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A FUNCTIONAL FOR THE MOMENTUM EQUATIONS OF INCOMPRESSIBLE VISCOUS FLOW

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Abstract. Vectorial mechanics and analytical mechanics are two time-honored forms of classical mechanics. Vectorial mechanics is mainly based on Newton's laws in a clear and simple mathematical form. It has achieved a high degree of sophistication and success in solid mechanics. Analytical mechanics is based on the principle of virtual work and D'Alembert's principle, which is highly universal. Often, the term vectorial mechanics is applied to the form based on Newton's work, to contrast it with analytical mechanics which uses two scalar properties of motion, the kinetic and potential energies, instead of vector forces, to analyze the motions. Analytical mechanics was primarily developed to extend the scope of classical mechanics in a systematic, generalized and efficient way to solve problems using the concept of constraints on systems and path integrals. In this paper, we give a functional of fluid in Lagrangian form. Then we demonstrate that the momentum equations of incompressible viscous flow can be achieved after several mathematical operations. At last, we show the Eulerian approximation of the energy functional under some assumptions. Our work lays a good foundation for our numerical methods.

1 INTRODUCTION

As well as in solid mechanics, there are also two time-honored approaches in hydrodynamics, namely, vectorial mechanics and analytical mechanics. Vectorial mechanics is mainly based on Newton's laws in a clear and simple mathematical form. Due to Newton's contribution to mechanics, vectorial mechanics has achieved an overwhelming advantage to analytical mechanics. The reference [1] gives several reasons why vectorial mechanics is preferred by engineers. In contrast to vectorial mechanics, analytical mechanics is based on the principle of virtual work and D'Alembert's principle, which are highly universal. That is, if one more physical effect is to consider, just incorporate the corresponding energy into the total energy functional. With the development of FEM (finite element method), analytical mechanics has been widely used in solid mechanics. However, there is relatively less scientific research on variational principles of incompressible viscous flow. In hydrodynamics, the study of variational principles lags behind although several researchers have studied the variational principles in fluid mechanics [3, 4, 5, 6]. As Chien Wei-zang [2] stated, most of the work were focusing on inviscid flow and external flow. Based on the weak form of the momentum equations for incompressible flow, Chien Wei-zang[2] established an energy

functional in Eulerian form. Unfortunately, he gave an integral formulation for the terms corresponding the convective terms without the specific functional. Lu Wentang [1] developed a functional in Lagrangian form and showed the equivalence to incompressible viscous flow based the principle of least action. In his paper, the work done by the viscous force is evaluated along the path line of a fluid particle, which is relatively difficult to use in numerical methods. Due to the convective terms in the equations of incompressible viscous in Eulerian form, the symmetry would not exist and the equations can often become unstable if the Galerkin method is used. That is, when the incompressible flow problems are solved by the standard Galerkin method which employs the equal order basis for the velocity and pressure field, we will not be able to obtain simultaneously the satisfactory results for velocity and pressure [8]. Zienkienwicz [7] proposed the CBS (characteristic-based split) algorithm to circumvent the BB restrictions. So, it is essential to establish a more meaningful and practical functional. It is just what we are trying to do in the present paper.

The outline of the present paper is arranged as follows. First, we will develop a functional for incompressible viscous flow in Lagrangian form. Every term in the energy functional stands for a more physical meaning. We will also demonstrate its equivalence to the momentum equations of incompressible viscous flow using the Lagrangian equations. Once this is done, the Eulerian approximation to the Lagrangian energy functional will be given in section 3. In section 4, several conclusions will be drawn.

2 THE ENERGY FUNCTIONAL IN LAGRANGIAN FORM

Let

$$\Pi \equiv \Pi_k + \Pi_p + \Pi_v + \Pi_b + \Pi_f$$

$$\Pi_k \equiv \int_\Omega \frac{1}{2}\rho v_i v_i d\Omega \qquad \Pi_b \equiv -\int_{\partial\Omega} p u_i n_i dS + \int_{\partial\Omega} \tau_{ij} u_i n_j dS \tag{1}$$

$$\Pi_v \equiv \int_\Omega \tau_{ij}\varepsilon_{ij} d\Omega \qquad \Pi_p \equiv \int_\Omega p u_{i,i} d\Omega \qquad \Pi_f \equiv \int_\Omega f_i u_i d\Omega$$

