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THE $v-$ INVARIANT χ^2 SEQUENCE SPACES

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Abstract. In this paper we define $v-$ invariatness of a double sequence space of χ and examine the $v-$ invariatness of the double sequence space of χ. Furthermore, we give duals of double sequence space of χ.

Keywords: Gai sequence, analytic sequence, modulus function, double sequences.

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1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial work on double sequence spaces is found in Bromwich [4]. Later on, they were investigated by Hardy [5], Moricz [9], Moricz and Rhoades [10], Basarir and Solankan [2], Tripathy [17], Turkmenoglu [19], and many others.

Let us define the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{(x_{mn}) \in w^2 : sup_{m,n\in N} |x_{mn}|^{t_{mn}} < \infty\right\},$$
$$\mathcal{C}_p(t) := \left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn} - l|^{t_{mn}} = 1 \, for\, some\, l \in \mathbb{C}\right\},$$
$$\mathcal{C}_{0p}(t) := \left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1\right\},$$
$$\mathcal{L}_u(t) := \left\{(x_{mn}) \in w^2 : \textstyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty\right\},$$
$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p-lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}$; $\mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [21,22] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [23] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [24] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation

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between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [25] and Mursaleen and Edely [26] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the $M-$core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the $M-$core of x. More recently, Altay and Basar [27] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also have examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Quite recently Basar and Sever [28] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and have examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [29] have studied the space $\chi^2_M(p, q, u)$ of double sequences and have given some inclusion relations.

Spaces are strongly summable sequences was discussed by Kuttner [31], Maddox [32], and others. The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [8] as an extension of the definition of strongly Cesàro summable sequences. Connor [33] further extended this definition to a definition of strong $A-$ summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong $A-$ summability, strong $A-$ summability with respect to a modulus, and $A-$ statistical convergence. In [34] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [35]-[38], and [39] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton. In their work and throughout this paper, the four dimensional matrices and double sequences have real-valued entries unless specified otherwise. In this paper we extend a few results known in the literature for ordinary(single) sequence spaces to multiply sequence spaces.

We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a + b)^p \leq a^p + b^p \tag{1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$ (see[1]).

A sequence $x = (x_{mn})$is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m + n)! |x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{all\, finite\, sequences\}$.

Consider a double sequence $x = (x_{ij})$. The $(m, n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \Im_{ij}$ for all $m, n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i, j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space(or a metric space)X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m, n \in \mathbb{N})$

are also continuous.
If X is a sequence space, we give the following definitions:

(i)$X^{'}=$ the continuous dual of X;

(ii)$X^{\alpha}=\left\{a=(a_{mn}):\sum_{m,n=1}^{\infty}|a_{mn}x_{mn}|<\infty,\, for\, each\, x\in X\right\}$;

(iii)$X^{\beta}=\left\{a=(a_{mn}):\sum_{m,n=1}^{\infty}a_{mn}x_{mn}\, is\, convegent,\, for\, each\, x\in X\right\}$;

(iv)$X^{\gamma}=\left\{a=(a_{mn}):sup_{mn}\geq 1\left|\sum_{m,n=1}^{M,N}a_{mn}x_{mn}\right|<\infty, for\, each\, x\in X\right\}$;

(v)$let\, X\, be an FK-space\supset\phi;\, then\, X^{f}=\left\{f(\Im_{mn}):f\in X^{'}\right\}$;

(vi)$X^{\delta}=\left\{a=(a_{mn}):sup_{mn}|a_{mn}x_{mn}|^{1/m+n}<\infty,\, for\, each\, x\in X\right\}$;

$X^{\alpha}.X^{\beta},X^{\gamma}$ are called $\alpha-(or K\ddot{o}the-Toeplitz)$dual of $X,\beta-(or\, generalized-K\ddot{o}the-Toeplitz)\, dual\, of X,\gamma-dual\, of\, X,\,\delta-dual\, of X\, respectively.X^{\alpha}$ is defined by Gupta and Kamptan [20]. It is clear that $x^{\alpha}\subset X^{\beta}$ and $X^{\alpha}\subset X^{\gamma}$, but $X^{\beta}\subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [30] as follows

