

## RUNS BASED ON DISCRETE ORDER STATISTICS

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ABSTRACT. In the present paper, we study runs based on discrete order statistics. Limit results for the total number of such runs and the length of the longest run are derived.

Keywords: Runs, order statistics, limit laws.

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### 1. INTRODUCTION

Runs are commonly used in different applications including reliability, quality control and molecular biology. They are intensively discussed in the statistical literature. For a thorough review on runs we refer to Balakrishnan and Koutras (2002), Fu and Lou (2003), and the references therein.

At the first stage of the research on runs, most studies were conducted for runs obtained from the sequences of integer-valued random variables. Attention has been turned later towards runs based on the sequences of continuous random variables. Among some recent publications in that direction we mention the papers of Eryilmaz (2005, 2007, 2008), Eryilmaz and Fu (2008), Fan et al. (2008), Eryilmaz and Stepanov (2008), and Stepanov (2011a, 2011b), where runs based on spacings and ratios of order statistics of continuous random variables have been investigated.

In the present work, we study runs based on order statistics obtained from discrete distributions. The problem discussed in our paper is new.

Let in the following,  $X_1, \dots, X_n$  be independent and identically distributed random variables taking the positive integers and having a distribution  $F(n) = P\{X \leq n\}$ . Let

$$-\infty = X_{0,n} < X_{1,n} \leq \dots \leq X_{n,n} < X_{n+1,n} = \infty$$

be the order statistics obtained from this sample. We say that the sequence of order statistics  $X_{i,n}, \dots, X_{j,n}$  forms a run of the length  $j - i - 1$  ( $j \geq i + 3$ ,  $i \geq 0$ ,  $j \leq n + 1$ ) if

$$X_{i,n} < X_{i+1,n} = X_{i+2,n} = \dots = X_{j-1,n} < X_{j,n}.$$

If  $j = i + 1$ , i.e. we have two order statistics  $X_{i,n} < X_{i+1,n}$ , we say that the order statistics  $X_{i,n}, X_{i+1,n}$  form a run of the length 1; if  $j = i + 2$  and  $X_{i,n} < X_{i+1,n} < X_{i+2,n}$ , we say that  $X_{i,n}, X_{i+1,n}, X_{i+2,n}$  form two runs, where each of these runs has the length 1.

Let us define auxiliary variables  $\xi_{i,n}$  by

$$\xi_{1,n} = 1, \quad \xi_{n+1,n} = 1,$$

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$$\xi_{i,n} = \begin{cases} 1, & \text{if } X_{i-1,n} < X_{i,n} \quad i = 2, \dots, n, \\ 0, & \text{otherwise.} \end{cases}$$

In terms of these variables, if

$$\xi_{i+1,n} = 1, \xi_{i+2,n} = \dots = \xi_{j-1,n} = 0, \xi_{j,n} = 1,$$

then we have a run of the length  $j - i - 1$ . We also define a variable  $R_n = 1, 2, \dots, n$  by

$$R_n = \sum_{l=1}^n \xi_{l,n}.$$

We refer to  $R_n$  as to the total number of runs based on the order statistics  $X_{1,n}, \dots, X_{n,n}$ . In the case of a single run, when  $\xi_i = 0$  ( $2 \leq i \leq n$ ), we have  $R_n = 1$ . Observe that the total number of runs indicates how many 'strong' (different) order statistics are in a discrete sample. For the length of the longest run we will use the following designation  $L_n$  ( $1 \leq L_n \leq n$ ).

The above notions are illustrated by the following example.

**Example 1.1.** Let us consider a sequence of integer order statistics:

$$\underbrace{2, 2, 2}_3, \underbrace{4}_2, \underbrace{5}_1, \underbrace{8, 8, 8, 8}_4, \underbrace{9, 9, 9}_3, \underbrace{10, 10}_2, \underbrace{14, 14, 14, 14, 14, 14}_6.$$

We have seven run groups, and  $R_{20} = 7$ ,  $L_{20} = 6$ .

The rest of this paper is organized as follows. In Section 2, we present distributional and limit results for  $R_n$  and  $L_n$ . In Subsection 2.1, we discuss these distributional and limit results in the finite case. In Subsection 2.2, we analyze the asymptotic behavior of runs based on discrete order statistics when the support is unbounded. In particular, we will show that for any unbounded support

$$R_n \rightarrow \infty \text{ a.s.}, \quad \frac{R_n}{n} \rightarrow_p 0,$$

$$\frac{R_n}{\sum_{i=1}^{\infty} (1 - (1 - p_i)^n)} \rightarrow_p 1 \quad \text{and} \quad \frac{L_n}{n} \rightarrow_p p_M,$$

where  $p_M = \max\{p_1, p_2, \dots\}$ .

