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COMPLETE FORMULATION OF THE SUBLOADING SURFACE MODEL

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Abstract. The subloading surface model is endowed the noticeable ability to describe the wide classes of irreversible mechanical behavior [1]. However, the past formulations of the subloading surface model have contained several inexact equations, which have been modified repeatedly after the concept of the subloading surface was proposed in 1977 [2]. The exact formulation is presented first in this article for the hypoelastic-based plasticity, which enjoys the distinguished superiority in the both aspects of the description of material behavior in high accuracy and of the numerical calculation in high efficiency.

1 INTRODUCTION

The subloading surface is based on the quite natural physical insight that the plastic strain rate develops continually as the stress approaches the yield surface. The formulation of the subloading surface model for the elastoplastic deformation has been modified repeatedly and developed from the initial ones [2,3,4,5]. Further, it has been applied to the descriptions of wide classes of irreversible mechanical phenomena, i.e. the monotonic and cyclic loadings of metals and soils, the viscoplastic deformation behavior 0, the damage behavior, the phase transformation behavior, the friction behavior and the crystal plasticity [3]. On the other hand, the other unconventional elastoplasticity models, e.g. the multi surface model [6], the two surface model [7] and the superposed kinematic hardening model [8], which does not assume that the inside of yield surface is an elastic domain but assume the existence of the small purely-elastic domain, have been formulated only for the description of cyclic loading behavior of metals. The extensive applicability of the subloading surface model to wide classes of irreversible mechanical behavior is based on the noticeable advantages such that it does not require the yield judgment on whether or not the stress reaches the yield surface and it is furnished with the controlling function such that the stress is pulled-back automatically to the yield surface when it goes out from the yield surface in numerical calculation due to finite incremental steps. However, the translation rules of the anisotropic hardening variable (back-stress) and the elastic-core, i.e. similarity-center of the yield and the subloading surfaces and the accurate expression of the Masing rule have been modified repeatedly but they have been formulated in the inexact forms in the past [1,3,9]. Now, the exact formulation of the subloading surface model will be attained in this article, passing near a half century after the concept of the subloading surface was proposed in 1977.

2 STRAIN RATE AND HYPOELASTIC RELATION

In the framework of the hypoelastic-based plasticity, incorporating velocity gradient $\boldsymbol{l} = \partial\mathbf{v}/\partial\mathbf{x}$, the strain rate $\mathbf{d} \equiv \mathrm{sym}[\boldsymbol{l}]$ and the continuum spin $\mathbf{w} \equiv \mathrm{ant}[\boldsymbol{l}]$ are used as the measures of the rates of deformation and rotation, respectively, where $\mathrm{sym}[\ ] = (\mathbf{t}+\mathbf{t}^T)/2$ and $\mathrm{ant}[\ ] = (\mathbf{t}-\mathbf{t}^T)/2$ stand for the symmetric and the anti-symmetric parts, respectively, for an arbitrary second-order tensor $\mathbf{t}$ and $(\ )^T$ for the transpose. Let the strain rate $\mathbf{d}$ be decomposed additively into elastic strain rate $\mathbf{d}^e$ and the plastic strain rate $\mathbf{d}^p$ as

$$\mathbf{d} = \mathbf{d}^e + \mathbf{d}^p \tag{1}$$

First, assume that the elastic strain rate is given by the hypoelastic relation on the premise that the elastic deformation is small compared with the plastic deformation:

$$\mathbf{d}^e = \mathbf{E}^{-1} : \overset{\circ}{\boldsymbol{\sigma}} \tag{2}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress. $\mathbf{E}$ is the elastic modulus tensor which is given by

$$E_{ijkl} = K\delta_{ij}\delta_{kl} + 2G\left\{\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{3}\delta_{ij}\delta_{kl}\right\} \tag{3}$$

for the Hooke's law, where K and G are the bulk modulus and the shear modulus, respectively. $(\overset{\circ}{\ })$ stands for the proper objective corotational rate, i.e.

$$\overset{\circ}{\mathbf{t}} \equiv \dot{\mathbf{t}} - \boldsymbol{\omega}\mathbf{t} + \mathbf{t}\boldsymbol{\omega} \tag{4}$$

for an arbitrary second order tensor $\mathbf{t}$, where $\boldsymbol{\omega}$ is the spin of substructure of material. The continuum spin $\mathbf{w}$ may be used for $\boldsymbol{\omega}$ up to a moderate deformation. Needless to say, the corotational rate is also used for tensor-valued internal variables to describe anisotropy.

The continuum spin $\mathbf{w}$ is additively decomposed into the elastic spin $\mathbf{w}^e$ and the *plastic spin* $\mathbf{w}^p$, i.e.

$$\mathbf{w} = \mathbf{w}^e + \mathbf{w}^p \tag{5}$$

Here, we adopt the *isoclinic concept* insisting that the rigid-body spin is involved in the elastic spin $\mathbf{w}^e$ under the postulate that the spin of the substructure is induced by the rotations due to the rigid-body motion and the elastic distortion. Then, the spin of substructure $\boldsymbol{\omega}$ is given by $\mathbf{w}^e$, i.e.

$$\boldsymbol{\omega} = \mathbf{w}^e = \mathbf{w} - \mathbf{w}^p \tag{6}$$

3 REFINEMENT FOR FORMULATION OF PLASTIC STRAIN RATE

The plastic strain rate will be formulated based on the concept of subloading surface in the following.

