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COUPLED ANALYSIS OF NONLINEAR STRUCTURAL MOTION AND FLUID SLOSHING

JOSÉ A. GONZÁLEZ* AND K. C. PARK†

*Escuela Técnica Superior de Ingeniería
Universidad de Sevilla
Camino de los Descubrimientos s/n, 41092 Sevilla, Spain
e-mail: japerez@us.es

†Division of Ocean Systems Engineering
Korea Advanced Institute of Science and Technology (KAIST)
335 Science Rd, Yuseong-gu, Daejeon 305-70, Republic of Korea
e-mail: kcpkaist@kaist.edu

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Abstract. Fluid sloshing in containers is modeled using a finite element formulation previously proposed by the authors for problems with moderate motions [1], extending in this work its application to arbitrarily large rotations and small deformations relative to a *floating frame of reference* moving with the fluid. This novel approach is used to investigate the coupling effects originated by the incidence of environmental sea waves on rigid floating vessels with internal flexible structural parts and fluids oscillating inside rigid or flexible tanks.

1 INTRODUCTION

This work presents a partitioned finite element formulation for the solution of structure-structure and fluid-structure coupled problems, under the assumption of relatively small deformations but arbitrarily large rotations, based on the the *floating frame of reference approach*. The main difference with the classical approach is in the algebraic separation of pure-deformational from pure-rigid modes, defining the position of the body-frame at any point of the undeformed state of the body. Pure-deformational modes are then measured with respect to this corotated configuration. The same concept was introduced by Fraeijs de Veubeke in [1], treating the case of a complete structure presenting coupled rigid-body and deformational motions. An important consequence of this definition of the reference frame is the uncoupling of rigid-body from deformational motions in the inertia mass matrix.

The mechanical response of the multibody system can also be coupled with a fluid domain. Fluid and structure systems are treaded separately and connected using *localized Lagrange multipliers* to a common frame tracking the interface motion. This coupling strategy permits easier parallelization and facilitates the enforcement of slip condition between the fluid and the structure walls. The fluid is modeled using a finite element formulation previously developed by the authors for sloshing problems with moderate motions, extending its application here to arbitrarily large rotations and small deformations relative to a floating frame of reference moving with the fluid. This novel approach is used to investigate the coupling effects of a rigid floating structure with flexible structural parts and a fluid oscillating inside rigid or flexible tanks due to global motion.

2 KINEMATICS

Two reference systems are introduced. The first one is a *fixed inertial-frame* and vectors expressed in this system are represented using capital letters. The second system is a *floating frame of reference* that is fixed to the body and moves with it, vectors defined in this system are written using lowercase. The position of an arbitrary point of the body is expressed in the inertial frame as:

$$\mathbf{X} = \mathbf{X}_0 + \mathbf{A}(\mathbf{r} + \mathbf{d}) \tag{1}$$

where $\mathbf{X}_0$ is the vector defining the position of a fixed point 0 of the body in the inertial frame, $\mathbf{A}$ is the rotation matrix of the body-frame, $\mathbf{r}$ is the vector defining the position of the point in its undeformed position expressed in the body-fixed frame and $\mathbf{d}$ is the deformational displacement vector of the same point expressed in the same frame.

Using the small deformational-displacements approximation, the position, velocity and acceleration of a particle expressed in the inertial frame can be simplified:

$$\begin{aligned}
\mathbf{X} &= \mathbf{X}_0 + \mathbf{A}(\mathbf{r} + \mathbf{d}) && (2)\\
\dot{\mathbf{X}} &= \dot{\mathbf{X}}_0 + \mathbf{A}\dot{\mathbf{d}} - \mathbf{A}\widetilde{\mathbf{r}}\boldsymbol{\omega} && (3)\\
\ddot{\mathbf{X}} &= \ddot{\mathbf{X}}_0 + \mathbf{A}\ddot{\mathbf{d}} - \mathbf{A}\widetilde{\mathbf{r}}\dot{\boldsymbol{\omega}} + \mathbf{A}\widetilde{\boldsymbol{\omega}}^2\mathbf{r} + 2\mathbf{A}\widetilde{\boldsymbol{\omega}}\dot{\mathbf{d}} && (4)
\end{aligned}$$

We can see in equation (4) that acceleration of a particle is obtained by composition of linear, deformational, angular, centrifugal and Coriolis acceleration terms.

Finally, given a set of virtual displacements $(\delta\mathbf{d}, \delta\mathbf{X}_0, \delta\boldsymbol{\theta})$, the virtual displacement $\delta\mathbf{X}$, expressed in the body-frame, is approximated :

$$\delta\mathbf{X} \;=\; \mathbf{A}\left[\begin{array}{ccc} \boldsymbol{\phi} & \mathbf{I} & -\widetilde{\mathbf{r}} \end{array}\right] \left\{\begin{array}{c} \delta\mathbf{d} \\ \mathbf{A}^{\mathsf{T}}\delta\mathbf{X}_0 \\ \delta\boldsymbol{\theta} \end{array}\right\} \tag{5}$$

where $\boldsymbol{\phi}$ represents the displacement interpolation matrix.

