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2D VELOCITY-VORTICITY VISCOUS INCOMPRESSIBLE FLOWS

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Abstract. The 2D unsteady Navier-Stokes equations in its velocity-vorticity formulation, after time discretization, leads to a nonlinear elliptic system which may be solved by iterative methods; however, decoupling the nonlinear vorticity equation from the vorticity in the velocity equations by linear interpolation, a *direct* method is applied, allowing the vorticity equation to be *linear*. Steady state convergent flows from the un–regularized unit driven cavity problem, which causes recirculation because of the nonzero boundary condition on the top wall, are reported for Reynolds numbers Re, $400 \leq Re \leq 4000$.

1 INTRODUCTION

The main goal of this paper is to present numerical results at moderate Reynolds numbers in the range of $400 \leq Re \leq 4000$. The results are obtained using a simple numerical scheme for the unsteady Navier-Stokes equations in velocity and vorticity variables. The numerical scheme consists in a direct method to solve the nonlinear steady subproblem that results after a convenient time discretization is applied. This is a novelty contribution since usually that kind of problems are solved by iterative techniques: for instance, [1] for isothermal problems and [2] for thermal ones.

The flows are obtained from the well known un-regularized driven (or lid-driven) cavity problem which originates recirculation phenomena due to the nonzero velocity boundary condition on the top wall: the recirculation is originated by the fluid flow coming from the upstream top corner, and then hitting the downstream top corner.

At moderate Reynolds numbers, say for instance $Re \leq 7500$, the flow approaches to an asymptotic steady state as t tends to ∞. The meshes in this work follow the size dictated by the thickness of the boundary layer (of order of $Re^{-\frac{1}{2}}$) and no refining on the mesh is required near the boundary. The results clearly show that as the Reynolds number increases the mesh has to be refined and this in turn leads to decrease the time step: numerically, by stability matters and physically, to capture the fast dynamics of the flow. At this stage the results that are shown are obtained with the contour values given by [3] which are easier to obtain than those given by [4].

2 THE CONTINUOUS PROBLEM AND THE NUMERICAL METHOD

Let $\Omega \subset R^N$ (N=2,3) be the region of the flow of a viscous incompressible fluid, and Γ its boundary. It is well known that this kind of unsteady flow is governed by the non-dimensional Navier-Stokes equations; in Ω, $t > 0$; given by

$$\mathbf{u}_t - \frac{1}{Re}\Delta\mathbf{u} + \nabla p + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{f} \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

where $\mathbf{u}$, and p are the velocity and pressure of the flow, respectively. The parameter $Re = UL/\nu$, ν=kinematic viscosity, is the Reynolds number. The momentum equation (1) must be supplemented with appropriate initial and boundary conditions, for instance $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$ in Ω $(\nabla \cdot \mathbf{u}_0 = 0)$ and $\mathbf{u} = \mathbf{f_1}$ on Γ, $t \geq 0$ $(\int \mathbf{f_1} \cdot \mathbf{n} d\Gamma = 0)$ respectively.

Taking twice the curl in the momentum equation in the primitive variables formulation of the Navier-Stokes equations given by (1), and restricting to the two-dimensional case, we obtain the velocity-vorticity formulation:

$$\omega_t - \frac{1}{Re}\nabla^2\omega + \boldsymbol{u}\cdot\nabla\omega = f, \tag{3}$$

for the vorticity ω and two Poisson equations for the velocity components

$$\nabla^2 u = -\frac{\partial \omega}{\partial y} \qquad \text{and} \qquad \nabla^2 v = \frac{\partial \omega}{\partial x}, \tag{4}$$

which are related, with (1), by

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}; \tag{5}$$

where $\boldsymbol{u} = (u, v)$ is the velocity with components u and v, see [1] for details for the general 3D case.