$$Z(\Delta)=\{x=(x_k)\in w:(\Delta x_k)\in Z\}$$

for $Z=c,c_0$ and ℓ_{∞}, where $\Delta x_k=x_k-x_{k+1}$ for all $k\in\mathbb{N}$.
Here c,c_0 and ℓ_{∞} denote the classes of convergent,null and bounded sclar valued single sequences respectively. The difference space bv_p of the classical space ℓ_p is introduced and studied in the case $1\leq p\leq\infty$ by BaŞar and Altay in [42] and in the case $0<p<1$ by Altay and BaŞar in [43]. The spaces $c(\Delta),c_0(\Delta),\ell_{\infty}(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\|=|x_1|+sup_{k\geq 1}|\Delta x_k| \text{ and } \|x\|_{bv_p}=\left(\sum_{k=1}^{\infty}|x_k|^p\right)^{1/p},(1\leq p<\infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta)=\left\{x=(x_{mn})\in w^2:(\Delta x_{mn})\in Z\right\}$$

where $Z=\Lambda^2,\chi^2$ and $\Delta x_{mn}=(x_{mn}-x_{mn+1})-(x_{m+1n}-x_{m+1n+1})=x_{mn}-x_{mn+1}-x_{m+1n}+x_{m+1n+1}$ for all $m,n\in\mathbb{N}$

2. Definitions and Preliminaries

Let $v=(v_{mn})$ be any fixed sequence of nonzero complex numbers satisfying
$\Lambda_v^2=\left\{x=(x_{mn}):sup_{mn}|v_{mn}x_{mn}|^{1/m+n}<\infty\right\}$
$\chi_v^2=\left\{x=(x_{mn}):((m+n)!\,|v_{mn}x_{mn}|)^{1/m+n}\rightarrow 0\, as\, m,n\rightarrow\infty\right\}$

In this paper Λ_v^2 and χ_v^2 will denote the sequence spaces of Pringsheim sense double analytic invariant and Pringsheim sense double gai invariant sequences respectively.

The space Λ_v^2 is a invariant metric space with the metric

$$d(x,y)=sup_{mn}\left\{|v_{mn}x_{mn}-v_{mn}y_{mn}|^{1/m+n}:m,n:1,2,3,...\right\} \qquad (2)$$

forall $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ in Λ^2_v.

The space χ^2_v is a invariant metric space with the metric

$$d(x, y) = sup_{mn} \left\{ ((m+n)! \, |v_{mn}x_{mn} - v_{mn}y_{mn}|)^{1/m+n} : m, n : 1, 2, 3, ... \right\} \quad (3)$$

forall $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ in χ^2_v.

Definition 2.1. *A sequence X is $v-$ invariant if $X_v = X$ where $X_v = \{x = (x_{mn}) : (v_{mn}x_{mn}) \in X\}$, where $X = \Lambda^2_v$ and χ^2_v.*

In this paper we define $v-$ invariantness of a sequence space X and give necessary and sufficient conditions for Λ^2_v and χ^2_v to $v-$ invariant. Now, if $X = \Lambda^2_v \, or \, \chi^2_v$ is $v-$ invariant sequence spaces then we have the following results.

3. Main Results

Theorem 3.1. *Let χ^2 be a $v-$invariant sequence space. Then (i) χ^2_v is a Banach invariant space if and only if χ^2_v is a Banach invariant metric space, (ii) χ^2_v is separable if and only if χ^2_v is separable.*

Proof. Let $u = (u_{mn})$ and $v = (v_{mn})$ be any fixed sequence of nonzero complex numbers such that

$$lim_{m.n \to \infty} sup \left((m+n)! \, |u_{mn} - 0|\right)^{1/m+n}$$

and

$$lim_{m.n \to \infty} sup \left((m+n)! \, |v_{mn} - 0|\right)^{1/m+n}$$

are positive (may be infinite).
If $v_{mn} = \lambda$ for every m, n, then obviously χ^2 is $v-$invariant, where λ is a scalar. This completes the proof. □

Theorem 3.2. *Let $w_{mn} = u_{mn}v^{-1}_{mn}$ for each $m, n \in \mathbb{N}$, where $v^{-1}_{mn} = \frac{1}{v_{mn}}$. Then (i) $\chi^2_v \subset \chi^2_u$ if and only if $sup_{mn} |w_{mn}| < \infty$. (ii) $\chi^2_v = \chi^2_u$ if and only if $0 < inf_{mn} |w_{mn}| \leq |w_{mn}| \leq sup_{mn} |s_{mn}| < \infty$.*

