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MULTIPLE SOLITON SOLUTIONS OF SECOND-ORDER BENJAMIN-ONO EQUATION

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Abstract. We employ the idea of Hirota's bilinear method, to obtain some new exact soliton solutions for high nonlinear form of Multiple soliton solutions of second-order Benjamin-Ono equation. Multiple singular soliton solutions were obtained by this method. Moreover, multiple singular soliton solutions were also derived.

Keywords: Hirota bilinear method, Benjamin-Ono equation, Multiple soliton solutions, Multiple singular soliton solutions.

AMS Subject Classification: 35C07, 35C08, 35A25

1. Introduction

Many important phenomena and dynamic processes in physics, mechanics, chemistry and biology can be represented by nonlinear partial differential equations. The study of exact solutions of nonlinear evolution equations plays an important role in soliton theory and explicit formulas of nonlinear partial differential equations play an essential role in the nonlinear science. Also, the explicit formulas may provide physical information and help us to understand the mechanism of related physical models.
In recent years, many kinds of powerful methods have been proposed to find solutions of nonlinear partial differential equations, numerically and/or analytically, e.g., the homogeneous balance method [1], the tanh-coth method [2], the Exp-function method [3], the decomposition method [4] and the improved tanh function method [5].
In this paper, by means of the Hirota's bilinear method, we will obtain some exact and new solutions for the second-order Benjamin-Ono equation. In the following section we have a brief review on the Hirota's bilinear method and in Section 3 and 4, we apply the Hirota's bilinear method to obtain multiple soliton solutions and multiple singular soliton solutions of the second-order Benjamin-Ono equation. Finally, the paper is concluded in Section 5.

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2. The Hirota bilinear method

To formally derive N-soliton solutions for completely integrable equations, we will use the Hirota's direct method combined with the simplified version of [6, 7, 8]. It was proved by many that soliton solutions are just polynomials of exponentials. This will be also confirmed in the coming discussions.
We first substitute

$$u(x,y,t) = \mathrm{e}^{kx+my-ct}, \tag{1}$$

into the linear terms of any equation under discussion to determine the relation between k, m and c. We then substitute the Cole-Hopf transformation

$$u(x,y,t) = R\,(\ln f(x,y,t))_{xx}, \tag{2}$$

into the equation under discussion, where the auxiliary function f, for the single soliton solution, is given by

$$f(x,y,t) = 1 + C_1\, f_1(x,y,t) = 1 + C_1\,\mathrm{e}^{\theta_1}. \tag{3}$$

The steps of the Hirota's method as summarized in [9, 10, 11, 12] are as follows:

(i) For the relation between k_i, m_i and c_i, we use

$$u(x,y,t) = \mathrm{e}^{\theta_i} \quad , \quad \theta_i = k_i\, x + m_i\, y - c_i\, t, \tag{4}$$

(ii) For single soliton, we use

$$f = 1 + C_1\,\mathrm{e}^{\theta_1}, \tag{5}$$

to determine R.
(iii) For two-soliton solutions, we use

$$f = 1 + C_1\,\mathrm{e}^{\theta_1} + C_2\,\mathrm{e}^{\theta_2} + C_1\,C_2\,a_{12}\,\mathrm{e}^{\theta_1+\theta_2}, \tag{6}$$

to determine the phase shift coefficient a_{12}, and hence can be generalized for a_{ij},$1 \le i < j \le 3$.
(iv) For three-soliton solutions, we use

$$\begin{aligned} f = {} & 1 + C_1\,\mathrm{e}^{\theta_1} + C_2\,\mathrm{e}^{\theta_2} + C_3\,\mathrm{e}^{\theta_3} + C_1\,C_2\,a_{12}\,\mathrm{e}^{\theta_1+\theta_2} \\ & + C_1\,C_3\,a_{13}\,\mathrm{e}^{\theta_1+\theta_3} + C_2\,C_3\,a_{23}\,\mathrm{e}^{\theta_2+\theta_3} \\ & + C_1\,C_2\,C_3\,b_{123}\,\mathrm{e}^{\theta_1+\theta_2+\theta_3}, \end{aligned} \tag{7}$$

to determine b_{123}. Pekcan proved in [13], $b_{123} = a_{12}\,a_{23}\,a_{13}$, then the equation gives rise to three-soliton solutions.
In the following, we will apply the aforementioned steps to second-order Benjamin-Ono equation. Multiple soliton solutions are obtained for $C_1 = C_2 = C_3 = 1$. However, multiple singular soliton solutions are obtained if $C_1 = C_2 = C_3 = -1$.