From equation (2) a function, called the streamfunction, is obtained, which using the relations

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \tag{6}$$

and, from (5), an elliptic equation for ψ is obtained:

$$\nabla^2 \psi = -\omega\,. \tag{7}$$

A description of the numerical method follows. The time derivative that appears in the vorticity equation (3) is approximated with the following scheme, which is unconditionally stable when it is implicitly combined with the Laplacian operator, and it is well behaved when $t \to +\infty$, [5],

$$f_t(\mathbf{x}, (n+1)\Delta t) = \frac{3f^{n+1} - 4f^n + f^{n-1}}{2\Delta t} + \mathcal{O}(\Delta t^2),\ n \geq 1, \tag{8}$$

where $f^n = f(\mathbf{x}, n\Delta t)$; then, replacing that derivative in (3) by (8) we have the following totally implicit approximate problem

$$-\nabla^2 u^{n+1} = \frac{\partial \omega^{n+1}}{\partial y} \tag{9}$$

$$-\nabla^2 v^{n+1} = -\frac{\partial \omega^{n+1}}{\partial x}, \qquad \mathbf{u}^{n+1}|_\Gamma = \mathbf{u}_{bc}^{n+1} \tag{10}$$

$$\alpha\omega^{n+1} - \nu\nabla^2\omega^{n+1} + \mathbf{u}^{n+1}\cdot\nabla\omega^{n+1} = f_\omega, \qquad \omega^{n+1}|_\Gamma = \omega_{bc}^{n+1} \tag{11}$$

where $\alpha = \frac{3}{2\Delta t}$, $f_\omega = \frac{4\omega^n - \omega^{n-1}}{2\Delta t}$ and $\frac{1}{Re}$ has been replaced by the kinematic viscosity coefficient ν, considering $U = L = 1$. On the other hand, $\mathbf{u}_{bc}$ and ω_{bc} denote boundary condition for $\mathbf{u}$ and ω.

We recall that the novelty here is to apply a direct method decoupling equations (9-10) from (11) with a linear interpolation from the values of ω known from the two previous time levels; then, in these conditions (9-10) are solved first, then with the values of u and v, (11) is solved as a linear equation.

Renaming $\{u^{n+1}, v^{n+1}, \omega^{n+1}\}$ as $\{u, v, \omega\}$, and holding the linear interpolation of the right hand sides of u and v mentioned above, one has the following linear, uncoupled elliptic equations

$$-\nabla^2 u = 2\frac{\partial \omega}{\partial y}^n - \frac{\partial \omega}{\partial y}^{n-1} \tag{12}$$

$$-\nabla^2 v = -2\frac{\partial \omega}{\partial x}^n + \frac{\partial \omega}{\partial x}^{n-1} \tag{13}$$

$$\alpha\omega - \nu\nabla^2\omega + \mathbf{u}\cdot\nabla\omega \ = f_\omega, \tag{14}$$

with the boundary condition mentioned in (9-11). To obtain (u^1, v^1, ω^1) in (9-11) or (12-14), a first order approximation is applied for the time derivatives in (3), among other options, through a subsequence with smaller time step; a elliptic problem of the form (12-14) is also obtained.

For the spatial discretization to solve elliptic problems of the form (12-14) it may be applied either Finite Differences (FD) or Finite Element (FE) if the region Ω is rectangular. For the (FD) case the following classic second order discretizations are used:

With $h > 0$,

$$\begin{aligned} f''(x) &= \frac{f(x-h_x) - 2f(x) + f(x+h_x)}{h_x^2} + \mathcal{O}(h_x^2), \\ f''(y) &= \frac{f(y-h_y) - 2f(y) + f(y+h_y)}{h_y^2} + \mathcal{O}(h_y^2); \\ f'(x) &= \frac{f(x+h_x) - f(x-h_x)}{2h_x} + \mathcal{O}(h_x^2), \\ f'(y) &= \frac{f(y+h_y) - f(y-h_y)}{2h_y} + \mathcal{O}(h_y^2). \end{aligned} \tag{15}$$

Remark 2. We can also approximate $f'(x)$ and $f'(y)$ with (7) replacing Δt by h_x and h_y.

