NUMERICAL COUPLING PROCEDURE
IN STEADY CONJUGATE HEAT TRANSFER PROBLEMS
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Abstract. This paper analyses the numerical stability of a coupling procedure between a CFD code and a conduction solver in a partitioned approach. A finite volume method is used in the fluid partition and a finite element method in the solid partition. Since our goal is to get a global fluid-solid solution, the analysis of the transient in the solid is not of particular interest. Consequently, the numerical method is based on the coupling of a steady state in the solid with a time-dependent solution in the fluid. At the shared interface, Dirichlet (on the fluid side) and Robin (on the solid side) conditions are applied. An interface stability study is performed according to the normal-mode analysis of the theory of Godunov-Ryabenkii. The existence of an optimal coupling parameter is highlighted.

1 INTRODUCTION

Conjugate heat transfer (CHT) problems are encountered in many real-world applications such as cooling systems, building heating, ventilating flows, heat-exchanger equipment, etc. They arise in situations where considering only heat transport in the fluid is not sufficient. In such cases we must take into account the fully coupled problem, including conduction in the solid wall adjacent to the fluid.

CHT is now widely used in engineering applications, but in most cases, arbitrary relaxation parameters or reference temperatures are used to stabilize the coupling procedure. This may have a significant negative impact on the convergence rate. Our intention in this paper is to recall that in a CHT problem based on Dirichlet-Robin interface conditions, there is an optimal coefficient in terms of stability and convergence. This coefficient has been highlighted recently [1] using a 1D model equation.

This article is organized as follows. First the model equations of the CHT procedure are described briefly. Then, the coupling interface conditions and the coupling algorithm are presented. After that, the stability analysis according to the theory of Godunov-Ryabenkii is summarized. This analysis provides an optimal coupling coefficient. This coefficient is given and its influence on stability and convergence is discussed.

2 MODEL EQUATIONS

2.1 Time-scale discontinuity

The CHT strategy presented in this paper is motivated by the desire to obtain rapidly a global fluid-structure steady solution. There is a significant discrepancy of the characteristic times, namely a fast transient process in the fluid, a very slow one in the structure. Hence, the thermal response of the solid can be very long (several hours or minutes). Since our goal is to

get a global fluid-solid solution, we must recognize that the analysis of the transient in the solid is not of particular interest.

2.2 Solid solution

If we want to obtain a steady-state solution of the heat equation in the solid domain, it is always possible to choose some initial data and march forward in time. But this approach is typically not an efficient way to compute the steady state solution, if this is all we want. Instead, if the boundary conditions are time independent and if we are interested only in computing the steady-state solution itself, then we can solve directly a second order ordinary differential equation (Laplace equation). As this procedure performs well in the solid when it is considered independently as a single subsystem, we should be able to use it in a coupled system.

2.3 Fluid solution

The same cannot be said for the fluid subsystem. In the vast majority of cases, the Navier-Stokes (NS) equations are solved to steady-state by a time marching scheme. As a consequence, a time marching scheme will be employed in the fluid domain in our CHT model. As physically realistic time-dependent solutions are not sought, the unsteady fluid and steady solid procedures will be interactively solved and coupled until a global fluid-solid solution is obtained.

2.4 Fluid-solid solution

The simulation of this type of problems is generally accomplished by partitioned staggered schemes [2][3]. As mentioned above, we just want to obtain a stable global solution by coupling a transient fluid solution with a steady solid state. But solving a Laplace equation at each time step could change the solution too rapidly for stability to be maintained. The goal of this paper is to provide an optimal coupling coefficient that never introduces stability restrictions.

