



A Simple Hysteretic Constitutive Model for Unsaturated Flow

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Abstract In this paper, we present a constitutive model to describe unsaturated flow that considers the hysteresis phenomena. This constitutive model provides simple mathematical expressions for both saturation and hydraulic conductivity curves, and a relationship between permeability and porosity. The model is based on the assumption that the porous media can be represented by a bundle of capillary tubes with throats or “ink bottles” and a fractal pore size distribution. Under these hypotheses, hysteretic curves are obtained for saturation and relative hydraulic conductivity in terms of pressure head. However, a non-hysteretic relationship is obtained when relative hydraulic conductivity is expressed as a function of saturation. The proposed relationship between permeability and porosity is similar to the well-known Kozeny–Carman equation but depends on the fractal dimension. The performance of the constitutive model is tested against different sets of experimental data and previous models. In all of the cases, the proposed expressions fit fairly well the experimental data and predicts values of permeability and hydraulic conductivity better than others models. □

Keywords Constitutive model · Unsaturated flow · Hysteresis phenomena · Saturation · Hydraulic conductivity

1 Introduction

Constitutive models for unsaturated flow provide relationships between saturation (or water content), hydraulic conductivity and pressure head. These relationships define the hydraulic behavior of soils and are necessary for the numerical resolution of the nonlinear Richards equation (Richards 1931). From a numerical point of view, it is desirable that the math-

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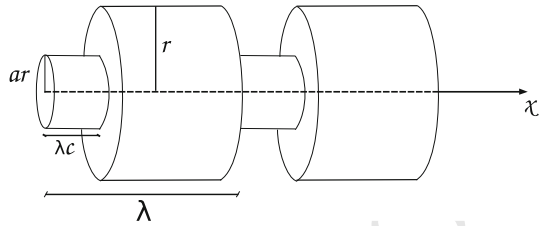
emathical expressions of the constitutive model have simple analytical forms with a small number of parameters in order to reduce the computational cost of each iteration of the linearization method. In the last decades, several empirical and semi-empirically models have been proposed, being the most widely used the [van Genuchten \(1980\)](#) and the [Brooks and Corey \(1964\)](#) models. Van Genuchten proposed an empirical relation for saturation to obtain a closed-form analytical expression for the hydraulic conductivity by using [Burdine \(1953\)](#) and [Mualem \(1976\)](#) predictive models. Similarly, the Brooks and Corey model combines a power-law relation for saturation with Burdine model to obtain a simple closed-form analytical expression for the hydraulic conductivity. More recently, [Assouline et al. \(1998\)](#) proposed a conceptual model for saturation based on the assumption that the soil structure results from a uniform random fragmentation process. Then, [Assouline \(2001\)](#) developed a model to predict the relative hydraulic conductivity based on the first two moments of the water retention curve. In the particular case of fractured rocks, a physical constitutive model based on fractal geometry has been proposed by [Guarracino \(2006\)](#) and [Monachesi and Guarracino \(2011\)](#).

Constitutive models describe hydraulic parameters at the representative elementary volume (REV) scale. The water flow in a REV is usually described by capillary tube models with different shapes and pore size distributions. Most models assume circular cross-sectional capillary tubes, but recently [Wang et al. \(2015\)](#) proposed a permeability model assuming arbitrary cross-sectional shapes of the tubes. Different approaches have been introduced to represent pore size distributions, e.g., multimodal, Gaussian and fractal distributions (e.g., [Rubin 1967](#); [Topp 1971](#); [Poulovassilis and Tzimas 1975](#); [Jerauld and Salter 1990](#); [Xu and Torres-Verdín 2013](#); [Guarracino et al. 2014](#)). Fractal distribution are commonly used to characterize porous media due to its simplicity and its capacity to describe a wide range of problems and soil textures (e.g., [Tyler and Wheatcraft 1990](#); [Yu et al. 2003](#); [Yu and Li 2001](#)). In particular, [Ghanbarian-Alavijeh et al. \(2011\)](#) propose a fairly recent review that illustrates the use of fractals to parameterize water retention curves.

Hydraulic properties of porous media present hysteresis phenomena which can significantly influence the flow and transport in partially saturated soils (e.g., [Rubin 1967](#); [Topp 1971](#); [Poulovassilis and Tzimas 1975](#); [Jerauld and Salter 1990](#)). Hysteresis refers to the non-unique relationships between pressure head and both saturation and hydraulic conductivity. This phenomena depends on the water movement history during the imbibition and drying processes and is mainly believed to be caused by irregularities in the cross section of the pores or “ink-bottle” effects, contact angle effects or entrapped air ([Jury et al. 1991](#); [Klausner 1991](#)). Modeling of hysteresis requires knowledge of at least one branch of the main hysteresis loop ([Mualem 1977](#)). In their review, [Pham et al. \(2005\)](#) divided hysteretic models into two main groups: domain models or physically based (e.g., [Néel 1942](#); [Mualem 1973](#)) and empirical models (e.g., [Feng and Fredlund 1999](#); [Karube and Kawai 2001](#)).