Considering the rectangular flow region $\Omega = (0, a) \times (0, b),\ a > 0,\ b > 0$, which for the unitary cavity, $a = b = 1$. With $M, N > 0$ integers, we denote $h_x = \frac{a}{M+1}$, $h_y = \frac{b}{N+1}$, $x_i = i * h_x$, $i = 0, 1, \cdots, M, M+1$, and $x_j = j * h_y$, $j = 0, 1, \cdots, N, N+1$; and by, say $\omega_{ij} = \omega(x_i, y_j)$. If $M = N$, $h_x = h_y = h$.

Then, with $M = N$,

$$\frac{\partial \omega}{\partial y} = \frac{\omega_{ij+1} - \omega_{ij-1}}{2h} \tag{16}$$

and

$$\frac{\partial \omega}{\partial x} = \frac{\omega_{i+1j} - \omega_{i-1j}}{2h}; \tag{17}$$

for $\alpha\omega - \nu\nabla^2\omega$ we have

$$\alpha\omega_{ij}^{n+1} - \nu\frac{\omega_{i-1j}^{n+1} + \omega_{i+1j}^{n+1} - 4\omega_{ij}^{n+1} + \omega_{ij-1}^{n+1} + \omega_{ij+1}^{n+1}}{h^2}. \tag{18}$$

Similarly for $-\nabla^2 u$ and $-\nabla^2 v$.

Obtaining, from (18), a constant matrix m for ω. If we keep $\mathbf{u}\cdot\nabla\omega$ in the left hand side, discretizating it like

$$\mathbf{u}\cdot(\nabla\omega) = u_{ij}^{n+1}\left[\frac{\omega_{i+1j}^{n+1}-\omega_{i-1j}^{n+1}}{2h}\right] - v_{ij}^{n+1}\left[\frac{\omega_{ij+1}^{n+1}-\omega_{ij-1}^{n+1}}{2h}\right] \tag{19}$$

and adding it to $\alpha\omega - \nu\nabla^2\omega$ a matrix depending on time is obtained, i.e., $m = m(t)$. The case $M = N = 3$ is shown next; it is split in two part for space reasons, the second part shows the last three columns.

$$\begin{pmatrix}
4\nu + \frac{h^2}{\Delta t} & -\nu + \frac{hv_{1,1}}{2} & 0 & -\nu + \frac{hu_{1,1}}{2} & 0 & 0 \\
-\nu - \frac{hv_{1,2}}{2} & 4\nu + \frac{h^2}{\Delta t} & -\nu + \frac{hv_{1,2}}{2} & 0 & -\nu + \frac{hu_{1,2}}{2} & 0 \\
0 & -\nu - \frac{hv_{1,3}}{2} & 4\nu + \frac{h^2}{\Delta t} & 0 & 0 & -\nu + \frac{hu_{1,3}}{2} \\
-\nu - \frac{hu_{2,1}}{2} & 0 & 0 & 4\nu + \frac{h^2}{\Delta t} & -\nu + \frac{hv_{2,1}}{2} & 0 \\
0 & -\nu - \frac{hu_{2,2}}{2} & 0 & -\nu - \frac{hv_{2,2}}{2} & 4\nu + \frac{h^2}{\Delta t} & -\nu + \frac{hv_{2,2}}{2} \\
0 & 0 & -\nu - \frac{hu_{2,3}}{2} & 0 & -\nu - \frac{hv_{2,3}}{2} & 4\nu + \frac{h^2}{\Delta t} \\
0 & 0 & 0 & -\nu - \frac{hu_{3,1}}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & -\nu - \frac{hu_{3,2}}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & -\nu - \frac{hu_{3,3}}{2}
\end{pmatrix}$$

$$\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
-\nu + \frac{hu_{2,1}}{2} & 0 & 0 \\
0 & -\nu + \frac{hu_{2,2}}{2} & 0 \\
0 & 0 & -\nu + \frac{hu_{2,3}}{2} \\
4\nu + \frac{h^2}{\Delta t} & -\nu + \frac{hv_{3,1}}{2} & 0 \\
-\nu - \frac{hv_{3,2}}{2} & 4\nu + \frac{h^2}{\Delta t} & -\nu + \frac{hv_{3,2}}{2} \\
0 & -\nu - \frac{hv_{3,3}}{2} & 4\nu + \frac{h^2}{\Delta t}
\end{pmatrix} \tag{20}$$

All calculations are made with $m = m(t)$, and so far it seems better than considering the constant matrix m, mentioned before, as it is usually done. As is already known the matrix is symmetric and pentadiagonal of size $N^2 \times N^2$. For N (or M) large enough it gives rise a sparse matrix.

