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MODEL PROBLEMS IN MAGNETO-HYDRODYNAMICS: INDIVIDUAL NUMERICAL CHALLENGES AND COUPLING POSSIBILITIES

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Abstract. In this work we discuss two model problems appearing in magneto-hydrodynamics (MHD), namely, the so called full MHD problem and the inductionless MHD problem. The first involves as unknowns the fluid velocity and pressure, the magnetic (induction) field and a pseudo-pressure introduced to impose the zero-divergence restriction of this last unknown. The building blocks of this model are the Stokes problem for the velocity and the pressure and the Maxwell problem for the magnetic field and pseudo-pressure. We discuss the numerical challenges of the approximation of these two model problems having in mind the need to couple them in the full problem, where additional coupling terms appear. The second model we consider is the inductionless MHD approximation. Instead of the magnetic induction and pseudo-pressure, the magnetic unknowns are now the current density and the electric potential. The building blocks are the Stokes problem for the fluid and the Darcy problem (in primal form) for the current density and electric potential. We discuss also the numerical challenges involved in the approximation of this last problem, particularly considering that it has to be coupled to the former. Once the building blocks have been analysed independently, the possibilities of dealing with the fully coupled problems are discussed. Iterative schemes that can be shown to be stable are presented in the stationary case, showing that a segregated solution for the flow and the magnetic problem is not possible. Most of the results presented are elaborated independently in previous works. Our objective in this paper is to present the different problems with a unified perspective.

1 INTRODUCTION

The objective of this work is to discuss some aspects related to the finite element approximation of two model problems in MHD, namely, the so called full MHD approximation and the inductionless model. The two main issues to be addressed are the compatibility conditions required for the unknowns and the iterative schemes that may be used (at least for the stationary problem), as well as the links between both aspects.

Let us present the two models to be discussed. The general MHD approximation can be stated as follows. Given a domain $\Omega \subset \mathbb{R}^d$ and a time interval $(0,T)$, find a velocity $\boldsymbol{u} : \Omega \times (0,T) \longrightarrow \mathbb{R}^d$, a pressure $p : \Omega \times (0,T) \longrightarrow \mathbb{R}$, a magnetic (induction) field $\boldsymbol{B} : \Omega \times (0,T) \longrightarrow \mathbb{R}^d$ and magnetic pseudo-pressure $r : \Omega \times (0,T) \longrightarrow \mathbb{R}$ as the solution of the problem:

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p - \frac{1}{\mu_{\rm m} \rho} (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} = \boldsymbol{f}_{\rm f}, \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2}$$

$$\partial_t \boldsymbol{B} + \frac{1}{\mu_{\rm m} \sigma} \nabla \times (\nabla \times \boldsymbol{B}) - \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \nabla r = \boldsymbol{f}_{\rm m}, \tag{3}$$

$$\nabla \cdot \boldsymbol{B} = 0. \tag{4}$$

In these equations, ρ is the fluid density, ν is the kinematic viscosity, $\boldsymbol{f}_{\rm f}$ and $\boldsymbol{f}_{\rm m}$ are the body forces for the momentum and magnetic field equations (zero in the applications), $\mu_{\rm m}$ is the the magnetic permeability and σ the conductivity. Appropriate initial and boundary conditions need to be appended. Note that the pseudo-pressure r has been added ($r = 0$ is the exact solution).

The second model to be considered is the inductionless MHD approximation. Now $\boldsymbol{B}$ is assumed to be given, causing an unknown current density $\boldsymbol{j} : \Omega \times (0,T) \longrightarrow \mathbb{R}^d$ and an unknown electric potential $\phi : \Omega \times (0,T) \longrightarrow \mathbb{R}$. The equations to be solved are:

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p - \frac{1}{\rho} (\boldsymbol{j} \times \boldsymbol{B}) = \boldsymbol{f}_{\rm f}, \tag{5}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{6}$$

$$\boldsymbol{j} + \sigma \nabla \phi - \sigma (\boldsymbol{u} \times \boldsymbol{B}) = \boldsymbol{0}, \tag{7}$$

$$\nabla \cdot \boldsymbol{j} = 0. \tag{8}$$

As before, appropriate initial and boundary conditions need to be appended.