3.1 Yield surface

The normal-yield surface with the isotropic and the kinematic hardenings is described as

$$f(\hat{\boldsymbol{\sigma}}) = F(H) \tag{7}$$

where H is the isotropic hardening variable and

$$\hat{\boldsymbol{\sigma}} \equiv \boldsymbol{\sigma} - \boldsymbol{\alpha} \tag{8}$$

$\boldsymbol{\alpha}$ ($\mathrm{tr}\boldsymbol{\alpha} = 0$) being the kinematic hardening variable, i.e. back stress. The rates of these internal variables can be described by

$$\left.\begin{array}{l} \dot{H} = f_{hd}(\boldsymbol{\sigma}, F;\ \mathbf{d}^p) = f_{hd}(\boldsymbol{\sigma}, F;\ \mathbf{d}^p/\|\mathbf{d}^p\|)\|\mathbf{d}^p\| \\ \mathring{\boldsymbol{\alpha}} = \mathbf{f}_{kd}(\boldsymbol{\sigma}, F, \boldsymbol{\alpha};\ \mathbf{d}^{p\prime}) = \mathbf{f}_{kd}(\boldsymbol{\sigma}, F, \boldsymbol{\alpha};\ \mathbf{d}^{p\prime}/\|\mathbf{d}^p\|)\|\mathbf{d}^p\| \end{array}\right\} \tag{9}$$

since they are homogeneous functions of $\mathbf{d}^p$ in degree-one since they are induced only in the plastic loading process $\mathbf{d}^p \neq \mathbf{O}$ and the first-order time-differential quantities, where $\| \ \|$ designates the magnitude and $(\ )'$ the deviatoric part, . Here, assume that $f(\hat{\boldsymbol{\sigma}})$ is the homogeneous function of $\hat{\boldsymbol{\sigma}}$ in degree-one and thus it follows by the Euler's theorem that

$$\frac{\partial f(\hat{\boldsymbol{\sigma}})}{\partial \hat{\boldsymbol{\sigma}}} : \hat{\boldsymbol{\sigma}} = f(\hat{\boldsymbol{\sigma}}) \tag{10}$$

3.2 Concept of subloading surface model

The plastic strain rate is induced explicitly when the stress lies on the yield surface. Here, in facts, the plastic strain rate is induced not suddenly at the moment when the stress reaches the yield surface but it is induced gradually as the stress approaches the yield surface. The accurate description of the plastic strain rate induced by the rate of stress inside the yield surface is required in order to predict the cyclic loading behavior of materials, although it has been ignored in the conventional elastoplasticity. Then, let the following postulate be incorporated, which is the basic concept of the subloading surface model [1,2,3,4,5].

Fundamental postulate of elastoplasticity (Subloading surface concept): *The plastic strain rate is induced when the stress approaches the yield surface but only the elastic strain rate is induced when the stress moves towards of the inside the yield surface* as shown in **Fig. 1**, while the stress rate causes the elastic strain rate inevitably. In other words, **the stress approaches the yield surface when a plastic strain rate is induced but it moves towards the inside of the yield surface when only an elastic strain rate is induced.**

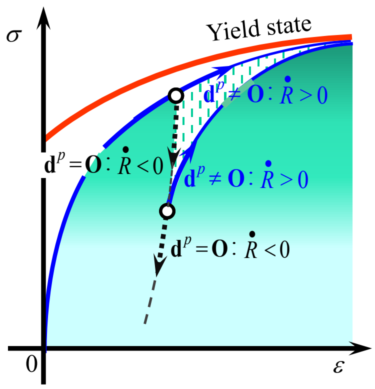

Fig. 1 Background of the subloading surface concept.

Then, it is first required to incorporate the measure which describes the approaching degree of the stress to the yield surface, renamed the *normal-yield surface*, in order to formulate the plastic strain rate based on the above-mentioned subloading surface concept.

Then, let the following ***subloading surface*** which passes through the current stress and keeps the similar shape and orientation to the normal-yield surface be introduced, which plays the general measure of approaching degree of the stress to the normal-yield surface (see **Fig. 2**).

$$f(\bar{\boldsymbol{\sigma}}) = RF(H) \tag{11}$$

where $R\,(0 \le R \le 1)$ is the ratio of the size of the subloading surface to that of the normal-yield surface and referred to as the *normal-yield ratio* which plays the role for the general measure to describe the approaching degree to the normal-yield surface.