3. Multiple soliton solutions of the second-order Benjamin-Ono equation:

In this paper, we investigate explicit formula of soliton solutions of the following high nonlinear form of second-order Benjamin-Ono equation given in [14],

$$u_{tt} + \alpha\,(u^2)_{xx} + \beta\, u_{xxxx} = 0 \tag{8}$$

where $u = u(x,t) : \mathbb{R}_x \times \mathbb{R}_t \to \mathbb{R}$.
To determine multiple-soliton solutions for Eq. (8), we follow the steps presented above. We first consider $C_1 = C_2 = C_3 = 1$. Substituting

$$u\,(x,t) = \mathrm{e}^{\theta_i}, \quad \theta_i = k_i x - w_i t \tag{9}$$

into the linear terms of Eq.(8) to find the relation

$$w_i = \pm\sqrt{-\beta} {k_i}^2, \qquad i = 1, 2, \ldots, N \tag{10}$$

and consequently, θ_i becomes

$$\theta_i = k_i x \pm \sqrt{-\beta}\, k_i^2\, t. \tag{11}$$

To determine R, we substitute

$$u(x,t) = R\,(\ln f(x,t))_{xx} \tag{12}$$

where

$$f(x,t) = 1 + f_1(x,t) = 1 + \mathrm{e}^{k_1 x \pm \sqrt{-\beta}\, {k_i}^2\, t}$$

into Eq.(8) and solve to find that $R = -\dfrac{6\,\beta}{\alpha}$.
This means that the single singular soliton solution is given by

$$u\,(x,t) = -\frac{6\,\beta}{\alpha}\left[\frac{k_1^2\,\mathrm{e}^{k_1 x \pm \sqrt{\beta} k_1^2 t}}{\left(1 + \mathrm{e}^{k_1 x \pm \sqrt{-\beta} k_1^2 t}\right)^2}\right]. \tag{13}$$

For the two-soliton solutions, we substitute

$$u(x,t) = -\frac{6\,\beta}{\alpha}\,(\ln f(x,t))_{xx}, \tag{14}$$

where

$$f(x,t) = 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + a_{12}\,\mathrm{e}^{\theta_1 + \theta_2}, \tag{15}$$

into Eq.(8), where θ_1 and θ_2 are given in Eq.(11) to obtain

$$a_{12} = \frac{(k_1 - k_2)^2}{k_1^2 + k_1 k_2 + k_2^2}, \tag{16}$$

and

$$w_s = \pm\sqrt{-\beta}\, k_s^2, \qquad s = 1, 2,$$

hence

$$a_{ij} = \frac{(k_i - k_j)^2}{k_i^2 + k_i k_j + k_j^2}, \tag{17}$$

and

$$w_s = \pm\sqrt{-\beta}\, k_s^2, \qquad s = 1, 2, 3.$$

This in turn gives

$$f\,(x,t) = 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + \frac{(k_1 - k_2)^2}{k_1^2 + k_1 k_2 + k_2^2}\,\mathrm{e}^{\theta_1 + \theta_2}, \tag{18}$$

where

$$\theta_i = k_i x \pm \sqrt{-\beta}\, k_i^2\, t, i = 1, 2, \tag{19}$$

which is a two soliton solution(Fig. 1).
Similarly, to determine the three soliton solutions, we set

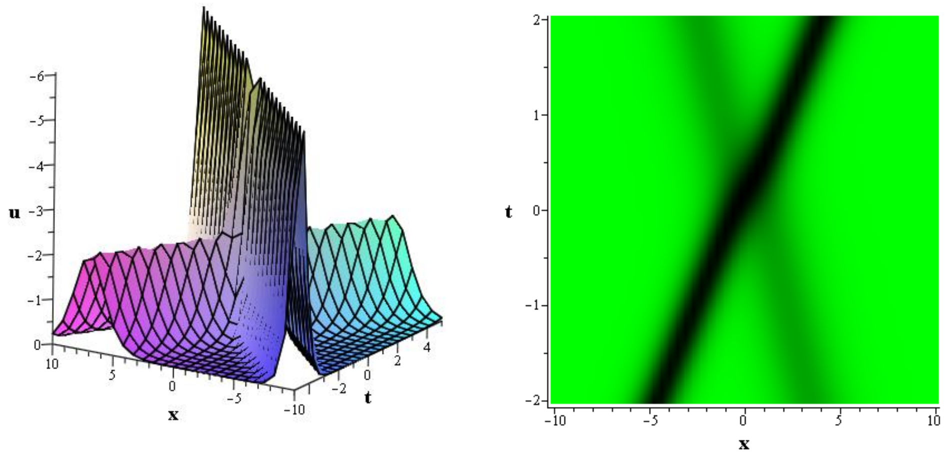


FIGURE 1. 2-soliton solution, the parameters: $\alpha = \beta = -1, k_1 = -1.2, k_2 = 2$.