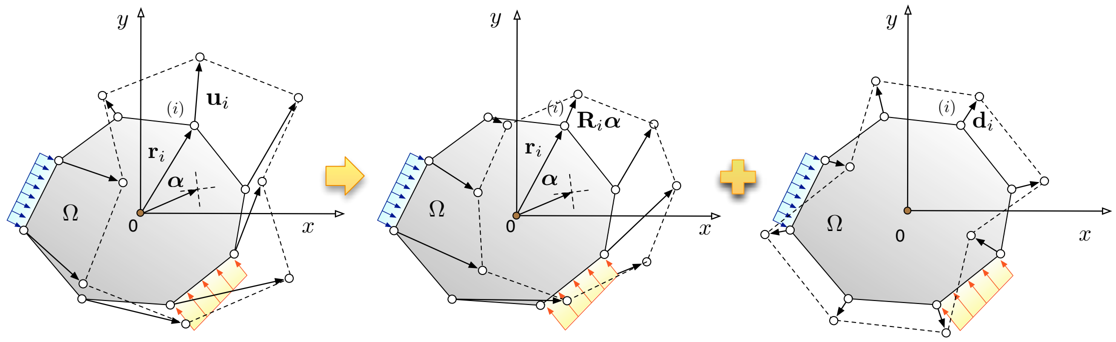


Figure 1: *Discretized free-floating substructure defined in the floating frame of reference. Decomposition of its local displacements in rigid-body and pure deformational displacement components.*

2.1 Description of the deformation

As explained earlier, the deformation of each floating substructure is described using a local frame of reference that translates and rotates following the undeformed picture of the solid. Merely condition to define this system is that the substructure displacement field does not present rigid body components or free modes.

Total displacements of a discretized free floating substructure can be separated into a pure deformational component plus a rigid-body part, as illustrated in Figure 1:

$$\mathbf{u} = \mathbf{d} + \mathbf{R}\boldsymbol{\alpha} \tag{6}$$

where $\mathbf{u}$ are the nodal displacements, $\mathbf{d}$ represents the vector of pure-deformational displacements, $\mathbf{R}$ is a basis of the rigid-body modes and vector $\boldsymbol{\alpha}$ collects the amplitudes of these rigid-body motions. Rigid-body matrix $\mathbf{R}$ is then a block-matrix composed of nodal contributions:

$$\mathbf{R}^{\mathsf{T}} = \begin{bmatrix} \mathbf{R}_1^{\mathsf{T}} & \ldots & \mathbf{R}_n^{\mathsf{T}} \end{bmatrix} \tag{7}$$

with sub-blocks that can be formed directly for each node of the mesh as:

$$\mathbf{R}_i = \begin{bmatrix} \mathbf{I} & -\widetilde{\mathbf{r}}_i \end{bmatrix} \quad (i = 1 \ldots n) \tag{8}$$

where $\mathbf{r}_i$ is the position vector of node i relative to a rotation center 0 and n is the total number of nodes of the free floating subdomain.

Once the rigid-body modes are obtained, separation of the total displacements into deformational and a rigid-body contributions can be done as described by Felippa and Park [2]. This is accomplished by using the projector:

$$\mathcal{P} = \mathbf{I} - \mathbf{M}\mathbf{R}\mathbf{M}_{\alpha}^{-1}\mathbf{R}^{\mathsf{T}} \tag{9}$$

where $\mathbf{M}$ is a symmetric definite positive mass matrix and $\mathbf{M}_{\alpha} = \mathbf{R}^{\mathsf{T}}\mathbf{M}\mathbf{R}$ is the principal mass matrix introduced by Park et al. in [3] a (6×6) matrix for a three-dimensional floating substructure. This operator presents the filtering properties $\mathcal{P}^{\mathsf{T}}\mathbf{R} = \mathbf{0}$ and $\mathcal{P}\mathbf{M}\mathbf{R} = \mathbf{0}$, allowing to separate pure deformational modes from rigid-body motions using the following expressions:

$$\mathbf{d} = \mathcal{P}^{\mathsf{T}}\mathbf{u}, \quad \mathbf{R}\boldsymbol{\alpha} = (\mathbf{I} - \mathcal{P}^{\mathsf{T}})\mathbf{u} \tag{10}$$

where, by definition, the projector $\mathcal{P}^{\mathsf{T}}$ performs an orthogonal projection in the subspace defined by the rigid-body modes and therefore is acting as a filter for the deformational component of displacements.

For our derivation of the equations of motion, it will be useful to separate the rigid-body modes $\mathbf{R}$ into its translational and rotational components:

$$\mathbf{R} = \left[\begin{array}{cc} \mathbf{R}_t & \mathbf{R}_r \end{array}\right] \tag{11}$$

using subscripts (t) and (r) to refer to the translational or the rotational component.

3 Variational formulation

The total virtual work for a group of deformable substructures undergoing arbitrary large-rotations is composed by the following terms:

$$\delta W_T = \delta W_i + \delta W_d + \delta W_f + \delta W_c \tag{12}$$

corresponding to the virtual work of inertia forces, internal forces, external forces and constraints.

