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GENERALIZED (2+1)-DIMENSIONAL BREAKING SOLITON EQUATION

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ABSTRACT. In this work, a general (2+1)-dimensional breaking soliton equation is investigated. The Hereman's simplified method is applied to derive multiple soliton solutions, hence to confirm the model integrability.

Keywords: Breaking soliton equation; multiple soliton solutions; multiple singular soliton solutions.

AMS Subject Classification: 35Q51, 35Q53, 37K10

1. Introduction

Many reliable methods are used in the literature to investigate completely integrable equations that admit multiple soliton solutions [1–6]. The algebraic-geometric method [2–4], the inverse scattering method, the Bäcklund transformation method, the Darboux transformation method, the Hirota bilinear method [7–14], and other methods are used to make progress and new developments in this filed. The Hirota's bilinear method is rather heuristic and possesses significant features that make it practical for the determination of multiple soliton solutions, and for multiple singular soliton solutions [15–23] for a wide class of nonlinear evolution equations in a direct method. Hereman et. al [10] developed a modified form of the Hirota's method that facilitates the computational work. The computer symbolic systems such as Maple, Mathematica can be used to overcome the tedious calculations.

In this work, we will study a generalized (2+1)-dimensional breaking soliton equation [1]

$$\left(u_{xt} - \beta(4u_{xy}u_x + 2u_{xx}u_y - u_{xxxy}) - \gamma(6u_xu_{xx} - u_{xxxx})\right)_x = -\alpha^2(\beta u_{yyy} + 3\gamma u_{xyy}). \quad (1)$$

For $\alpha = 0$, we obtain the (2+1)-dimensional equation

$$u_{xt} - \beta(4u_{xy}u_x + 2u_{xx}u_y - u_{xxxy}) - \gamma(6u_xu_{xx} - u_{xxxx}) = 0, \quad (2)$$

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which describes the interaction between the Riemann wave propagation along the y-axis and the long wave propagating along the x-axis [1–3]. For $\alpha = 0$ and $\gamma = 0$, Eq. (1) reduces to the (2+1)-dimensional breaking soliton equation

$$u_{xt} - \beta(4u_{xy}u_x + 2u_{xx}u_y - u_{xxxy}) = 0. \tag{3}$$

Eq. (3) was proved in [2] to be completely integrable equation.

Our aim from this work is to derive multiple regular soliton solutions and multiple singular soliton solutions for the (2+1)-dimensional equation (1). The modified form of the Hirota's bilinear method, that was established by Hereman et. al. [11] will be used to achieve the goal set for this work. The Hereman's method is now well-known in the literature, for more details see [2,10–23].

2. Multiple soliton solutions

In this section we will apply the Hereman's method which is a simplified form of the Hirota's bilinear method to study a generalized (2+1)-dimensional breaking soliton equation [1]

$$\left(u_{xt} - \beta(4u_{xy}u_x + 2u_{xx}u_y - u_{xxxy}) - \gamma(6u_xu_{xx} - u_{xxxx})\right)_x = -\alpha^2(\beta u_{yyy} + 3\gamma u_{xyy}). \tag{4}$$

To determine the dispersion relation we substitute

$$u(x,y,t) = e^{\theta_i}, \theta_i = k_ix + r_iy - \omega_i\, t, \tag{5}$$

into the linear terms of (4), and solving the resulting equation for ω_i, we find the dispersion relation is defined by

$$\omega_i = \frac{k_i^4(\beta r_i + \gamma k_i) + \alpha^2 r_i^2(\beta r_i + 3\gamma k_i)}{k_i^2}, i = 1, 2, \cdots N, \tag{6}$$

and hence θ_i becomes

$$\theta_i(x,y,t) = k_ix + r_iy - \frac{k_i^4(\beta r_i + \gamma k_i) + \alpha^2 r_i^2(\beta r_i + 3\gamma k_i)}{k_i^2}t, i = 1, 2, \cdots N. \tag{7}$$

We next substitute

$$u(x,y,t) = R\frac{\partial \ln f(x,y,t)}{\partial x} = R\frac{f_x(x,y,t)}{f(x,y,t)}, \tag{8}$$

where R is a constant that should be determined, and the auxiliary function $f(x,y,t)$ reads

$$f(x,y,t) = 1 + e^{k_1x + r_1y - \frac{k_1^4(\beta r_1 + \gamma k_1) + \alpha^2 r_1^2(\beta r_1 + 3\gamma k_1)}{k_1^2}t}, \tag{9}$$

into Eq. (4) and solve to find that

$$R = -2. \tag{10}$$

Substituting (9) and (10) into (8) gives the single soliton solution

$$u(x,y,t) = -\frac{2k_1 e^{k_1x + r_1y - \frac{k_1^4(\beta r_1 + \gamma k_1) + \alpha^2 r_1^2(\beta r_1 + 3\gamma k_1)}{k_1^2}t}}{1 + e^{k_1x + r_1y - \frac{k_1^4(\beta r_1 + \gamma k_1) + \alpha^2 r_1^2(\beta r_1 + 3\gamma k_1)}{k_1^2}t}}. \tag{11}$$