Apart from the complex physics represented by equations (1)-(4) and (5)-(8), its finite element approximation has several difficulties. The purpose of this paper is to touch two of them, as mentioned above. We will present here a summary of previous works, presented in a unified manner and showing their computational implications. In particular, basic compatibility conditions and the use of stabilized formulations as a means to avoid them are analyzed in [3, 1, 2], the finite element approximation of the general MHD problem in [4] and of the inductionless model in [5]. The reader is referred to these articles for additional references, where appropriate credit to previous developments is indicated. Due to space restrictions, no further references will be included.

2 BUILDING BLOCKS

Let us first consider problem (1)-(4). Deleting nonlinearities, coupling terms and time derivatives, we are left with a Stokes problem for $\boldsymbol{u}$ and p and a Maxwell problem for $\boldsymbol{B}$ and r. Therefore, the inf-sup conditions for both problems are *necessary* conditions to be met when the Galerkin finite element approximation of the problem is undertaken. It is easily shown that these conditions are also *sufficient*.

Let us turn our attention to (5)-(8). Deleting again nonlinearities, coupling terms and time derivatives, the problems we now find are a Stokes problem for $\boldsymbol{u}$ and p and a Darcy problem for $\boldsymbol{j}$ and ϕ. The inf-sup conditions associated to both problems are required if the problem is approximated using the standard Galerkin method.

From these observations it is clear that the building blocks of a finite element approximation of the general MHD model and the inductionless approximation are the Stokes, the Maxwell and the Darcy problems. These are the problems whose approximation is discussed in this section. First of all, let us write them in a unified format. They consist in finding $\boldsymbol{u} : \Omega \longrightarrow \mathbb{R}^d$ and $p : \Omega \longrightarrow \mathbb{R}$ such that

Stokes:

$$\begin{aligned} -\nu\Delta\boldsymbol{u} + \nabla p &= \boldsymbol{f}, \\ \nabla\cdot\boldsymbol{u} &= 0. \end{aligned}$$

Maxwell:

$$\begin{aligned} \lambda\nabla\times\nabla\times\boldsymbol{u} + \nabla p &= \boldsymbol{f}, \\ \nabla\cdot\boldsymbol{u} &= 0. \end{aligned}$$

Darcy:

$$\begin{aligned} \sigma\boldsymbol{u} + \nabla p &= \boldsymbol{f}, \\ \nabla\cdot\boldsymbol{u} &= 0. \end{aligned}$$

In these equations, ν, λ and σ are appropriate physical parameters.

To write down the variational formulation of these problems, let $V_X \times Q_X$ be the functional spaces where the pair $[\boldsymbol{u}, p]$ is sought, and let

$$a_X(\boldsymbol{u},\boldsymbol{v}) = \begin{cases} a_S(\boldsymbol{u},\boldsymbol{v}) := \nu(\nabla\boldsymbol{u},\nabla\boldsymbol{v}) & \text{for the Stokes problem} \\ a_M(\boldsymbol{u},\boldsymbol{v}) := \lambda(\nabla\times\boldsymbol{u},\nabla\times\boldsymbol{v}) & \text{for the Maxwell problem} \\ a_D(\boldsymbol{u},\boldsymbol{v}) := \sigma(\boldsymbol{u},\boldsymbol{v}) & \text{for the Darcy problem} \end{cases}$$

Spaces V_X and Q_X are determined by requiring that

	V_X	Q_X	$\langle \nabla p, \boldsymbol{v} \rangle$
Stokes	$V_S = H_0^1(\Omega)^d$	$Q_S = L_0^2(\Omega)$	$-(p, \nabla \cdot \boldsymbol{v})$
Maxwell	$V_{M1} = H_0(\mathbf{curl}; \Omega)$	$Q_{M1} = H_0^1(\Omega)$	$+(\nabla p, \boldsymbol{v})$
	$V_{M2} = V_{M1} \cap H(\mathrm{div}; \Omega)$	$Q_{M2} = L_0^2(\Omega)$	$-(p, \nabla \cdot \boldsymbol{v})$
Darcy	$V_{D1} = L^2(\Omega)^d$	$Q_{D1} = H^1(\Omega)$	$+(\nabla p, \boldsymbol{v})$
	$V_{D2} = H_0(\mathrm{div}; \Omega)$	$Q_{D2} = L_0^2(\Omega)$	$-(p, \nabla \cdot \boldsymbol{v})$

