

ENERGY PRESERVING METHODS FOR VOLTERRA LATTICE EQUATION

B. KARASÖZEN¹, Ö. ERDEM², §

ABSTRACT. We investigate linear energy preserving methods for the Volterra lattice equation as non-canonical Hamiltonian system. The averaged vector field method was applied to the Volterra lattice equation in bi-Hamiltonian form with quadratic and cubic Poisson brackets. Numerical results confirm the excellent long time preservation of the Hamiltonians and the polynomial integrals.

Keywords: Energy preserving integrators, Runge-Kutta methods, Bi-Hamiltonian systems, Poisson structure.

AMS Subject Classification: 65P10, 65L06, 37K10

1. INTRODUCTION

Traditionally, numerical integration of ordinary and partial differential equations concerns itself with the construction of numerical methods to minimize the global error, to ensure the numerical stability and to control the time step. In the last two decades, the concept of the design of numerical integrators shifted to preserve the intrinsic geometric properties, like to preserve the symplectic structure, symmetries, conserved quantities, the volume and phase space structure. These methods are known as geometric or structure preserving integrators. Most of the studies are concentrated in construction of symplectic and multisymplectic integrators for Hamiltonian ordinary and partial differential equations [12, 18, 24]. It is known that all Runge-Kutta methods preserve the linear integrals of the associated differential equations, and that only symplectic Runge-Kutta methods preserve the quadratic integrals. No Runge-Kutta method preserves higher order polynomial or nonlinear integrals. The symplectic structure and Hamiltonians can not preserved simultaneously, unless the integrator produces exact solutions (see [[12], page 379]).

Integrators that preserve the the energy or integrals have been considered since several decades. The development of the energy preserving methods started with the work of Courant, Friedrichs and Lewy [6]. They again gain attraction in recent years and several integral preserving methods for ordinary differential equations using discrete gradients and discrete variational derivatives [2, 3, 11, 21, 22, 23]. Discrete gradient methods were also applied to nonlinear evolutionary partial differential equations [7, 10]. There is also class of integrators which preserve exactly the energy of polynomial Hamiltonian systems

¹ Department of Mathematics and Institute of Applied Mathematics, Middle East Technical University, 06800 Ankara, Turkey,
e-mail: bulent@metu.edu.tr

² Max-Planck Institute, Mathematics in the Sciences, 04103 Leipzig, Germany.

§ Manuscript received 11 September 2011.

TWMS Journal of Applied and Engineering Mathematics Vol.1 No.2 © Işık University, Department of Mathematics 2011; all rights reserved.

[14, 15], which are known as "extended collocation methods" and "Hamiltonian Boundary Value Methods".

All methods mentioned above are non-linear, so that the numerical solution is not invariant with respect to the time transformation. Recently the so called average vector field method (AVF) attracted a lot of interest [19, 22] which is extension of the implicit mid-point rule. Higher order AVF methods are constructed by using the Gaussian quadrature and they are interpreted as Runge-Kutta method with continuous stages. The AVF methods of arbitrarily higher order were developed and analyzed for canonical and non-canonical Hamiltonian systems in [13, 5]. A relation between the energy preservation and symplecticity is established by so called B-series. A B-series for a system of ordinary differential equations is formal power series of the step size and elementary differential equations [2]. The discrete gradient methods for the integration of Hamiltonian systems can not be expanded in a B-series. Energy preserving B-series methods for canonical Hamiltonian systems were characterized in [1, 9]. The AVF method and its high order extension for canonical and non-canonical Hamiltonian systems [13, 5] are B-series methods. Using the B-series, it can be shown that the AVF method for canonical and non-canonical Hamiltonian systems is conjugate to symplectic or Poisson integrators [13, 5]. Many partial differential equations like the Korteweg de Vries equation, non-linear Schrödinger equation, sine-Gordon equation can be recast as non-canonical Hamiltonian or Poisson systems. In the last two decades various symplectic and multi-symplectic integrators were successfully applied to Hamiltonian partial differential equations by preserving the integrals very accurately in long term. Application of the AVF method for various nonlinear evolutionary partial differential equations was given in [4].