Proof. Sufficiency is trivial, since

$$|u_{mn}x_{mn}|^{1/m+n} = |w_{mn}|^{1/m+n} |v_{mn}x_{mn}|^{1/m+n} \quad (4)$$

For the necessity suppose that $\chi^2_v \subset \chi^2_u$ but $sup_{mn} = \infty$. Then there exists a strictly increasing sequence $(w_{m_in_i}) > i$ we put

$$((m+n)! \, |x_{mn}v_{mn}|)^{1/m+n} = \begin{cases} 0 & \text{if } m, n \neq m_in_i \\ \frac{i}{u_{m_in_i}} & \text{if } m, n = m_in_i \end{cases} \quad (5)$$

Then we have $((m+n)! \, |x_{mn}v_{mn}|)^{1/m+n} < 1$ and $((m+n)! \, |x_{mn}u_{mn}|)^{1/m+n} = i$, where $m, n = m_in_i$. When $x \in \chi^2_v - \chi^2_u$ contrary to the assumption that $\chi^2_v \subset \chi^2_u$.
(ii) To prove this, it is enough to show that $\chi^2_u \subset \chi^2_v$ if and only if $inf_{mn} |w_{mn}| > 0$. It is obvious that $inf_{mn} |w_{mn}| > 0$ if and only if $sup_{mn} \left|\frac{1}{w_{mn}}\right| < \infty$. Hence the result follows from proof (i). □

Theorem 3.3. *(i) $\chi^2 \subset \chi^2_v$ if and only if $sup_{mn} |v_{mn}| < \infty$, (ii) $\chi^2_v \subset \chi^2$ if and only if $inf_{mn} |v_{mn}| > 0$, (iii)$\chi^2_v = \chi^2$ if and only if $0 < inf_{mn} |v_{mn}| \leq v_{mn} \leq \infty \leq sup_{mn} |v_{mn}| < \infty$.*

Proof. Taking $v = \begin{pmatrix} 1 & 1 & 1\cdots & 1 & 1 \\ 1 & 1 & 1\cdots & 1 & 1 \\ \vdots \\ 1 & 1 & 1\cdots & 1 & 1 \end{pmatrix}$ upto $(m,n)^{th}$ term and replacing u by v in Theorem 3.2 (i).
It is trivial that $inf_{mn}\left|v_{mn}\right| > 0$ if and only if $sup_{mn}\left|\frac{1}{v_{mn}}\right| < \infty$.
Hence taking $u = \begin{pmatrix} 1 & 1 & 1\cdots & 1 & 1 \\ 1 & 1 & 1\cdots & 1 & 1 \\ \vdots \\ 1 & 1 & 1\cdots & 1 & 1 \end{pmatrix}$ upto $(m,n)^{th}$ term in Theorem 3.2 (i), we get Theorem 3.3(ii).
Finally, taking Taking $u = \begin{pmatrix} 1 & 1 & 1\cdots & 1 & 1 \\ 1 & 1 & 1\cdots & 1 & 1 \\ \vdots \\ 1 & 1 & 1\cdots & 1 & 1 \end{pmatrix}$ upto $(m,n)^{th}$ term in Theorem 3.2 (ii), since clearly $inf_{mn}\frac{1}{v_{mn}} > 0$ if and only if $sup_{mn}\left|v_{mn}\right| < \infty$, we get (iii). □

Corollary 3.1. *If* χ^2 *is* $v-$ *invariant if and only if* $0 < inf_{mn}\left|v_{mn}\right| \le \left|v_{mn}\right| \le sup_{mn}\left|v_{mn}\right| < \infty$.

Proof. Follows from Theorem 3.3 (iii). □

Theorem 3.4. *(i)* $\chi^2_v \subset \chi^2_u$ *if and only if* $w = (w_{mn}) \in \chi^2$, *(ii)* $\chi^2_v = \chi^2_u$ *if and only if* $w \notin \chi^2$.

