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APPLICATION OF THE HOMOTOPY PERTURBATION METHOD (HPM) AND VARIATIONAL ITERATION METHOD (VIM) TO GAS DYNAMIC EQUATION

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Abstract. A gas-dynamic control system is one where the path of an object in flight is controlled by either the generation or redirection of gas flow out of an orifice rather than with the traditional movable control surfaces. In this paper considering Hes Homotopy perturbation and Variational iteration methods are calculated the homogeneous gas dynamics equations. The exact (analytic) solution of the equation is calculated in the form of a series with easily computable components like to Adomians decomposition method. The Comparison of result of HPM and VIM with Adomians decomposition method show that they are agreement with them, and HPM, VIM can solve large class of nonlinear problems.

Keywords: Gas dynamics equation, Adomians Decomposition method, Homotopy Perturbation Method (HPM), Variational Iteration Method (VIM).

AMS Subject Classification: 35A15

1. Introduction

The recent years have seen significant development in the use of various methods for the numerical and analytic solution of the partial differential equation. The aim of the present paper is to implement the homotopy perturbation and variational iteration methods [1220] for the homogeneous gas dynamics equations to an approximation solution. The solution of this method is derived in the form of power series with easy computable components.

Over the last decades several analytical/approximate methods have been developed to solve linear and nonlinear ordinary and partial differential equations. Some of these techniques include variational iteration method (VIM) (He [16,17]; Ganji et al., [12,14,15];), homotopy perturbation method (HPM) (He [2,8,9,10,11]; Ganji et al., [5,6,7,12,14,15];) etc. Linear and Nonlinear phenomena play an important role in varios fields of science and engineering. Most models of real life problems are still very difficult to solve. Therefore, approximate analytical solutions such as homotopy perturbation method [2-15] were

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introduced. This method is the most effective and convenient ones for both linear and nonlinear equations.

Perturbation method is based on assuming a small parameter. The majority of nonlinear problems, especially those having strong nonlinearity, have no small parameters at all and the approximate solutions obtained by the perturbation methods, in most cases, are valid only for small values of the small parameter. Generally, the perturbation solutions are uniformly valid as long as a scientific system parameter is small. However, we cannot rely fully on the approximations, because there is no criterion on which the small parameter should exists. Thus, it is essential to check the validity of the approximations numerically and/or experimentally. To overcome these difficulties, HPM have been proposed recently.

Recently, He (1999) proposed a variational iteration method based on the use of restricted variations and correction functionals which has found a wide application for the solution of nonlinear ordinary and partial differential equations. This method does not require the presence of small parameters in the differential equation, and provides the solution (or an approximation to it) as a sequence of iterates. The method does not require that the nonlinearities be differentiable with respect to the dependent variable and its derivatives.

In this paper we will apply the homotopy perturbation method and variational iteration method to gas dynamics equation and comparing the result of HPM, VIM with ADM [21-23]

2. Basic Idea of Homotopy-Perturbation Method

To explain this method, let us consider the following function:

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{1}$$

With the boundary conditions of:

$$B(u, \frac{\partial u}{\partial n}) = 0, \quad r \in \Gamma, \tag{2}$$

Where $A, B, f(r)$ and Γ are a general differential operator, a boundary operator, a known analytical function and the boundary of the domain Ω, respectively. Generally speaking the operator A can be divided in to o linear part L and a nonlinear part $N(u)$. Eq.(1) can therefore, be written as:

$$L(u) + N(u) - f(r) = 0 \tag{3}$$

By the homotopy technique, we construct a homotopy
$u(r, p) : \Omega \times [0, 1] \to \mathbb{R}$ Which satisfies:

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \tag{4}$$

$$p \in [0, 1], \quad r \in \Omega,$$

or

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(u) - f(r)] = 0, \tag{5}$$

where $p \in [0, 1]$ is an embedding parameter, while u_0 is an initial approximation of Eq. (1), which satisfies the boundary conditions. Obviously, from Eqs. (4) and (5) we will have:

$$H(v, 0) = L(v) - L(u_0) = 0, \tag{6}$$

$$H(v,1) = A(v) - f(r) = 0, \tag{7}$$

The changing process of p from zero to unity is just that of $v(r,p)$ from u_0 to $u(r)$. In topology, this is called deformation, while $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopy. According to the HPM, we can first use the embedding parameter p as a small parameter, and assume that the solutions of Eqs. (4), (5) can be written as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + \cdots , \tag{8}$$