For non-canonical Hamiltonian systems, i.e. for Poisson systems with the non-constant skew symmetric structure matrix, general structure preserving integrators are not available [17]. For example the implicit-mid point method which preserves the symplectic structure for canonical Hamiltonian systems is not a Poisson integrator. There exist only some geometric integrators which preserve the Poisson structure for some differential equations. For example the symplectic Euler method and partitioned Lobatto IIIA-IIIIB methods preserve the Poisson structure of the Volterra lattice equation [8] and the integrable discretization of nonlinear Schrödinger equation [25]. In this paper we will apply the AVF method for non-canonical Hamiltonian systems in [5] to the Volterra lattice in bi-Hamiltonian form. Numerical results show the excellent energy preserving properties of the second and fourth order AVF method and fourth order energy preserving trapezoidal rule [14, 15] for the Volterra lattice equation with quadratic and cubic Poisson brackets.

The outline of the paper is as follows. In the next Section various energy preserving methods and their properties are given. In Section 3, we give the application of the AVF method to the Volterra lattice equation. Section 4 consists of numerical results with one and two stage AVF method and with the fourth order trapezoidal rule. The paper ends with some conclusions and suggestions for a future work.

2. ENERGY PRESERVING INTEGRATORS

We consider the system of ordinary differential equations

$$y' = f(y), \quad y(0) = y_0 \in \mathbb{R}^n \quad (1)$$

with $f(y) = J(y)\nabla H$, where $J(y)$ is a $n \times n$ skew-symmetric structure matrix and H is an invariant, for example, the Hamiltonian or energy of the system. The energy preserving

discrete gradient methods rely on appropriate approximations of $\nabla H(y)$ and $J(y)$.

Symmetric discrete gradient methods for Poisson systems are given as [21, 22]

$$\frac{y^{n+1} - y^n}{h} = J \left(\frac{y^n + y^{n+1}}{2} \right) \cdot \bar{\nabla} H_s(y^n, y^{n+1}) \quad (2)$$

$$\bar{\nabla} H_s(y^n, y^{n+1}) := \frac{1}{2} (\bar{\nabla} H(y^n, y^{n+1}) + \bar{\nabla} H(y^{n+1}, y^n))$$

with the coordinate increments

$$\bar{\nabla} H_s(y^n, y^{n+1}) := \begin{pmatrix} \frac{H(y_1^{n+1}, y_2^n, \dots, y_i^n, \dots, y_m^n) - H(y_1^n, y_2^n, \dots, y_i^n, \dots, y_m^n)}{y_1^{n+1} - y_1^n} \\ \dots \\ \frac{H(y_1^{n+1}, y_2^{n+1}, \dots, y_i^{n+1}, \dots, y_m^n) - H(y_1^{n+1}, y_2^{n+1}, \dots, y_i^n, \dots, y_m^n)}{y_i^{n+1} - y_i^n} \\ \dots \\ \frac{H(y_1^{n+1}, y_2^{n+1}, \dots, y_i^{n+1}, \dots, y_m^{n+1}) - H(y_1^{n+1}, y_2^{n+1}, \dots, y_i^{n+1}, \dots, y_m^n)}{y_m^{n+1} - y_m^n} \end{pmatrix}.$$

Another class of energy preserving methods systems for polynomial Hamiltonians are s -stage the trapezoidal methods defined as [14, 15]

$$y^{n+1} = y^n + h \sum_{i=1}^s b_i J(K_i) \nabla H(K_i). \quad (3)$$

For $s = 1$ it reduces to the implicit mid-point rule. The fourth order 3-stage method is given as [15]

$$\begin{aligned} y^{n+1} &= y^n \\ &+ \frac{h}{6} \left(J(y^n) \nabla H(y^n) + 4J \left(\frac{y^n + y^{n+1}}{2} \right) \nabla H \left(\frac{y^n + y^{n+1}}{2} \right) + J(y^{n+1}) \nabla H(y^{n+1}) \right). \end{aligned}$$

These methods are typically non-linear, so that the numerical solutions are not invariant with respect to the linear transformations, and in general can not be expanded as B-series, i.e. formal series for the system (1) in powers of the step size h and elementary differentials. The averaged vector field method for (1)

$$y^{n+1} = y^n + h \int_0^1 f(y^n + \tau(y^{n+1} - y^n)) d\tau, \quad n = 0, 1, \dots \quad (4)$$

is an extension of Runge-Kutta methods and admits B-series expansion. It is invariant with respect to linear transformations and it reduces to the discrete gradient method for canonical Hamiltonian systems.