where Ω is an arbitrary system full of material points, $\partial\Omega$ is the boundary of the domain , $\mathbf{X}$ are the curvilinear coordinates attached to the material while $\mathbf{x}$ are the spatial coordinates, $u_i = u_i\left(\mathbf{x}\left(\mathbf{X};t\right);t\right)$ and $v_i = v_i\left(\mathbf{x}\left(\mathbf{X};t\right);t\right)$ are components of the displacement and velocity of material point $\mathbf{X}$ at time t respectively, p is the pressure, ε_{ij} and τ_{ij} are the strain and stress respectively, f_i is the body force (per unit volume), n_i is the unit outwards normal vector to the boundary. In incompressible flow, the density ρ is assumed constant and the velocity v_i is subjected to the following constraints

$$v_{i,i} = 0 \tag{2}$$

The boundary consists of two kinds of boundaries, namely velocity boundary S_1 and stress boundary S_2. The conditions for velocity v_i and traction t_i are introduced as follows:

$$\begin{aligned} &v_i = \hat{v}_i \qquad on \quad S_1 \\ &t_i = \left\{-p\delta_{ij} + \mu\left(v_{i,j} + v_{j,i}\right)\right\} n_j = \hat{t}_i \quad on \quad S_2 \end{aligned} \tag{3}$$

where the superposed ˆ denotes the functions which are given on the boundary.

In addition, the initial conditions consist of specifying the values of velocity and pressure at the initial time:

$$\begin{aligned} v_i\left(x_i,0\right) &= v_i^{(0)}\left(x_i\right) \\ p\left(x_i,0\right) &= p^{(0)}\left(x_i\right) \end{aligned} \tag{4}$$

The Lagrangian law states that the motion of a system subjected to complete constrains is governed by the following equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial T}{\partial q_k}=Q_k \quad \left(k=1,2,\cdots,N\right) \tag{5}$$

The above equations are called Lagrangian equations. Effectively, the Lagrangian law is also applied to a continuum. Now we will show how to get the momentum equation using the Lagrangian law.

2.1 Kinetic energy

Considering that the density ρ is constant, we know

$$\frac{d}{dt}\left(\frac{\partial \prod_k}{\partial v_i}\right)=\frac{d}{dt}\int_\Omega \rho v_i d\Omega=\int_\Omega \frac{\partial v_i}{\partial t}\rho d\Omega+\int_\Omega \rho v_j v_{i,j} d\Omega \tag{6}$$

In the above equation, the Reynolds theorem and law of mass conservation are used.

2.2 Pressure energy

Given the Guass's theorem, the energy $\prod_p$ can be written in the following form

$$\prod_p=\int_\Omega p u_{i,i} d\Omega=\int_\Omega (pu_i)_{,i} d\Omega-\int_\Omega p_{,i}u_i d\Omega=\int_{\partial\Omega} p u_i n_i dS-\int_\Omega p_{,i}u_i d\Omega \tag{7}$$

2.3 Viscous energy

For Newtonian fluid, the work done by the viscous stress can be achieved by the following formulation

$$\begin{aligned} \prod_v &= -\int_\Omega \tau_{ij}\varepsilon_{ij} dV=-\mu\int_\Omega v_{i,j}u_{i,j}d\Omega-\mu\int_\Omega v_{i,j}u_{j,i}d\Omega \\ &= -\mu\int_{\partial\Omega} v_{i,j}u_i n_j dS-\mu\int_{\partial\Omega} v_{j,i}u_i n_j dS+\mu\int_\Omega v_{i,jj}u_i d\Omega+\mu\int_\Omega v_{j,ij}u_i d\Omega \\ &= -\int_{\partial\Omega}\tau_{ij}u_i n_j dS+\int_\Omega \tau_{ij,j}u_i d\Omega \end{aligned} \tag{8}$$

where μ is the dynamics viscosity.

2.4 Summary of the equations

Substituting $\prod_k$, $\prod_p$, $\prod_v$, $\prod_b$ and $\prod_f$ into the Lagrangian equations, we get

$$\int_\Omega \frac{\partial v_i}{\partial t}\rho d\Omega+\int_\Omega \rho v_j v_{i,j} d\Omega+\int_\Omega p_{,i}d\Omega-\int_\Omega \tau_{ij,j}d\Omega-\int_\Omega f_i d\Omega=0 \tag{9}$$

Considering that the Ω is arbitrary, the momentum equations of incompressible flow can be achieved as shown

$$\rho\frac{\partial v_i}{\partial t}+\rho v_j v_{i,j}+p_{,i}-\tau_{ij,j}-f_i=0 \tag{10}$$

The above equation (10) or (11) is completed with the boundary conditions (2) and initial conditions (3).