Virtual work of inertia forces δW_i for a free-floating substructure is obtained integrating in the volume V the product of particle acceleration (4) times virtual displacement (5):

$$\delta W_i = \int_V \rho \ddot{\mathbf{X}} \cdot \delta \mathbf{X} \, dV \tag{13}$$

Introducing a FEM discretization, and considering equations (4) and (5), evaluation of the virtual work of inertia forces yields:

$$\begin{aligned}\delta W_i = \left\{ \begin{array}{c} \delta \mathbf{d} \\ \mathbf{A}^{\mathsf{T}} \delta \mathbf{X}_0 \\ \delta \boldsymbol{\theta} \end{array} \right\}^{\mathsf{T}} \{ & \left[\begin{array}{ccc} \mathbf{M} & \mathbf{S}_t & \mathbf{S}_r \\ \mathbf{S}_t^{\mathsf{T}} & \mathbf{M}_t & -m\widetilde{\mathbf{r}_G} \\ \mathbf{S}_r^{\mathsf{T}} & -m\widetilde{\mathbf{r}_G}^{\mathsf{T}} & \mathbf{M}_r \end{array} \right] \left\{ \begin{array}{c} \ddot{\mathbf{d}} \\ \mathbf{A}^{\mathsf{T}} \ddot{\mathbf{X}}_0 \\ \dot{\boldsymbol{\omega}} \end{array} \right\} + \\ & \left\{ \begin{array}{c} \mathbf{g}_d^{cen}(\boldsymbol{\omega}) \\ m\widetilde{\boldsymbol{\omega}}^2 \mathbf{r}_G \\ \widetilde{\boldsymbol{\omega}} \mathbf{M}_r \boldsymbol{\omega} \end{array} \right\} + \left\{ \begin{array}{c} \mathbf{g}_d^{cor}(\boldsymbol{\omega}, \dot{\mathbf{d}}) \\ \mathbf{g}_t^{cor}(\boldsymbol{\omega}, \dot{\mathbf{d}}) \\ \mathbf{g}_r^{cor}(\boldsymbol{\omega}, \dot{\mathbf{d}}) \end{array} \right\} \} \end{aligned} \tag{14}$$

where m is the total mass of the body, $\mathbf{r}_G$ is the position vector of the body COG in local coordinates, $\mathbf{M}_t = m\mathbf{I}_3$ is the (3x3) translational mass matrix, $\mathbf{M}_r$ is the (3x3) inertia tensor and $\mathbf{M} = \int_V \rho \boldsymbol{\phi}^{\mathsf{T}} \boldsymbol{\phi} \, dV$ is the finite-element mass matrix.

Coupling inertia terms in (14) take the form:

$$\mathbf{S}_t = \int_V \rho \boldsymbol{\phi}^\mathsf{T} \, dV, \quad \mathbf{S}_r = \int_V \rho \boldsymbol{\phi}^\mathsf{T} \widetilde{\mathbf{r}}^\mathsf{T} \, dV \tag{15}$$

and the quadratic velocity terms due to centrifugal and Coriolis accelerations:

$$\mathbf{g}_d^{cen}(\boldsymbol{\omega}) = \int_V \rho \boldsymbol{\phi}^\mathsf{T} \widetilde{\boldsymbol{\omega}}^2 \mathbf{r} \, dV, \quad \mathbf{g}_d^{cor}(\boldsymbol{\omega}, \dot{\mathbf{d}}) = 2 \int_V \rho \boldsymbol{\phi}^\mathsf{T} \widetilde{\boldsymbol{\omega}} \boldsymbol{\phi} \dot{\mathbf{d}} \, dV \tag{16}$$

with contributions of the Coriolis acceleration to the translational and rotational rigid-body equations given by:

$$\mathbf{g}_t^{cor}(\boldsymbol{\omega}, \dot{\mathbf{d}}) = 2\widetilde{\boldsymbol{\omega}} \mathbf{S}_t^\mathsf{T} \dot{\mathbf{d}}, \quad \mathbf{g}_r^{cor}(\boldsymbol{\omega}, \dot{\mathbf{d}}) = 2 \int_V \rho \widetilde{\mathbf{r}} \widetilde{\boldsymbol{\omega}} \boldsymbol{\phi} \dot{\mathbf{d}} \, dV \tag{17}$$

It is observed that the rigid-deformational inertia coupling terms can be expressed as the product of the deformational mass matrix and the rigid-body modes as $\mathbf{S}_t = \mathbf{M}\mathbf{R}_t$ and $\mathbf{S}_r = \mathbf{M}\mathbf{R}_r$. Similarly, for the rigid-body mass matrix terms of (14), we have:

$$\mathbf{M}_t = \mathbf{R}_t^\mathsf{T} \mathbf{M} \mathbf{R}_t, \quad \mathbf{M}_r = \mathbf{R}_r^\mathsf{T} \mathbf{M} \mathbf{R}_r, \quad -m\widetilde{\mathbf{r}_G} = \mathbf{R}_t^\mathsf{T} \mathbf{M} \mathbf{R}_r \tag{18}$$

relations previously derived by Park et al. in [3].