Linear energy preserving average vector field integrators for Poisson systems are constructed [5]

$$y^{n+1} = y^n + hJ \left(\frac{y^n + y^{n+1}}{2} \right) \int_0^1 \nabla H(y^n + \tau(y^{n+1} - y^n)) d\tau \quad (5)$$

as an extension of averaged vector field integrators for canonical Hamiltonian systems [13]

$$y^{n+1} = y^n + hJ \int_0^1 \nabla H(y^n + \tau(y^{n+1} - y^n)) d\tau. \quad (6)$$

The AVF methods in [13, 5] are constructed by using the Gaussain quadrature similar to the symplectic Gauss-Legendre Runge-Kutta methods. They are interpreted as continuous-stage Runge-Kutta methods [13]

$$Y_\tau = y^n + h \int_0^1 a_{\tau,\sigma} f(Y_\sigma) d\sigma, \quad y^{n+1} = y^n + h \int_0^1 b_\sigma f(Y_\sigma) d\sigma, \tag{7}$$

for $\tau \in (0, 1)$. Here, $Y_\tau \approx y(t_n + c_\tau h)$, where $c_\tau = \int_0^1 a_{\tau,\sigma} d\sigma$. The integral stages Y_σ correspond to the values of the polynomial $y(t_n + \tau h)$ and the coefficients are given by

$$c_\tau = \tau, \quad a_{\tau,\sigma} = \sum_{i=1}^s \frac{1}{b_i} \int_0^\tau l_i(\alpha) d\alpha l_i(\sigma), \quad b_\sigma = 1. \tag{8}$$

with the basis polynomials of Lagrange interpolation at the collocation points c_1, \dots, c_s

$$l_i(\tau) = \prod_{j=1, j \neq i}^s \frac{\tau - c_j}{c_i - c_j}, \quad b_i = \int_0^1 l_i(\tau) d\tau.$$

Higher order linear integral preserving AVF methods, constructed using Gaussian quadrature for canonical and non-canonical Hamiltonian systems have even order $2s$ [13, 5].

For example, the AVF method (5-7) with $c_1 = 1/2$ as one stage Gaussian collocation method is an extension of the implicit mid-point rule. The two stage, fourth order Gaussian AVF integrator with

$$c_{1,2} = 1/2 \mp \sqrt{3}/6, \quad l_1(\tau) = (\tau - c_2)/(c_1 - c_2), \quad l_2(\tau) = (\tau - c_1)/(c_2 - c_1)$$

is given as [5]

$$\begin{aligned} Y_1 &= y^n + h \int_0^1 \left(\frac{1}{2} l_1(\sigma) B(Y_1) + \left(\frac{1}{2} - \frac{\sqrt{3}}{3} \right) l_2(\sigma) B(Y_2) \right) \nabla H(Y_\sigma) d\sigma, \\ Y_2 &= y^n + h \int_0^1 \left(\left(\frac{1}{2} + \frac{\sqrt{3}}{3} \right) l_1(\sigma) B(Y_1) + \frac{1}{2} l_2(\sigma) B(Y_2) \right) \nabla H(Y_\sigma) d\sigma, \\ y^{n+1} &= y^n + h \int_0^1 (l_1(\sigma) B(Y_1) + l_2(\sigma) B(Y_2)) \nabla H(Y_\sigma) d\sigma. \end{aligned} \tag{9}$$

Gaussian AVF methods are time-symmetric and conjugate to symplectic integrators. A numerical one-step method $y^{n+1} = \Phi_h(y^n)$ time-symmetric such that $y^n = \Phi_{-h}(y^{n+1})$ when applied to reversible systems (e.g. Hamiltonian systems). AVF methods are time-symmetric [13]. B-series integrators like the AVF method, which preserve exactly conserve the energy for Hamiltonian systems are not symplectic. But the AVF methods of order $2s$ are conjugate-symplectic up to order $2s + 2$, i.e. there exist a change of coordinates $\Psi(y) = y + \mathcal{O}(h^{2s})$, such that $\Psi_h^{-1} \circ \phi_h \circ \Psi_h$ is symplectic integrator [13].