For the two soliton solutions, we substitute

$$u(x,y,t) = -2\frac{\partial \ln f(x,y,t)}{\partial x}, \tag{12}$$

where the auxiliary function $f(x,y,t)$ for the two soliton solutions reads

$$f(x,y,t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}, \tag{13}$$

into Eq. (4), where θ_1 and θ_2 are given above in (7) to obtain the phase shift a_{12} by

$$a_{12} = \frac{k_1^2k_2^2(k_1-k_2)^2 - \alpha^2(k_1r_2-k_2r_1)^2}{k_1^2k_2^2(k_1+k_2)^2 - \alpha^2(k_1r_2-k_2r_1)^2}, \tag{14}$$

and this can be generalized for the phase shifts by

$$a_{ij} = \frac{k_i^2k_j^2(k_i-k_j)^2 - \alpha^2(k_ir_j-k_jr_i)^2}{k_i^2k_j^2(k_i+k_j)^2 - \alpha^2(k_ir_j-k_jr_i)^2}, 1 \le i < j \le 3. \tag{15}$$

It is interesting to point out that the phase shifts do not depend on the parameters β and γ. Only the parameter α affects these phase shifts. Moreover, for $\alpha = 0$, the phase shifts reduce to

$$a_{ij} = \frac{(k_i-k_j)^2}{(k_i+k_j)^2}, 1 \le i < j \le 3, \tag{16}$$

that is consistent with the result obtained in [2]. The last result depends only on the coefficients of the space variable x only.

The auxiliary function $f(x,y,t)$ for the two soliton solutions is given by

$$\begin{aligned} f(x,y,t) &= 1 + e^{k_1x+r_1y-\frac{k_1^4(\beta r_1+\gamma k_1)+\alpha^2r_1^2(\beta r_1+3\gamma k_1)}{k_1^2}t} + e^{k_2x+r_2y-\frac{k_2^4(\beta r_2+\gamma k_2)+\alpha^2r_2^2(\beta r_2+3\gamma k_2)}{k_2^2}t} \\ &+ \frac{k_1^2k_2^2(k_1-k_2)^2-\alpha^2(k_1r_2-k_2r_1)^2}{k_1^2k_2^2(k_1+k_2)^2-\alpha^2(k_1r_2-k_2r_1)^2} \\ &\times e^{(k_1+k_2)x+(k_1+k_2)y-\left(\frac{k_1^4(\beta r_1+\gamma k_1)+\alpha^2r_1^2(\beta r_1+3\gamma k_1)}{k_1^2}+\frac{k_2^4(\beta r_2+\gamma k_2)+\alpha^2r_2^2(\beta r_2+3\gamma k_2)}{k_2^2}\right)t}. \end{aligned} \tag{17}$$

To determine the two soliton solutions explicitly, we substitute (17) into (8).

To determine the three soliton solutions, we use the auxiliary function

$$\begin{aligned} f(x,y,t) &= 1 + \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3) \\ &+ a_{12}\exp(\theta_1+\theta_2) + a_{23}\exp(\theta_2+\theta_3) + a_{13}\exp(\theta_1+\theta_3) \\ &+ b_{123}\exp(\theta_1+\theta_2+\theta_3), \end{aligned} \tag{18}$$

and proceed as before to find that

$$b_{123} = a_{12}a_{13}a_{23}, \tag{19}$$

where the phase shifts a_{ij} are defined in (15). To determine the three soliton solutions explicitly, we substitute the last result for $f(x,y,t)$ into (8). The higher level soliton solutions, for $N \ge 4$ can be obtained in a parallel manner. This confirms the fact that the (2+1)-dimensional breaking soliton equation (4) is completely integrable and gives rise to multiple soliton solutions of any order.

3. Multiple singular soliton solutions

In this section we will proceed as before study the multiple singular soliton solutions for a generalized (2+1)-dimensional breaking soliton equation

$$\left(u_{xt} - \beta(4u_{xy}u_x + 2u_{xx}u_y - u_{xxxy}) - \gamma(6u_xu_{xx} - u_{xxxx})\right)_x = -\alpha^2(\beta u_{yyy} + 3\gamma u_{xyy}). \tag{20}$$

The dispersion relation is the same as derived before, hence we set

$$\omega_i = \frac{k_i^4(\beta r_i + \gamma k_i) + \alpha^2 r_i^2(\beta r_i + 3\gamma k_i)}{k_i^2}, i = 1, 2, \cdots N, \tag{21}$$

and θ_i becomes

$$\theta_i(x, y, t) = k_i x + r_i y - \frac{k_i^4(\beta r_i + \gamma k_i) + \alpha^2 r_i^2(\beta r_i + 3\gamma k_i)}{k_i^2} t, i = 1, 2, \cdots N. \tag{22}$$

