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## ON $\Phi$ -SCHAUDER FRAMES

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ABSTRACT. In this short note we introduce and study a particular type of Schauder frames, namely,  $\Phi$ -Schauder frames.

Keywords: Frames, retro Banach frames, Schauder frames.

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### 1. INTRODUCTION AND PRELIMINARIES

Frames for Hilbert spaces were introduced by Duffin and Schaeffer [6] in 1952, while addressing some deep problems in non-harmonic Fourier series. Later, in 1986, Daubechies, Grossmann and Meyer [5] found new applications to wavelets and Gabor transforms in which frames played an important role.

Today, frames play important roles in many applications in mathematics, science and engineering . In particular frames are widely used in sampling theory, wavelet theory, wireless communication, signal processing, image processing , differential equations, filter banks, geophysics, quantum computing, wireless sensor network, multiple-antenna code design and many more. Reason is that frames provides both great liberties in the design of vector space decompositions, as well as quantitative measure on the computability and robustness of the corresponding reconstructions. In the theoretical direction, powerful tools from operator theory and Banach spaces are being employed to study frames. For a nice and comprehensive survey on various types of frames, one may refer to [1, 4, 10] and the references therein.

Coifman and Weiss [3] introduced the notion of atomic decomposition for function spaces. Later , Feichtinger and Gröchenig [7] extended this idea to Banach spaces. This concept was further generalized by Gröchenig [8] who introduced the notion of Banach frames for Banach spaces. Casazza, Han and Larson [2] also carried out a study of atomic decompositions and Banach frames. Recently, various generalization of frames in Banach spaces have been introduced and studied. Han and Larson [9] introduced Schauder frames for Banach spaces. Schauder frames were further studied in [13, 14]. The notion of retro Banach frames in Banach spaces introduced and studied in [12].

Throughout this note  $E$  will denote an infinite dimensional Banach space and  $E^*$  the conjugate space of  $E$ .

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**Definition 1.1** ([9]). A pair  $(x_n, f_n)$  ( $\{x_n\} \subset E, \{f_n\} \subset E^*$ ) is called a Schauder frame for  $E$  if

$$x = \sum_{n=1}^{\infty} f_n(x)x_n, \quad \text{for all } x \in E, \tag{1}$$

where the series converges in the norm topology of  $E$ .

**Definition 1.2** ([12]). Let  $E$  be a Banach space and let  $E_d$  be an associated Banach space of scalar-valued sequences indexed by  $\mathbb{N}$ . Let  $\{x_n\} \subset E$  and  $T : (E^*)_d \rightarrow E^*$  be given. The pair  $(\{x_n\}, T)$  is called a retro Banach frame for  $E^*$  with respect to  $(E^*)_d$  if :

- (i)  $\{f(x_n)\} \in (E^*)_d$ , for all  $f \in E^*$ .
- (ii) There exist positive constants  $A$  and  $B$  with  $0 < A \leq B < \infty$  such that

$$A\|f\|_{E^*} \leq \|\{f(x_n)\}\|_{(E^*)_d} \leq B\|f\|_{E^*}, \quad \text{for all } f \in E^*. \tag{2}$$

- (iii)  $T$  is a bounded linear operator such that  $T(\{f(x_n)\}) = f$ , for all  $f \in E^*$ .

The positive constants  $A$  and  $B$  are called the lower and upper frame bounds, of the retro Banach frame  $(\{x_n\}, T)$ , respectively. The operator  $T : (E^*)_d \rightarrow E^*$  is called the reconstruction operator (or the pre-frame operator) and the inequality (2) is called the retro frame inequality.

**Lemma 1.1.** Let  $E$  be a Banach space and  $\{f_n\} \subset E^*$  be a sequence such that  $\{x \in E : f_n(x) = 0, \text{ for all } n \in \mathbb{N}\} = \{0\}$ . Then  $E$  is linearly isometric to the Banach space  $X = \{\{f_n(x)\} : x \in E\}$ , where the norm is given by  $\|\{f_n(x)\}\|_X = \|x\|_E, x \in E$ .

In this note we introduce and study a particular type of Schauder frames, namely,  $\Phi$ -Schauder frames. Necessary and / or sufficient conditions for a Schauder frame to be  $\Phi$ -Schauder frame have been obtained.

## 2. $\Phi$ -SCHAUDER FRAMES

**Definition 2.1.** A Schauder frame  $(x_n, f_n)$  ( $\{x_n\} \subset E, \{f_n\} \subset E^*$ ) for a Banach space  $E$  is said to be  $\Phi$ -Schauder frame if  $\inf_{1 \leq n < \infty} \|f_n\| > 0$  and there exists a functional  $\Phi \in E^*$  such that  $\Phi(x_n) = 1$ , for all  $n \in \mathbb{N}$ .

The functional  $\Phi$  is called the *associated functional* of the Schauder frame  $(x_n, f_n)$ . To show existence of  $\Phi$ -Schauder frames, we have following example.