3. VOLTERRA LATTICE EQUATIONS

The m -dimensional Volterra lattice

$$\dot{y}_i = y_i(y_{i+1} - y_{i-1}), \quad i = 1, \dots, m \tag{10}$$

with periodic boundary conditions $y_{m+i} = y_i$, represents an integrable discretization of the Korteweg de Vries (KdV) equation and of inviscid Burger's equation and describes many phenomena such as the vibrations of particles on lattices, waves in plasmas and the

evolution of populations in a hierarchical system of competing species [8]. The Volterra lattice equations (10) have a bi-Hamiltonian structure [8]

$$\dot{y} = J_0(y)\nabla H_1 = J_1(y)\nabla H_0 \quad (11)$$

with respect to the quadratic and the cubic Poisson brackets

$$\{y_i, y_{i+1}\}_0 = y_i y_{i+1}, \quad \{y_i, y_{i+1}\}_1 = y_i y_{i+1} (y_i + y_{i+1}), \quad \{y_i, y_{i+2}\}_1 = y_i y_{i+1} y_{i+2}. \quad (12)$$

The corresponding Hamiltonians are

$$H_1 = \sum_{i=1}^m y_i, \quad H_0 = \frac{1}{2} \sum_{i=1}^m \log(y_i). \quad (13)$$

The structure matrix $J_0(x)$ corresponding to the quadratic is a cyclic tridiagonal matrix.

$$J_0(y) = \begin{pmatrix} 0 & y_1 y_2 & \dots & \dots & -y_1 y_m \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \dots & -y_{i-1} y_i & 0 & y_i y_{i+1} & \dots \\ \vdots & \vdots & \dots & \vdots & \vdots \\ y_1 y_m & \dots & \dots & -y_{m-1} y_m & 0 \end{pmatrix}.$$

Similarly the structure matrix $J_1(x)$ corresponding to the cubic Poisson bracket is a cyclic pentadiagonal matrix.

If the Poisson brackets are compatible, i.e. the sum of $\{\cdot, \cdot\}_0 + \{\cdot, \cdot\}_1$ is again a Poisson bracket, then the bi-Hamiltonian system (1) has a finite number of functionally independent first integrals $I_i, i = 1, \dots, m$ in involution i.e. $\{I_i, I_j\}_k = 0$ for $k = 0, 1$ and $i \neq j$, i.e. the Volterra lattice is completely integrable with respect to both brackets.

Additional conserved quantities of the Volterra lattice (10) are the quadratic and cubic integrals [8].

$$I_q = \sum_{i=1}^m \frac{1}{2} y_i^2 + y_i y_{i+1}, \quad I_c = \sum_{i=1}^m \frac{1}{3} y_i^3 + y_i y_{i+1} (y_i + y_{i+1} + y_{i+2}). \quad (14)$$

In discretized form, the Hamiltonian H_1, I_q, I_c correspond to the mass, momentum and energy integrals of the KdV equation.

A function $P(y)$ is called a distinguished or Casimir function, if $\{P, F\} = 0$ for any function $F(y)$. The Hamiltonian H_0 is a Casimir with respect to the Poisson bracket $\{\cdot, \cdot\}_0$, i.e. $\{H_0, F\}_0 = 0$. It was shown in [5] that the AVF integrators based on the Gaussian quadrature exactly preserves the quadratic Casimirs $P(y) = y^T A y$ with a constant symmetric matrix A .

When the symmetrized discrete gradient method (2), the AVF method (6) and one stage trapezoidal method is applied to the Volterra lattice (11) with with quadratic brackets, i.e. with $J_0(y)$ and $H_1(y)$, the implicit mid-point method is obtained.

Nonlinear implicit equations resulting by from application of the AVF method (6) to the Volterra lattice (11)

$$y^{n+1} - y^n - hF(y^{n+1}, y^n) = 0$$

are solved usually by Newton's method

$$(I - \frac{h}{2}F'(y_k^n))(y_{k+1}^{n+1} - y_k^{n+1}) = -hF(y_k^n), \quad k = 0, 1, \dots$$

where k denotes the iteration index, and $F'(y_k^n)$ the Jacobian evaluated at y_k^n . As starting value in the Newton iteration $y_0^{n+1} = y^n$ is taken.