In this study, we derive a constitutive model for unsaturated flow assuming a porous media conceptualized as a bundle of constrictive capillary tubes with a fractal pore size distribution. The tubes present pore throats or “ink bottles” that allow to introduce the hysteresis in a simple form and also to characterize soils with high porosity and low permeability. Analytical closed-form expressions are obtained for saturation and hydraulic conductivity curves which are easy to evaluate and show a good agreement with experimental data. The proposed expressions have four independent physical and geometrical parameters: the fractal dimension of the pore size distribution, a radial factor that characterize the size of the pore throat, and the maximum and minimum values of pressure head. In addition, an expression for the permeability as a function of porosity is obtained from the proposed model which becomes similar to the Kozeny–Carman equation but shows a better agreement with different experimental data sets.

Fig. 1 Pore geometry of a single capillary tube with periodic pore throats



2 Constitutive Model

In this section, we derive closed-form analytical expressions for saturation and hydraulic conductivity curves. First, we present the pore geometry of the proposed model and we derive some hydraulic properties which are valid for a single pore. Then, assuming a cylindrical REV of porous media with a fractal pore size distribution, we obtain expressions for porosity, saturated hydraulic conductivity, and saturation and relative hydraulic conductivity curves.

2.1 Hydraulic Description at Pore Scale

The porous media is represented by a bundle of constrictive capillary tubes. Each pore is conceptualized as a cylindrical tube of radius r and length L with periodically throats represented by a segment of the tube with a smaller radius, as illustrated in Fig. 1.

Assuming that the pore geometry has a wavelength λ and that the length of the tube L contains an integer number N of wavelengths, the pore radius along the tube can be described as follows:

$$r(x) = \begin{cases} ar & \text{if } x \in [0 + 2\pi n, \lambda c + 2\pi n) \\ r & \text{if } x \in [\lambda c + 2\pi n, \lambda + 2\pi n) \end{cases} \quad (1)$$

where a is the radial factor ($0 \leq a \leq 1$), c is the length factor of the pore throat ($0 \leq c \leq 1$) and $n = 0, 1, \dots, N - 1$. The parameter a represents the ratio in which the radius is reduced, and the parameter c represents the fraction of λ with a narrow neck. Note that if $c = 1$ or $c = 0$ we obtain a straight tube with radii ar or r , respectively.

Based on the above assumptions, the volume of a single tube can be calculated by integrating the cross-sectional area over the length L as follows:

$$V_p(r) = \int_0^L \pi r^2(x) dx = N \left[\int_0^{\lambda c} \pi r^2 dx + \int_{\lambda c}^{\lambda} \pi (ra)^2 dx \right] = L \pi r^2 f_v(a, c), \quad (2)$$

where

$$f_v(a, c) = a^2 c + 1 - c, \quad (3)$$

f_v is a factor that varies between 0 and 1, and quantifies the reduction in pore volume due to the constrictivity of the tube. A density plot of f_v is shown in Fig. 2a. Note that low values of parameter c or large values of parameter a produce small variations of the pore volume.

Under the assumption of laminar flow and ignoring the convergence and divergence of the flow, the volumetric flow rate of a pore with periodical varying aperture $Q_p(r)$ can be approximated with (Bodurtha 2003; Bousfield and Karles 2004):

$$Q_p(r) = \frac{\rho g}{\mu} \left[\frac{1}{L} \int_0^L \frac{8}{\pi r^4(x)} dx \right]^{-1} \frac{\Delta h}{L}, \quad (4)$$

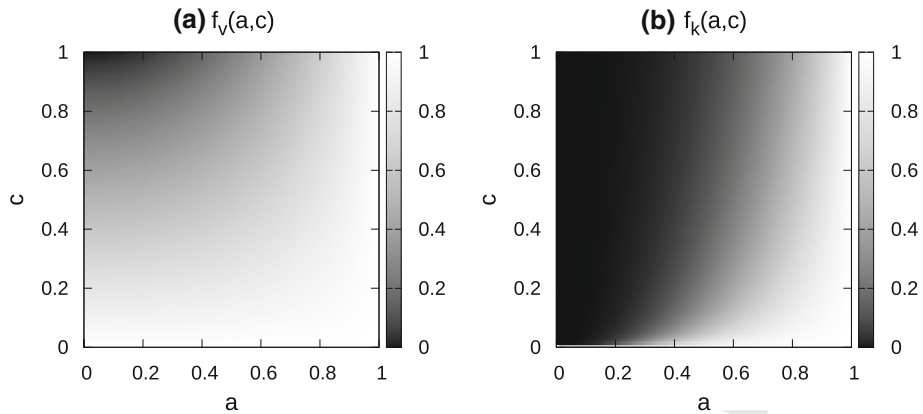


Fig. 2 Dimensionless factors $f_v(a, c)$ and $f_k(a, c)$. These factors control the pore volume and the volumetric flow rate at pore scale, and the porosity and saturated hydraulic conductivity at REV scale

