrm{i}}$, and r_{i} stands for the (scalar invariant) loading rates which are written in the form:

$$r_{\mathrm{i}} = \frac{\partial \varphi_{\mathrm{i}}}{\partial \mathbf{e}} : \mathrm{L_V}\mathbf{e} + \frac{\partial \varphi_{\mathrm{i}}}{\partial \mathrm{T}}\dot{\mathrm{T}}, \tag{6}$$

where φ_{i} is the expression for the loading surface associated with the index i in terms of the spatial variables. The (spatial) loading – unloading criteria follow naturally from Equations (5) as:

$$\begin{cases} \text{If } \mathrm{j_{adm}} = \varnothing: & \text{elastic state.} \\ \text{If } \mathrm{j_{adm}} \neq \varnothing \text{ and } \mathrm{j_{act}} = \varnothing: & \\ \quad \text{i. If } \mathrm{r}_\alpha < 0 \text{ for all } \alpha \in \mathrm{j_{adm}}: & \text{elastic unloading,} \\ \quad \text{ii. If } \mathrm{r}_\alpha = 0 \text{ for at least one } \alpha \in \mathrm{j_{adm}}: & \text{neutral loading,} \\ \text{If } \mathrm{j_{adm}} \neq \varnothing \text{ and } \mathrm{j_{act}} \neq \varnothing: & \text{inelastic loading.} \end{cases} \tag{7}$$

where the sets j_{adm} and j_{act} are now defined in terms of the spatial variables as: $j_{adm} = \{\alpha \in \{1,2,...,n\} / h_\alpha > 0\}$ and $j_{act} = \{\alpha \in J_{adm} / r_\alpha > 0\}$.

3 STRESS AND RATE OF DEFORMATION DECOMPOSITIONS

In this section, we derive decompositions of the rate of the Kirchhoff stress tensor and of the rate of deformation tensor into mechanical and thermal parts. The approach presented herein extends the work of Marsden, Hughes and Pister, which has been carried out within the theory of finite deformation nonlinear elasticity (see Marsden and Hughes [5], pp. 204 – 206). Their work has itself generalized the so called "Duhamel – Neumann hypothesis" for linearized (infinitesimal) elasticity (Sokolnikoff [9], p. 359). Our derivations are for inelastic materials with internal variables and in particular for shape memory alloys, within the framework of finite strains and rotations. Our results are given by Theorems 1 and 2. In Theorem 1 the Kirchhoff stress tensor is decomposed in terms of the rate of deformation tensor and the rate of temperature. Our proof is based on a manipulation of the constitutive equation for the Kirchhoff stress tensor $\boldsymbol{\tau}$, which is defined as:

$$\boldsymbol{\tau} = \rho_{\text{ref}} \frac{\partial \psi}{\partial \mathbf{e}}, \tag{8}$$

where ψ is the free energy in spatial coordinates. The Kirchhoff stress tensor is related to Cauchy stress $\boldsymbol{\sigma}$ by $\boldsymbol{\tau} = J\boldsymbol{\sigma}$, where J is the determinant of the deformation gradient **F,** and can be also thought of as the push – forward onto the spatial configuration of the second Piola

– Kirchhoff stress tensor. In our derivation we will employ the concept of Lie derivative, a Legendre transformation and the rate Equations (5). The procedure is the following:

By considering an inelastic process and taking Lie derivatives of both members of Equation (8) and noting that the material time derivative of the mass density in the reference configuration is zero, $\dot{\rho}_{\mathrm{ref}} = 0,$ we derive:

$$\mathrm{L}_{\mathbf{v}}\boldsymbol{\tau} = \rho_{\mathrm{ref}}\left(\frac{\partial^2\psi}{\partial\mathbf{e}^2}:\mathrm{L}_{\mathbf{v}}\mathbf{e} + \frac{\partial^2\psi}{\partial\mathbf{e}\partial\mathrm{T}}\dot{\mathrm{T}} + \frac{\partial^2\psi}{\partial\mathbf{e}\partial\mathbf{q}}:\mathrm{L}_{\mathbf{v}}\mathbf{q}\right), \tag{9}$$