Application of two stage linear energy preserving AVF method (10) to the Volterra lattice (11) gives

$$\begin{aligned} Y_1 &= y_0 + h(a_{11}B(Y_1) + a_{12}B(Y_2)) \\ Y_2 &= y_0 + h(a_{21}B(Y_1) + a_{22}B(Y_2)) \\ a_{11} &= \frac{1}{2} \left(\frac{1}{2} \frac{1}{c_1 - c_2} - \frac{c_2}{c_1 - c_2} \right), & a_{12} &= \left(\frac{1}{2} - \frac{\sqrt{3}}{2} \right) \left(\frac{1}{2} \frac{1}{c_2 - c_1} - \frac{c_1}{c_2 - c_1} \right), \\ a_{21} &= \left(\frac{1}{2} + \frac{\sqrt{3}}{2} \right) \left(\frac{1}{2} \frac{1}{c_1 - c_2} - \frac{c_2}{c_1 - c_2} \right), & a_{22} &= \frac{1}{2} \left(\frac{1}{2} \frac{1}{c_2 - c_1} - \frac{c_1}{c_2 - c_1} \right). \end{aligned}$$

Application of the Newton method results in

$$\begin{aligned} &\begin{pmatrix} I - ha_{11}F'(Y_{1,k}) & -ha_{12}F'(Y_{2,k}) \\ -ha_{21}F'(Y_{1,k}) & I - ha_{22}F'(Y_{2,k}) \end{pmatrix} \begin{pmatrix} Y_{1,k+1} - Y_{1,k} \\ Y_{2,k+1} - Y_{2,k} \end{pmatrix} \\ &= -h \begin{pmatrix} a_{11}B(Y_{1,k}) + a_{12}B(Y_{2,k}) \\ a_{21}B(Y_{1,k}) + a_{22}B(Y_{2,k}) \end{pmatrix}, \quad k = 0, 1, \dots \end{aligned}$$

with the starting values $Y_{1,0} = Y_{2,0} = y^n$. Compared with the one stage method, a linear system with a two by two block matrix have to solved at each iteration step, which shows that higher order Gaussian AVF methods are computationally expensive.

4. NUMERICAL RESULTS

For the periodic Volterra lattice we have used the following initial condition

$$y(x_i) = 1 + (1/2m^2) \operatorname{sech}^2(x_i), \quad x_i = -1 + (i - 1)(1/2m^2), \quad i = 1, \dots, m$$

All computations are done with a constant time step $h = 0.1$ for a Volterra lattice of dimension $m = 40$. In the case of quadratic brackets, the unknowns are in quadratic form. Therefore one Newton iteration was sufficient to solve for the unknowns. In the case of the cubic Poisson bracket, several Newton iterations are needed to solve the implicit equations accurately. Also the integral terms will be undefined when y^n is used as a starting value in the Newton iteration, therefore the limit of the integral is used for the first Newton step.

Long term preservation of the linear and logarithmic Hamiltonians (14), quadratic and cubic integrals (15) computed with the one stage linear energy preserving integrator (6) are shown in In Figure 1 & 2. In Table 1, the errors in the mean square root norm are given $\sqrt{\sum_{i=1}^N (I^i - I_0)^2}/N$, where I^i denote the computed Hamiltonians or first integrals at time step t_i and N is the number of time steps. Table 1 shows that the Hamiltonians and integrals are preserved within the machine accuracy for the quadratic brackets. The conserved quantities for the cubic Poisson bracket are preserved less accurately than for the quadratic brackets. But there is no drift in long term integration using both brackets in Figure 1 & 2.

In Table 2, in Figure & 4, numerical results with two stage AVF method (10) and fourth order trapezoidal method (4) are given for the quadratic bracket. These results shows the

excellent energy preservation of higher order AVF methods.

The logarithmic Hamiltonian H_0 as a Casimir for the Volterra lattice with quadratic bracket is also well preserved in Figure 1 and 3 in long term integration. Therefore the one and two stage AVF integrators (6) and (10) are conjugate to a Poisson integrator.

A comparison of results obtained with the symplectic Euler and Lobatto IIIA-B in [8] shows, that the AVF method preserve the Casimir H_0 and the quadratic and cubic integrals I_q, I_c more accurately.