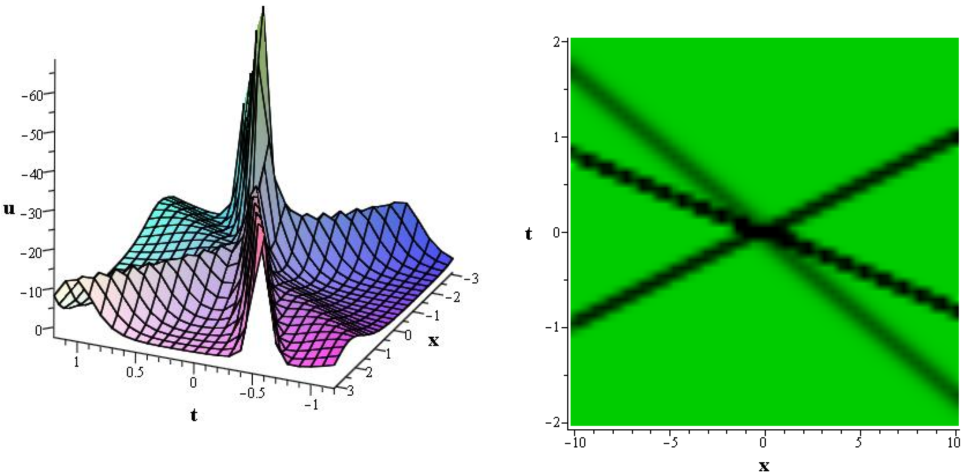


FIGURE 2. 3-soliton solution, the parameters: $\alpha = \beta = -1, k_1 = -5, k_2 = 3, k_3 = -2$.

$$f(x,t) = 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + \mathrm{e}^{\theta_3} + a_{12}\,\mathrm{e}^{\theta_1+\theta_2} + a_{13}\,\mathrm{e}^{\theta_1+\theta_3} + a_{23}\,\mathrm{e}^{\theta_2+\theta_3} + a_{12}\,a_{23}\,a_{13}\,\mathrm{e}^{\theta_1+\theta_2+\theta_3}. \tag{20}$$

To determine the three soliton solutions explicitly, we substitute the last result for $f(x,t)$ into Eq. (14), (See Fig. 2).

The higher level soliton solutions, for $n \geq 4$ can be obtained in a parallel manner. The obtained results confirm that the second-order Benjamin-Ono equation is completely integrable and possesses multiple soliton solutions of any order.

4. Multiple singular soliton solutions of the second-order Benjamin-Ono equation :

We first consider $C_1 = C_2 = C_3 = -1$. Substituting

$$u\,(x,t) = \mathrm{e}^{\theta_i}, \quad \theta_i = k_i x - w_i t \tag{21}$$

into the linear terms of Eq.(8) to find the relation

$$w_i = \pm\sqrt{-\beta}{k_i}^2, \qquad i = 1, 2, \ldots, N \tag{22}$$

and consequently, θ_i becomes

$$\theta_i = k_i x \pm \sqrt{-\beta}\, k_i^2\, t. \tag{23}$$

To determine R, we substitute

$$u(x,t) = R\,(\ln f(x,t))_{xx} \tag{24}$$

where

$$f(x,t) = 1 - f_1(x,t) = 1 - \mathrm{e}^{k_1 x \pm\sqrt{-\beta}\, {k_i}^2\, t}$$

into Eq.(8) and solve to find that $R = -\dfrac{6\,\beta}{\alpha}$.

This means that the single singular soliton solution is given by

$$u\,(x,t) = -\frac{6\,\beta}{\alpha}\left[\frac{k_1^2\,\mathrm{e}^{k_1 x\pm\sqrt{\beta}k_1^2 t}}{\left(1 + \mathrm{e}^{k_1 x\pm\sqrt{-\beta}k_1^2 t}\right)^2}\right]. \tag{25}$$

For the two-soliton solutions, we substitute

$$u(x,t) = -\frac{6\,\beta}{\alpha}\,(\ln f(x,t))_{xx}, \tag{26}$$

where

$$f(x,t) = 1 - \mathrm{e}^{\theta_1} - \mathrm{e}^{\theta_2} + a_{12}\,\mathrm{e}^{\theta_1+\theta_2}, \tag{27}$$