Table 1: Functional setting

	$\vert\!\vert\!\vert [\boldsymbol{v}, q] \vert\!\vert\!\vert_X$
Stokes	$\nu^{\frac{1}{2}} \Vert \nabla \boldsymbol{v} \Vert + \nu^{-\frac{1}{2}} \Vert q \Vert$
Maxwell 1	$\lambda^{\frac{1}{2}} \Vert \nabla \times \boldsymbol{v} \Vert + \frac{\lambda^{\frac{1}{2}}}{L_0} \Vert \boldsymbol{v} \Vert + \lambda^{-\frac{1}{2}} \Vert q \Vert + \lambda^{-\frac{1}{2}} L_0 \Vert \nabla q \Vert$
Maxwell 2	$\lambda^{\frac{1}{2}} \Vert \nabla \times \boldsymbol{v} \Vert + \lambda^{\frac{1}{2}} \Vert \nabla \cdot \boldsymbol{v} \Vert + \frac{\lambda^{\frac{1}{2}}}{L_0} \Vert \boldsymbol{v} \Vert + \lambda^{-\frac{1}{2}} \Vert q \Vert$
Darcy, primal	$\sigma^{\frac{1}{2}} \Vert \boldsymbol{v} \Vert + \sigma^{-\frac{1}{2}} \Vert \nabla q \Vert$
Darcy, dual	$\sigma^{\frac{1}{2}} \Vert \boldsymbol{v} \Vert + \sigma^{\frac{1}{2}} L_0 \Vert \nabla \cdot \boldsymbol{v} \Vert + \frac{\sigma^{-\frac{1}{2}}}{L_0} \Vert q \Vert$

Table 2: Working norms

- $a_X(\boldsymbol{u}, \boldsymbol{v})$ is continuous.
- The term $\langle \nabla p, \boldsymbol{v} \rangle$, obtained by testing ∇p by $\boldsymbol{v}$, is well defined under the minimum regularity conditions.

In case of the Stokes problem, the first condition implies that $\boldsymbol{u}$, $\boldsymbol{v}$ need to be in $H^1(\Omega)^d$, and thus the minimum regularity for p corresponds to take $\langle \nabla p, \boldsymbol{v} \rangle = -(p, \nabla \cdot \boldsymbol{v})$, which requires $p \in L^2(\Omega)$. However, for both the Maxwell and the Darcy problem we may choose either $\langle \nabla p, \boldsymbol{v} \rangle = -(p, \nabla \cdot \boldsymbol{v})$, $p \in L^2(\Omega)$, $\nabla \cdot \boldsymbol{v} \in L^2(\Omega)$, or $\langle \nabla p, \boldsymbol{v} \rangle = (\nabla p, \boldsymbol{v})$, $\nabla p \in L^2(\Omega)^d$, $\boldsymbol{v} \in L^2(\Omega)^d$. The possibilities for the functional setting of the different problems are summarized in Table 1. The choice of the functional setting has important practical consequences, both physical and numerical.

The norms in the product space $V_X \times Q_X$ depending on the choice of the functional setting are indicated in Table 2. All have been written to ensure a correct scaling. Note that the two possibilities for the Maxwell problem have been simply indicated as Maxwell 1 and 2, whereas for the Darcy problem they correspond to the well known primal and dual formulations [3].

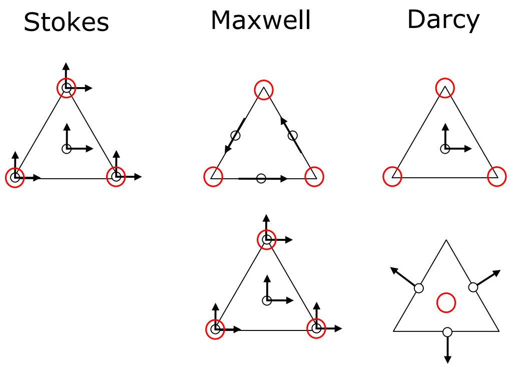

Figure 1: inf-sup stable elements

3 GALERKIN VS STABILIZED FINITE ELEMENT APPROXIMATIONS

When the Galerkin finite element approximation of the problem is attempted, the finite element spaces $V_h \subset V_X$ and $Q_h \subset Q_X$ need to satisfy the discrete counterpart of the inf-sup conditions that hold at the continuous level. These inf-sup conditions are different for the three problems considered, leading to different requirements for V_h and Q_h. The simplest inf-sup stable elements in 2D are schematically shown in Fig. 1, where nodes to interpolate the velocity components are printed in black and pressure nodes in red. For the Maxwell and the Darcy problems, the first row corresponds to the choice $p \in H^1(\Omega)$ and the second to $p \in L^2(\Omega)$.