TABLE 1. Average errors of the Hamiltonians and first integrals

Δt	quadratic Poisson structure				cubic Poisson structure			
	H_1	H_0	I_q	I_c	H_1	H_0	I_q	I_c
1/5	1.7e-16	8.7e-17	5.3e-16	2.4e-15	2.6e-8	1.3e-8	7.5e-8	2.6e-7
1/10	2.2e-16	1.1e-16	6.7e-16	2.2e-15	1.9e-9	9.5e-10	5.7e-9	1.9e-8
1/20	1.5e-16	6.9e-17	4.4e-16	1.4e-15	1.3e-8	6.7e-9	3.9e-8	1.3e-7

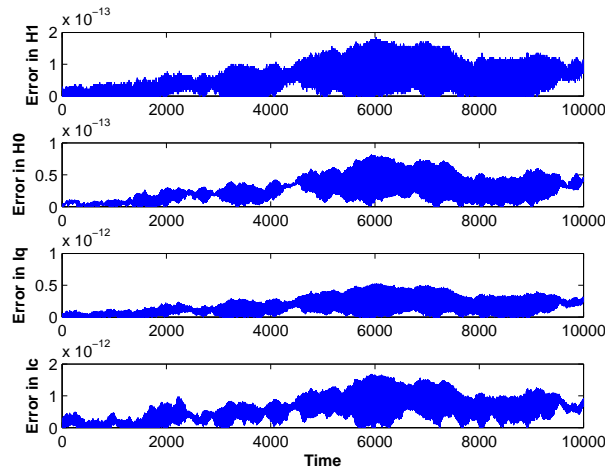


FIGURE 1. One stage AVF with quadratic Poisson brackets

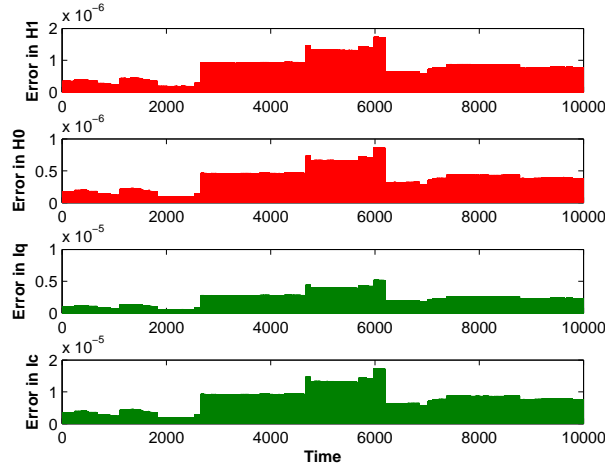


FIGURE 2. One stage AVF method with cubic Poisson brackets

	two stage trapezoidal rule				Two stage AVF method			
Δt	H_1	H_0	I_q	I_c	H_1	H_0	I_q	I_c
1/5	1.3e-16	7.1e-17	8.5e-16	6.7e-15	2.1e-20	2.1e-16	6.0e-16	3.1e-15
1/10	1.6e-16	6.4e-17	4.6e-16	1.6e-15	2.0e-20	4.0e-16	1.2e-15	4.4e-15
1/20	9.4e-17	4.4e-17	2.6e-16	8.8e-16	2.3e-21	9.3e-17	2.6e-16	9.3e-16