Setting $p = 1$ yields in the approximate solution of Eq. (4) to:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots , \tag{9}$$

The combination of the perturbation method and homotopy method is called the HPM, which eliminates the drawbacks of the traditional perturbation methods while keeping all its advantage. The series (9) is convergent for most cases. However, the convergent rate depends on the nonlinear operator $A(v)$. Moreover, He (1999) made the following suggestions:

- The second derivative of $N(v)$ with respect to v must be small because the parameter may be relatively large, i.e.$p \to 1$.
- The norm of $L^{-1}\frac{\partial N}{\partial v}$ must be smaller than one so that the series converges.

3. Basic Idea of Variational Iteration Method:

To clarify the basic ideas of VIM, we consider the following differential equation:

$$Lu + Nu = g(t), \tag{10}$$

Where L is a linear operator, N is a nonlinear operator and $g(t)$ is a homogeneous term. According to VIM, we can write down a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(Lu_n(\tau) + N\tilde{u}(\tau) - g(\tau))d\tau \tag{11}$$

Where λ is a general lagrangian multiplier which can be identified optimally via the variational theorem, The subscript n indicates the nth approximation and u_n is considered as a restricted variation, i.e.$\delta\tilde{u} = 0$.

4. The Gas Dynamic Equation in General:

The classical gas dynamic equation is given by the formula [1]:

$$\frac{\partial}{\partial t}u(x,t) + \frac{u^2(x,t)}{2}\frac{\partial}{\partial x}u(x,t) - u(x,t)(1 - u(x,t)) = 0; \tag{12}$$

$u(x,t)$ =Degree of tempreture,
t =Time from Zero Point, seconds,
x =Distance from zero to 1,

Consider Eq. (12), and assume that initial condition as follow and assume that we have not any boundary condition,

$$u(x,0) = g(x) \tag{13}$$

Where $g(x)$ =Initial condition of gas equation,
The exact solution by Adomians Decomposition method [12] has been expressed as follows:

$$v(x,t)=a(1-e^{-x})+at(1-e^{-x})-a^2t(1-e^{-x})^2-a^2t(e^{-x}-e^{-2x})+ \\ +\frac{a^2t^2}{2}(1-e^{-x})+\frac{a^2t^2}{2}(1+e^{-x}-e^{-2x}+ae^{-2x})+\cdots \tag{14}$$

5. Application of Homotopy Perturbation Method:

We consider Eq. (12) for gas dynamic equation with initial conditions as follows:
To solve Eq. (12) by means of HPM, we consider the following process after separating the linear and nonlinear parts of the equation.
A homotopy can be constructed as follows [20]:

$$H(u,p)=(1-p)(\frac{\partial u(x,t)}{\partial t})+p(\frac{\partial u(x,t)}{\partial t}+\frac{u^2(x,t)}{2}\frac{\partial u(x,t)}{\partial x}-u(x,t)(1-u(x,t))). \tag{15}$$

Substituting $u=u_0+pu_1+\cdots$ in to Eq. (15) and rearranging the resultant equation based on powers of $p-$terms, one has:

$$p^0:\frac{\partial}{\partial t}u_0(x,t)=0, \tag{16}$$

$$p^1:\frac{1}{2}u_0^2(x,t)\frac{\partial}{\partial x}u_0(x,t)+u_0^2(x,t)+\frac{\partial}{\partial t}u_1(x,t)-u_0(x,t)=0. \tag{17}$$

$$p^2:2u_0(x,t)u_1(x,t)+\frac{\partial}{\partial t}u_2(x,t)+\frac{1}{2}u_0^2(x,t)\frac{\partial}{\partial x}u_1(x,t)- \\ -u_1(x,t)+u_0(x,t)u_1(x,t)\frac{\partial}{\partial x}u_0(x,t)=0 \tag{18}$$