$$\left.\begin{aligned} &\hat{\mathbf{c}} \equiv \mathbf{c} - \boldsymbol{\alpha} = (\mathbf{c} - \bar{\boldsymbol{\alpha}})/R \\ &\bar{\boldsymbol{\alpha}} \equiv \mathbf{c} - R\hat{\mathbf{c}} \\ &\bar{\boldsymbol{\sigma}} \equiv \boldsymbol{\sigma} - \bar{\boldsymbol{\alpha}}, \quad \tilde{\boldsymbol{\sigma}} \equiv \boldsymbol{\sigma} - \mathbf{c} \end{aligned}\right\} \tag{12}$$

leading to the expressions

$$\bar{\boldsymbol{\sigma}}\,(= \boldsymbol{\sigma} - \bar{\boldsymbol{\alpha}} = \boldsymbol{\sigma} - (\mathbf{c} - R\,\hat{\mathbf{c}}\,)) = \tilde{\boldsymbol{\sigma}} + R\hat{\mathbf{c}} \tag{13}$$

$\mathbf{c}$ represents the center of similarity of the normal-yield and the subloading surfaces, i.e. the similarity-center, while let it be called the *elastic-core* since the most elastic deformation behavior is induced when the stress lies on it. $\bar{\boldsymbol{\alpha}}$ stands for the conjugate (similar) point in the subloading surface to the point $\boldsymbol{\alpha}$ in the normal-yield surface. All of the relations of variables in Eq. (12) hold by virtue of the similarity of the subloading surface to the normal-yield surface as known from Fig. 2.

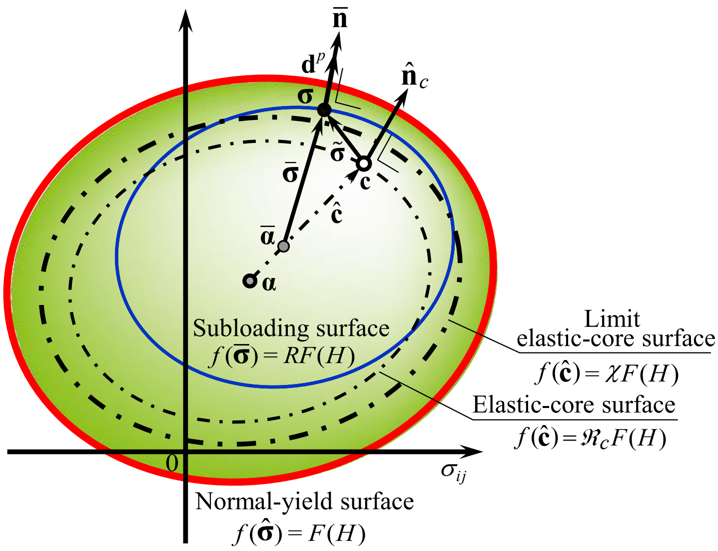

Fig. 2 Normal-yield, subloading and elastic-core surfaces.

3.3 Evolution rule of normal-yield ratio

Based on the afore-mentioned fundamental postulate of elastoplasticity, the rate of the normal-yield ratio must satisfy the following conditions.

$$\dot{R}\begin{cases}\to\infty \text{ for } R=0\\ >0 \text{ for } R<1\\ =0 \text{ for } R=1\\ (<0 \text{ for } R>1)\end{cases} \quad \text{for } \mathbf{d}^p\neq\mathbf{O} \tag{14}$$

$$\dot{R}\begin{cases}=0 \text{ for } \mathbf{d}^e=\mathbf{O}\\ <0 \text{ for } \mathbf{d}^e\neq\mathbf{O}\end{cases} \quad \text{for } \mathbf{d}^p=\mathbf{O} \tag{15}$$

Taking account of the requirement in Eq. (14), let the evolution rule of the normal-yield ratio in the plastic deformation process be formulated as follows:

$$\dot{R} = U(R)\|\mathbf{d}^p\| \text{ for } \mathbf{d}^p\neq\mathbf{O} \tag{16}$$

where $U(R)$ is the monotonically-increasing function of R fulfilling the conditions (**Fig. 3**).

$$U(R)\begin{cases}\to+\infty \text{ for } R=0 \text{ (quasi-elastic state)}\\ >0 \text{ for } R<1 \text{ (subyield state)}\\ =0 \text{ for } R=1 \text{ (normal-yield state)}\\ <0 \text{ for } R>1 \text{ (over normal-yield state)}\end{cases} \tag{17}$$

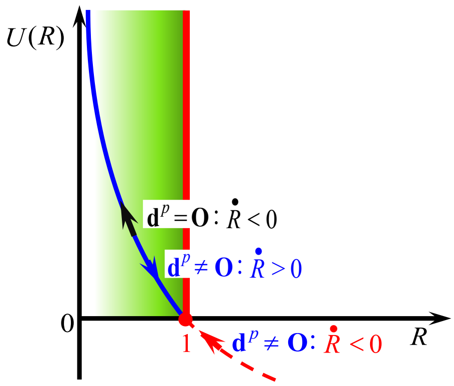

Fig. 3 Function $U(R)$ in the evolution rule of normal-yield ratio.