100 where ρ is the water density, g gravity, μ water viscosity and Δh the head drop across the
 101 tube.

102 Substituting Eq. (1) in Eq. (4) yields:

$$103 \quad Q_p(r) = \frac{\rho g \pi r^4}{\mu} f_k(a, c) \frac{\Delta h}{L}, \quad (5)$$

104 where

$$105 \quad f_k(a, c) = \frac{a^4}{c + a^4(1 - c)}, \quad (6)$$

106 f_k is a factor that quantifies the volumetric flow rate reduction due to pore throats and varies
 107 between 0 and 1. Figure 2b shows the variation of f_k as a function of the radial factor a and
 108 the length factor c . As it can be expected, low values of a drastically reduce the volumetric
 109 flow rate of the tube.

110 If we compare Fig. 2a, b, it can be noticed that the values of the parameters a and c modify
 111 the volume and the volumetric flow rate of the tube in different ways. For example, for low
 112 values of a and c , the volume of the pore is slightly affected while the volumetric flow of
 113 the pore significantly decreases. Also note that for $a = 1$ or $c = 0$, $f_v = f_k = 1$, and the
 114 expressions obtained for Eqs. (2) and (5) represent the volume and the volumetric flow rate
 115 of a straight tube of radius r , respectively.

116 2.2 Hydraulic Description at REV Scale

117 To derive the expressions for saturation and hydraulic conductivity, we consider as a REV a
 118 straight circular cylinder of radius R and length L . The choice of the REV geometry is based
 119 on the shape of soil samples commonly used in laboratory tests. Other geometries, such as
 120 rectangular REV, can also be considered by introducing minor changes in model derivation.
 121 The pore structure of the REV is represented by a bundle of constrictive tubes (as described
 122 in the previous section) with a fractal pore size distribution. We also assume that the pore
 123 radius r varies from a minimum value r_{\min} to a maximum value r_{\max} .

124 The cumulative size distribution of pores is assumed to obey the following fractal law
 125 (e.g., Tyler and Wheatcraft 1990; Yu 2008; Guarracino 2007):

$$126 \quad N(r) = \left(\frac{r}{R}\right)^{-D}, \quad (7)$$

127 where D ($1 < D < 2$) is the fractal dimension. Note that if $r_{\max} = R$, $N = 1$ and the REV is
 128 fully occupied by a single pore. On the other hand, if $r_{\min} = 0$, the REV contains an infinite
 129 number of pores.

130 Differentiating Eq. (7) with respect to r , we obtain the number of pores whose sizes are
 131 within the infinitesimal range r and $r + dr$:

$$132 \quad dN(r) = -DR^D r^{-D-1} dr. \quad (8)$$

133 The negative sign in Eq. (8) implies that the pore number decreases with the increase in the
 134 pore size (Yu et al. 2003).

135 The porosity ϕ of the REV can be computed from its definition:

$$136 \quad \phi = \frac{\text{Volume of pore space}}{\text{Volume of REV}} = \frac{\int_{r_{\min}}^{r_{\max}} V_p(r) dN(r)}{\pi R^2 L}. \quad (9)$$

137 Replacing Eqs. (2) and (8) into Eq. (9), the porosity of the REV can be expressed as:

$$138 \quad \phi = f_v \phi^{\text{ST}}, \quad (10)$$

139 where

$$140 \quad \phi^{\text{ST}} = \frac{D}{R^{(2-D)}(2-D)} \left[r_{\max}^{2-D} - r_{\min}^{2-D} \right] \quad (11)$$

141 is the porosity of the REV considering straight tubes (i.e., $a = 1$).