There are several inconveniences in the use of the interpolations of Fig. 1. For example, if we consider a combined problem of the form

$$
\begin{aligned}
-\nu\Delta \boldsymbol{u} + \lambda\nabla\times\nabla\times\boldsymbol{u} + \sigma\boldsymbol{u} + \nabla p &= \boldsymbol{f}, \\
\nabla\cdot\boldsymbol{u} &= 0,
\end{aligned}
$$

it is clear that spaces satisfying the inf-sup condition for the Stokes problem need to be used if $\nu > 0$. However, we might be interested in letting $\nu \to 0$, or $\lambda \to 0$ or $\sigma \to 0$. From the numerical point of view, oscillations will show up if the correct interpolation is not used in each case.

Another inconvenience is faced in the case of a coupled problem, such as the MHD models discussed earlier. From the implementation point of view, it is much simpler to have all the unknowns at the same nodes of the finite element mesh.

As an alternative to the use of inf-sup stable elements, our approach is the use of stabilized formulations, in which *any conforming* $\boldsymbol{u}$*-*p *interpolation is allowed.* No stability problems will be found in the limits $\nu \to 0$, $\sigma \to 0$, $\lambda \to 0$ and, if Lagrangian interpolations are used, coupling of different problems will be easy. For example, in contrast to the different interpolations shown in Fig. 1, it will be possible to use the simplest continuous P_1 interpolation for $\boldsymbol{u}$ and p.

The key ingredients to design the stabilization methods presented in the following are

- A two scale decomposition of $\boldsymbol{u}$ and p, within the variational multiscale framework (VMS).
- A proper scaling of the problem, which requires the introduction of a length scale.
- A closed form expression for the subscales based on an approximate Fourier analysis of the problem.

These ingredients will not be elaborated here. The methods proposed will be stated without (heuristic) derivation. For simplicity, we will take $\boldsymbol{f} \in L^2(\Omega)^d$.

4 STOKES' PROBLEM

This is the problem for which stabilized finite element methods are best known. Let us start writing the variational form of the problem, which is: find $[\boldsymbol{u}, p] \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ such that

$$B_S([\boldsymbol{u}, p], [\boldsymbol{v}, q]) = L_S([\boldsymbol{v}, q]) = (\boldsymbol{f}, \boldsymbol{v}),$$

for all $[\boldsymbol{v}, q]$, where

$$B_S([\boldsymbol{u}, p], [\boldsymbol{v}, q]) = \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (p, \nabla \cdot \boldsymbol{v}) + (q, \nabla \cdot \boldsymbol{u}),$$

The stabilized finite element approximation we propose is: find $[\boldsymbol{u}_h, p_h] \in V_h \times Q_h$ such that

$$B_{S,h}([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) = L_{S,h}([\boldsymbol{v}_h, q_h]) \quad \forall [\boldsymbol{v}_h, q_h] \in V_h \times Q_h,$$

where $B_{S,h}$ and $L_{S,h}$ depend on the particular stabilized formulation. In particular, for the so called Algebraic Subgrid Scale (ASGS) method $B_{S,h}$ and $L_{S,h}$ are given by:

$$\begin{aligned} B_{S,h}([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) &= B_S([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) + \sum_K \tau_p \langle \nabla \cdot \boldsymbol{u}_h, \nabla \cdot \boldsymbol{v}_h \rangle_K \\ &\quad + \sum_K \tau_u \langle -\nu \Delta \boldsymbol{u}_h + \nabla p_h, \nu \Delta \boldsymbol{v}_h + \nabla q_h \rangle_K, \\ L_{S,h}([\boldsymbol{v}_h, q_h]) &= (\boldsymbol{f}, \boldsymbol{v}_h) + \sum_K \tau_u \langle \boldsymbol{f}, \nu \Delta \boldsymbol{v}_h + \nabla q_h \rangle_K, \end{aligned}$$

where τ_p and τ_u are the stabilization parameters, that we compute as

$$\tau_p = c_1 \nu, \quad \tau_u = (c_1 \nu)^{-1} h^2,$$

with c_1 an algorithmic constant and h the element size of the mesh, which we consider quasi-uniform for simplicity.