The function $U(R)$ is given by the following equation, where u is the material parameter.

$$U(R)=u\cot\left(\frac{\pi}{2}R\right) \tag{18}$$

Equation (17) should be extended to the following equation for metals in which plastic strain rate is not induced until the normal-yield ratio R reaches a certain value R_e (<1) which is the material parameter.

$$U(R)\begin{cases}\to+\infty \text{ for } 0\le R\le R_e \text{ (quasi-elastic state)}\\ >0 \text{ for } R_e<R<1 \text{ (subyield state)}\\ =0 \text{ for } R=1 \text{ (normal-yield state)}\\ <0 \text{ for } R>1 \text{ (over normal-yield state)}\end{cases} \tag{19}$$

Eqs. (18) is modified for Eq. (19) as follows:

$$U(R) = u\cot\left(\frac{\pi}{2}\frac{\langle R - R_e\rangle}{1 - R_e}\right) \tag{20}$$

where $\langle\ \rangle$ is the Macauley's bracket. If u is fixed to be constant, Eq. (16) with Eq. (20) can be integrated analytically as

$$R = \frac{2}{\pi}(1 - R_e)\cos^{-1}\left[\cos\left(\frac{\pi}{2}\frac{R_0 - R_e}{1 - R_e}\right)\exp\left(-u\frac{\pi}{2}\frac{\varepsilon^p - \varepsilon_0^p}{1 - R_e}\right)\right] + R_e \quad \text{for } R_0 \geq R_e \tag{21}$$

under the initial condition $\varepsilon^p = \varepsilon_0^p : R = R_0$, where $\varepsilon^p \equiv \int \|\mathbf{d}^p\| dt$ (t: time), whilst one must set $R_0 = R_e$ for $R_0 < R_e$. The use of the analytical integration in Eqs. (21) contributes to the enhancement of the numerical calculation in the return-mapping projection. However, it spoils the automatic controlling function to attract the stress to the normal-yield surface, which is inevitable in the numerical calculation in the forward-Euler method.

The subloading surface model possesses the following distinguished abilities.

1) Smooth transition from elastic to plastic state is described, fulfilling always the smoothness condition 0 as shown schematically in **Fig. 4**.

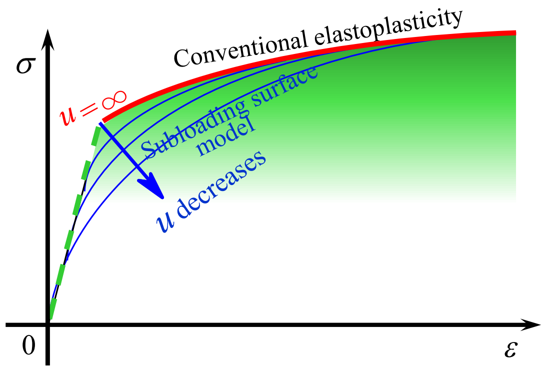

Fig. 4 Influence of material parameter u on curvature of stress vs. strain curve.

Then, we don't need suffer from the determination of an offset-value of strain for yielding. On the other hand, the determination of an offset value is required in the conventional elastoplasticity with an abrupt elastic-plastic transition, although it is accompanied with an arbitrariness. Smoother stress-strain curve is described for smaller value of the material parameter u.

2) Plastic strain rate can be described even for the loadings under a low stress level and a small stress amplitude since a purely-elastic domain is not assumed.

3) The yield-judgment whether or not the stress reaches the yield surface is unnecessary for the loading criterion, since the plastic strain rate develops continuously as the stress approaches the normal-yield surface.

4) The stress is automatically attracted to the normal-yield surface in the plastic loading process. Therefore, in the numerical calculation due to the forward-Euler method, it is pulled back automatically to the normal-yield surface when it goes out from the normal-yield surface in numerical calculation because of $\dot{R} < 0$ for $R > 1$ from Eq. (16) with Eq.

$(17)_4$ or $(19)_4$ as seen in **Fig. 5**.

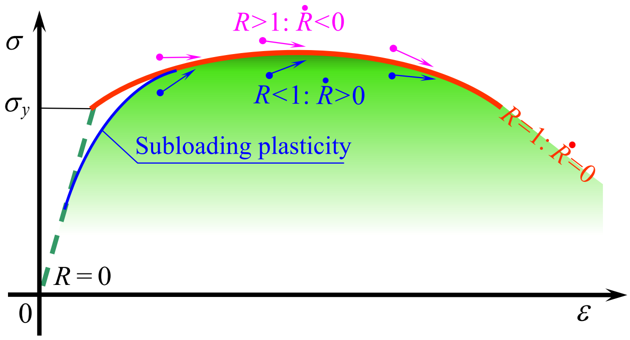

Fig. 5 Stress is automatically attracted to normal-yield surface in plastic loading process.

Consequently, the realistic description of mechanical behavior is attained by virtue of 1) and 2) and further the efficient numerical calculation is realized by virtue of 3) and 4). Thus, the subloading surface model possesses the noticeable advantages in both aspects of the physical description and the numerical calculation.