With $\alpha = 0$ and $\nu = 1$, it is clear that the above matrix is similar to $-\nabla^2 u$ and $-\nabla^2 v$, but constants; these latter matrices are diagonally dominant whereas the previous one is strictly diagonally dominant if Δt, in α, is sufficiently small, [8].

It should be noted that the above constant matrix m is the consequence of taking $\mathbf{u} \cdot \nabla\omega$, in (14), to the right hand side with the linear interpolation of the two previous time levels, i.e.,

$$2(\mathbf{u} \cdot \nabla\omega)^n - (\mathbf{u} \cdot \nabla\omega)^{n-1},$$

Therefore, in each time step, three algebraic linear equations systems, associated with the semi-discrete system for u, v, and ω in $(12 - 14)$, will be solved, that is, algebraic systems of the form $\mathcal{A}\ x = b$, with $\mathcal{A} = m(t)$ $(or\ m)$. Up to now these algebraic linear systems have been solved with LinearSolve, storing previously the matrix as a SparseArray, both Mathematica 8 Commands. The results are reported through the contour of ω and ψ; concerning ψ, this is obtained, in the final time in which ω has been already computed solving the equation for ψ given by (7).

3 NUMERICAL EXPERIMENTS AND DISCUSSION

The spatial discretization must be supplemented with boundary conditions, for each equation, on Γ for all $t \geq 0$, in this case for the driven cavity problem, we have $(u, v) = (1, 0)$ on the moving top wall $y = 1$ and $(u, v) = (0, 0)$ elsewhere; this problem, as mentioned above causes recirculation. Then in terms of this boundary condition for $\mathbf{u}$, the one given by u, v and ω reads

$$\begin{array}{lll} u = 0, v = 0; & \omega = \frac{\partial v}{\partial x} & on\ \ \Gamma_{x=0} \\ u = 0, v = 0; & \omega = \frac{\partial v}{\partial x} & on\ \ \Gamma_{x=1} \\ u = 0, v = 0; & \omega = -\frac{\partial u}{\partial y} & on\ \ \Gamma_{y=0} \\ u = 1, v = 0; & \omega = -\frac{\partial u}{\partial y} & on\ \ \Gamma_{y=1} \end{array} \qquad (21)$$

Besides, $\omega(\mathbf{x}, 0) = \omega_0(\mathbf{x})$ denotes the vorticity initial condition which, by (5), has to satisfy $\omega_0 = \frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y}$ if $\mathbf{u_0} = (u_0, v_0)$ is the initial velocity.

For moderate Reynolds numbers Re, the following values are reported: Re=400, 1000 and 4000; actually, $Re = 400$ is not shown but the calculation agrees perfectly with the one we are comparing with.. The results, for each case of Re that is considered, correspond to steady state flows, the iso-contours for the vorticity ω are shown first and the streamlines for the streamfunction ψ right below. The cases Re=400 and 1000 agree perfectly with the ones reported by [3] who solve the stationary problem using the streamfunction and vorticity formulation; they use the contour values $vo = \{-5., -3, -1, 1, 3, 5.\}$ for vorticity and

$vz = \{-0.11, -0.10, -0.08, -0.06, -0.04, -0.02, -0.01\}$ for the streamfunction, which give rise the streamlines. Other kind of contour values are $v1 = \{-3., -2., -1., -.5, 0., .5, 1., 2., 3., 4., 5.\}$ for vorticity given by [4] which are harder to obtain than those of [3], as has already already pointed out in earlier works of the third author. The case $Re = 4000$ does not agree at all with the one [3] but it does with the one in [1], where some explanation is given for this small disagreement.