For the Orthogonal Subgrid Scales (OSS) method $B_{S,h}$ and $L_{S,h}$ are given by:

$$\begin{aligned} B_{S,h}([\boldsymbol{u}_h,p_h],[\boldsymbol{v}_h,q_h]) &= B_S([\boldsymbol{u}_h,p_h],[\boldsymbol{v}_h,q_h]) \\ &\quad + \sum_K \tau_p \langle P_h^\perp(\nabla\cdot\boldsymbol{u}_h), P_h^\perp(\nabla\cdot\boldsymbol{v}_h)\rangle_K \\ &\quad + \sum_K \tau_u \langle P_h^\perp(-\nu\Delta\boldsymbol{u}_h+\nabla p_h), P_h^\perp(\nu\Delta\boldsymbol{v}_h+\nabla q_h)\rangle_K, \\ L_{S,h}([\boldsymbol{v}_h,q_h]) &= (\boldsymbol{f},\boldsymbol{v}_h), \end{aligned}$$

where $P_h^\perp$ is the projection orthogonal to the finite element space and τ_p and τ_u are the same as for the ASGS method.

The numerical analysis of both the ASGS and the OSS methods shows that they have the same stability and convergence properties. Let us define the mesh dependent norm:

$$|\!|\!|[\boldsymbol{v}_h,q_h]|\!|\!|_{S,h}^2 = \nu\|\nabla\boldsymbol{v}_h\|^2 + \frac{1}{\nu}\|q_h\|^2 + \frac{h^2}{\nu}\sum_K \|\nabla q_h\|_K^2.$$

We also define the error function

$$E_S^2(h) = \nu\varepsilon_1^2(\boldsymbol{u}) + \frac{1}{\nu}\varepsilon_0^2(p),$$

where $\varepsilon_i(\cdot)$ denotes the interpolation error in the $H^i(\Omega)$-seminorm. We then have:

Theorem 4.1 (Stability) *Suppose that the constant c_1 is large enough. Then, there exists a constant $C>0$ such that*

$$\inf_{[\boldsymbol{u}_h,p_h]} \sup_{[\boldsymbol{v}_h,q_h]} \frac{B_{S,h}([\boldsymbol{u}_h,p_h],[\boldsymbol{v}_h,q_h])}{|\!|\!|[\boldsymbol{u}_h,p_h]|\!|\!|_{S,h}|\!|\!|[\boldsymbol{v}_h,q_h]|\!|\!|_{S,h}} \geq C > 0.$$

Theorem 4.2 (Convergence) *Let $[\boldsymbol{u},p]$ be the solution of the continuous problem and $[\boldsymbol{u}_h,p_h]$ the solution of the discrete one. Suppose that c_1 is large enough. Then*

$$|\!|\!|[\boldsymbol{u}-\boldsymbol{u}_h,p-p_h]|\!|\!|_{S,h} \lesssim E_S(h).$$

5 MAXWELL'S PROBLEM

The variational formulation of Maxwell's problem can be written as: find $[\boldsymbol{u},p]\in V\times Q$ such that

$$B_M([\boldsymbol{u},p],[\boldsymbol{v},q]) = L_M([\boldsymbol{v},q]) = (\boldsymbol{f},\boldsymbol{v}), \tag{9}$$

for all $[\boldsymbol{v}, q]$, where

$$B_M([\boldsymbol{u}, p], [\boldsymbol{v}, q]) = \lambda(\nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{v}) + \langle \nabla p, \boldsymbol{v} \rangle - \langle \nabla q, \boldsymbol{u} \rangle,$$

and $\langle \nabla q, \boldsymbol{v} \rangle$ has two possible expressions according to the functional setting chosen:

$$\text{Formulation M1}: \ \langle \nabla q, \boldsymbol{v} \rangle = (\nabla q, \boldsymbol{v}), \ V = H_0(\mathbf{curl}, \Omega), \ Q = H_0^1(\Omega).$$

$$\text{Formulation M2}: \ \langle \nabla q, \boldsymbol{v} \rangle = -(q, \nabla \cdot \boldsymbol{v}), \ V = H_0(\mathbf{curl}, \Omega) \cap H(\mathrm{div}, \Omega), \ Q = L^2(\Omega).$$