Observe that we have defined $\mathbf{d}$ in (14) as a pure deformational mode expressed in the body reference frame, but no mechanism has been introduced to enforce such condition. This is done introducing projector (9) and replacing the deformation vector $\mathbf{d}$ by its filtered counterpart $\mathcal{P}^\mathsf{T}\mathbf{d}$, arriving to the final expression for the vital work of ineria forces:

$$\begin{aligned} \delta W_i = \left\{ \begin{array}{c} \delta \mathbf{d} \\ \mathbf{A}^\mathsf{T} \delta \mathbf{X}_0 \\ \delta \boldsymbol{\theta} \end{array} \right\}^\mathsf{T} \{ \left[\begin{array}{ccc} \mathbf{M}_\mathcal{P} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_t & -m\widetilde{\mathbf{r}_G} \\ \mathbf{0} & -m\widetilde{\mathbf{r}_G}^\mathsf{T} & \mathbf{M}_r \end{array} \right] \left\{ \begin{array}{c} \ddot{\mathbf{d}} \\ \mathbf{A}^\mathsf{T} \ddot{\mathbf{X}}_0 \\ \dot{\boldsymbol{\omega}} \end{array} \right\} + \\ \left\{ \begin{array}{c} \mathcal{P} \mathbf{M} \boldsymbol{\Omega} \mathbf{R}_r \boldsymbol{\omega} \\ m \widetilde{\boldsymbol{\omega}}^2 \mathbf{r}_G \\ \widetilde{\boldsymbol{\omega}} \mathbf{M}_r \boldsymbol{\omega} \end{array} \right\} + \left\{ \begin{array}{c} 2\mathcal{P} \boldsymbol{\Omega} \mathbf{M} \dot{\mathbf{d}} \\ \mathbf{0} \\ 2\mathbf{R}_r^\mathsf{T} \boldsymbol{\Omega} \mathbf{M} \dot{\mathbf{d}} \end{array} \right\} \} \end{aligned} \tag{19}$$

$$\mathbf{M}_\mathcal{P} = \mathcal{P} \mathbf{M} \mathcal{P}^\mathsf{T}, \quad \boldsymbol{\Omega} = \mathbf{diag}(\widetilde{\boldsymbol{\omega}})$$

where deformational and rigid-body inertia terms are uncoupled.

Assuming that the reference point of the body is located at the COG, i.e. $\mathbf{r}_G = \mathbf{0}$, the discrete approximation of the variational is expressed:

$$\delta W_i = \delta \mathbf{X}_0^\mathsf{T} \{ \mathbf{M}_t \ddot{\mathbf{X}}_0 \} + \delta \mathbf{q}^\mathsf{T} \{ \mathbf{M}_\mathbf{G} \ddot{\mathbf{q}} - 2\mathbf{M}_{\dot{\mathbf{G}}} \mathbf{q} \} + \delta \mathbf{d}^\mathsf{T} \{ \mathbf{M}_\mathcal{P} \ddot{\mathbf{d}} + \mathbf{g}_d \} \tag{20}$$

$$\mathbf{M}_\mathbf{G} = \mathbf{G}^\mathsf{T} \mathbf{M}_r \mathbf{G}, \quad \mathbf{M}_{\dot{\mathbf{G}}} = \dot{\mathbf{G}}^\mathsf{T} \mathbf{M}_r \dot{\mathbf{G}}$$

with rotations parametrized using *Euler-parameters* denoted by the quaternion $\mathbf{q}$.

To compute the virtual work due to deformations, the substructure is discretized using the classical linear FEM approximation, where the assembly of element contributions by the direct stiffness method leads to the semi-discrete equations of motion:

$$\delta W_d = \delta \mathbf{d}^\mathsf{T}\{\mathbf{K}_\mathcal{P}\mathbf{d}\} \tag{21}$$

$$\mathbf{K}_\mathcal{P} = \mathcal{P}\mathbf{K}\mathcal{P}^\mathsf{T}$$

where $\mathbf{K}$ is the small-displacements finite element stiffness matrix, presenting a null-space given by the rigid-body modes, that is, $\mathbf{KR} = \mathbf{0}$ and where the projector operator has been introduced to eliminate the rigid-body component.

The virtual work produced by body loads and boundary tractions acting on the body is approximated by the discrete equation:

$$\delta W_f = -\delta \mathbf{d}^\mathsf{T}\{\mathcal{P}\mathbf{f}\} - \delta \mathbf{X}_0^\mathsf{T}\{\mathbf{A}\mathbf{R}_t^\mathsf{T}\mathbf{f}\} - \delta \mathbf{q}^\mathsf{T}\{\mathbf{G}^\mathsf{T}\mathbf{R}_r^\mathsf{T}\mathbf{f}\} \tag{22}$$

provided that the total external force vector is expressed as $\mathbf{f}$.

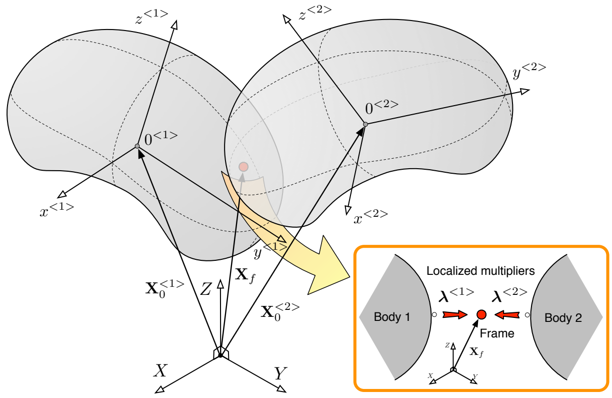


Figure 2: *Flexible substructures connected to an intermediate frame. The frame is endowed with independent displacement degrees of freedom and body-frame displacement compatibility condition is enforced using localized Lagrange multipliers.*

Flexible substructures are then connected using classical Lagrange multipliers, see Figure 2, that appear in the system as internal forces to satisfy compatibility conditions expressed in the body frame of reference.