3.4 Associated flow rule for subloading surface

The associated flow rule for the subloading surface is adopted:

$$\mathbf{d}^p = \dot{\lambda}\bar{\mathbf{n}} \quad (\dot{\lambda}>0) \quad (\|\mathbf{d}^p\| = \dot{\lambda}) \tag{22}$$

where 10

$$\bar{\mathbf{n}} \equiv \frac{\partial f(\bar{\boldsymbol{\sigma}})}{\partial \boldsymbol{\sigma}} \Big/ \left\| \frac{\partial f(\bar{\boldsymbol{\sigma}})}{\partial \boldsymbol{\sigma}} \right\| = \frac{\partial f(\bar{\boldsymbol{\sigma}})}{\partial \bar{\boldsymbol{\sigma}}} \Big/ \left\| \frac{\partial f(\bar{\boldsymbol{\sigma}})}{\partial \bar{\boldsymbol{\sigma}}} \right\| \quad (\|\bar{\mathbf{n}}\|=1) \tag{23}$$

designating the magnitude and the direction of plastic strain rate by $\dot{\lambda}$ and $\bar{\mathbf{n}}$, respectively. Then, $\dot{H}$ and $\mathring{\boldsymbol{\alpha}}$ in Eq. (9) are expressed as follows:

$$\dot{H} = \dot{\lambda} f_{hn}(\boldsymbol{\sigma}, F; \bar{\mathbf{n}}), \quad \mathring{\boldsymbol{\alpha}} = \dot{\lambda}\,\mathbf{f}_{kn}(\boldsymbol{\sigma}, F, \boldsymbol{\alpha}; \bar{\mathbf{n}}) \tag{24}$$

where f_{hn} and $\mathbf{f}_{\alpha n}$ are the homogeneous functions of $\bar{\mathbf{n}}$ in degree-one.

3.5 Unified nonlinear kinematic hardening rule

The nonlinear kinematic-hardening rule of Armstrong and Frederick [10] is inapplicable to the general anisotropic hardening material with the plastically-compressibility causing the rotational-hardening as known from the fact that the anisotropic hardening is not induced by the isotropic plastic deformation but is induced by the deviatoric plastic deformation. Then, let the generalized evolution rule of anisotropic hardening variable $\boldsymbol{\alpha}$ $(\mathrm{tr}\boldsymbol{\alpha} = \mathrm{tr}\mathring{\boldsymbol{\alpha}} = 0)$ be given as follows:

$$\mathring{\boldsymbol{\alpha}} = c_k\left(\mathbf{d}^{p\prime} - \frac{1}{b_k}\|\mathbf{d}^{p\prime}\|\boldsymbol{\alpha}\right) = \dot{\lambda}\,\mathbf{f}_{kn}, \quad \mathbf{f}_{kn} = c_k\left(\bar{\mathbf{n}}' - \frac{1}{b_k}\|\bar{\mathbf{n}}'\|\boldsymbol{\alpha}\right) \tag{25}$$

where c_k is the material constant but b_k is the material function in general. Equation (25) is applicable not only to the kinematic hardening in metals but also to the rotational hardening, i.e. the rotation of yield surface in soils [1].

3.6 Translation rule of elastic-core

The most elastic deformation behavior is induced in the state that the stress lies on the similarity-center, i.e. $\boldsymbol{\sigma} = \mathbf{c}$ leading to $R = 0$. Then, the similarity-center $\mathbf{c}$ in the mathematical sense is interpreted physically as the most elastic stress state so that let it be called the *elastic-core* or *elastic-center*. Here, note that the elastic-core $\mathbf{c}$ approaches the normal-yield surface, following the stress $\boldsymbol{\sigma}$ in the plastic loading process. However, from the physical point of view the elastic-core should not approach the normal-yield surface without limitation as known from the fact that the abrupt transition from the elastic to the plastic state is predicted if the elastic-core lies on the normal-yield surface. On the other hand, the small yield surface enclosing a purely-elastic region is allowed to contact with the yield surface in the cyclic kinematic-hardening models [6,7] predicting the abrupt elastic-plastic transition. In addition, from the mathematical point of view the subloading surface is not determined uniquely if the stress coincides with the similarity-center lying just on the normal-yield surface.

Now, let the following *elastic-core surface* be introduced, which always passes through the elastic-core $\mathbf{c}$ and keeps the similar shape and orientation to the normal-yield surface with respect to the kinematic-hardening variable $\boldsymbol{\alpha}$.

$$f(\hat{\mathbf{c}}) = \mathcal{R}_c F(H), \text{ i.e. } \mathcal{R}_c = f(\hat{\mathbf{c}}) / F(H) \tag{26}$$

where $\mathcal{R}_c$ designates the ratio of the size of the elastic-core surface to the normal-yield surface (see Fig. 2) so that let it be called the *elastic-core yield ratio*. Here, the elastic-core should not reach (lie on) the normal-yield surface as described above so that the elastic-core does not go over the following *limit elastic-core surface.*

$$f(\hat{\mathbf{c}}) = \chi F(H) \tag{27}$$

where $\chi(<1)$ is material parameter and the following inequality must be satisfied.

$$f(\hat{\mathbf{c}}) \le \chi F(H), \text{ i.e. } \mathcal{R}_c \le \chi \tag{28}$$