We will refer to M1 as the *curl formulation* and to M2 as the *curl-div formulation.* The main theoretical interest of Maxwell's problem is that there are solutions that can be found approximating M1 but not from the approximation of M2. These are the so called *singular solutions*. This fact, known as *the corner paradox*, follows from the following results:

Lemma 5.1 *If Ω is not convex, $H^1(\Omega)^d$ is a closed proper subspace of $H(\mathbf{curl}, \Omega) \cap H(\mathrm{div}, \Omega)$ (all with tangential boundary conditions).*

Corollary 5.1 *If Ω is not convex, $\boldsymbol{u}$ is the solution of (9) and $\boldsymbol{u}_h$ its finite element approximation, then*

$$\lim_{h \to 0} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{H(\mathbf{curl},\Omega) \cap H(\mathrm{div},\Omega)} \neq 0,$$

in general.

One could argue whether this Lemma implies an approximability problem when using standard Lagrangian finite elements. The reason is that it can be shown that

$$\text{If } \boldsymbol{u}_h \in H^1(\Omega)^d \ \Rightarrow \ \|\nabla \boldsymbol{u}_h\| \lesssim \|\nabla \times \boldsymbol{u}_h\| + \|\nabla \cdot \boldsymbol{u}_h\|.$$

Thus, if $\boldsymbol{u}_h$ is C^0, its H^1 norm will be bounded, and thus $\boldsymbol{u}_h$ will converge to a function in $H^1(\Omega)^d$, whereas the exact solution may belong to $H(\mathbf{curl}, \Omega) \cap H(\mathrm{div}, \Omega)$.

The problem however cannot be attributed to C^0 interpolations, but to the curl-div formulation. It is not true that C^0 spaces cannot approximate the solution to (9). This only happens if, for some reason, $\nabla \cdot \boldsymbol{u}_h$ happens to be uniformly bounded. Thus, we have proposed a stabilized finite element method using C^0 spaces but able to reproduce the curl formulation [1]. This reads as follows: find $[\boldsymbol{u}_h, p_h] \in V_h \times Q_h$ such that

$$B_{M,h}([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) = L_{M,h}([\boldsymbol{v}_h, q_h]) \quad \forall [\boldsymbol{v}_h, q_h] \in V_h \times Q_h,$$

where

$$\begin{aligned} B_{M,h}([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) &= B_M([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) \\ &\quad + \sum_K \tau_p \langle \tilde{P}(\nabla \cdot \boldsymbol{u}_h), \tilde{P}(\nabla \cdot \boldsymbol{v}_h) \rangle_K \\ &\quad + \sum_K \tau_u \langle \tilde{P}(\nabla p_h), \tilde{P}(\nabla q_h) \rangle_K, \\ L_{M,h}([\boldsymbol{v}_h, q_h]) &= (\boldsymbol{f}, \boldsymbol{v}_h) + \sum_K \tau_u \langle \tilde{P}(\boldsymbol{f}), \tilde{P}(\nabla q_h) \rangle_K, \end{aligned}$$

and where

$$\tilde{P} = \begin{cases} I & \text{for the ASGS method} \\ P_h^{\perp} & \text{for the OSS method} \end{cases}$$

The stabilization parameters are given by:

$$\tau_p = c_2 \lambda \frac{h^2}{\ell^2}, \qquad \tau_u = \frac{\ell^2}{\lambda},$$

where

$$\ell = \begin{cases} L_0 \text{ (characteristic length)} & \text{for the curl formulation (M1)} \\ h & \text{for the curl-div formulation (M2)} \end{cases}$$

It can be shown that *it is possible to switch from the functional setting M1 to M2 just by a proper scaling of the stabilization parameters.*

The numerical analysis we have performed shows that the formulations proposed are stable and optimally convergent in the norm

$$\|\![\boldsymbol{v}_h, q_h]\|\!|_{M,h} = \lambda^{\frac{1}{2}} \|\nabla \times \boldsymbol{v}_h\| + \lambda^{\frac{1}{2}} \frac{h}{\ell} \|\nabla \cdot \boldsymbol{v}_h\| + \frac{\ell}{\lambda^{\frac{1}{2}}} \|\nabla q_h\|.$$

Note that

- $\ell = L_0$ yields the discrete $H(\mathbf{curl}, \Omega) \times H^1(\Omega)$ norm.
- $\ell = h$ yields the discrete $H(\mathbf{curl}, \Omega) \cap H(\mathrm{div}, \Omega) \times L^2(\Omega)$ norm.