3   COUPLING CONDITIONS AT THE FLUID-SOLID INTERFACE

3.1 Discretized model

The 1D discrete model is presented in Figure 1. This model is based on a finite-volume (FV) procedure on the fluid-side ($x > 0$) and a finite element (FE) procedure on the solid-side ($x < 0$).
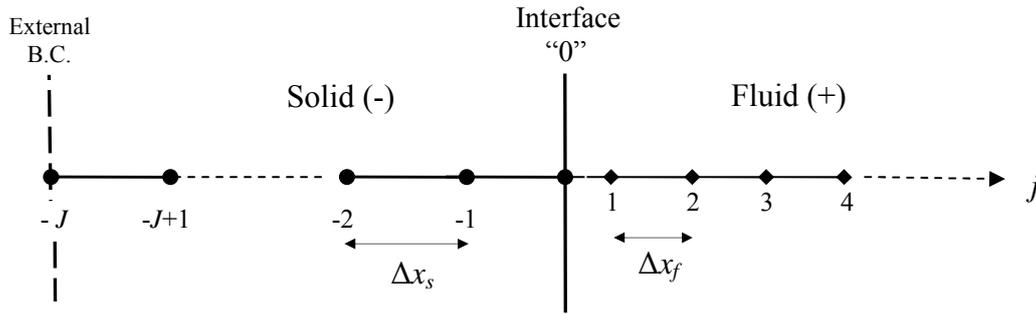
Figure 1 : Schematic of the fluid ($x > 0$) and solid ($x < 0$) domains for the discretized model

3.2 Dirichlet-Robin procedure

At the common interface (x = 0), coupling conditions are applied. Roux & Garaud [4] studied recently the behavior of interface conditions in a steady CHT problem. They first generalized the result obtained by Giles [5] and confirmed that Dirichlet conditions must be imposed in the fluid domain. Then, they pointed out from the Schur complement that the Dirichlet-Neumann condition may suffer considerably from destabilizing effects. On the contrary, it is indicated in the same work, that the Robin condition has many attractive features. First it can always be formulated in such a way that the associated local problem is well posed even though the Neumann problem is not. Second, the use of such a condition introduces an interface stiffness forcing the boundary to behave in the same way as the boundary of the other domain. Finally, it provides much better stability properties. Consequently a Dirichlet-Robin procedure will be considered here. In other words, in the first step of the coupling procedure in the interval $\left[t^n, t^{n+1}\right]$, the temperature coming from the solid is applied on the interface "0+ " of the fluid domain

$$T_{0+}^{n+1} = T_{0-}^{n} \tag{1}$$

A numerical Robin condition is in turn used as a boundary condition for the interface "0-" of the solid domain

$$q_s^{n+1} - \alpha_f^{n+1} T_{0-}^{n+1} = -q_f^{n+1} - \alpha_f^{n+1} T_{0+}^{n+1} \tag{2}$$

The subscripts f and s denote the fluid and solid domain respectively. q is the heat flux (W.m^{-2}). The general Dirichlet-Robin condition considered in this paper introduces the numerical coupling parameter α_f (W.m^{-2}.K^{-1}) the choice of which directly influences the stability of the coupled process. The expression of an optimal coefficient will be given in this paper. The coupling procedure is briefly described in the next section.

3   CSS PROCEDURE

Many partitioned staggered procedures have been proposed to solve fluid-structure interaction problems. They can be categorized as either strongly-coupled or loosely-coupled. The strongly-coupled schemes involve predictor-corrector iterations and then increase the complexity of the implementation of a CHT problem as well as the computational cost at each time step. In this paper, we have employed the basic loosely-coupled conventional serial staggered (CSS) algorithm [6] whose generic cycle is described in Fig. 1 in the time interval $\left[t^{n}, t^{n+1}\right]$ with time step size $\Delta t = t^{n+1} - t^{n}$, where n corresponds to the coupling time level. This procedure is repeated until a global steady-state solution is obtained.