## 2. RESULTS

Let  $p_i = P\{X = i\}$  and  $N$  ( $1 \leq N \leq \infty$ ) be the smallest number such that  $p_i = 0$  ( $i > N$ ).

**2.1. Finite case.** In this subsection we assume that  $N < \infty$ . The probability mass function of  $\xi_{i,n}$  and the expected value of  $R_n$  can be written as

$$P\{\xi_{i,n} = 1\} = \sum_{j=1}^{N-1} P\{j = X_{i-1,n} < X_{i,n}\} =$$

$$\frac{n!}{(n-i+1)!} \sum_{j=1}^{N-1} (1 - F(j))^{n-i+1} \sum_{l=1}^{i-1} \frac{p_j^l F^{i-l-1}(j-1)}{l!(i-l-1)!} =$$

$$\frac{n!}{(n-i+1)!(i-1)!} \sum_{j=1}^{N-1} (1 - F(j))^{n-i+1} [F^{i-1}(j) - F^{i-1}(j-1)]$$

and

$$ER_n = N - F^n(N - 1) - \sum_{j=1}^{N-1} (1 - p_j)^n. \tag{1}$$

It follows from (1) that  $ER_n \rightarrow N$  as  $n \rightarrow \infty$ . Generally speaking, in the finite case we have a Bernoulli scheme, where  $R_n \rightarrow N$  a.s. and  $L_n/n \rightarrow p_M$  a.s. for  $p_M = \max\{p_1, \dots, p_N\}$ .

**2.2. Infinite case.** Let  $N = \infty$ . The probability mass function of  $\xi_{i,n}$  and the joint mass function of  $\xi_{i,n}, \xi_{k,n}$  are given here by

$$P\{\xi_{i,n} = 1\} = \frac{n!}{(n - i + 1)!(i - 1)!} \sum_{j=1}^{\infty} (1 - F(j))^{n-i+1} [F^{i-1}(j) - F^{i-1}(j - 1)] \tag{2}$$

and

$$\begin{aligned} P\{\xi_{i,n} = 1, \xi_{k,n} = 1\} &= \quad (2 \leq i < k \leq n) \\ \sum_{j=1}^{\infty} \sum_{t=j+1}^{\infty} P\{X_{i-1,n} = j, X_{k-1,n} = t\} &= \frac{n!}{(n - k + 1)!} \sum_{j=1}^{\infty} \sum_{l=1}^{i-1} \frac{p_j^l F^{i-l-1}(j - 1)}{l!(i - l - 1)!} \\ &\quad \sum_{t=j+1}^{\infty} (1 - F(j))^{n-k+1} \sum_{m=1}^{k-i} \frac{p_t^m [F(t - 1) - F(j)]^{k-i-m}}{m!(k - i - m)!} = \\ &\quad \frac{n!}{(n - k + 1)!} \sum_{j=1}^{\infty} \frac{F^{i-1}(j) - F^{i-1}(j - 1)}{(i - 1)!} \\ &\quad \sum_{t=j+1}^{\infty} (1 - F(j))^{n-k+1} \frac{[F(t) - F(j)]^{k-i} - [F(t - 1) - F(j)]^{k-i}}{(k - i)!}. \end{aligned} \tag{3}$$

The conditional mass function of  $R_{n+1}$  given  $\{R_n = k\}$  can be written as:

$$\begin{aligned} P\{R_{n+1} = k + 1 | R_n = k\} &= P\{X_{n+1} \neq X_i\} = \quad (i = 1, \dots, n) \\ &\quad \sum_{j=1}^{\infty} p_j (1 - p_j)^n. \end{aligned}$$

Then

$$E\{R_{n+1} | R_n = k\} = k + \sum_{j=1}^{\infty} p_j (1 - p_j)^n.$$

From the properties of the conditional expectation, one can obtain

$$ER_n = \sum_{j=1}^{\infty} [1 - (1 - p_j)^n] = E_n. \tag{4}$$

It should be mentioned that (4) can be derived directly from (2). It follows from (4) that  $ER_n \rightarrow \infty$ .

Let us consider other properties of  $R_n$ . It is easily seen that  $R_{n+1} \geq R_n$  a.s. This observation leads us to the following limit result.