which, on substituting for the Lie derivative of **q** from Equation (5) yields:

$$\begin{aligned}\mathrm{L}_{\mathbf{v}}\boldsymbol{\tau} &= \rho_{\mathrm{ref}}\left[\frac{\partial^2\psi}{\partial\mathbf{e}^2}:\mathrm{L}_{\mathbf{v}}\mathbf{e} + \frac{\partial^2\psi}{\partial\mathbf{e}\partial\mathrm{T}}\dot{\mathrm{T}} + \frac{\partial^2\psi}{\partial\mathbf{e}\partial\mathbf{q}}:\sum_{\mathrm{i}=1}^{\mathrm{n}} h_{\mathrm{i}}\mathbf{l}_{\mathrm{i}}(\mathbf{e},\mathrm{T},\mathbf{q},\mathbf{F})\left(\frac{\partial\varphi_{\mathrm{i}}}{\partial\mathbf{e}}:\mathrm{L}_{\mathbf{v}}\mathbf{e} + \frac{\partial\varphi_{\mathrm{i}}}{\partial\mathrm{T}}\dot{\mathrm{T}}\right)\right] = \\ &= \rho_{\mathrm{ref}}\left[\frac{\partial^2\psi}{\partial\mathbf{e}^2} + \frac{\partial^2\psi}{\partial\mathbf{e}\partial\mathbf{q}}:\sum_{\mathrm{i}=1}^{\mathrm{n}} h_{\mathrm{i}}\mathbf{l}_{\mathrm{i}}(\mathbf{e},\mathrm{T},\mathbf{q},\mathbf{F})\frac{\partial\varphi_{\mathrm{i}}}{\partial\mathbf{e}}\right]:\mathrm{L}_{\mathbf{v}}\mathbf{e} + \left[\frac{\partial^2\psi}{\partial\mathbf{e}\partial\mathrm{T}} + \frac{\partial^2\psi}{\partial\mathbf{e}\partial\mathbf{q}}:\sum_{\mathrm{i}=1}^{\mathrm{n}} h_{\mathrm{i}}\mathbf{l}_{\mathrm{i}}(\mathbf{e},\mathrm{T},\mathbf{q},\mathbf{F})\frac{\partial\varphi_{\mathrm{i}}}{\partial\mathrm{T}}\right]\dot{\mathrm{T}}.\end{aligned} \tag{10}$$

Note, that the double contraction between a fourth and a second order tensor, denoted by the symbol **:**, produces a second order tensor. We now define the *tangential stiffness* (a fourth order tensor) and the *thermal stress coefficient tensor* (a second order tensor) as

$$\mathbf{a} = \frac{\partial^2\psi}{\partial\mathbf{e}^2} + \frac{\partial^2\psi}{\partial\mathbf{e}\partial\mathbf{q}}:\sum_{\mathrm{i}=1}^{\mathrm{n}} h_{\mathrm{i}}\mathbf{l}_{\mathrm{i}}(\mathbf{e},\mathrm{T},\mathbf{q},\mathbf{F})\frac{\partial\varphi_{\mathrm{i}}}{\partial\mathbf{e}},\ \mathbf{m} = \frac{\partial^2\psi}{\partial\mathbf{e}\partial\mathrm{T}} + \frac{\partial^2\psi}{\partial\mathbf{e}\partial\mathbf{q}}:\sum_{\mathrm{i}=1}^{\mathrm{n}} h_{\mathrm{i}}\mathbf{l}_{\mathrm{i}}(\mathbf{e},\mathrm{T},\mathbf{q},\mathbf{F})\frac{\partial\varphi_{\mathrm{i}}}{\partial\mathrm{T}}, \tag{11}$$

respectively. It is instructive and helpful to give here their indicial expression also:

$$a_{\mathrm{abcd}} = \frac{\partial^2\psi}{\partial e_{\mathrm{ab}}\partial e_{\mathrm{cd}}} + \frac{\partial^2\psi}{\partial e_{\mathrm{ab}}\partial q_{\mathrm{r_1\ldots r_m}}}\sum_{\mathrm{i}=1}^{\mathrm{n}} h_{\mathrm{i}} l_{\mathrm{ir_1\ldots r_m}}\frac{\partial\varphi_{\mathrm{i}}}{\partial e_{\mathrm{cd}}},\ \mathrm{m}_{\mathrm{ab}} = \frac{\partial^2\psi}{\partial e_{\mathrm{ab}}\partial\mathrm{T}} + \frac{\partial^2\psi}{\partial e_{\mathrm{ab}}\partial q_{\mathrm{r_1\ldots r_m}}}\sum_{\mathrm{i}=1}^{\mathrm{n}} h_{\mathrm{i}} l_{\mathrm{ir_1\ldots r_m}}\frac{\partial\varphi_{\mathrm{i}}}{\partial\mathrm{T}}. \tag{12}$$

By taking into account these definitions and by noting that the Lie derivative of the Almansi strain tensor equals the rate of deformation tensor **d** (e.g., see Holzapfel [7], p. 107), we can state the following theorem:

Theorem 1: *For the rate – independent SMA material with internal variables, whose evolution in the course of martensitic transformations is described by the rate Equations (5) (or equivalently by Equations (3)), the thermomechanical constitutive equation relating the (objective) rate of the Kirchhoff stress tensor to the rate of deformation tensor and to the rate of the temperature is given by*

$$\mathrm{L_v}\boldsymbol{\tau} = \rho_{\mathrm{ref}}(\mathbf{a} : \mathbf{d} + \mathbf{m}\dot{\mathrm{T}}), \tag{13}$$

where the tangential stiffness **a** *and the thermal coefficients* **m** *are dependent on the martensitic phase transformation properties, through their dependence on the internal variables.*

We will now obtain the counterpart of Equation (13), i.e., an expression for the rate of deformation tensor in terms of the rate of the Kirchhoff stress tensor and the temperature rate. This will be accomplished as follows. First, we define the *complementary free energy,* by using the Legendre transformation technique, as:

$$\chi = \hat{\chi}(\boldsymbol{\tau}, \mathrm{T}, \mathbf{q}) = \frac{1}{\rho_{\mathrm{ref}}}(\boldsymbol{\tau} : \mathbf{e}) - \psi(\mathbf{e}, \mathrm{T}, \mathbf{q}). \tag{14}$$

We assume that the change of variables from **e** to $\boldsymbol{\tau}$ is invertible and that the evolution of the internal variables can be given in the stress space in the course *of inelastic loading* by a rate equation of the form

$$\mathrm{L_V}\mathbf{q} = \sum_{\mathrm{i}=1}^{\mathrm{n}} t_{\mathrm{i}}\mathbf{c}_{\mathrm{i}}(\boldsymbol{\tau}, \mathrm{T}, \mathbf{q}, \mathbf{F})[\mathbf{x}_{\mathrm{i}}(\boldsymbol{\tau}, \mathrm{T}, \mathbf{q}, \mathbf{F}) : \mathrm{L_V}\boldsymbol{\tau} + y_{\mathrm{i}}(\boldsymbol{\tau}, \mathrm{T}, \mathbf{q}, \mathbf{F})\dot{\mathrm{T}}], \tag{15}$$

where the functions t_{i}, $\mathbf{c}_{\mathrm{i}}$ have an analogous meaning with the functions h_{i}, $\mathbf{l}_{\mathrm{i}}$ in the rate Equations (5) and $\mathbf{x}_{\mathrm{i}}$, y_{i} are tensorial functions of the (new) state variables. By taking partial derivatives in Equation (14) with respect to the Kirchhoff stress we obtain:

$$\rho_{\mathrm{ref}}\frac{\partial\hat{\chi}}{\partial\boldsymbol{\tau}}=\mathbf{e}+\boldsymbol{\tau}:\frac{\partial\mathbf{e}}{\partial\boldsymbol{\tau}}-\rho_{\mathrm{ref}}\frac{\partial\psi}{\partial\mathbf{e}}\frac{\partial\mathbf{e}}{\partial\boldsymbol{\tau}},\tag{16}$$

which in light of the constitutive equation for the Kirchhoff stress tensor (Equation (8)) yields the constitutive relation for the Almansi strain tensor as a function of the Kirchhoff stress tensor:

$$\mathbf{e}=\rho_{\mathrm{ref}}\frac{\partial\hat{\chi}}{\partial\boldsymbol{\tau}}.\tag{17}$$

By taking the Lie derivative of both hand sides in Equation (17) we obtain

$$\mathrm{L}_{\mathbf{v}}\mathbf{e}=\rho_{\mathrm{ref}}\left(\frac{\partial^2\chi}{\partial\boldsymbol{\tau}^2}:\mathrm{L}_{\mathbf{v}}\boldsymbol{\tau}+\frac{\partial^2\chi}{\partial\boldsymbol{\tau}\partial\mathrm{T}}\dot{\mathrm{T}}+\frac{\partial^2\chi}{\partial\boldsymbol{\tau}\partial\mathbf{q}}:\mathrm{L}_{\mathbf{v}}\mathbf{q}\right),\tag{18}$$

which in turn, upon substitution of the rate Equation (15) takes the form

$$\begin{aligned}\mathrm{L}_{\mathbf{v}}\mathbf{e}&=\rho_{\mathrm{ref}}\Big(\frac{\partial^2\chi}{\partial\boldsymbol{\tau}^2}:\mathrm{L}_{\mathbf{v}}\boldsymbol{\tau}+\frac{\partial^2\chi}{\partial\boldsymbol{\tau}\partial\mathrm{T}}\dot{\mathrm{T}}+\frac{\partial^2\chi}{\partial\boldsymbol{\tau}\partial\mathbf{q}}:\sum_{\mathrm{i}=1}^{\mathrm{n}}t_{\mathrm{i}}\mathbf{c}_{\mathrm{i}}(\boldsymbol{\tau},\mathrm{T},\mathbf{q},\mathbf{F})[\mathbf{x}_{\mathrm{i}}(\boldsymbol{\tau},\mathrm{T},\mathbf{q},\mathbf{F}):\mathrm{L}_{\mathbf{v}}\boldsymbol{\tau}+y_{\mathrm{i}}(\boldsymbol{\tau},\mathrm{T},\mathbf{q},\mathbf{F})\dot{\mathrm{T}}\Big)=\\&\rho_{\mathrm{ref}}\Big[\frac{\partial^2\chi}{\partial\boldsymbol{\tau}^2}+\frac{\partial^2\chi}{\partial\boldsymbol{\tau}\partial\mathbf{q}}:\sum_{\mathrm{i}=1}^{\mathrm{n}}t_{\mathrm{i}}\mathbf{c}_{\mathrm{i}}(\boldsymbol{\tau},\mathrm{T},\mathbf{q},\mathbf{F})\mathbf{x}_{\mathrm{i}}(\boldsymbol{\tau},\mathrm{T},\mathbf{q},\mathbf{F})\Big]:\mathrm{L}_{\mathbf{v}}\boldsymbol{\tau}+\\&+\Big[\frac{\partial^2\chi}{\partial\boldsymbol{\tau}\partial\mathrm{T}}+\frac{\partial^2\chi}{\partial\boldsymbol{\tau}\partial\mathbf{q}}:\sum_{\mathrm{i}=1}^{\mathrm{n}}t_{\mathrm{i}}\mathbf{c}_{\mathrm{i}}(\boldsymbol{\tau},\mathrm{T},\mathbf{q},\mathbf{F})y_{\mathrm{i}}(\boldsymbol{\tau},\mathrm{T},\mathbf{q},\mathbf{F})\Big]\dot{\mathrm{T}}.\end{aligned}\tag{19}$$