The virtual work done by the constraints δW_c can be expressed:

$$\delta W_c = \int_{\Gamma_c} \delta\{(\mathbf{A}\boldsymbol{\lambda}) \cdot (\mathbf{X} - \mathbf{X}_f)\}\, dV \tag{23}$$

where $\boldsymbol{\lambda}$ represents the localized Lagrange multipliers attached to each substructure and expressed in the body frame of reference, used to enforce the kinematic compatibility condition between the body and a frame with position $\mathbf{X}_f$.

After discretization, equation (23) transforms into:

$$\begin{aligned}\delta W_c = \;& \delta\boldsymbol{\lambda}^{\mathsf{T}}\{\mathbf{B}^{\mathsf{T}}(\mathbf{R}_t\mathbf{A}^{\mathsf{T}}\mathbf{X}_0 + \mathbf{r} + \mathcal{P}^{\mathsf{T}}\mathbf{d}) - \mathbf{L}_f\mathbf{A}^{\mathsf{T}}\mathbf{X}_f\} + \delta\mathbf{X}_0^{\mathsf{T}}\{\mathbf{A}\mathbf{R}_t^{\mathsf{T}}\mathbf{B}\boldsymbol{\lambda}\} + \\ & \delta\mathbf{q}^{\mathsf{T}}\{\mathbf{G}^{\mathsf{T}}\mathbf{R}_r^{\mathsf{T}}\mathbf{B}\boldsymbol{\lambda}\} + \delta\mathbf{d}^{\mathsf{T}}\{\mathbf{B}_{\mathcal{P}}\boldsymbol{\lambda}\} - \delta\mathbf{X}_f^{\mathsf{T}}\{\mathbf{A}\mathbf{L}_f^{\mathsf{T}}\boldsymbol{\lambda}\}\end{aligned} \tag{24}$$

with $\mathbf{B}_{\mathcal{P}} = \mathcal{P}\mathbf{B}$, providing independent Lagrange multipliers for each substructure. This approach allows to express substructural constraint equations in the particular local system of each body involving only the unknowns of one body and its frame.

Finally, an additional constraint enforcing the unity norm of the quaternion should be added. This is done introducing a new Lagrangian multiplier μ to enforce this condition and adding to (24) a new term:

$$\delta W_q = \delta\{\mu(\mathbf{q}^{\mathsf{T}}\mathbf{q} - 1)\} \tag{25}$$

4 PARTITIONED EQUATIONS OF MOTION

The total virtual-work of a FEM partitioned system undergoing arbitrarily large rotations with small deformations is derived from (12), (20), (21), (22), (24) and (25):

$$\begin{aligned}\delta W_T = \;& \delta\mathbf{d}^{\mathsf{T}}\{\mathbf{M}_{\mathcal{P}}\ddot{\mathbf{d}} + \mathbf{K}_{\mathcal{P}}\mathbf{d} + \mathbf{B}_{\mathcal{P}}\boldsymbol{\lambda} + \mathcal{P}\mathbf{M}\boldsymbol{\Omega}\mathbf{R}_r\mathbf{G}\dot{\mathbf{q}} + 2\mathcal{P}\boldsymbol{\Omega}\mathbf{M}\dot{\mathbf{d}} - \mathcal{P}\mathbf{f}\} + \\ & \delta\boldsymbol{\lambda}^{\mathsf{T}}\{\mathbf{B}^{\mathsf{T}}(\mathbf{R}_t\mathbf{A}^{\mathsf{T}}\mathbf{X}_0 + \mathbf{r} + \mathcal{P}\mathbf{d}) - \mathbf{L}_f\mathbf{A}^{\mathsf{T}}\mathbf{X}_f\} + \\ & \delta\mathbf{X}_0^{\mathsf{T}}\{\mathbf{M}_t\ddot{\mathbf{X}}_0 + \mathbf{A}\mathbf{R}_t^{\mathsf{T}}\mathbf{B}\boldsymbol{\lambda} - \mathbf{A}\mathbf{R}_t^{\mathsf{T}}\mathbf{f}\} + \delta\mu\{2\mathbf{q}^{\mathsf{T}}\dot{\mathbf{q}}\} - \delta\mathbf{X}_f^{\mathsf{T}}\{\mathbf{A}\mathbf{L}_f^{\mathsf{T}}\boldsymbol{\lambda} \\ & \delta\mathbf{q}^{\mathsf{T}}\{\mathbf{M}_{\mathbf{G}}\ddot{\mathbf{q}} - 2\mathbf{M}_{\dot{\mathbf{G}}}\mathbf{q} + 2\mu\mathbf{q} + \mathbf{G}^{\mathsf{T}}\mathbf{R}_r^{\mathsf{T}}\mathbf{B}\boldsymbol{\lambda} + 2\mathbf{R}_r^{\mathsf{T}}\boldsymbol{\Omega}\mathbf{M}\dot{\mathbf{d}} - \mathbf{G}^{\mathsf{T}}\mathbf{R}_r^{\mathsf{T}}\mathbf{f}\}\}\end{aligned} \tag{26}$$

and from the stationary-point condition of total virtual-work, equations of motion are obtained:

$$\begin{bmatrix} (\mathbf{M}_{\mathcal{P}}\frac{d^2}{dt^2} + \mathbf{K}_{\mathcal{P}}) & \mathbf{B}_{\mathcal{P}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{\mathcal{P}}^{\mathsf{T}} & \mathbf{0} & \mathbf{B}^{\mathsf{T}}\mathbf{R}_t\mathbf{A}^{\mathsf{T}} & \mathbf{0} & \mathbf{0} & -\mathbf{L}_f\mathbf{A}^{\mathsf{T}} \\ \mathbf{0} & \mathbf{A}\mathbf{R}_t^{\mathsf{T}}\mathbf{B} & \mathbf{M}_t\frac{d^2}{dt^2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{G}^{\mathsf{T}}\mathbf{R}_r^{\mathsf{T}}\mathbf{B} & \mathbf{0} & (\mathbf{M}_{\mathbf{G}}\frac{d^2}{dt^2} - 2\mathbf{M}_{\dot{\mathbf{G}}}) & 2\mathbf{q} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 2\dot{\mathbf{q}}^{\mathsf{T}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{A}\mathbf{L}_f^{\mathsf{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{d} \\ \boldsymbol{\lambda} \\ \mathbf{X}_0 \\ \mathbf{q} \\ \boldsymbol{\mu} \\ \mathbf{X}_f \end{Bmatrix}$$

$$= \begin{Bmatrix} \mathcal{P}(\mathbf{f} - \mathbf{M}\boldsymbol{\Omega}\mathbf{R}_r\mathbf{G}\dot{\mathbf{q}} - 2\boldsymbol{\Omega}\mathbf{M}\dot{\mathbf{d}}) \\ -\mathbf{B}^{\mathsf{T}}\mathbf{r} \\ \mathbf{A}\mathbf{R}_t^{\mathsf{T}}\mathbf{f} \\ \mathbf{G}^{\mathsf{T}}\mathbf{R}_r^{\mathsf{T}}\mathbf{f} - 2\mathbf{R}_r^{\mathsf{T}}\boldsymbol{\Omega}\mathbf{M}\dot{\mathbf{d}} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \tag{27}$$

where all terms can be clearly identified; first equation is the deformational part of the FEM elastic equations, second equation imposes the interface compatibility condition between the substructure boundary and the frame, third equation represents the translational global equilibrium condition expressed in the inertial frame of reference, fourth equation represents the rotational equilibrium condition expressed in the floating-frame of reference, fifth equation enforces unity of the quaternions and last equation represents the frame equilibrium condition expressed in the inertial frame.

5 FLUID FORMULATION

The fluid is considered to be inviscid and incompressible, confined in a carrier structure under the action of gravity field $\mathbf{g}$, initially at rest with density ρ^0 and with initial hydrostatic pressure due to gravity $p^0 = p_{ext} + \rho^0 g(H - z)$. Next, we define in the body-frame a Lagrangian deformational displacement field $\mathbf{d}$ following the fluid particle and assume small deviations from this equilibrium position, as represented in Figure 3.

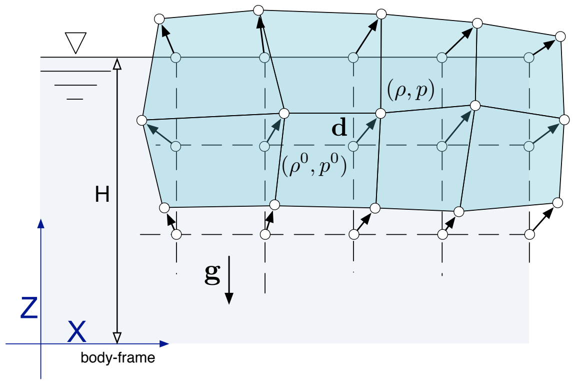


Figure 3: *Initial state of the fluid and deformation under the action of gravity and inertia forces. Discretization and Lagrangian description of motion using a floating frame of reference.*

According to González and Park [4], the virtual work of one fluid finite element expressed in the body-frame and including the gravity body force, can be expressed as:

$$\begin{aligned} \delta W_d^{(e)} &= \rho^0 c^2 V_f^{(e)} (\bar{\boldsymbol{\nabla}} \cdot \mathbf{d})^{(e)} (\bar{\boldsymbol{\nabla}} \cdot \delta \mathbf{d})^{(e)} + \int_{V_f^{(e)}} p^0 \mathbf{I} \colon (\boldsymbol{\nabla}^{\mathsf{T}} \mathbf{d} \boldsymbol{\nabla} \delta \mathbf{d})\, dV - \\ & \int_{V_f^{(e)}} \rho (\mathbf{A} - \mathbf{I}) \mathbf{g}\, dV \end{aligned} \tag{28}$$

a linearized approximation around the initial hydrostatic equilibrium state, assuming that the fluid deviations from equilibrium are small and where the last term accounts for the rotation of the gravity field.