**Theorem 2.1.** *For any discrete  $F$  such that  $N = \infty$ , we have*

$$R_n \rightarrow \infty \quad a.s.$$

*Proof.* Since  $R_n$  is monotone, it converges almost surely either to a finite limit or to infinity. Thus, it is sufficient to show that  $R_n$  converges in probability to infinity.

The event  $\{R_n = k\}$  means that the variables  $X_1, \dots, X_n$  take exactly  $k$  different positive integer values, i.e.

$$P\{R_n \leq m\} = \sum_{k=1}^m P\{R_n = k\} = \sum_{k=1}^m \frac{n!}{(n-k+1)!} \sum_{i_1=1}^{\infty} p_{i_1} \sum_{i_2=i_1+1}^{\infty} p_{i_2} \cdots \sum_{i_k=i_{k-1}+1}^{\infty} p_{i_k} (p_{i_1} + p_{i_2} + \dots + p_{i_k})^{n-k} \leq \sum_{k=1}^m \frac{n!}{(n-k+1)!} p_{sup,k}^{n-k} \rightarrow 0, \tag{5}$$

where  $p_{sup,k} = \sup_{i_1, i_2, \dots, i_k} (p_{i_1} + p_{i_2} + \dots + p_{i_k}) < 1$ . □

In light of the previous result it is interesting to trace the limit behavior of the ratio  $\frac{ER_n}{n}$ . It is easily seen that

$$\frac{ER_n}{n} > \frac{ER_{n+1}}{n+1} \quad (n \geq 1).$$

The following asymptotic result holds.

**Theorem 2.2.** *For any discrete  $F$  such that  $N = \infty$ , we have*

$$\frac{ER_n}{n} \rightarrow 0 \quad \text{and} \quad \frac{R_n}{n} \rightarrow_p 0.$$

*Proof.* Indeed,

$$\frac{ER_n}{n} = o_n(1) + \frac{1}{n} \sum_{j=J}^{\infty} [1 - (1 - p_j)^n].$$

By the inequality

$$n(a-b)b^{n-1} < a^n - b^n < n(a-b)a^{n-1} \quad (a > b > 0), \tag{6}$$

we get

$$\frac{ER_n}{n} < o_n(1) + 1 - F(J-1).$$

Choosing  $J$  as large one can make the right-hand side of the last inequality smaller than any  $\varepsilon > 0$ . The first statement of Theorem 2.2 readily follows.

The second statement of Theorem 2.2 follows from Chebyshev's inequality and the first statement. □

We see that the sequence  $R_n$  converges to infinity, and  $R_n/n$  converges to zero. Properly normalized  $R_n$  can tend to one.

**Theorem 2.3.** *For any discrete  $F$  such that  $N = \infty$ , we have*

$$\frac{R_n}{E_n} \rightarrow_p 1$$

*Proof.* By Chebyshev's inequality,

$$P\{|R_n - ER_n| > \varepsilon ER_n\} \leq \frac{Var R_n}{\varepsilon^2 E_n^2}.$$

Let us estimate  $Var R_n$ . We have

$$Var R_n = 2 \sum_{i=2}^{n-1} \sum_{k=i+1}^n E \xi_{i,n} \xi_{k,n} + 3E_n - E_n^2.$$

It follows from (3) that

$$\sum_{i=2}^{n-1} \sum_{k=i+1}^n E\xi_{i,n}\xi_{k,n} = 1 - E_n + \sum_{j=1}^{\infty} \sum_{l=j+1}^{\infty} (1 - (1 - p_j)^n - (1 - p_l)^n + (1 - p_j - p_l)^n)$$

and

$$\begin{aligned} VarR_n &= 2 + E_n - E_n^2 + 2 \sum_{j=1}^{\infty} \sum_{l=j+1}^{\infty} (1 - (1 - p_j)^n - (1 - p_l)^n + (1 - p_j - p_l)^n) = \\ &= 2 + \sum_{j=1}^{\infty} ((1 - p_j)^n - (1 - 2p_j)^n) + \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} ((1 - p_j - p_l)^n - (1 - p_j - p_l + p_j p_l)^n) \leq \\ &= 2 + \sum_{j=1}^{\infty} ((1 - p_j)^n - (1 - 2p_j)^n). \end{aligned}$$

Then

$$\frac{VarR_n}{E_n^2} = \frac{\sum_{j=1}^{\infty} ((1 - p_j)^n - (1 - 2p_j)^n)}{\sum_{j=1}^{\infty} (1 - (1 - p_j)^n) E_n}.$$

By (5),

$$\frac{VarR_n}{E_n^2} < \frac{n \sum_{j=1}^{\infty} p_j (1 - p_j)^{n-1}}{\sum_{j=1}^{\infty} (1 - (1 - p_j)^n) E_n} < \frac{1}{E_n} \rightarrow 0.$$

□

In the above theorem we found that  $\frac{R_n}{E_n} \rightarrow_p 1$ . What one can say about the limiting behavior of  $E_n$ ? Obviously,  $E_n \leq n$ . In the following result we find the lower bound for  $E_n$ .