Proof. **(i)** The sufficiency is trivial by an equation (4). For the necessity suppose that $\chi^2_v \subset \chi^2_u$ but $w \notin \chi^2$. Then, either $w \in \Lambda^2_v$ (or)$w \notin \Lambda^2_v$ Now we put $\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n} = \left(w_{mn} \times \frac{1}{(u_{mn})^{1/m+n}}\right) = \frac{1}{(v_{mn})^{1/m+n}}$. Then

$\left((m+n)!\left|x_{mn}v_{mn}\right|\right)^{1/m+n} = \begin{pmatrix} 1 & 1 & 1\cdots & 1 & 1 \\ 1 & 1 & 1\cdots & 1 & 1 \\ \vdots \\ 1 & 1 & 1\cdots & 1 & 1 \end{pmatrix}$ and

$\left((m+n)!\left|x_{mn}u_{mn}\right|\right)^{1/m+n} = (w_{mn})$. Whence $x \in \chi^2_v - \chi^2_u$, contrary to the assumption that $\chi^2_v \subset \chi^2_u$. Hence we obtain the necessity.
(ii) Sufficiency, let $w \in \chi^2_v \subset \chi^2_u$ by (i).
Let $x \in \chi^2_u$, so that $\left((m+n)!\left|x_{mn}u_{mn}\right|\right)^{1/m+n} \in \chi^2$. Now, since $w \in \chi^2$, $lim_{mn}\frac{1}{w_{mn}} = 0$. Therefore, from the equality
$\left((m+n)!\left|x_{mn}v_{mn}\right|\right)^{1/m+n} = \left((m+n)!\left|x_{mn}u_{mn}\frac{1}{w_{mn}}\right|\right)^{1/m+n}$, we have
$\left((m+n)!\left|x_{mn}v_{mn}\right|\right)^{1/m+n} \in \chi^2$ and hence $\chi^2_u \subset \chi^2_v$.
Necessity: Suppose that $\chi^2_v = \chi^2_u$, that is $\chi^2_v \subset \chi^2_u$ and $\chi^2_u \subset \chi^2_v$. Then $lim_{mn}w_{mn} = lim_{mn}u_{mn} \times \frac{1}{v_{mn}}$ and $lim_{mn}\frac{1}{w_{mn}} = lim_{mn}\frac{1}{u_{mn}} \cdot \frac{1}{v_{mn}} = 0$. It is trivial that $lim_{mn}\frac{1}{w_{mn}} = 0$ if and only if $lim_{mn}w_{mn} \neq 0$. Hence $w \notin \chi^2$. □

Theorem 3.5. *(i)* $\chi^2 \subset \chi^2_v$ *if and only if* $v \in \chi^2$ *(ii)* $\chi^2_v = \chi^2$ *if and only if* $v \notin \chi^2$ *and* $lim_{mn}v_{mn} \neq 0$.

Proof. Taking $v = \begin{pmatrix} 1 & 1 & 1\cdots & 1 & 1 \\ 1 & 1 & 1\cdots & 1 & 1 \\ \vdots & & & & \\ 1 & 1 & 1\cdots & 1 & 1 \end{pmatrix}$ and replacing u by v in Theorem 3.4 (i), we obtain (i). Theorem 3.4 (ii) gives us (ii) for $u = \begin{pmatrix} 1 & 1 & 1\cdots & 1 & 1 \\ 1 & 1 & 1\cdots & 1 & 1 \\ \vdots & & & & \\ 1 & 1 & 1\cdots & 1 & 1 \end{pmatrix}$. □

Remark 3.1. *If $v \in \chi^2$ and $lim_{mn} v_{mn} = 0$ that is $v \in \chi^2$, then $\chi^2 \subset \chi_v^2$.*

Proposition 3.1. $\chi_v^2 \subset \Gamma_v^2$.

Proof. Let $x \in \chi_v^2$.
Then we have $((m+n)!\,|x_{mn}v_{mn}|)^{1/m+n} \to 0\, as\, m,n \to \infty$.
Here, we get $|x_{mn}v_{mn}|^{1/m+n} \to 0\, as\, m,n \to \infty$. Thus we have $x \in \Gamma_v^2$ and so $\chi^2 \subset \Gamma_v^2$. □

Proposition 3.2. $\left(\Gamma_v^2\right)^\beta \overset{\subset}{\neq} \Lambda_v^2$.