**Example 2.1.** Let  $E = l^1$  and  $\{e_n\} \subset E$  be the sequence of canonical unit vectors.

- (a) Let  $\{x_n\} \subset E, \{f_n\} \subset E^*$  be sequences defined by

$$\left. \begin{aligned} x_1 &= \frac{1}{2}e_1, & x_2 &= \frac{1}{2}e_1, & x_n &= e_{n-1}, \\ f_1(x) &= \xi_1, & f_2(x) &= \xi_1, & f_n(x) &= \xi_{n-1}, & x = \{\xi_n\} \in E \end{aligned} \right\}, \quad n = 3, 4, 5, \dots$$

Then  $(x_n, f_n)$ , is a Schauder frame for  $E$  with  $\inf_{1 \leq n < \infty} \|f_n\| > 0$ . Now  $\Phi = (2, 1, 1, 1, 1, \dots) \in E^*$  is such that  $\Phi(x_n) = 1$ , for all  $n \in \mathbb{N}$ . Hence  $(x_n, f_n)$  is a  $\Phi$ -Schauder frame of for  $E$ .

- (b) Let  $\{x_n\} \subset E, \{f_n\} \subset E^*$  be sequences defined by

$$\left. \begin{aligned} x_1 &= e_1, & x_{n-1} &= e_{n-1} \\ f_1(x) &= 0, & f_{n-1}(x) &= \xi_{n-1}, & x = \{\xi_n\} \in E \end{aligned} \right\}, \quad n = 2, 3, 4, \dots$$

Then,  $(x_n, f_n)$  is a Schauder frame for  $E$ . But  $\inf_{1 \leq n < \infty} \|f_n\| = 0$ . Thus,  $(x_n, f_n)$  is not a  $\Phi$ -Schauder frame for  $E$ .

**Remark 2.1.** A  $\Phi$ -Schauder frame  $(x_n, f_n)$  for a Banach space  $E$  also depends on  $E$ . Indeed, the pair  $(x_n, f_n)$  in Example 2.1(a) is a  $\Phi$ -Schauder frame for  $l^1$  but not for  $c_0$ .

The following theorem gives sufficient conditions for a Schauder frame to be  $\Phi$ -Schauder frame.

**Theorem 2.1.** Let  $(x_n, f_n)$  ( $\{x_n\} \subset E, \{f_n\} \subset E^*$ ) be a Schauder frame for a Banach space  $E$  with  $\inf_{1 \leq n < \infty} \|f_n\| > 0$  and  $z_0$  a given non-zero vector in  $E$ . If there exists no reconstruction operator  $W$  such that  $(\{x_n + z_0\}, W)$  is a retro Banach frame for  $E^*$ , then  $(x_n, f_n)$  is  $\Phi$ -Schauder frame.

*Proof.* If there exists no reconstruction operator  $W$  such that  $(\{x_n + z_0\}, W)$  is a retro Banach frame for  $E^*$ , then by Lemma 1.1 there exists a non-zero functional  $\phi$  in  $E^*$  such that  $\phi(x_n + z_0) = 0$ , for all  $n \in \mathbb{N}$ . Since  $(x_n, f_n)$  is Schauder frame for  $E$ , so  $\phi(x_k) \neq 0$  for some  $k$ . Thus,  $\phi(z_0) \neq 0$ . Put  $\Phi = -\frac{\phi}{\phi(z_0)}$ . Then,  $\Phi$  is a functional in  $E^*$  such that  $\Phi(x_n) = 1$ , for all  $n \in \mathbb{N}$ . Hence  $(x_n, f_n)$  is a  $\Phi$ -Schauder frame . □

**Remark 2.2.** The conditions in Theorem 2.1 are not necessary. Indeed, let  $(x_n, f_n)$  be the  $\Phi$ -Schauder frame for  $E = l^1$  given in Example 2.1(a) and  $z_0 = -e_1$ . Then, by Lemma 1.1 there exists a reconstruction operator  $W : (E^*)_d = \{\{f(x_n + z_0)\} : f \in E^*\} \rightarrow E^*$  such that  $(\{x_n + z_0\}, W)$  is a retro Banach frame for  $E^*$  with bounds  $A = B = 1$ .

**Remark 2.3.** The conditions in Theorem 2.1 turns out to be necessary provided  $\Phi(z_0) = -1$ . Under this condition there exists no reconstruction operator  $\Theta_0$  such that  $(\{x_n + z_0\}, \Theta_0)$  is a retro Banach frame for  $E^*$ . Since otherwise by retro frame inequality for  $(\{x_n + z_0\}, \Theta_0)$  and using  $\Phi(x_n + z_0) = 0$ , for all  $n \in \mathbb{N}$ , we obtain  $\Phi = 0$ , a contradiction.

Thus, we have following theorem.

**Theorem 2.2.** Let  $(x_n, f_n)$  be a  $\Phi$ -Schauder frame for a Banach space  $E$  and that  $z_0$  be a non zero vector in  $E$  such that  $\Phi(z_0) = -1$ . Then, there exists no reconstruction operator  $\Theta_0$  such that  $(\{x_n + z_0\}, \Theta_0)$  is a retro Banach frame for  $E^*$ .

The following theorem shows that if  $E$  and  $F$  are Banach spaces having  $\Phi$ -Schauder frames, then their product space  $E \times F$  with a suitable norm also has a  $\Phi$ -Schauder frame .