We now define the fourth and second order *material compliance* tensors **r** and **s** as

$$\mathbf{r}=\frac{\partial^2\chi}{\partial\boldsymbol{\tau}^2}+\frac{\partial^2\chi}{\partial\boldsymbol{\tau}\partial\mathbf{q}}:\sum_{\mathrm{i}=1}^{\mathrm{n}}t_{\mathrm{i}}\mathbf{c}_{\mathrm{i}}(\boldsymbol{\tau},\mathrm{T},\mathbf{q},\mathbf{F})\mathbf{x}_{\mathrm{i}}(\boldsymbol{\tau},\mathrm{T},\mathbf{q},\mathbf{F}),\ \mathbf{s}=\frac{\partial^2\chi}{\partial\boldsymbol{\tau}\partial\mathrm{T}}+\frac{\partial^2\chi}{\partial\boldsymbol{\tau}\partial\mathbf{q}}:\sum_{\mathrm{i}=1}^{\mathrm{n}}t_{\mathrm{i}}\mathbf{c}_{\mathrm{i}}(\boldsymbol{\tau},\mathrm{T},\mathbf{q},\mathbf{F})y_{\mathrm{i}}(\boldsymbol{\tau},\mathrm{T},\mathbf{q},\mathbf{F}),\tag{20}$$

respectively, with components

$$r_{\mathrm{abcd}}=\frac{\partial^2\chi}{\partial\tau_{\mathrm{ab}}\partial\tau_{\mathrm{cd}}}+\frac{\partial^2\chi}{\partial\tau_{\mathrm{ab}}\partial q_{\mathrm{r_1\ldots r_n}}}\sum_{\mathrm{i}=1}^{\mathrm{n}}t_{\mathrm{i}}c_{\mathrm{ir_1\ldots r_n}}x_{\mathrm{icd}},\ s_{\mathrm{ab}}=\frac{\partial^2\chi}{\partial\tau_{\mathrm{ab}}\partial\mathrm{T}}+\frac{\partial^2\chi}{\partial\tau_{\mathrm{ab}}\partial q_{\mathrm{r_1\ldots r_n}}}\sum_{\mathrm{i}=1}^{\mathrm{n}}t_{\mathrm{i}}c_{\mathrm{ir_1\ldots r_n}}y_{\mathrm{i}}.\tag{21}$$

Therefore, and since the Lie derivative of $\boldsymbol{e}$ is equal to $\boldsymbol{\tau}$, Equation (19) with the help of Equations (20) yields for the rate of deformation tensor

$$\mathbf{d} = \rho_{\mathrm{ref}}(\mathbf{r} : \mathrm{L}_{\mathrm{V}}\boldsymbol{\tau} + \mathbf{s}\dot{\mathrm{T}}). \tag{22}$$

We have therefore proved the following theorem:

Theorem 2: *For a rate – independent shape memory alloy material with internal variables, whose evolution in the course of martensitic transformations is described in strain – space and in the current configuration by the rate Equations (5) (or equivalently in the reference configuration by Equations (3)), its rate of deformation tensor can be decomposed in terms of the Lie derivative of its stress tensor and the rate of its temperature, according to Equation (22).*

4 CONCLUSIONS

The formulation of a finite strain multi surface generalized plasticity theory in strain space and in referential and current configurations for the description of phase transformations, as well as the important loading-unloading criteria have been reviewed. The main thrust of this work is the *derivation* of a decomposition of the rate of the Kirchhoff stress tensor and of the rate of deformation tensor into mechanical and thermal rates, for materials with internal variables, for arbitrary strains and rotations and for non isothermal conditions.

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