Proof. Let $y = (y_{mn})$ be an arbitrary point in $\left(\Gamma_v^2\right)^\beta$. If y is not in Λ_v^2, then for each natural number p, we can find an index $m_p n_p$ such that

$$\left|y_{m_p n_p}\right|^{1/m_p+n_p} > p v_{mn}, (p = 1, 2, 3, \cdots) \tag{6}$$

Define $x = \{x_{mn}\}$ by

$$x_{mn} = \begin{cases} \frac{1}{p^{m+n} v_{mn}}, & \text{for } (m,n) = (m_p, n_p) \text{ for some} p \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

Then x is in Γ_v^2, but for infinitely mn,

$$|y_{mn}x_{mn}| > 1. \tag{8}$$

Consider the sequence $z = \{z_{mn}\}$, where $z_{11} = x_{11}v_{11} - s$ with

$$s = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{mn}v_{mn}, z_{mn} = x_{mn}v_{mn}. \tag{9}$$

Then z is a point of Γ_v^2. Also, $\sum\sum z_{mn} = 0$. Hence, z is in Γ_v^2; but, by (8), $\sum\sum z_{mn}y_{mn}$ does not converge:

$$\Rightarrow \sum\sum x_{mn}y_{mn}\, diverges. \tag{10}$$

Thus, the sequence y would not be in $\left(\Gamma_v^2\right)^\beta$. This contradiction proves that

$$\left(\Gamma_v^2\right)^\beta \subset \Lambda_v^2. \tag{11}$$

Let $y_{1n}v_{1n} = x_{1n}v_{1n} = 1$ and $y_{mn}v_{mn} = x_{mn}v_{mn} = 0\,(m > 1)$ for all n, then obviously $x \in \Gamma_v^2$ and $y \in \Lambda_v^2$, but

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{mn}y_{mn} = \infty. Hence,\, y \notin \left(\Gamma_v^2\right)^\beta \tag{12}$$

From (11) and (12), we are granted $\left(\Gamma_v^2\right)^\beta \overset{\subset}{\neq} \Lambda_v^2$. □

Proposition 3.3. *The $\beta-$ dual space of χ_v^2 is Λ^2.*

Proof. First, we observe that $\chi_v^2 \subset \Gamma_v^2$, by Proposition 3.1. Therefore $\left(\Gamma_v^2\right)^\beta \subset \left(\chi_v^2\right)^\beta$. But $\left(\Gamma_v^2\right)^\beta \overset{\subset}{\neq} \Lambda_v^2$, by Proposition 3.2. Hence

$$\Lambda_v^2 \subset \left(\chi_v^2\right)^\beta \tag{13}$$

Next we show that $\left(\chi_v^2\right)^\beta \subset \Lambda_v^2$. Let $y = (y_{mn}) \in \left(\chi_v^2\right)^\beta$. Consider $f(x) = \sum_{m=1}^\infty \sum_{n=1}^\infty x_{mn} y_{mn}$ with $x = (x_{mn}) \in \chi_v^2$
$x = [(\Im_{mn} - \Im_{mn+1}) - (\Im_{m+1n} - \Im_{m+1n+1})]$

$$= \begin{pmatrix} 0, & 0, & ...0, & 0, & ... & 0 \\ 0, & 0, & ...0, & 0, & ... & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0, & 0, & ...\frac{1}{(m+n)!(v_{mn})^{1/m+n}}, & \frac{-1}{(m+n)!(v_{mn})^{1/m+n}}, & ... & 0 \\ 0, & 0, & ...0, & 0, & ... & 0 \end{pmatrix} - \begin{pmatrix} 0, & 0, & ...0, & 0, & ... & 0 \\ 0, & 0, & ...0, & 0, & ... & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0, & 0, & ...\frac{1}{(m+n)!(v_{mn})^{1/m+n}}, & \frac{-1}{(m+n)!(v_{mn})^{1/m+n}}, & ... & 0 \\ 0, & 0, & ...0, & 0, & ... & 0 \end{pmatrix}$$

$$\left\{((m+n)!\,|x_{mn} v_{mn}|)^{1/m+n}\right\} = \begin{pmatrix} 0, & 0, & ...0, & 0, & ... & 0 \\ 0, & 0, & ...0, & 0, & ... & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0, & 0, & ...\frac{1}{(m+n)!(v_{mn})^{1/m+n}}, & \frac{-1}{(m+n)!(v_{mn})^{1/m+n}}, & ... & 0 \\ 0, & 0, & ...\frac{-1}{(m+n)!(v_{mn})^{1/m+n}}, & \frac{1}{(m+n)!(v_{mn})^{1/m+n}}, & ... & 0 \\ 0, & 0, & ...0, & 0, & ... & 0 \end{pmatrix}.$$

Hence converges to zero.