3 AN EULERIAN APPROXIMATION TO THE ENERGY FUNCTIONAL

For any time interval $[\tau,T]$, there is a one-to-one transformation as shown

$$\begin{aligned}\mathbf{x}&=\mathbf{x}(\mathbf{X};t)\\ \mathbf{X}&=\mathbf{x}(\mathbf{X};\tau)\end{aligned} \tag{11}$$

Therefore, the velocity of fluid particle $\mathbf{X}$ is given by

$$v\big(\mathbf{x}(\mathbf{X};t);t\big)=\left.\frac{\partial\mathbf{x}(\mathbf{X};t)}{\partial t}\right|_{\mathbf{X}} \tag{12}$$

Based on the work of Duarte [9], we can build an approximation to $\mathbf{x}$ as follows

$$\mathbf{x}=\mathbf{X}+(t-\tau)\mathbf{v}\big(\mathbf{x}(\mathbf{X};\tau);\tau\big) \tag{13}$$

It can be easily shown that this approximation will result in a first-order error of $O(t-\tau)$.

For simplicity, we introduce a general variable φ. We denote Ω_τ and Ω_t, respectively , as the spatial domain occupied by a definitive system of fluid particles at time τ and t. Hence, we can obtain

$$\begin{aligned}\int_{\mathbf{X}\in\Omega_\tau}\varphi\big(\mathbf{x}(\mathbf{X};t);t\big)d\Omega_\tau&=\int_{\mathbf{X}\in\Omega_\tau}\varphi\big(\mathbf{X}+(t-\tau)\mathbf{v}\big(\mathbf{x}(\mathbf{X};\tau);\tau\big);t\big)d\Omega_\tau+O\Big[(t-\tau)^2\Big]\\ &=\int_{\mathbf{X}\in\Omega_\tau}\varphi\big|_{\mathbf{x}=\mathbf{X},\,t=\tau}\,d\Omega_\tau+\int_{\mathbf{X}\in\Omega_\tau}\left.\frac{\partial\varphi}{\partial t}\right|_{\mathbf{x}=\mathbf{X},\,t=\tau}(t-\tau)d\Omega_\tau\\ &\quad+\int_{\mathbf{X}\in\Omega_\tau}\left.\left(v_k\frac{\partial\varphi}{\partial x_k}\right)\right|_{\mathbf{x}=\mathbf{X},\,t=\tau}(t-\tau)d\Omega_\tau+O\Big[(t-\tau)^2\Big]\end{aligned} \tag{14}$$

Given the Guass's theorem and the constraint of incompressibility, we can get the following approximation

$$\begin{aligned}\int_{\mathbf{X}\in\Omega_\tau}\varphi\big(\mathbf{x}(\mathbf{X};t);t\big)d\Omega_\tau&=\int_{\mathbf{X}\in\Omega_\tau}\varphi\big|_{\mathbf{x}=\mathbf{X},\,t=\tau}\,d\Omega_\tau+\int_{\mathbf{X}\in\Omega_\tau}\left.\frac{\partial\varphi}{\partial t}\right|_{\mathbf{x}=\mathbf{X},\,t=\tau}(t-\tau)d\Omega_\tau\\ &\quad+\int_{\mathbf{X}\in\partial\Omega_\tau}(\varphi v_k)\big|_{\mathbf{x}=\mathbf{X},\,t=\tau}(t-\tau)n_k dS_\tau\end{aligned} \tag{15}$$

From the viewpoint of numerical methods, the Equation (16) is always rewritten as

$$\int_{\mathbf{X}\in\Omega_\tau}\varphi\big(\mathbf{x}(\mathbf{X};t);t\big)d\Omega_\tau=\int_{\mathbf{X}\in\Omega_\tau}\left.\left(\varphi+\frac{\partial\varphi}{\partial t}(t-\tau)\right)\right|_{\mathbf{x}=\mathbf{X},\,t=\tau}d\Omega_\tau+\int_{\mathbf{X}\in\partial\Omega_\tau}(\varphi u_k)\big|_{\mathbf{x}=\mathbf{X},\,t=\tau}n_k dS_\tau \tag{16}$$