The mesh size is denoted by h and the time step by Δt and they are specified in each case under study. Figures 1 and 2 pictures de iso-vorticity contours and streamlines for $Re = 1000$ respectively, obtained with $h = 1/256$ and $\Delta t = 0.005$. Figures 3 and 4 those for $Re = 4000$, obtained with $h = 1/600$ and $\Delta t = 0.0025$.

4 CONCLUSIONS

We have presented results for moderate Reynolds numbers that are obtained from a direct numerical method applied to the Navier-Stokes equations in its velocity-vorticity formulation, which is not a common formulation to use. Besides the novelty of the direct method the results become from a variable matrix depending on time and just using Mathematica 8. We are working to improve the scheme to be able to handle high Reynolds numbers as well as to extend it to thermal problems like those in [2] and in [7].

REFERENCES

[1] Nicolás, A. and Bermúdez, B., Viscous incompressible flows by the velocity-vorticity Navier-Stokes equations. *Computer Modeling in Engineering & Sciences* (2007) **20**:73–83.

[2] Nicolás, A. and Bermúdez B., 2D mixed convection viscous incomprensible flows with velocity-vorticity variables. *Computer Modeling in Engineering & Sciences* (2011) **82**:163–178.

[3] Schreiber R. and Keller H.B., Driven cavity flow by efficient numerical techniques. *J. Comp. Phy.* (1983) **40**:310–310.

[4] Ghia U., Ghia K.N. and Shin C.T., Driven cavity flow by efficient numerical techniques. *J. Comp. Phy.* (1982) **48**:387–411.

[5] Glowinski, R. *Numerical methods for nonLinear variational problems.* Springer-Verlag, (1984).

[6] Nicolás, A. and Bermúdez, B., 2D thermal/isothermal incompressible viscous flows. *Int. J. for Num. Meth. in Fluids* (2005) **48**:349–366.

[7] Nicolás, A. and Bermúdez, B., Thermal recirculation of viscous incomprensible flows in enclosures. *European Congress on Computacional Methods in Applied Sciences*

and Engineering(ECCOMAS 2012) (2012); J. Eberhardsteiner et al. (eds.), Vienna, Austria. September 10–14.

[8] Burden R. L. and Faires J. D. *Numerical Analysis*. PWS, Boston, USA., (1985)