Fluid solver
(2)   Transient computation

n $n+1$

(1) Apply
$(T_{0+}^{n+1} = T_{0-}^{n})$

(3) Apply
$(q_{s}^{n+1} - \alpha_{f}^{n+1} T_{0-}^{n+1} = -q_{f}^{n+1} - \alpha_{f}^{n+1} T_{0+}^{n+1})$
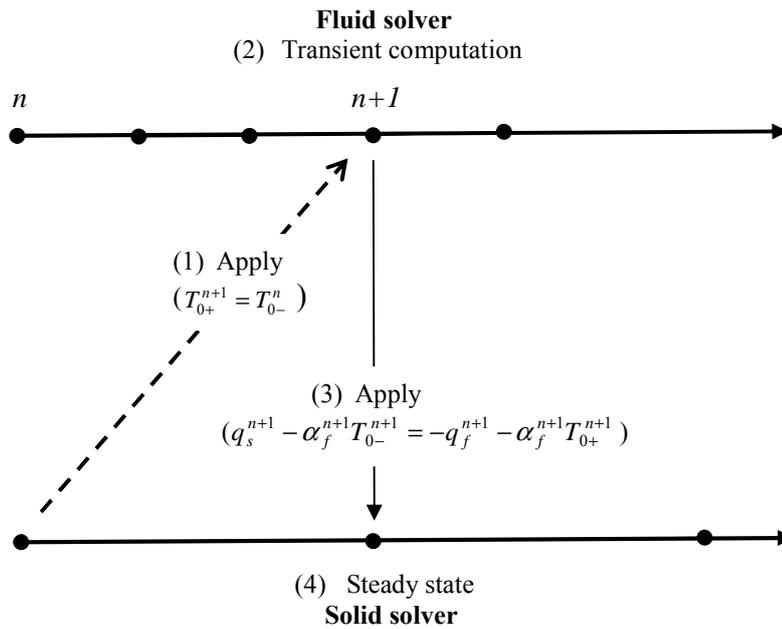
(4)   Steady state
Solid solver

Figure 2 : CSS procedure

Our goal is not to discuss the pros and cons of coupling algorithms, but to present a stability analysis and a resulting ideal coupling parameter. For this, we need a basic algorithm and naturally, we choose the simplest. Developments that follow remain valid if we adopt another one. However, the results may vary slightly.

4 NUMERICAL TREATMENTS

4.1 Numerical schemes

As mentioned previously, the fluid domain is discretized with a FV scheme and the solid domain with a FE scheme. On the interior mesh points we employ an implicit Euler-backward scheme for the time derivative. The diffusive term (heat flux) is computed in the same manner as the viscous terms in the NS equations, that is a second order accurate central difference formulation. The numerical treatments are summarized in Table 1.

Table 1 : Numerical treatments of the CHT procedure

	FLUID DOMAIN	SOLID DOMAIN
Time stepping	Implicit backward	Implicit backward
Numerical approach	Finite Volume	Finite Element
Spatial scheme	2nd order central	2nd order central
Fluid-Solid Interface	Dirichlet	Robin

4.2 Discretized system

The numerical discrete CHT system is obtained after discretization of the diffusion equation & Dirichlet conditions on the fluid side and the Laplace equation & Robin conditions on the solid side. The resulting discrete system is obtained

$$
\begin{cases}
\dfrac{\rho\,C_f}{\Delta t}(T_j^{n+1} - T_j^n) = \dfrac{k_f}{\Delta x_f^2}(T_{j+1}^{n+1} - 2T_j^{n+1} + T_{j-1}^{n+1}) & j > 0 \quad\quad \text{a)}\\[2mm]
T_{0+}^{n+1} = T_{0-}^n & j = 0+ \quad\quad \text{b)}\\[2mm]
T_{j+1}^{n+1} - 2T_j^{n+1} + T_{j-1}^{n+1} = 0 & j < 0 \quad\quad \text{c)}\\[2mm]
\dfrac{k_s}{\Delta x_s}(T_{0-}^{n+1} - T_{-1}^{n+1}) + \alpha_f T_{0-}^{n+1} = \dfrac{k_f}{\Delta x_f/2}(T_1^{n+1} - T_{0+}^{n+1}) + \alpha_f T_{0+}^{n+1} & j = 0- \quad\quad \text{d)}
\end{cases}
\tag{3}
$$

where $T_j^n = T(j\Delta x, n\Delta t)$. ρ is the density, C_f is the specific heat, k_f is the thermal conductivity of the fluid.