Let the translation rule of elastic-core be formulated as

$$\overset{\circ}{\mathbf{c}} = c\left(\mathbf{d}^p - \frac{\mathcal{R}_c}{\chi}\|\mathbf{d}^p\|\hat{\mathbf{n}}_c\right) = \dot{\lambda}\,\mathbf{f}_{cn}, \quad \mathbf{f}_{cn} = c\left(\bar{\mathbf{n}} - \frac{\mathcal{R}_c}{\chi}\hat{\mathbf{n}}_c\right) \tag{29}$$

$$\left(= \begin{cases} c\mathbf{d}^p \text{ for } \mathcal{R}_c = 0 \\ \left.\begin{array}{l} \mathbf{O} \text{ for } \bar{\mathbf{n}} = \hat{\mathbf{n}}_c \\ 2c\mathbf{d}^p \text{ for } \bar{\mathbf{n}} = -\hat{\mathbf{n}}_c \end{array}\right\} \text{ for } \mathcal{R}_c = \chi \end{cases} \right)$$

where c is a material constant or material parameter and

$$\hat{\mathbf{n}}_c \equiv \frac{\partial f(\hat{\mathbf{c}})}{\partial \mathbf{c}} \Big/ \left\|\frac{\partial f(\hat{\mathbf{c}})}{\partial \mathbf{c}}\right\| = \frac{\partial f(\hat{\mathbf{c}})}{\partial \hat{\mathbf{c}}} \Big/ \left\|\frac{\partial f(\hat{\mathbf{c}})}{\partial \hat{\mathbf{c}}}\right\| \quad (\|\hat{\mathbf{n}}_c\|=1) \tag{30}$$

Here, it follows from Eq. (29) that

$$\hat{\mathbf{n}}_c : \overset{\circ}{\mathbf{c}} = c\,\dot{\lambda}\left(\hat{\mathbf{n}}_c : \bar{\mathbf{n}} - \frac{\mathcal{R}_c}{\chi}\right) < 0 \text{ for } \mathcal{R}_c > \chi \tag{31}$$

Therefore, the evolution rule of the elastic-core in Eq. (29) is furnished with the distinguished ability in numerical calculation that the elastic-core is automatically pulled-back to the limit elastic-core surface when it goes out from that surface by the input of finite numerical increment as shown in **Fig. 6**.

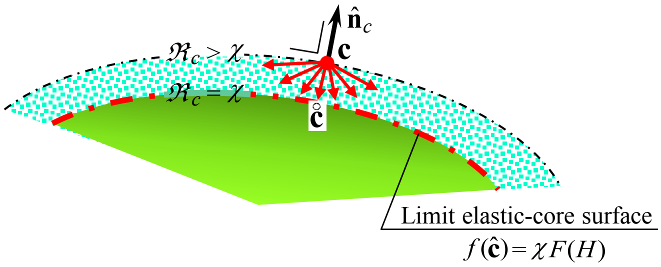

Fig. 6 Elastic-core is automatically attracted to the limit elastic-core surface when it goes out from the surface.

3.7 Plastic strain rate

The material-time derivative of Eq. (11) leads to the consistency condition of the subloading surface in the corotational time-derivative:

$$\frac{\partial f(\bar{\boldsymbol{\sigma}})}{\partial \bar{\boldsymbol{\sigma}}} : \mathring{\boldsymbol{\sigma}} - \frac{\partial f(\bar{\boldsymbol{\sigma}})}{\partial \bar{\boldsymbol{\sigma}}} : \mathring{\bar{\boldsymbol{\alpha}}} - \dot{R} F - R \dot{F} = 0 \tag{32}$$

It holds for Eq. (11) that

$$\frac{\partial f(\bar{\boldsymbol{\sigma}})}{\partial \bar{\boldsymbol{\sigma}}} : \bar{\boldsymbol{\sigma}} = RF \tag{33}$$

instead of Eq. (10), and then it follows that

$$\bar{\mathbf{n}} : \bar{\boldsymbol{\sigma}} = \frac{\partial f(\bar{\boldsymbol{\sigma}})}{\partial \bar{\boldsymbol{\sigma}}} : \bar{\boldsymbol{\sigma}} \Big/ \left\| \frac{\partial f(\bar{\boldsymbol{\sigma}})}{\partial \bar{\boldsymbol{\sigma}}} \right\| = RF \Big/ \left\| \frac{\partial f(\bar{\boldsymbol{\sigma}})}{\partial \bar{\boldsymbol{\sigma}}} \right\| \tag{34}$$

leading to

$$1 \Big/ \left\| \frac{\partial f(\bar{\boldsymbol{\sigma}})}{\partial \bar{\boldsymbol{\sigma}}} \right\| = \frac{\bar{\mathbf{n}} : \bar{\boldsymbol{\sigma}}}{RF} \tag{35}$$

The substitution of Eq. (35) into Eq. (32) leads to

$$\bar{\mathbf{n}} : \mathring{\boldsymbol{\sigma}} - \bar{\mathbf{n}} : \left[\left(\frac{\dot{F}}{F} + \frac{\dot{R}}{R} \right) \bar{\boldsymbol{\sigma}} + \mathring{\bar{\boldsymbol{\alpha}}} \right] = 0 \tag{36}$$