Moreover, when $\ell = L_0$ if the continuous solution is singular ($\boldsymbol{u} \in H^r(\Omega)^d, r < 1$), $\boldsymbol{u}_h \to \boldsymbol{u}$ in H^r.

6 DARCY'S PROBLEM

As for the Maxwell problem, there are two possible functional settings, now called primal and dual. The variational formulation of the primal problem reads: find $\boldsymbol{u} \in L^2(\Omega)^d$ and $p \in H^1(\Omega)$ such that

$$\begin{aligned} \sigma(\boldsymbol{u}, \boldsymbol{v}) + (\nabla p, \boldsymbol{v}) &= (\boldsymbol{f}, \boldsymbol{v}), && \forall \boldsymbol{v} \in L^2(\Omega)^d, \\ -(\nabla q, \boldsymbol{u}) &= (g, q), && \forall q \in H^1(\Omega), \end{aligned}$$

whereas for the dual problem the variational formulation is: find $\boldsymbol{u} \in H_0(\mathrm{div}; \Omega)$ and $p \in L_0^2(\Omega)$ such that

$$\begin{aligned} \sigma(\boldsymbol{u}, \boldsymbol{v}) - (p, \nabla \cdot \boldsymbol{v}) &= (\boldsymbol{f}, \boldsymbol{v}), && \forall \boldsymbol{v} \in H_0(\mathrm{div}; \Omega), \\ (q, \nabla \cdot \boldsymbol{u}) &= (g, q), && \forall q \in L_0^2(\Omega). \end{aligned}$$

A forcing term g has been included in the continuity equation for generality. Let B_D and L_D the appropriate forms of the problem, written as find $[\boldsymbol{u}, p] \in V_D \times Q_D$ such that

$$B_D([\boldsymbol{u}, p], [\boldsymbol{v}, q]) = L_D([\boldsymbol{v}, q]) \qquad \forall [\boldsymbol{v}, q] \in V_D \times Q_D.$$

The stabilized approximations that we propose read as follows: find $[\boldsymbol{u}_h, p_h] \in V_{D,h} \times Q_{D,h}$ such that

$$B_{D,h}([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) = L_{D,h}([\boldsymbol{v}_h, q_h]) \quad \forall [\boldsymbol{v}_h, q_h] \in V_{D,h} \times Q_{D,h}$$

where

$$\begin{aligned} B_{D,h}([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) &= B_D([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) \\ &+ \sum_K \tau_p \langle \tilde{P}(\nabla \cdot \boldsymbol{u}_h), \tilde{P}(\nabla \cdot \boldsymbol{v}_h) \rangle_K \\ &+ \sum_K \tau_u \langle \tilde{P}(\sigma \boldsymbol{u}_h + \nabla p_h), \tilde{P}(-\sigma \boldsymbol{v}_h + \nabla q_h) \rangle_K \\ L_{D,h}([\boldsymbol{v}_h, q_h]) &= (\boldsymbol{f}, \boldsymbol{v}_h) + (g, q_h) + \sum_K \tau_p \langle \tilde{P}(g), \tilde{P}(\nabla \cdot \boldsymbol{v}_h) \rangle_K \\ &+ \sum_K \tau_u \langle \tilde{P}(\boldsymbol{f}), \tilde{P}(-\sigma \boldsymbol{v}_h + \nabla q_h) \rangle_K, \end{aligned}$$

and where $\tilde{P}$ is defined as for Maxwell's problem. The stabilization parameters are computed as

$$\tau_p = c_3 \sigma \ell^2, \quad \tau_u = (c_3 \sigma \ell^2)^{-1} h^2$$

where

$$\ell = \begin{cases} h & \text{for the primal formulation (D1).} \\ L_0 \text{ (characteristic length)} & \text{for the dual formulation (D2).} \end{cases}$$

We will be able *to switch from the functional setting D1 to D2 just by a proper scaling of the stabilization parameters.*

The numerical analysis indicates that the formulations proposed are stable and optimally convergent in the norm

$$|||[\boldsymbol{v}_h, q_h]|||_{D,h}^2 = \sigma \|\boldsymbol{v}_h\|^2 + \sigma \ell^2 \|\nabla \cdot \boldsymbol{v}_h\|^2 + \frac{1}{\sigma L_0^2} \|q_h\|^2 + \frac{h^2}{\sigma \ell^2} \sum_K \|\nabla q_h\|_K^2.$$

Note that

- $\ell = L_0$ yields the discrete $H(\mathrm{div}, \Omega) \times L^2(\Omega)$ norm (dual problem).