We next substitute

$$u(x, y, t) = R\frac{\partial \ln f(x, y, t)}{\partial x} = R\frac{f_x(x, y, t)}{f(x, y, t)}, \tag{23}$$

where R is a constant that should be determined, and the auxiliary function $f(x, y, t)$ for the singular case reads

$$f(x, y, t) = 1 - e^{k_1 x + r_1 y - \frac{k_1^4(\beta r_1 + \gamma k_1) + \alpha^2 r_1^2(\beta r_1 + 3\gamma k_1)}{k_1^2} t}, \tag{24}$$

into Eq. (20) and solve to find that

$$R = -2. \tag{25}$$

Substituting (24) and (25) into (23) gives the single singular soliton solution

$$u(x, y, t) = \frac{2k_1 e^{k_1 x + r_1 y - \frac{k_1^4(\beta r_1 + \gamma k_1) + \alpha^2 r_1^2(\beta r_1 + 3\gamma k_1)}{k_1^2} t}}{1 - e^{k_1 x + r_1 y - \frac{k_1^4(\beta r_1 + \gamma k_1) + \alpha^2 r_1^2(\beta r_1 + 3\gamma k_1)}{k_1^2} t}}. \tag{26}$$

For the two soliton solutions, we substitute

$$u(x, y, t) = -2\frac{\partial \ln f(x, y, t)}{\partial x}, \tag{27}$$

where the auxiliary function $f(x, y, t)$ for the two soliton solutions is given by

$$f(x, y, t) = 1 - e^{\theta_1} - e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2}, \tag{28}$$

into Eq. (20), where θ_1 and θ_2 are given above in (22) to obtain the phase shift a_{12} by

$$a_{12} = \frac{k_1^2 k_2^2 (k_1 - k_2)^2 - \alpha^2 (k_1 r_2 - k_2 r_1)^2}{k_1^2 k_2^2 (k_1 + k_2)^2 - \alpha^2 (k_1 r_2 - k_2 r_1)^2}, \tag{29}$$

and this can be generalized for the phase shifts by

$$a_{ij} = \frac{k_i^2 k_j^2 (k_i - k_j)^2 - \alpha^2 (k_i r_j - k_j r_i)^2}{k_i^2 k_j^2 (k_i + k_j)^2 - \alpha^2 (k_i r_j - k_j r_i)^2}, 1 \le i < j \le 3. \tag{30}$$

The phase shifts do not depend on the parameters β and γ, but depend on α only.

The auxiliary function $f(x, y, t)$ for the two soliton solutions is given by

$$\begin{aligned} f(x, y, t) &= 1 - e^{k_1 x + r_1 y - \frac{k_1^4(\beta r_1 + \gamma k_1) + \alpha^2 r_1^2(\beta r_1 + 3\gamma k_1)}{k_1^2} t} - e^{k_2 x + r_2 y - \frac{k_2^4(\beta r_2 + \gamma k_2) + \alpha^2 r_2^2(\beta r_2 + 3\gamma k_2)}{k_2^2} t} \\ &+ \frac{k_1^2 k_2^2 (k_1 - k_2)^2 - \alpha^2 (k_1 r_2 - k_2 r_1)^2}{k_1^2 k_2^2 (k_1 + k_2)^2 - \alpha^2 (k_1 r_2 - k_2 r_1)^2} \\ &\times e^{(k_1 + k_2)x + (k_1 + k_2)y - (\frac{k_1^4(\beta r_1 + \gamma k_1) + \alpha^2 r_1^2(\beta r_1 + 3\gamma k_1)}{k_1^2} + \frac{k_2^4(\beta r_2 + \gamma k_2) + \alpha^2 r_2^2(\beta r_2 + 3\gamma k_2)}{k_2^2}) t}. \end{aligned} \tag{31}$$

To determine the two singular soliton solutions explicitly, we substitute (31) into (23).

To determine the three soliton solutions, we use the auxiliary function

$$\begin{aligned} f(x,y,t) &= 1-\exp(\theta_1)-\exp(\theta_2)-\exp(\theta_3) \\ &+ a_{12}\exp(\theta_1+\theta_2)+a_{23}\exp(\theta_2+\theta_3)+a_{13}\exp(\theta_1+\theta_3) \\ &+ b_{123}\exp(\theta_1+\theta_2+\theta_3), \end{aligned} \tag{32}$$

and proceed as before to find that

$$b_{123} = -a_{12}a_{13}a_{23}, \tag{33}$$

where the phase shifts a_{ij} are defined in (30). To determine the three singular soliton solutions explicitly, we substitute the last result for $f(x,y,t)$ into (23). The higher level singular soliton solutions, for $N \geq 4$ can be obtained in a parallel manner.

4. Discussion

In this work we proved the integrability of a generalized (2+1)-dimensional breaking soliton equation. Multiple soliton solutions were formally derived for this equation. Moreover, multiple singular soliton solutions were derived as well. The Hereman's method shows effectiveness and reliability in handling nonlinear evolution equations.

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