Here, the rate variable $\mathring{\bar{\boldsymbol{\alpha}}}$ is described from Eq. $(12)_2$ as

$$\mathring{\bar{\boldsymbol{\alpha}}} = R \mathring{\hat{\boldsymbol{\alpha}}} + (1-R) \mathring{\hat{\mathbf{c}}} - \dot{R} \hat{\mathbf{c}} \tag{37}$$

The substitution of Eq. (37) into Eq. (36) leads to

$$\bar{\mathbf{n}} : \mathring{\boldsymbol{\sigma}} - \bar{\mathbf{n}} : \left[\frac{\dot{F}}{F} \bar{\boldsymbol{\sigma}} + \frac{\dot{R}}{R} (\bar{\boldsymbol{\sigma}} - R \hat{\mathbf{c}}) + R \mathring{\hat{\boldsymbol{\alpha}}} + (1-R) \mathring{\hat{\mathbf{c}}} \right] = 0 \tag{38}$$

Further, noting the relation

$$\bar{\boldsymbol{\sigma}} - R\hat{\mathbf{c}} = \boldsymbol{\sigma} - \bar{\boldsymbol{\alpha}} - (\mathbf{c} - \bar{\boldsymbol{\alpha}}) = \tilde{\boldsymbol{\sigma}} \tag{39}$$

it follows from Eq. (38) that

$$\bar{\mathbf{n}} : \overset{\circ}{\boldsymbol{\sigma}} - \bar{\mathbf{n}} : \left[\frac{\dot{F}}{F} \bar{\boldsymbol{\sigma}} + R \overset{\circ}{\bar{\boldsymbol{\alpha}}} + \frac{\dot{R}}{R} \tilde{\boldsymbol{\sigma}} + (1-R) \overset{\circ}{\mathbf{c}} \right] = 0 \tag{40}$$

Substituting Eqs. (16), (25) and (29) into Eq. (40), one has

$$\bar{\mathbf{n}} : \overset{\circ}{\boldsymbol{\sigma}} - \bar{\mathbf{n}} : \left[\frac{F'}{F} \dot{\lambda} f_{hn} \bar{\boldsymbol{\sigma}} + R \dot{\lambda} \mathbf{f}_{kn} + \frac{U}{R} \dot{\lambda} \tilde{\boldsymbol{\sigma}} + c(1-R) \dot{\lambda} \mathbf{f}_{cn} \right] = 0 \tag{41}$$

where $F' \equiv dF/dH$.

Further substituting the associated flow rule in Eq. (22) into Eq. (41), one has

$$\bar{\mathbf{n}} : \overset{\circ}{\boldsymbol{\sigma}} - \bar{M}^p \dot{\lambda} = 0 \tag{42}$$

where

$$\bar{M}^p = \bar{\mathbf{n}} : \left[\frac{F'}{F} f_{hn} \bar{\boldsymbol{\sigma}} + c_k R \left(\bar{\mathbf{n}} - \frac{1}{b_k} \boldsymbol{\alpha} \right) + \frac{U}{R} \tilde{\boldsymbol{\sigma}} + c(1-R) \left(\bar{\mathbf{n}} - \frac{\mathfrak{R}_c}{\chi} \hat{\mathbf{n}}_c \right) \right] \tag{43}$$

by Eqs. (25) and (29).

The plastic multiplier $\dot{\lambda}$ and the plastic strain rate $\mathbf{d}^p$ are given from Eq. (42) and (22) as follows:

$$\dot{\lambda} = \frac{\bar{\mathbf{n}} : \overset{\circ}{\boldsymbol{\sigma}}}{\bar{M}^p}, \quad \mathbf{d}^p = \frac{\bar{\mathbf{n}} : \overset{\circ}{\boldsymbol{\sigma}}}{\bar{M}^p} \bar{\mathbf{n}} \tag{44}$$

3.8 Stress-strain relations

The strain rate is given by substituting Eqs. (2) and $(44)_2$ into Eq. (1) as follows:

$$\mathbf{d} = \mathbf{E}^{-1} : \overset{\circ}{\boldsymbol{\sigma}} + \frac{\bar{\mathbf{n}} : \overset{\circ}{\boldsymbol{\sigma}}}{\bar{M}^p} \bar{\mathbf{n}} \tag{45}$$

from which the proportionality factor described in terms of the strain rate, denoted by $\dot{\Lambda}$ instead of $\dot{\lambda}$, in the flow rule (22) is given as follows:

$$\dot{\Lambda} = \frac{\bar{\mathbf{n}} : \mathbf{E} : \mathbf{d}}{\bar{M}^p + \bar{\mathbf{n}} : \mathbf{E} : \bar{\mathbf{n}}}, \quad \mathbf{d}^p = \frac{\bar{\mathbf{n}} : \mathbf{E} : \mathbf{d}}{\bar{M}^p + \bar{\mathbf{n}} : \mathbf{E} : \bar{\mathbf{n}}} \bar{\mathbf{n}} \tag{46}$$

The stress rate is given from Eq. (45) by use of Eq. $(46)_1$ instead of Eq. $(44)_1$, as follows:

$$\overset{\circ}{\boldsymbol{\sigma}} = \mathbf{E} : \mathbf{d} - \frac{\bar{\mathbf{n}} : \mathbf{E} : \mathbf{d}}{\bar{M}^p + \bar{\mathbf{n}} : \mathbf{E} : \bar{\mathbf{n}}} \mathbf{E} : \bar{\mathbf{n}} \tag{47}$$