**Proposition 2.1.** *For any  $n \geq 1$*

$$E_n \geq \sum_{j=1}^n \frac{(1 - F(1))^{j-1}}{j}.$$

*Proof.* Observe that

$$E_n = \sum_{i=1}^{\infty} p_i + \sum_{i=1}^{\infty} p_i(1 - p_i) + \dots + \sum_{i=1}^{\infty} p_i(1 - p_i)^{n-1}.$$

However,

$$\sum_{i=1}^{\infty} p_i(1 - p_i)^j \geq \sum_{i=1}^{\infty} (F(i) - F(i - 1))(1 - F(i))^j \geq \int_{F(1)}^1 (1 - x)^j dx = \frac{(1 - F(1))^j}{j + 1}.$$

□

It is reasonable to assume that for some distributions the sum  $E_n$  can tend to finite limit as  $n \rightarrow \infty$  and for other distributions it can tend to infinity. We focus now on these matters.

In Eisenberg *et al.* (1993) and Brands *et al.* (1994) the asymptotic behavior of the number of maxima in the discrete case is discussed. The following limit

$$\lim_{i \rightarrow \infty} \frac{p_{i+1}}{p_i} = p \in [0, 1] \tag{7}$$

is used for distribution tail classification.

**Theorem 2.4.** 1) Let the limit in (7) exist and  $p \in [0, 1)$ , then  $E_n \rightarrow E < \infty$ .

1) Let the limit in (7) exist and  $p = 1$ , then  $E_n \rightarrow \infty$ .

*Proof.* Observe first that if the limit in (7) exist and  $p \in [0, 1]$ , then for any  $n \geq 1$

$$\lim_{i \rightarrow \infty} \frac{1 - (1 - p_{i+1})^n}{1 - (1 - p_i)^n} = p \in [0, 1].$$

2) Let us choose  $\varepsilon > 0$  and  $I$  such that

$$\frac{1 - (1 - p_{i+1})^n}{1 - (1 - p_i)^n} < p + \varepsilon < 1 \quad (i > I).$$

Then

$$\sum_{i=I+1}^{\infty} [1 - (1 - p_i)^n] < \frac{1 - (1 - p_{I+1})^n}{1 - p - \varepsilon}.$$

Obviously,  $E_n$  is bounded and increasing, which implies the first statement of Theorem 2.4.

2) Choose  $\varepsilon > 0$  and  $I$  such that

$$\frac{1 - (1 - p_{i+1})^n}{1 - (1 - p_i)^n} > 1 - \varepsilon \quad (i > I).$$

Then

$$\sum_{i=I+1}^{\infty} [1 - (1 - p_i)^n] > \frac{1 - (1 - p_{I+1})^n}{\varepsilon}.$$

Taking  $\varepsilon$  as small we can do the right-hand side of the last inequality as great. This implies the truth of the second statement of Theorem 2.4.  $\square$

The last result can be commented in the following way. When  $p \in [0, 1)$ , i.e. the distribution tail is not heavy, the order statistics are concentrated in the "beginning" of the tail, each positive integer is occupied by many sample observations and, consequently, we have a finite number of runs. When the tail is heavy, i.e.  $p = 1$ , the sample disperse is great. We have many "strong" (isolated) order statistics, which give us the infinite number of runs.

In the conclusion, we present a limit result for the longest run.

**Proposition 2.2.** For any discrete  $F$  such that  $N = \infty$ , we have

$$\frac{L_n}{n} \rightarrow_p p_M,$$

where  $p_M = \max\{p_1, p_2 \dots\}$ .

*Proof.* We apply the same argument that was already used in the end of Subsection 2.1. Choose  $K$  rather large such that  $K > J \geq 1$ , where  $p_M = p_J$ , and  $p_M > \tilde{p}_K = p_K + p_{K+1} + \dots$ . Now we can suppose that we have a finite case, where  $X = 1, \dots, K - 1, K$  with  $p_1, \dots, p_{K-1}, \tilde{p}_K$ . The argument that was once used for a Bernoulli scheme can be applied now again.  $\square$

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