Method	Primal mixed $k=0$, $l=1$	Stabilized $k=l=1$	Dual mixed $k=1$, $l=0$
$\Vert \boldsymbol{e}_u \Vert$	h	h^2	h
$\Vert e_p \Vert$	h^2	h^2	h
$\Vert \nabla \cdot \boldsymbol{e}_u \Vert$	1	h	h
$\Vert \nabla e_p \Vert$	h	h	1

Table 3: Convergence order using the lowest order interpolations

- $\ell = h$ yields the discrete $L^2(\Omega)^d \times H^1(\Omega)$ norm (primal problem).
- $\ell = (hL_0)^{1/2}$ yields a norm that happens to be optimal for equal order interpolations.

Just as an example, in Table 3 we have indicated the convergence orders that can be found using the Galerkin method and inf-sup stable elements for the primal formulation (left column), for the dual formulation (right column) and using the stabilized method we propose (central column). In this table, k and l refer to the interpolation order of $\boldsymbol{u}$ and p, respectively. The gain in accuracy using the stabilized formulation we propose is clear.

7 COUPLING AND CONCLUSIONS

Let us consider now the linearization of the stationary version of problems (1)-(4) and (5)-(8), restricting our attention to fixed-point-type schemes. Starting with the former, it can be shown that the only stable scheme is [4]:

$$\boldsymbol{a} \cdot \nabla \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p - \frac{1}{\mu_{\mathrm{m}} \rho} (\nabla \times \boldsymbol{B}) \times \boldsymbol{b} = \boldsymbol{f}_{\mathrm{f}}, \tag{10}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{11}$$

$$\frac{1}{\mu_{\mathrm{m}} \sigma} \nabla \times (\nabla \times \boldsymbol{B}) - \nabla \times (\boldsymbol{u} \times \boldsymbol{b}) + \nabla r = \boldsymbol{f}_{\mathrm{m}}, \tag{12}$$

$$\nabla \cdot \boldsymbol{B} = 0. \tag{13}$$

where $\boldsymbol{a}$ is the velocity and $\boldsymbol{b}$ the magnetic field of the previous iteration. It is observed that all the variables *need* to be computed in a coupled way.

Let us move our attention to problem (5)-(8). Now it can be shown that the only stable scheme is [5]:

$$\boldsymbol{a} \cdot \nabla \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p - \frac{1}{\rho} (\boldsymbol{j} \times \boldsymbol{B}) = \boldsymbol{f}_{\mathrm{f}}, \tag{14}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{15}$$

$$\boldsymbol{j} + \sigma \nabla \phi - \sigma (\boldsymbol{u} \times \boldsymbol{B}) = 0, \tag{16}$$

$$\nabla \cdot \boldsymbol{j} = 0. \tag{17}$$

Remember that now $\boldsymbol{B}$ is given. Again, all the variables have to be computed in a coupled way.

Let us conclude with some remarks related to the finite element approximation of problems (10)-(13) and (14)-(17). The first point to remark is that the stabilized finite element approximation that we propose allows the use of arbitrary, and in particular equal, interpolation for all the unknowns. This implies an important ease of implementation, since only one data structure (nodal connections, derivatives of shape functions) is required for all the variables. For example, when computing the element matrices arising from (10), contributions multiplying the arrays of $\boldsymbol{u}$, p and $\boldsymbol{B}$ can be computed within the same loops, over the elements, over the nodes and over the integration points.

An aspect that we have not explored here is that the extension of the stabilized formulation to (10)-(13) and (14)-(17) allows to deal with combined problems and all the range of physical parameters. In particular, instabilities due to small viscosity (or large magnetic permeability or conductivity) are avoided.

A very important aspect of our formulation is the adaptation to the appropriate functional setting by a proper design of the scaling in the stabilization parameters. This is not a mathematical divertimento, but has important consequences. For example, for the magnetic field we may want or need to capture singular solutions. For problem (14)-(17) we may choose a better approximation for ϕ or for $\boldsymbol{j}$. Nevertheless, we have also shown that there is a version of the stabilization parameters that yields better accuracy that both the primal and the dual formulation of the Darcy problem of which ϕ and $\boldsymbol{j}$ are solution.

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