Therefore $[(\Im_{mn} - \Im_{mn+1}) - (\Im_{m+1n} - \Im_{m+1n+1})] \in \chi_v^2$.

Hence $d\left((\Im_{mn} - \Im_{mn+1}) - (\Im_{m+1n} - \Im_{m+1n+1}), 0\right) = 1$. But

$|y_{mn} v_{mn}|^{1/m+n} \leq \|f\|\, d\left((\Im_{mn} - \Im_{mn+1}) - (\Im_{m+1n} - \Im_{m+1n+1}), 0\right) \leq \|f\| \cdot 1 < \infty$ for each m, n. Thus (y_{mn}) is a double invariant bounded sequence and hence an invariant analytic sequence. In other words $y \in \Lambda_v^2$. But $y = (y_{mn})$ is arbitrary in $\left(\chi_v^2\right)^\beta$. Therefore

$$\left(\chi_v^2\right)^\beta \subset \Lambda_v^2 \tag{14}$$

From (13) and (14) we get $\left(\chi_v^2\right)^\beta = \Lambda_v^2$. □

Proposition 3.4. $\Lambda-$ *dual of* χ_v^2 *is* Λ_v^2.

Proof. Let $y \in \Lambda-$ dual of χ_v^2. Then $|x_{mn}y_{mn}| \leq \frac{M^{m+n}}{v_{mn}}$ for some constant $M > 0$ and for each $x \in \chi_v^2$. Therefore $|y_{mn}v_{mn}| \leq M^{m+n}$ for each m, n by taking

$$x = \Im_{mn} = \begin{pmatrix} 0, & 0, & ...0, & 0, & ... & 0 \\ 0, & 0, & ...0, & 0, & ... & 0 \\ . \\ . \\ . \\ 0, & 0, & ...\frac{1}{(m+n)!(v_{mn})^{1/m+n}}, & 0, & ... & 0 \\ 0, & 0, & ...0, & 0, & ... & 0 \end{pmatrix}.$$

This shows that $y \in \Lambda_v^2$. Then

$$\left(\chi_v^2\right)^\Lambda \subset \Lambda_v^2 \tag{15}$$

On the other hand, let $y \in \Lambda_v^2$. Let $\epsilon > 0$ be given. Then $|y_{mn}v_{mn}| < M^{m+n}$ for each m, n and for some constant $M > 0$. But $x \in \chi_v^2$. Hence $((m+n)! \, |x_{mn}v_{mn}|) < \epsilon^{m+n}$ for each m, n and for each $\epsilon > 0$. i.e $|x_{mn}| < \frac{\epsilon^{m+n}}{(m+n)!(v_{mn})^{1/m+n}}$. Hence

$$|x_{mn}y_{mn}| = |x_{mn}| \, |y_{mn}| < \frac{\epsilon^{m+n}}{(m+n)!(v_{mn})^{1/m+n}} M^{m+n} = \frac{(\epsilon M)^{m+n}}{(m+n)!(v_{mn})^{1/m+n}}$$

$\Rightarrow y \in \left(\chi_v^2\right)^\Lambda$

$$\Lambda_v^2 \subset \left(\chi_v^2\right)^\Lambda \tag{16}$$

From (15) and (16) we get $\left(\chi_v^2\right)^\Lambda = \Lambda_v^2$. □

Proposition 3.5. *Let* $\left(\chi_v^2\right)^*$ *denote the dual space of* χ_v^2. *Then we have* $\left(\chi_v^2\right)^* = \Lambda_v^2$.

Proof. We recall that

$$x = \Im_{mn} = \begin{pmatrix} 0, & 0, & ...0, & 0, & ... & 0 \\ 0, & 0, & ...0, & 0, & ... & 0 \\ . \\ . \\ . \\ 0, & 0, & ...\frac{1}{(m+n)!(v_{mn})^{1/m+n}}, & 0, & ... & 0 \\ 0, & 0, & ...0, & 0, & ... & 0 \end{pmatrix}.$$

with $\frac{1}{(m+n)!(v_{mn})^{1/m+n}}$ in the $(m, n)^{th}$ position and zero otherwise, with