With the following initial conditions:

$$u_0(x,0)=g(x)=a(1-e^{-x}), \\ u_i(0,t)=0, \quad i=1,2,\cdots \tag{19}$$

$u(x,t)$ may be written as follows by solving the Eqs (16), (17) and (18):

$$u_0(x,t)=a(1-e^{-x}) \tag{20}$$

$$u_1(x,t)=-\frac{a}{2}(a^2e^{-x}-2a^2e^{-2x}+a^2e^{-3x}+2a-4ae^{-x}+2ae^{-2x}-2+2e^{-x})t \tag{21}$$

$$u_2(x,t)=-\frac{at^2}{8}(-8a^2+12a+32a^2e^{-x}+a^4e^{-x}+4e^{-x}-24ae^{-x}-8a^4e^{-2x} \\ -40a^2e^{-2x}+12a^3e^{-4x}+16a^2e^{-3x}+12ae^{-2x}+18a^4e^{-3x}-16a^4e^{-4x} \\ +5a^4e^{-5x}-4-12a^3e^{-x}+36a^3e^{-2x}-36a^3e^{-3x}) \tag{22}$$

In the same manner, the rest of components were obtained using the maple package.
According to the HPM, we can conclude that:

$$u(x,t) = \lim_{p\to 1} u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+\cdots. \tag{23}$$

Therefore, substituting the values of $u_0(x,t)$, $u_1(x,t)$ and $u_2(x,t)$ from Eqs. (20), (21) and (22) in to Eq. (23) yields:

$$\begin{aligned} u(x,t) = {} & a(1-e^{-x}) - \frac{a}{2}(a^2e^{-x} - 2a^2e^{-2x} + a^2e^{-3x} + 2a - 4ae^{-x} + 2ae^{-2x} - 2 + 2e^{-x})t \\ & - \frac{at^2}{8}(-8a^2 + 12a + 32a^2e^{-x} + a^4e^{-x} + 4e^{-x} - 24ae^{-x} - 8a^4e^{-2x} - 40a^2e^{-2x} \\ & + 12a^3e^{-4x} + 16a^2e^{-3x} + 12ae^{-2x} + 18a^4e^{-3x} - 16a^4e^{-4x} + 5a^4e^{-5x} - 4 \\ & - 12a^3e^{-x} + 36a^3e^{-2x} - 36a^3e^{-3x}) \end{aligned} \tag{24}$$

Fig. 1 shows the results of gas dynamics equation by HPM and difference with ADM result after substituting $a = .01$ and plotting for $0 < x < 1$ and $0 < t < 1$. And the variational iteration formula is obtained in the form:

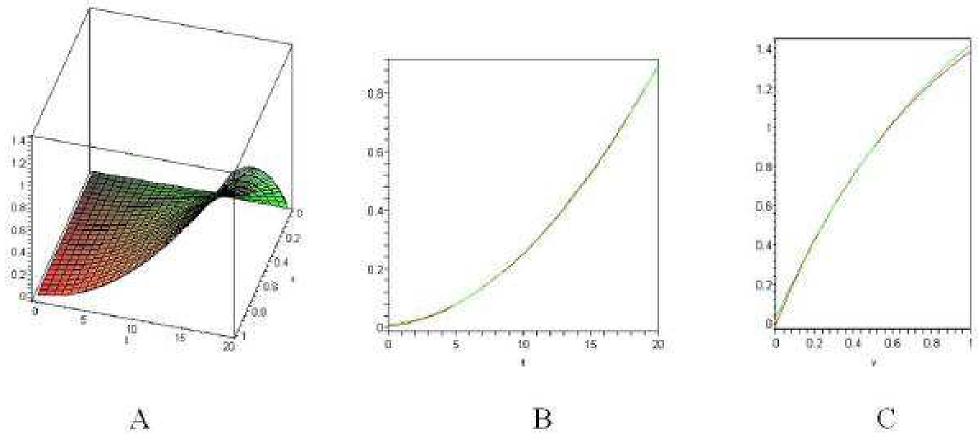


Figure 1: Result of gas dynamics equation using (HPM) method [$0 < x < 1$ and $0 < t < 20$]. ($A \cdots 3D$, $B \cdots x = 0.5$, $C \cdots t = 20$)

6. Application of Variational Iteration Method:

In this section, variational iteration method is developed for solving gas dynamic equation. Consider gas dynamics equation in Eq.12.