Element displacements are discretized as $\mathbf{d} = \boldsymbol{\phi} \mathbf{d}^{(e)}$, where $\boldsymbol{\phi}$ collects the element shape functions while $\mathbf{d}^{(e)}$ gathers nodal values of the element. The element stiffness matrix is then composed of two terms:

$$\mathbf{K}^{(e)} = \mathbf{K}^{(e)}_{ac} + \mathbf{K}^{(e)}_{geo}, \tag{29}$$

$$\mathbf{K}^{(e)}_{ac} = \frac{\rho^0 c^2}{V^{(e)}_f} \int_{V^{(e)}_f} (\boldsymbol{\nabla} \cdot \boldsymbol{\phi})^{\mathsf{T}} \, dV \int_{V^{(e)}_f} (\boldsymbol{\nabla} \cdot \boldsymbol{\phi}) \, dV \tag{30}$$

$$\mathbf{K}^{(e)}_{geo} = \int_{V^{(e)}_f} p^{\mathcal{T}}_f (\boldsymbol{\nabla}\boldsymbol{\phi})^{\mathsf{T}} (\boldsymbol{\nabla}\boldsymbol{\phi}) \, dV. \tag{31}$$

where $\mathbf{K}^{(e)}_{ac}$ is the acoustic stiffness matrix and $\mathbf{K}^{(e)}_{geo}$ is the geometrical stiffness matrix.

Upon assembling the element matrices, we arrive to a discrete variational for the complete fluid mesh analog to (21):

$$\delta W_d = \delta \mathbf{d}^{\mathsf{T}} \{ \mathbf{K}_{\mathcal{P}} \mathbf{d} - \mathcal{P}(\mathbf{A} - \mathbf{A}^0)^{\mathsf{T}} \mathbf{F}_g \} = \delta \mathbf{d}^{\mathsf{T}} \{ \mathbf{K}_{\mathcal{P}} \mathbf{d} - \mathbf{f} \} \tag{32}$$

in which $\mathbf{K}_{\mathcal{P}}$ is the projected stiffness matrix of the fluid, $\mathbf{d}$ the vector of fluid deformational displacements and $\mathbf{f}$ the nodal forces increment due to gravity expressed in the body-frame.

6 FLOATING STRUCTURES

We study the particular case of a fluid contained in a flexible tank that is transported by a marine vessel. The hull of the ship is treated as a rigid body connected to the internal flexible structure and the influence of the external fluid into the system is introduced as an input. It is important to mention that the only external effect considered in the simulations are the buoyancy restoring forces and moments. Other hydrodynamic effects due to the interaction with the external fluid like, diffraction forces, added-mass, or viscous damping effects, are not considered and could be important in some applications.

The static stability condition of the ship under the effect of restoring forces, which are buoyancy and weight, is called the metacentric stability. Considering the roll motion of the ship, see Figure 4, there is a restoring moment in the form:

$$m_\phi = -\rho g \triangledown |GM_T| \sin(\phi) \tag{33}$$

where ρ is the density of water, $\triangledown$ is the displaced volume of water, GM_T is the transverse metacentric height, and ϕ is the roll angle. Similarly, there is a longitudinal metacenter, GM_L, which acts as center of rotation in case of disturbance in the pitch degree of freedom.

Heave motion of the ship is dominated by the effect of restoring forces. If the hull is assumed to have a shape of a rectangular prism, and a constant waterline area A_w, which is the area enclosed by the curve at the intersection of the body and the water surface, then the vertical force that forms as a result of the deviation in the vertical position of the ship with respect to the equilibrium position, can be given by

$$F_z = -\rho g A_w \delta Z_0 \tag{34}$$

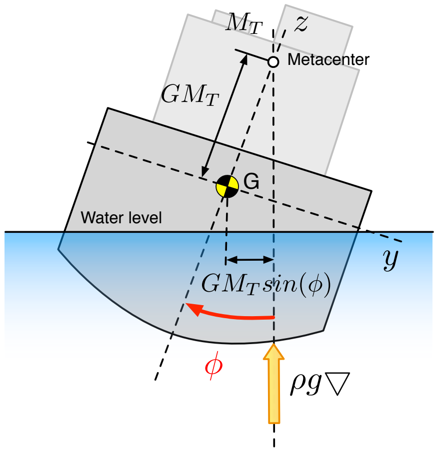


Figure 4: *Transversal restoring force on a floating structure due to roll rotation. Restoring moment is proportional to the distance between the center of gravity (G) and the transversal metacentre (M_T) in the y-z plane.*

Finally, the restoring forces vector may be obtained by the collection of these force components as follows:

$$\mathbf{f}_h(\mathbf{X}_0, \mathbf{q}) = -\rho g A_w \mathbf{A}^{\mathsf{T}} \mathbf{A}_Z (\mathbf{X}_0 - \mathbf{X}_0^0), \quad \mathbf{A}_Z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{35}$$

The buoyancy moments can be expressed:

$$\mathbf{m}_h(\mathbf{q}) = -\rho g \bigtriangledown \left\{ \begin{array}{c} |GM_T|\, \mathbf{q}^{\mathsf{T}} \mathbf{A}_\phi \mathbf{q} \\ |GM_L|\, \mathbf{q}^{\mathsf{T}} \mathbf{A}_\theta \mathbf{q} \\ 0 \end{array} \right\} \tag{36}$$

with constant matrices

$$\mathbf{A}_\phi = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad ; \quad \mathbf{A}_\theta = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