$$x = \Im_{mn}, \left\{ ((m+n)! \, |x_{mn}v_{mn}|)^{1/m+n} \right\}$$

$$= \begin{pmatrix} 0^{1/2}, & 0, & ...0, & 0, & ... & 0^{1/1+n} \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0^{1/m+1}, & 0, & ...\left(\frac{(m+n)!v_{mn}}{(m+n)!v_{mn}}\right)^{1/m+n}, & 0, & ... & 0^{1/m+n+1} \\ 0^{1/m+2}, & 0, & ...0, & 0, & ... & 0^{1/m+n+2} \end{pmatrix}.$$

$$= \begin{pmatrix} 0, & 0, & ...0, & 0, & ... & 0 \\ 0, & 0, & ...0, & 0, & ... & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0, & 0, & ...1^{1/m+n}, & 0, & ... & 0 \\ 0, & 0, & ...0, & 0, & ... & 0 \end{pmatrix}.$$

which is a double χ sequence. Hence $\Im_{mn} \in \chi_v^2$. Let us take $f(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{mn}y_{mn}$ with $x \in \chi_v^2$ and $f \in \left(\chi_v^2\right)^*$. Take $x = (x_{mn}) = \Im_{mn} \in \chi_v^2$. Then

$$|y_{mn}v_{mn}|^{1/m+n} \leq \|f\|\, d\left(\Im_{mn}, 0\right) < \infty \text{ for each } m, n$$

Thus (y_{mn}) is a bounded invariant sequence and hence an double analytic invariant sequence. In other words $y \in \Lambda_v^2$. Therefore $\left(\chi_v^2\right)^* = \Lambda_v^2$. □

Proposition 3.6. $\left(\Lambda_v^2\right)^\beta = \Lambda_v^2$.

Proof. Step 1: Let $(x_{mn}) \in \Lambda_v^2$ and let $(y_{mn}) \in \Lambda_v^2$. Then we get $|y_{mn}v_{mn}|^{1/m+n} \leq M$ for some constant $M > 0$.
Also $(x_{mn}v_{mn}) \in \Lambda_v^2 \Rightarrow (|x_{mn}v_{mn}|)^{1/m+n} \leq \epsilon = \frac{1}{2M}$
$\Rightarrow |x_{mn}| \leq \frac{1}{2^{m+n}M^{m+n}v_{mn}}$.
Hence $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|x_{mn}y_{mn}| \leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|x_{mn}|\,|y_{mn}|$

$$< \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{2^{m+n}}\frac{1}{M^{m+n}}M^{m+n}\frac{1}{(v_{mn})^2}$$

$$< \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{2^{m+n}}\frac{1}{(v_{mn})^2} < \infty.$$

Therefore, we get that $(x_{mn}) \in \left(\Lambda_v^2\right)^\beta$ and so we have

$$\Lambda_v^2 \subset \left(\Lambda_v^2\right)^\beta \tag{17}$$

Step 2: Let $(x_{mn}) \in \left(\Lambda_v^2\right)^\beta$. This says that

$$\Rightarrow \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|x_{mn}y_{mn}| < \infty\, for\, each\ (y_{mn}) \in \Lambda_v^2 \tag{18}$$

Assume that $(x_{mn}) \notin \Lambda_v^2$, then there exists a sequence of positive integers $(m_p + n_p)$ strictly increasing such that

$$\left|x_{m_p+n_p}\right| > \frac{1}{(2v)^{m_p+n_p}}\,(p = 1, 2, 3, \cdots)$$

Take

$$y_{m_p,n_p} = (2v)^{m_p+n_p}\,(p = 1, 2, 3, \cdots)$$

and

$$y_{mn} = 0 \text{ otherwise}$$

Then $(y_{mn}) \in \Lambda_v^2$. But

$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn} y_{mn}| = \sum \sum_{p=1}^{\infty} \left| x_{m_p n_p} y_{m_p n_p} \right| > 1 + 1 + 1 + \cdots$.

We know that the infinite series $1 + 1 + 1 + \cdots$ diverges. Hence $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn} y_{mn}|$ diverges. This contradicts (18). Hence $(x_{mn}) \in \Lambda_v^2$. Therefore

$$\left(\Lambda_v^2\right)^{\beta} \subset \Lambda_v^2 \tag{19}$$

From (17) and (19) we get $\left(\Lambda_v^2\right)^{\beta} = \Lambda_v^2$. □

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