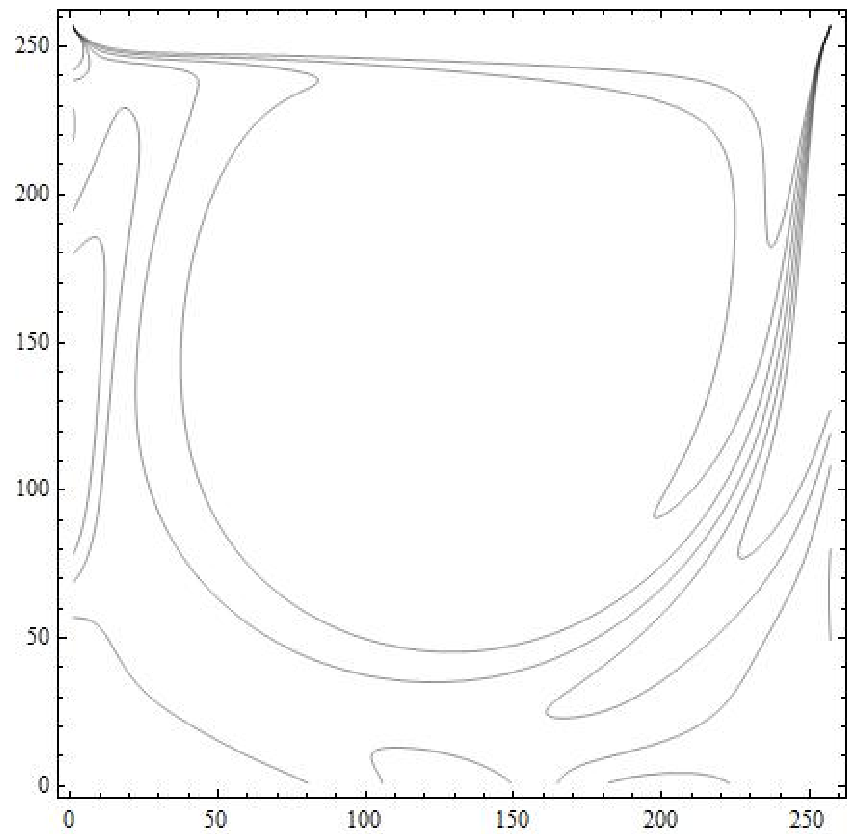

Figure 1: ω: Re=1000 (Keller); h=1/256, dt=.005

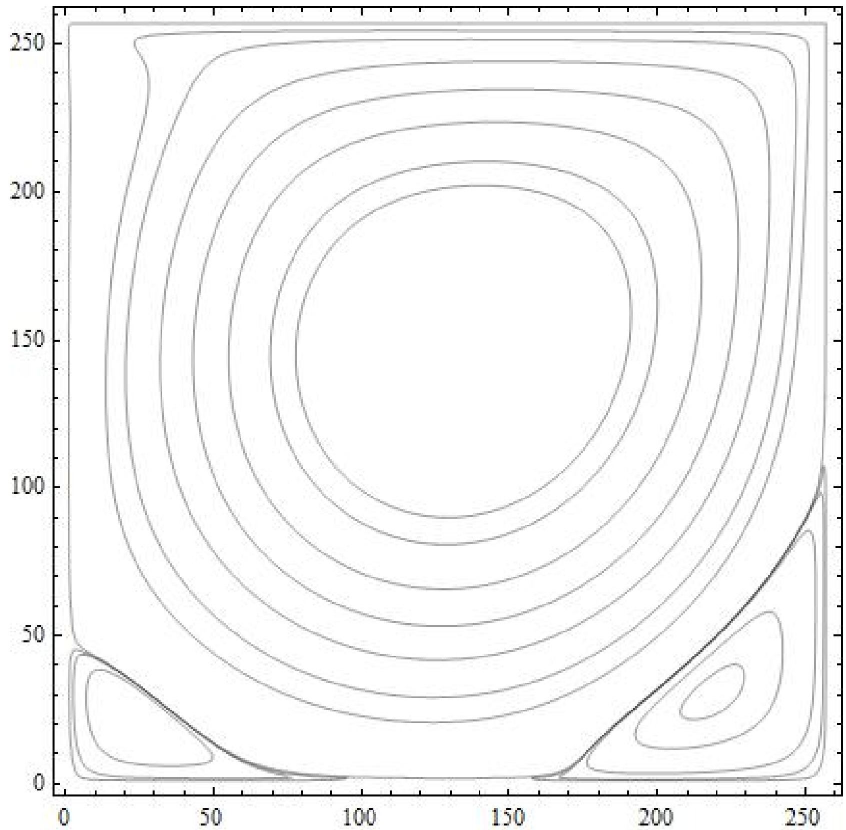

Figure 2: ψ: Re=1000 (Keller); h=1/256, dt=.005