Variation of the buoyancy forces and moments is obtained by differentiation from (35) and (36):

$$\Delta \mathbf{f}_h(\mathbf{X}_0, \mathbf{q}) = -\rho g A_w \mathbf{A}^{\mathsf{T}} \mathbf{A}_Z \Delta \mathbf{X}_0 - \rho g A_w \widetilde{\mathbf{f}_h} \mathbf{G} \Delta \mathbf{q} \tag{37}$$

and the moments increment are obtained from:

$$\Delta \mathbf{m}_h(\mathbf{q}) = -2\rho g \bigtriangledown \left[\left\{ \begin{array}{c} |GM_T| \\ 0 \\ 0 \end{array} \right\} \mathbf{q}^{\mathsf{T}} \mathbf{A}_\phi + \left\{ \begin{array}{c} 0 \\ |GM_L| \\ 0 \end{array} \right\} \mathbf{q}^{\mathsf{T}} \mathbf{A}_\theta \right] \Delta \mathbf{q} \tag{38}$$

7 SIMULATION OF SLOSHING IN FLOATING STRUCTURES

In this Section, a two-body problem is used to validate the proposed computational technique and demonstrate its potential. We consider the case of a rigid floating structure connected to a rigid tank transporting 75 m^3 of water. The geometrical properties and dimensions of the ship, including the exact position of the tank inside the ship, are presented in Table 1.

m	Mass	2433 ton
I_ϕ	Rolling inertia	$2433 \times 10^4\ m^4$
I_θ	Pitching inertia	$2433 \times 10^6\ m^4$
I_z	Pitching inertia	$2433 \times 10^6\ m^4$
$\bigtriangledown$	Displaced volume of water	6000 m^3
A_w	Water plane area	500 m^2
GM_t	Transverse metacentric height	3 m
GM_l	Longitudinal metacentric height	30 m
ρ	Fluid density	1000 kg/m^3
T_z	Heave period	4.425 s
T_ϕ	Roll period	2.332 s
T_θ	Pitch period	7.375 s

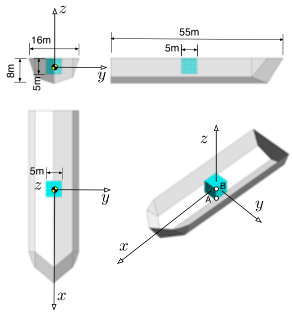


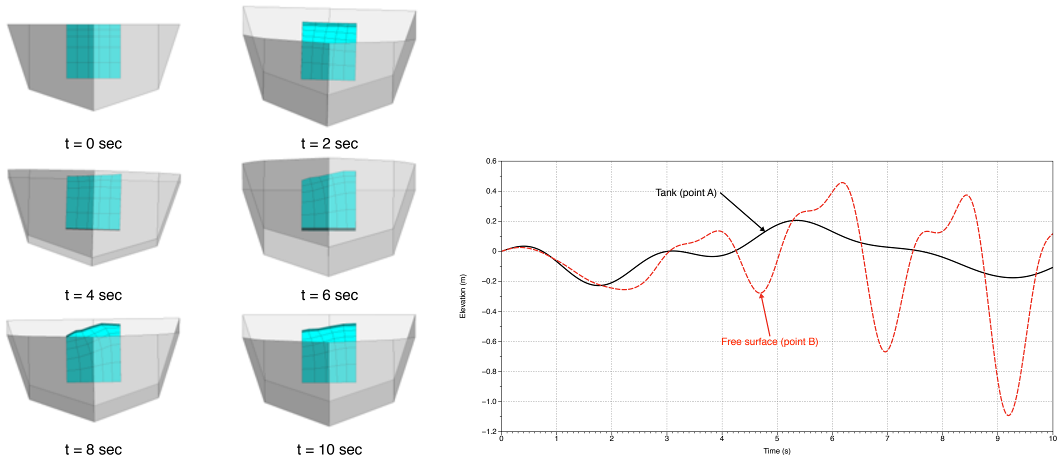
Table 1: Ship design parameters. 3D cubical tank configuration. Mesh, dimensions and relative position of the tank inside the marine vessel.

Ship motion is induced by imposing an initial rigid-body rotational velocity to the ship $\boldsymbol{\omega}^0 = \{0.1, 0.05, 0\}^\mathsf{T} rad/s$. The transported fluid, with properties $\rho = 1000kg/m^3$ and $c = 1500m/s$, is modeled using 125 hexahedral finite elements with three degrees of freedom per node.

Implicit time integration of the equations of motion is performed by using a fixed time step $\Delta t = 0.01s$, with Newmark parameters $(\gamma = \frac{1}{2}, \beta = \frac{1}{4})$ combined with a dissipation parameter $\alpha = 0.01$. Figure 5 shows different positions in time of the hull and the free-surface of the fluid due to the prescribed rolling-pitching initial velocity. Time history of the tank displacement and free-surface elevation is given in Figure 5 for two points A and B aligned in the vertical direction near the wall.

8 CONCLUSIONS

- A fully implicit partitioned finite-element formulation for the analysis of structure-structure and fluid-structure systems presenting large rotations and small deformations has been presented. The proposed computational framework is based on the *floating frame of reference* approach separating rigid-body motions from de-



(a) (b)

Figure 5: (a) Sloshing motion inside the tank. Instantaneous ship positions and deformed configurations of the fluid. (b) Evolution of the vertical displacement at the bottom of the tank and elevation of the free-surface.

formational displacements and uses localized Lagrange multipliers to satisfy the constraints.

- It has been demonstrated that this new procedure is very well suited for modeling moderate sloshing phenomena in coupled carrier-internal fluid problems where the carrier presents large translations and rotations while deformations in the fluid can be considered small. A numerical example is used to demonstrate the accuracy, robustness, and efficiency of the proposed solution algorithm.

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