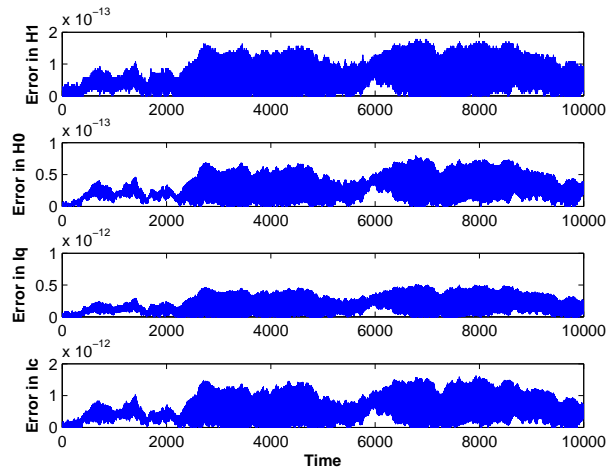


FIGURE 3. Two stage trapezoidal method

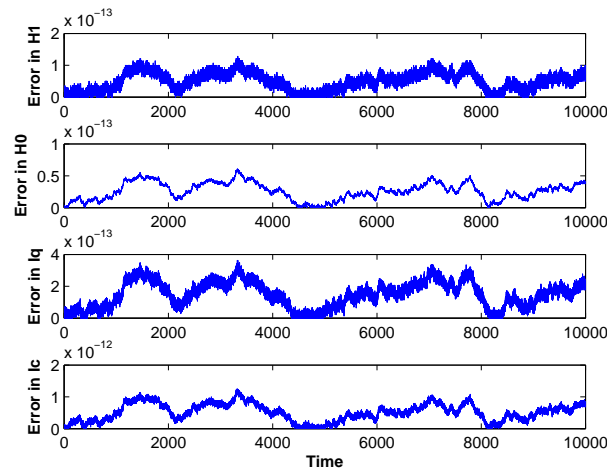


FIGURE 4. Two stage AVF method

5. CONCLUSIONS

All computations show that the Hamiltonians, quadratic and cubic integrals are preserved with a high accuracy, for the quadratic Poisson bracket better than for the cubic Poisson bracket. It is known that the Hamiltonians and the Poisson structure can not be preserved by any integrator at the same time. But, we can conclude that the AVF for the Volterra lattice equation are conjugate to a Poisson integrator, because there is no drift in long term preservation of the Hamiltonians.

One of the problem of the AVF method is the computation of the integrals. For polynomial vector fields the integrals can be computed exactly at the begin of integration and the complexity of the AVF method is comparable with the implicit Runge–Kutta methods. For the Volterra lattice, for the second Hamiltonian formulation, the logarithmic terms were integrated exactly too. Because the energy preserving methods are implicit as symplectic and multisymplectic integrators, the resulting nonlinear equations must be solved to within round-off error, to preserve symplecticity or the energy. This limits the applicability of these methods to large scale systems. In [20] a Newton-Taylor iteration was proposed to solve the implicit equations of symplectic Runge-Kutta methods, using the Jacobian of the vector field. This method preserve the symplectic structure at large time steps using only matrix-vector multiplications in each iteration. Similarly for energy preserving method, linearly implicit are used which require the solution of precisely one linear system of equations in each time step [7] and makes the energy preserving methods more efficient.

Another interesting lattice equation is the Ablowitz-Ladik integrable discretization of non-linear Schrödinger (IDNLS) equation [25] which has quadratic Poisson bracket as the Volterra lattice and the Hamiltonian contains quadratic and logarithmic terms. We plan to apply the AVF method to IDNLS equation.

Acknowledgement This paper was completed when the first author was visiting Fulbright scholar at Rice University. He is very grateful to Fulbright for the support and to the Department of Computational and Applied Mathematics for the hospitality.