To solve Eq. (12) via VIM, one has to find the Langrangian multiplier, which can be identified by substituting Eq. (12) into Eq. (11), upon making it stationary leads to the following:

$$\begin{aligned} 1 - \lambda'|_{\tau=t} &= 0 \\ \lambda|_{\tau=t} &= 0 \\ \lambda''|_{\tau=t} &= 0 \end{aligned} \tag{25}$$

Solving the system of Eq. (25), Yields:

$$\lambda(\tau) = -1, \tag{26}$$

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t -\left\{ \frac{du_n(x,\tau)}{d\tau} + \frac{1}{2}u_n^2(x,\tau)\frac{du_n(x,\tau)}{d\tau} - u_n(x,\tau)(1-u_n(x,\tau)) \right\} dt \tag{27}$$

Now, we assume that the initial approximation has the form:

$$u_0(x,t) = a(1 - e^{-x}) \tag{28}$$

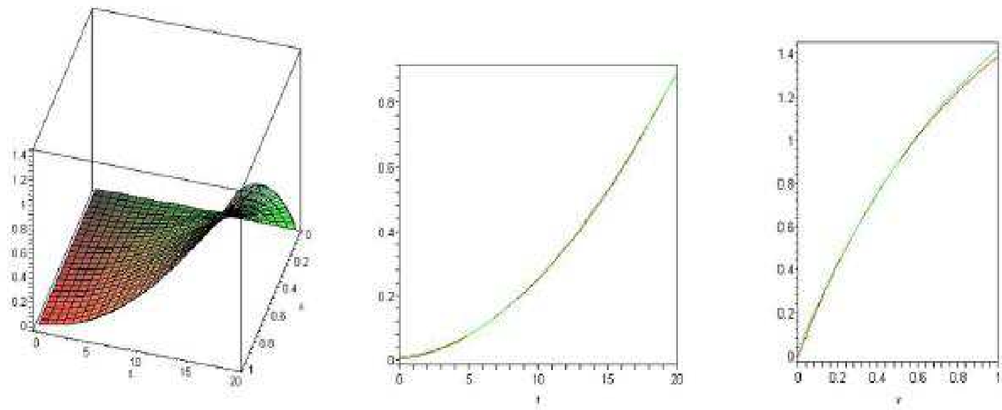
Using the above variational formula (27), we have:

$$u_1(x,t) = u_0(x,t) + \int_0^t -\left\{ \frac{du_0(x,\tau)}{d\tau} + \frac{1}{2}u_0^2(x,\tau)\frac{du_0(x,\tau)}{d\tau} - u_0(x,\tau)(1 - u_0(x,\tau)) \right\} dt \tag{29}$$

Substituting Equation (28) in to Equation (29) and after simplification, we have:

$$u_1(x,t) = \frac{a}{2}(2 - 2e^{-x} - a^2te^{-x} + 2a^2te^{-2x} - a^2te^{-3x} + 2t - 2at + 4ate^{-x} - 2te^{-x} - 2ate^{-2x}) \tag{30}$$

In the same way, we obtain $u_2(x,t)$ that because it is too big we neglect to write that, Fig. 2 shows results of gas dynamics equation by HPM and difference with ADM result, after substituting $a = .01$ and plotting for $0 < x < 1$ and $t > 0$.



A B C

Figure 2: Result of gas dynamic equation using (VIM) method $[0 < x < 1$ and $t > 0]$. $(A \cdots 3D, \quad B \cdots x = 0.5, \quad C \cdots t = 20)$

7. Conclusions

In this paper, variational Iteration and homotopy perturbation methods have been successfully applied to find the solution of gas dynamics equation. Solution of gas dynamic equation shows that the results of proposed methods are in agreement with each other. The homotopy perturbation method which was used to solve gas dynamic equation seems to be very easy. There is less computation needed in comparison with the other methods (numerical and analytical methods). The results obtained here clearly show, that the variational iteration method, is capable of solving gas dynamic equation, with a rapid convergent successive approximation, without any restrictive assumptions or transformations, that may change the physical behavior of the problem.both of VIM and HPM solution are in agreement with decomposition method.

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