**Theorem 2.3.** Let  $(x_n, f_n)$  ( $\{x_n\} \subset E, \{f_n\} \subset E^*$ ) and  $(y_n, g_n)$  ( $\{y_n\} \subset F, \{g_n\} \subset F^*$ ) be Schauder frames for Banach spaces  $E$  and  $F$ , respectively. Then there exist sequences  $\{z_n\} \subset E \times F$  and  $\{h_n\} \subset (E \times F)^*$  such that  $(z_n, h_n)$  is a Schauder frame for  $E \times F$ . Furthermore, if  $(x_n, f_n)$  and  $(y_n, g_n)$  are  $\Phi$ -Schauder frames, then  $(z_n, h_n)$  is also a  $\Phi$ -Schauder frame.

*Proof.* Let  $\{z_n\} \subset E \times F$  and  $\{h_n\} \subset (E \times F)^*$  be sequences defined by

$$\begin{cases} z_{2n} = (0, y_n), \\ z_{2n-1} = (x_n, 0) \end{cases} \quad n \in \mathbb{N},$$

$$\begin{cases} h_{2n}(x, y) = g_n(y), \\ h_{2n-1}(x, y) = f_n(x) \end{cases} \quad n \in \mathbb{N}.$$

Then

$$\begin{aligned}(x, y) &= \left( \sum_{n=1}^{\infty} f_n(x)x_n, \sum_{n=1}^{\infty} g_n(y)y_n \right) \\ &= \sum_{n=1}^{\infty} h_n(x, y)z_n, \quad \text{for all } (x, y) \in E \times F.\end{aligned}$$

Hence  $(z_n, h_n)$  is a Schauder frame for  $E \times F$ .

Further, suppose  $(x_n, f_n)$  and  $(y_n, g_n)$  are  $\Phi$ -Schauder frames. Then, by nature of construction of  $(z_n, h_n)$ , there exists a functional  $\Phi_0 \in (E \times F)^*$  such that  $\Phi_0(z_n) = 1$ , for all  $n \in \mathbb{N}$  and  $\inf_{1 \leq n < \infty} \|\Phi_0\| > 0$ . Therefore,  $(z_n, h_n)$  is  $\Phi_0$ -Schauder frame.  $\square$

### 3. CONCLUDING REMARKS

Let  $(x_n, f_n)$  be a Schauder frame for  $E$ . Then, there exist a reconstruction operator  $\Theta_0$  such that  $(\{x_n\}, \Theta_0)$  is a retro Banach frame for  $E^*$ . We say that  $(\{x_n\}, \Theta_0)$  is an *associated retro Banach frame* of the Schauder frame  $(x_n, f_n)$ .

Recently, retro Banach frames of type  $P$  for Banach spaces introduced and studied in [15]: A retro Banach frame  $(\{x_n\}, T)$  for  $E^*$  is said to be of *type  $P$*  if it is exact and there exists a functional  $\Psi \in E^*$  such that  $\Psi(x_n) = 1$ , for all  $n \in \mathbb{N}$ .

Conclusion: For a given  $\Phi$ -Schauder frame of  $E$ , its associated retro Banach frame need not be of type  $P$  and vice-versa. This is given in the form of remarks.

**Remark 3.1.** Let  $(x_n, f_n)$  be a  $\Phi$ -Schauder frame for  $E$ . Then, associated retro Banach frame  $(\{x_n\}, \Theta_0)$ , in general, not of type  $P$ . Indeed, let  $(x_n, f_n)$  ( $\{x_n\} \subset E, \{f_n\} \subset E^*$ ) be a system for  $E = l^1$  given in Example 2.1(a). Then,  $(x_n, f_n)$  is a  $\Phi$ -Schauder frame for  $E$  but the associated retro Banach frame  $(\{x_n\}, \Theta_0)$  is not of type  $P$ .

**Remark 3.2.** Let  $(\{x_n\}, \Theta)$  be a retro Banach frame of type  $P$  for  $E$ . Then, in ganeral,  $E$  has no Schauder frame. Furthermore, if  $E$  admit an associated Schauder frame, say  $(x_n, f_n)$ . Then, in general,  $(x_n, f_n)$  is not a  $\Phi$ -Schauder frame for  $E$ . Indeed, let  $E = l^1$  and  $\{x_n\} \subset E, \{f_n\} \subset E^*$  be sequences defined by

$$\left. \begin{aligned}x_n &= ne_n, \\ f_n(x) &= \frac{1}{n}\xi_n, \quad x = \{\xi_n\} \in E\end{aligned} \right\}, \quad n = 1, 2, 3, \dots$$

Then,  $(x_n, f_n)$  is a Schauder frame for  $E$ . Also, there exists a reconstruction operator  $\Theta_0$  such that  $(\{x_n\}, \Theta_0)$  is a retro Banach frame [associated retro Banach frame of the Schauder frame  $(x_n, f_n)$ ] for  $E^*$  which is of type  $P$ . But  $(x_n, f_n)$  is not a  $\Phi$ -Schauder frame for  $E$ .

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