REFERENCES

- [1] Chartier, P., Faou, E. and Murua, A., (2006), An algebraic approach to invariant preserving integrators: the case of quadratic and Hamiltonian invariants, *Numer. Math.*, 103, 575–590.
- [2] Celledoni, E., McLachlan, R. I., Owren, B. and Quispel, G. R. W., (2010), Structure of B-series for Some Classes of Geometric Integrators, *Found. Comput. Math.*, 10, 673–693.
- [3] Celledoni, E., McLachlan, R. I., McLaren, D. I., Owren, B., Quispel, G. R. W. and Wright, W.M., (2009), Energy-preserving Runge-Kutta methods, *ESAIM: Mathematical Modelling and Numerical Analysis* 43, 645-649.
- [4] Celledoni, E., Grimm, V., McLachlan, R. I., McLaren, D. I., O’Neale, D. R. J., Owren, B., and Quispel, G. R. W., (2009), Preserving energy resp. dissipation in numerical pdes, using the Average Vector Field method. Technical Report 7/2009, Norwegian University of Science and Technology Trondheim, Norway.
- [5] Cohen, D. and Hairer, E., (2011), Linear energy-preserving integrators for Poisson systems, *BIT*, 51, 91-101.
- [6] Courant, R., Friedrichs, K., and Lewy, H., (1967), On the partial difference equations of mathematical physics. *IBM J.*, 11:215-234.
- [7] Dahlby, M. and Owren, B., (2010), A general framework for deriving integral preserving numerical methods for PDEs, *SIAM J. Sci. Comput.*, 33, 2318-2340.
- [8] Ergenç, T. and Karasözen, B., (2006), Poisson integrators for Volterra lattice equations, *Applied Numerical Mathematics*, 56, 879-887.
- [9] Faou, E., Hairer, E., and Pham, T. L., (2004), Energy conservation with non-symplectic methods: examples and counter-examples, *BIT*, 44, 699-709.
- [10] Furihata, D. and Matsuo, T., (2010), *Discrete Variational Derivative Method: A Structure- Preserving Numerical Method for Partial Differential Equations*, Chapman and Hall.
- [11] Gonzalez, O., (1996), Time integration and discrete Hamiltonian systems, *J. Nonlinear Sci.*, 6, 449-467
- [12] Hairer, E., Lubich, C., and Wanner, G., (2006), *Geometric Numerical Integration. Structure-Preserving Algorithms for Ordinary Differential Equations*, Springer Series in Computational Mathematics 31, Springer-Verlag, Berlin, 2nd edition.
- [13] Hairer, E., (2010), Energy-preserving variant of collocation methods, *J. Numer. Anal. Ind. Appl. Math.*, 5, 73-84.
- [14] Iavernaro, F. and Pace, B., (2007), s-stage trapezoidal methods for the conservation of Hamiltonian functions of polynomial type, *AIP Conf. Proc.*, 936, 603–606.
- [15] Iavernaro F. and Trigiante, D., (2009), High-order symmetric schemes for the energy conservation of polynomial Hamiltonian problems, *J. Numer. Anal. Ind. Appl. Math.*, 4, 87-101.
- [16] Kac, M. and van Moerbeke, P., (1975), On explicit soluble system of nonlinear differential equations related to certain Toda lattices, *Advances in Mathematics*, 16, 160–169.
- [17] Karasözen, B., (2004), Poisson integrators, *Mathematical Modelling and Computation* 40, 1225–1244.
- [18] Leimkuhler, B. and Reich, S., (2004), *Simulating Hamiltonian Dynamics*, Cambridge Univesity Press.
- [19] McLachlan, R. I., Quispel, G. R. W. and Robidoux, N., (1999), Geometric integration using discrete gradients, *R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci.*, 357, 1021-1045.
- [20] McLachlan, R. I., (2007), A new implementation of symplectic Runge-Kutta methods, *SIAM J. Sci. Comput.*, 29, 1637–1649.
- [21] McLaren, D. I. and Quispel, G. R. W., (2004), Integral-preserving integrators, *J. Phys. A: Math. Gen.* 37, 489-495.
- [22] Quispel, G. R. W. and McLaren, D. I., (2008), A new class of energy-preserving numerical integration methods, *J. Phys. A: Math. Theor.* 41, 045206.
- [23] Quispel, G. R. W. and Capel, H. W., (1996), Solving ODEs numerically while preserving a first integral. *Phys. Lett. A*, 218, 223-228.
- [24] Sanz-Serna, J. M. and Calvo, M. P., (1994), *Numerical Hamiltonian Problems*, Chapman and Hall.
- [25] Suris, Y. B., (1997), A note on an integrable discretization of the nonlinear Schrödinger equation, *Inverse Problems*, 13, 1211–1236.



Bülent Karasözen studied Mechanical Engineering at Berlin Technical University. He received his PhD in System and Control Engineering in 1981 from Berlin Technical University. He joined in 1982 to Mathematics Department of Middle East Technical University (METU). Since 2002 he is the director of the Scientific Computing Program at Institute of Applied Mathematics, METU. He organized ten international conferences and coordinated five international projects. He received DAAD (German Academic Exchanges Services) and Fulbright scholarships. He was visiting researcher at Darmstadt Technical University, University of Geneva, Carnegie Mellon University and Rice University. His research interests are structure preserving integration of ordinary and partial differential equations, continuous optimization, finite elements and optimal control of partial differential equations.



Özge Erdem studied Mathematics at Middle East Technical University (METU) in Ankara. She received her master degree in 2010 from Scientific Computing department at Insitute of Applied Mathematics in METU. She studied eigenvalue algorithms on Laplacian matrices of undirected graphs during her masters education. She was a student assistant in 2008. In 2009, she was a visiting graduate student at Max Planck Institute in Mathematics in Leipzig in Germany. She joined as a Ph.D student to the Dynamical Systems and Networks reserach group of Jrgen Jost in 2010. Her research interests include graph theory, dynamics of networks, dynamical systems and synchronization.