5   OPTIMAL COUPLING COEFFICIENT

5.1 Stability analysis and amplification factor

This stability analysis is very similar to the standard Fourier stability method except that the Fourier analysis ignores boundary conditions and as these may affect the stability, the theory of Godunov & Ryabenkii (G-R) [7][8] is preferable. We introduce the normal mode solution for the case defined by the discretized system (3) by considering eigensolutions of the form

$$T_j^n = \begin{cases} z^{n-1} \kappa_f^j, & j > 0 \\ z^n \kappa_s^j, & j \le 0 \end{cases} \tag{4}$$

where z is the "temporal amplification factor" and κ is the "spatial amplification factor". The discretized model (3) is stable in the sense of G-R if no solutions of the form (4) are admitted with $|\kappa_f| < 1$, $|\kappa_s| > 1$, and $|z| > 1$ as $j \to \pm\infty$. Moreover, we exclude modes (4) with $|z| = 1$ (neutrally stable modes).

Introducing (4) into the interface conditions (3), after elementary transformations, we obtain the following temporal amplification factor

$$z = g(z) = \left[\frac{k_f}{\Delta x_f} \left(\frac{z-1}{Dz} - \sqrt{\left(2 + \frac{z-1}{Dz} \right)^2 - 4} \right) + \alpha_f \right] \bigg/ \left(\frac{\beta k_s}{\Lambda_s} + \alpha_f \right) \tag{5}$$

where D is the Fourier number defined as follows

$$D = \frac{k_f \Delta t}{\rho C_f \Delta x_f^2} \tag{6}$$

Without going into too much detail, let us mention that the parameter β in Eq. (5) accounts for the contributions to physical and geometric solid characteristics and controls the external boundary condition (see Fig. 1). The influence of this parameter on the global CHT procedure is crucial.

5.2 Stability zones

The function $\max_{z \ge 1} \{|g|\}$ has been plotted in Figure 3 in terms of α_f for different Fourier numbers D. We have chosen typical fluid and solid physical parameters corresponding to air and steel respectively. We can observe that this function is defined and continuous and that each curve is composed of two half-lines with a singular point similar to a cusp at the intersection. At this point, $\max_{z \ge 1} \{|g|\}$ attains its minimum value.
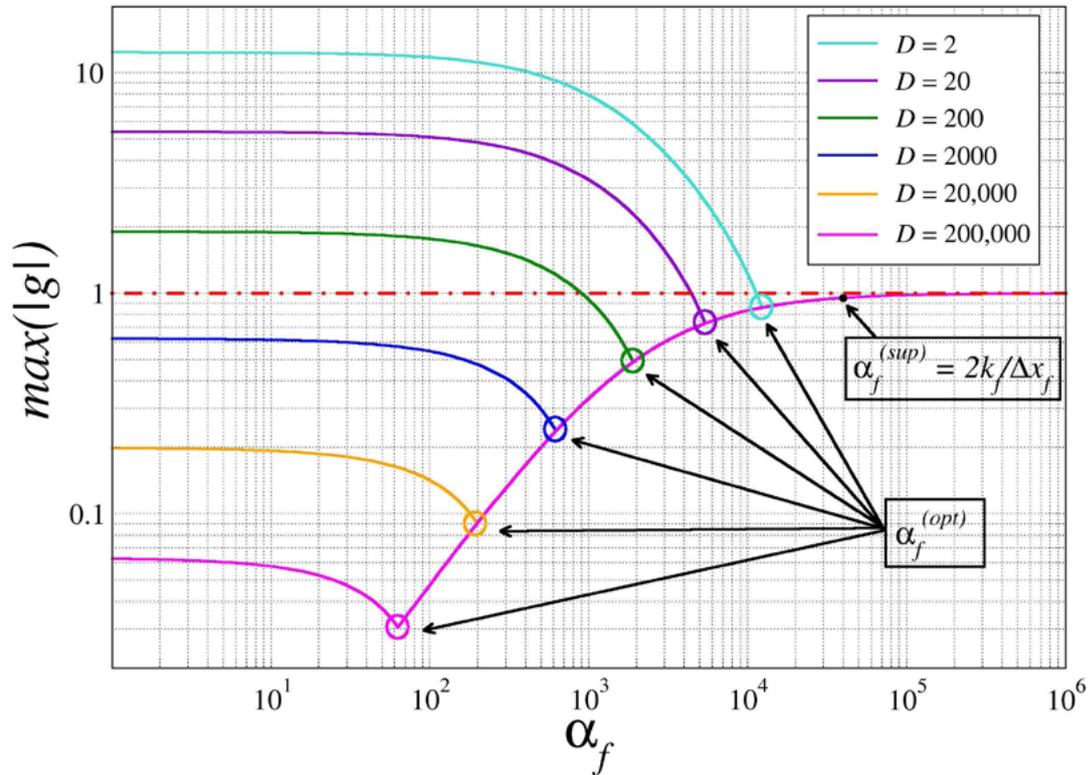
Figure 3 : $\max\{|g|\}$ vs α_f for various Fourier numbers