into Eq.(8), where θ_1 and θ_2 are given in Eq.(23) to obtain

$$a_{12} = \frac{(k_1 - k_2)^2}{k_1^2 + k_1 k_2 + k_2^2}, \tag{28}$$

and

$$w_s = \pm\sqrt{-\beta}\, k_s^2, \qquad s = 1, 2,$$

hence

$$a_{ij} = \frac{(k_i - k_j)^2}{k_i^2 + k_i k_j + k_j^2}, \tag{29}$$

and

$$w_s = \pm\sqrt{-\beta}\, k_s^2, \qquad s = 1, 2, 3.$$

This in turn gives

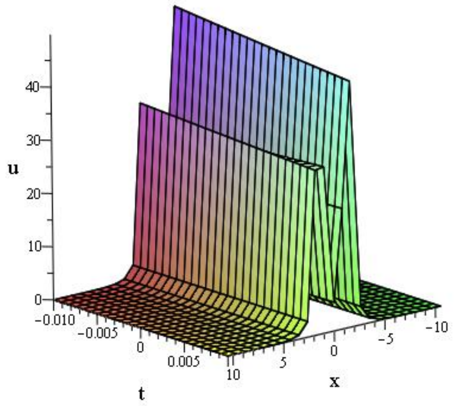
$$f\,(x,t) = 1 - \mathrm{e}^{\theta_1} - \mathrm{e}^{\theta_2} + \frac{(k_1 - k_2)^2}{k_1^2 + k_1 k_2 + k_2^2}\,\mathrm{e}^{\theta_1+\theta_2}, \tag{30}$$

where

$$\theta_i = k_i x \pm \sqrt{-\beta}\, k_i^2\, t, i = 1, 2, \tag{31}$$

which is a two soliton solution(Fig. 3).

Similarly, to determine the three soliton solutions, we set



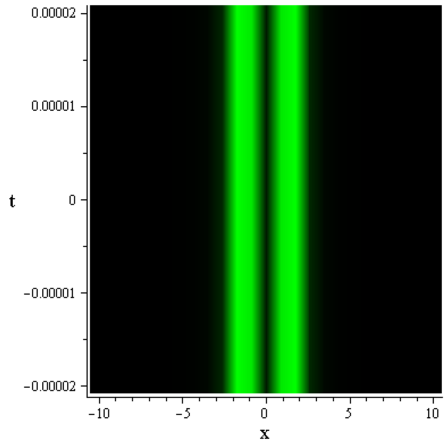
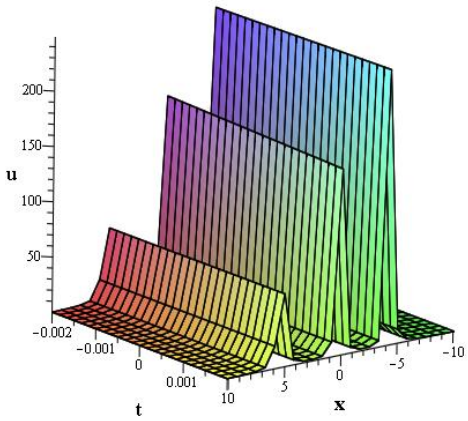


FIGURE 3. 2-soliton solution, the parameters: $\alpha = \beta = -0.9, k_1 = 1.1, k_2 = -1.2$.



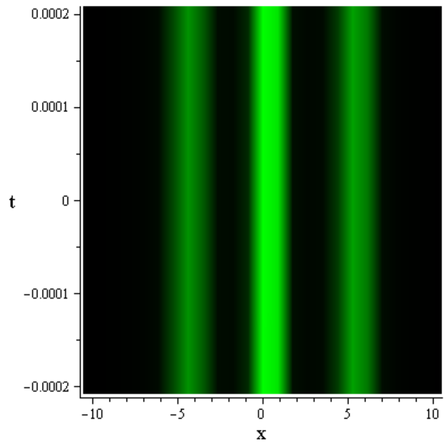


FIGURE 4. 3-soliton solution, the parameters: $\alpha = \beta = -1, k_1 = -0.5, k_2 = 0.3, k_3 = 1.2$.

$$\begin{aligned} f(x,t) &= 1 - \mathrm{e}^{\theta_1} - \mathrm{e}^{\theta_2} - \mathrm{e}^{\theta_3} + a_{12}\,\mathrm{e}^{\theta_1+\theta_2} + a_{13}\,\mathrm{e}^{\theta_1+\theta_3} + a_{23}\,\mathrm{e}^{\theta_2+\theta_3} \\ &\quad - a_{12}\,a_{23}\,a_{13}\,\mathrm{e}^{\theta_1+\theta_2+\theta_3}. \end{aligned} \tag{32}$$

To determine the three soliton solutions explicitly, we substitute the last result for $f(x,t)$ into Eq. (26), (See Fig. 4).

The higher level soliton solutions, for $n \geq 4$ can be obtained in a parallel manner. The obtained results confirm that the second-order Benjamin-Ono equation is completely integrable and possesses multiple soliton solutions of any order.

5. Conclusion

The study of exact soliton solutions of nonlinear evolution equations plays an important role in soliton theory and explicit formulas of nonlinear partial differential equations play an essential role in the nonlinear science. In this paper, by using the Hirota bilinear method, we obtained some explicit formulas of soliton solutions for the second-order Benjamin-Ono equation. Multiple soliton solutions were formally derived. Moreover, multiple singular soliton solutions of any order was derived as well.

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