In principle, $\prod_b$ should be expressed in Eulerian form. However, we still evaluate the integral over Ω_τ instead of Ω_t since the time interval $\Delta t = T - \tau$ is always sufficiently small. For the same reason, the second term on the right-hand side of Equation (12) is also ommitted. Hence, we obtain the Eulerian approximation to the energy functional:

$$\begin{aligned}\prod_{Euler} &= \int_{\Omega_\tau} \left(\frac{1}{2}\rho v_i v_i + p u_{i,i} + \tau_{ij}\varepsilon_{ij} + f_i u_i \right) d\Omega_\tau \\ &+ \int_{\partial\Omega_\tau} \left(\frac{1}{2}\rho v_i v_i + p u_{i,i} + \tau_{ij}\varepsilon_{ij} + f_i u_i \right) u_k n_k dS_\tau - \int_{\partial\Omega_\tau} p u_i n_i dS_\tau + \int_{\partial\Omega_\tau} \tau_{ij} u_i n_j dS_\tau\end{aligned} \tag{17}$$

Using the Lagrangian equations on the Eulerian energy functional, we will get the momentum equations:

$$\frac{d}{dt}\left(\frac{\partial \prod_{Euler}}{\partial v_i} \right) = \int_{\Omega_\tau} \frac{\partial v_i}{\partial t} \rho d\Omega + \int_{\Omega_\tau} \rho v_j v_{i,j} d\Omega + \int_{\partial\Omega_\tau} \frac{d}{dt}(\rho v_i) u_k n_k dS_\tau \tag{18}$$

$$\begin{aligned}\frac{\partial \prod_{Euler}}{\partial u_i} &= \int_{\Omega_\tau} -p_{,i} d\Omega + \int_{\Omega_\tau} \tau_{ij,j} d\Omega + \int_{\Omega_\tau} f_i d\Omega \\ &+ \int_{\partial\Omega_\tau} \left(\frac{1}{2}\rho v_i v_i + p u_{i,i} + \tau_{ij}\varepsilon_{ij} + f_i u_i \right) n_i dS_\tau \\ &+ \int_{\partial\Omega_\tau} -p_{,i} u_k n_k dS_\tau + \int_{\partial\Omega_\tau} \tau_{ij,j} u_k n_k dS_\tau + \int_{\partial\Omega_\tau} f_i u_k n_k dS_\tau \\ &- \int_{\partial\Omega_\tau} p n_i dS_\tau + \int_{\partial\Omega_\tau} \tau_{ij} n_j dS_\tau\end{aligned} \tag{19}$$

Inside the integration domain, the momentum equations can be obtained as follows

$$\int_{\Omega_\tau} \frac{\partial v_i}{\partial t} \rho d\Omega + \int_{\Omega_\tau} \rho v_j v_{i,j} d\Omega + \int_{\Omega_\tau} p_{,i} d\Omega - \int_{\Omega_\tau} \tau_{ij,j} d\Omega - \int_{\Omega_\tau} f_i d\Omega = 0 \tag{20}$$

On the boundaries, the momentum equations satisfy:

$$\begin{aligned}&\int_{\partial\Omega_\tau} \frac{d}{dt}(\rho v_i) u_k n_k dS_\tau - \int_{\partial\Omega_\tau} \left(\frac{1}{2}\rho v_i v_i + p u_{i,i} + \tau_{ij}\varepsilon_{ij} + f_i u_i \right) n_i dS_\tau \\ &+ \int_{\partial\Omega_\tau} p_{,i} u_k n_k dS_\tau - \int_{\partial\Omega_\tau} \tau_{ij,j} u_k n_k dS_\tau - \int_{\partial\Omega_\tau} f_i u_k n_k dS_\tau + \int_{\partial\Omega_\tau} p n_i dS_\tau - \int_{\partial\Omega_\tau} \tau_{ij} n_j dS_\tau = 0\end{aligned} \tag{21}$$

4 CONCLUSIONS

In this article, we give an energy functional of fluid in Lagrangian form and demonstrate the equavalence to the equations of incompressible viscous flow. In addition, under several assumptions, an approximation of the functional in Eulerian form is established. This functional lays a good foundation to develop a new numerical method of incompressible viscous flow.

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