3.9 Loading criterion

The loading criterion is given as follows :

$$\left. \begin{array}{l} \mathbf{d}^p \neq \mathbf{O} \ \text{for} \ \dot{\Lambda} > 0 \\ \mathbf{d}^p = \mathbf{O} \ \text{for} \ \dot{\Lambda} \leq 0 \end{array} \right\} \quad \text{or} \quad \left. \begin{array}{l} \mathbf{d}^p \neq \mathbf{O} \ \text{for} \ \bar{\mathbf{n}} : \mathbf{E} : \mathbf{d} > 0 \\ \mathbf{d}^p = \mathbf{O} \ \text{for} \ \bar{\mathbf{n}} : \mathbf{E} : \mathbf{d} \leq 0 \end{array} \right\} \tag{48}$$

where the judgment whether or not the stress reaches the yield surface is not required since the plastic strain rate is induced continuously as the stress approaches the normal-yield surface. It should be noted that the loading judgment by the plastic multiplier $\dot{\lambda}$ in terms of the stress rate in Eq. $(44)_1$ cannot be used for the softening behavior and even for the hardening behavior since the stress is required to be pulled-back to the normal-yield surface leading to the contraction of the subloading surface after it goes over the normal-yield surface in the plastic loading process.

3.10 Calculation of normal-yield ratio

Substituting Eq. (13) into Eq. (11), the subloading surface is described as follows:

$$f(\tilde{\boldsymbol{\sigma}} + R\hat{\mathbf{c}}) = RF(H) \tag{49}$$

from which the normal-yield ratio R is calculated by substituting the updated values of $\boldsymbol{\sigma}$, $\boldsymbol{\alpha}$, $\mathbf{c}$, F.

The normal-yield ratio can be calculated by the following two methods:

1) We calculate it from Eq. (49) in both of the plastic (loading) and the elastic (unloading) processes after all the other variables are calculated.

2) We calculate it by Eq. (49) in the elastic (unloading) process and in the state $R \le R_e$ but we calculate it by the time-integration of Eq. (16) in the plastic (loading) process. Here, the analytical time-integration in Eq. (21) is beneficial to the enhancement of numerical analysis in the return-mapping projection 0. However, its use spoils the controlling function to pull-back the stress to the normal-yield surface numerical analysis in the forward-Euler method.

The second method 2) would be superior to the first method 1), since the normal-yield ratio is calculated directly from the plastic strain rate.

3.11 Expression of Masing rule

Note the following facts:

1) The difference between the curvatures in the reloading and the reverse loading curves becomes larger as the plastic deformation proceeds, which is called the Masing rule.

2) The elastic core approaches the normal-yield surface, following the current stress, when the plastic deformation proceeds continuously, and the approaching degree of the elastic core to the normal-yield surface is expressed by the elastic-core yield ratio $\mathfrak{R}_c$ in Eq. (26).

3) The transition from the elastic to plastic state is more abrupt, i.e. the curvature of stress–strain curve is greater for a larger value of the material parameter u in the function $U(R)$ in Eq. (16). Therefore, the increase in the curvature of stress-strain curve can be described by giving a larger value to the material parameter u.

4) By the facts 1)-3), the difference between the values u in the reloading process and the reverse loading process is greater for a larger value of $\mathfrak{R}_c$.

5) The direction $\bar{\mathbf{n}}$ of plastic strain rate is near to the outward-normal $\hat{\mathbf{n}}_c$ of the elastic core surface (Fig. 2) in the reloading process but it is far from $\hat{\mathbf{n}}_c$ in the unloading process. Then, the degree how the process is near the reloading process can be expressed by the following scalar product of these unit tensors:

$$C_\sigma \equiv \hat{\mathbf{n}}_c : \bar{\mathbf{n}} \quad (-1 \le C_\sigma \le 1) \tag{50}$$

Eventually, introducing the variables $\Re_c$ and C_σ, let the material parameter u in Eq. (18) or Eq. (20) be extended as follows:

$$u = \bar{u}\exp(u_c \Re_c C_\sigma) \tag{51}$$

where $\bar{u}$ (average value of u) and u_c are the material constant. u is the continuous function of the variables $\Re_c$ and C_σ. $C_\sigma = 1$, 0 and -1 designate the states that the current stress has the outward-normal, tangential and inward-normal directions, respectively, of the similarity-center surface. Then, u increases in the loading direction but inversely it decreases in the opposite direction.

3.12 Plastic spin

The plastic spin in Eq. (6) is given following Zbib and Aifantis [11] by

$$\mathbf{w}^p = \eta^p(\boldsymbol{\sigma}\mathbf{d}^p - \mathbf{d}^p\boldsymbol{\sigma}) = \eta^p \dot{\bar{\lambda}}(\boldsymbol{\sigma}\bar{\mathbf{n}} - \bar{\mathbf{n}}\boldsymbol{\sigma}) \tag{52}$$

where η^p is the material parameter.

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