In other words, the existence of a transition value for α_f is highlighted. At this transition value, the shape of the curve switches and turns back abruptly. This value is denoted $\alpha_f^{(opt)}$.

5.3 Optimal coefficient

It can be shown [1] that the transition occurs at a unique and remarkable value $\alpha_f^{(opt)}$ whose exact expression is given by

$$\alpha_f^{(opt)} = \frac{k_f}{\Delta x_f}\left(\sqrt{(1+\frac{1}{D})^2 - 1} - \frac{1}{D}\right) \tag{7}$$

When the optimal value defined by (7) is employed, we obtain the best-case scenario with no additional computational effort. The point $\alpha_f^{(opt)}$ is the intersection of two opposite zones. The left half-line ($\alpha_f < \alpha_f^{(opt)}$) is controlled by Neumann conditions. It is a fast process prone to instability. The right half-line ($\alpha > \alpha_f^{(opt)}$) is controlled by Dirichlet conditions. It is a low but always stable process. The intersection $\alpha_f^{(opt)}$ is a perfect equilibrium between both.

5.4 Energy, stability and optimal coefficient

First of all, let us recall that the Fourier number used in unsteady-state flow problems, whose expression is given by (6), is a dimensionless number that characterizes heat conduction:

$$D = \frac{\text{heat conduction rate}}{\text{rate of thermal energy storage}}$$

The various curves in Fig. 3 can be interpreted easily in terms of the Fourier number D and the coupling coefficient α_f.

Low values of α_f $(\alpha_f < \alpha_f^{(opt)})$ will result in a rapid convergence, but in this case however D must be sufficiently large to allow heat diffusion on the fluid side. Otherwise, a low Fourier number D will soak up a lot of heat. It will be then necessary to enhance stability by increasing the coupling parameter. Thus, the energy transfer will be "frozen" by an increase of α_f.

But, likewise, large values of α_f $(\alpha_f >> \alpha_f^{(opt)})$ will always lead to an extremely slow convergence. This is only consistent with small values of D and corresponds to a slow diffusion of heat through the fluid subsystem. But it should be pointed out that relatively large Fourier numbers D indicate fast propagation and energy in this case will be unnecessarily frozen by α_f. As a result, this will have a very negative impact on the computing time.

All these situations might happen in the same coupled computation. It is the reason why it is crucial to use a local coupling coefficient and it has been shown that $\alpha_f = \alpha_f^{(opt)}$ is the optimal choice in the case of the model equation adopted herein.

6   CONCLUDING REMARKS

We have shortly presented a stability analysis of a CHT problem using a Dirichlet-Robin procedure. An optimal coupling coefficient has been highlighted and discussed. It is a dynamic parameter with no increase in computational effort. The resulting coupling method can be regarded as an adaptive procedure to always obtain the fastest rate of convergence and the best stability properties.

There are many other points that would have deserved special attention. First of all, the influence of the parameter β that governs the external boundary condition and mimics the geometric and physical behavior of the solid domain. This key parameter has a direct impact on the global stability. The other important study to be carried out is to extend the present analysis to a general Robin-Robin procedure. These points, not presented here, have been thoroughly considered in a recent paper in the *Journal of Computational Physics* [1].

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