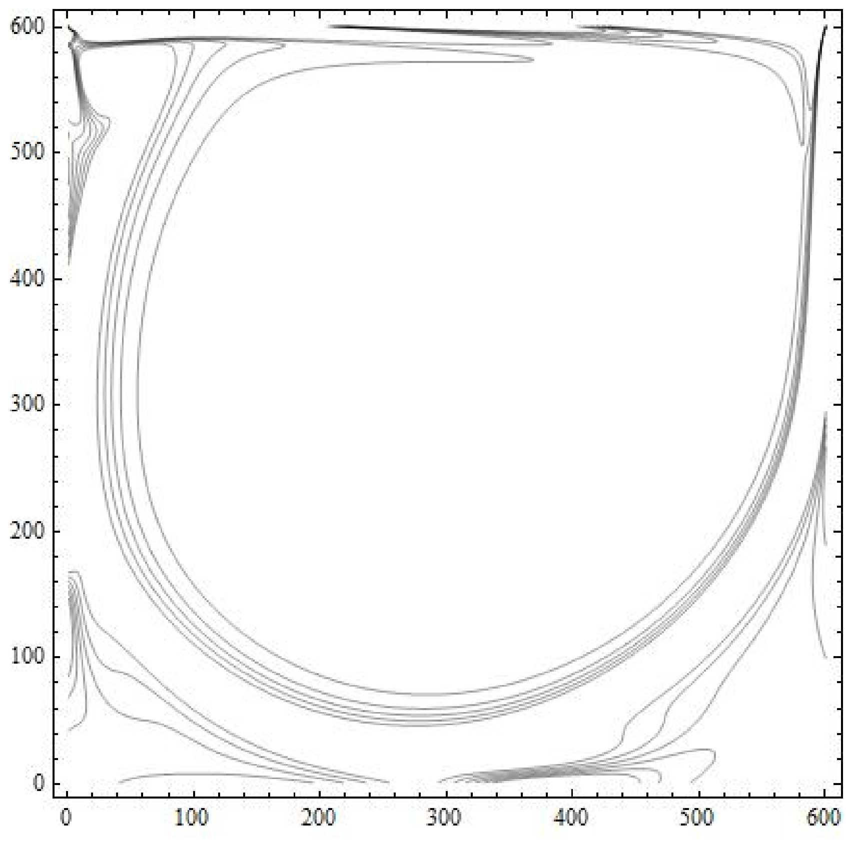

Figure 3: ω: Re=4000 (Keller); h=1/600, dt=.0025

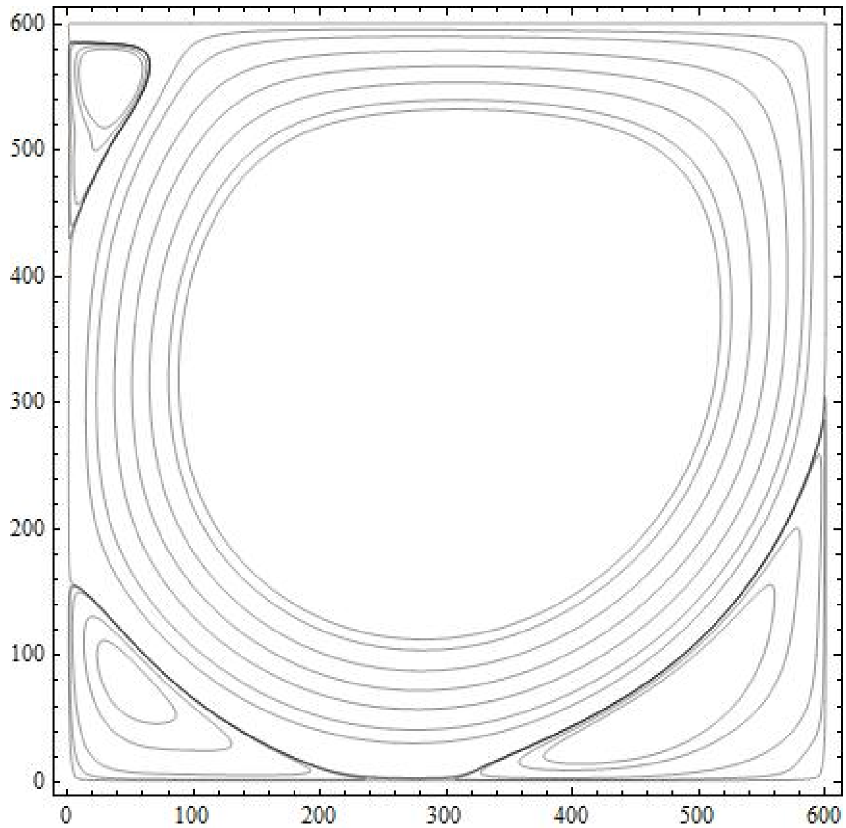

Figure 4: ψ: Re=4000 (Keller); h=1/600, dt=.0025