142 The volumetric flow rate Q at REV scale can be obtained by integrating all the pores
 143 volumetric flow rates given by Eq. (5) over the entire range of pore sizes:

$$144 \quad Q = \int_{r_{\min}}^{r_{\max}} Q_p(r) dN(r) = \frac{\rho g}{\mu} \frac{f_k}{8} \frac{\Delta h}{L} \frac{\pi D R^D}{(4-D)} \left[r_{\max}^{4-D} - r_{\min}^{4-D} \right]. \quad (12)$$

145 On the other hand, on the basis of Darcy's law (1856), the volumetric flow rate through the
 146 REV can be expressed as:

$$147 \quad Q = K_s \frac{\Delta h}{L} \pi R^2, \quad (13)$$

148 where K_s is the saturated hydraulic conductivity. Combining Eqs. (12) and (13), an expression
 149 for K_s is obtained:

$$150 \quad K_s = f_k K_s^{\text{ST}}, \quad (14)$$

151 where

$$152 \quad K_s^{\text{ST}} = \frac{\rho g}{\mu} \frac{1}{8} \frac{D}{R^{(2-D)}(4-D)} \left[r_{\max}^{4-D} - r_{\min}^{4-D} \right] \quad (15)$$

153 is the saturated hydraulic conductivity of the REV considering straight tubes.

154 By inspection of Eqs. (10) and (14), it can be noticed that the factors f_v and f_k produce
 155 different changes in the macroscale properties of the REV ϕ and K_s , respectively. It can
 156 be demonstrated that for every value of parameters a and c , f_k is always smaller than f_v

Author Proof

157 allowing us to represent media with high porosity, low permeability and low specific surface
 158 area. Our model is also able to describe media which have the same porosity but different
 159 permeabilities. For example, clay and sand soils have typically similar porosities, but their
 160 hydraulic conductivities differ by several orders of magnitude (e.g., [Carsel and Parrish 1988](#)).

161 For most porous media, $r_{\min}/r_{\max} \simeq 10^{-2}$ ([Yu and Li 2001](#)); then, we can assume that
 162 $r_{\min} \ll r_{\max}$. Under the above assumption, the terms r_{\min}^{2-D} and r_{\min}^{4-D} in Eqs. (11) and (15)
 163 can be considered negligible. Then, combining the resulting expressions, we can obtain the
 164 following relationship between K_s and ϕ :

$$165 \quad K_s = \alpha f_k \left[\frac{\phi}{f_v} \right]^{\left(\frac{4-D}{2-D}\right)}, \quad (16)$$

166 where

$$167 \quad \alpha = \frac{\rho g}{\mu} \frac{DR^2}{8(4-D)} \left[\frac{2-D}{D} \right]^{\left(\frac{4-D}{2-D}\right)}. \quad (17)$$

168 Note that the exponent of porosity in Eq. (16) $(4-D)/(2-D)$ is greater than 3. In the
 169 limit case of a cubic exponent, Eq. (16) becomes similar to the KC equation. This issue will
 170 be further analyzed in Sect. 3.1 where Eq. (16) is tested against experimental data sets.

171 2.3 Saturation and Relative Hydraulic Conductivity Curves

172 In this section, we derive the saturation and relative hydraulic conductivity curves of the
 173 constitutive model. Due to the varying aperture of the pores, the retention curves obtained
 174 from drainage and imbibition tests are expected to be different. The hysteresis phenomena
 175 can be easily introduced in the model thanks to the pore geometry illustrated in Fig. 1 and
 176 described by Eq. (1).

177 For a straight tube, we can relate the radius of the water-filled pore r_h to the pressure head
 178 h by the following equation ([Jurin 1717](#); [Bear 1998](#)):

$$179 \quad h = \frac{2\sigma \cos(\beta)}{\rho g r_h}, \quad (18)$$

180 where σ is the surface tension of the water and β the contact angle.

181 To obtain the main drying saturation curve, we consider that the REV is initially fully
 182 saturated and is drained by a pressure head h . We assume that a tube becomes fully desaturated
 183 if the radius of the pore throat ar is greater than the radius r_h given by Eq. (18). Then it is
 184 reasonable to also assume that pores with radii r between r_{\min} and r_h/a will remain fully
 185 saturated. Therefore, according to Eqs. (2) and (8), the drying saturation curve S_e^d can be
 186 computed by:

$$187 \quad S_e^d = \frac{\int_{r_{\min}}^{r_h/a} V_p(r) dN(r)}{\int_{r_{\min}}^{r_{\max}} V_p(r) dN(r)} = \frac{\left(\frac{r_h}{a}\right)^{2-D} - r_{\min}^{2-D}}{r_{\max}^{2-D} - r_{\min}^{2-D}}. \quad (19)$$

188 Substituting Eq. (18) into Eq. (19) yields

$$189 \quad S_e^d(h) = \begin{cases} 1 & \text{if } h \leq \frac{h_{\min}}{a} \\ \frac{(ha)^{D-2} - h_{\max}^{D-2}}{h_{\min}^{D-2} - h_{\max}^{D-2}} & \text{if } \frac{h_{\min}}{a} \leq h \leq \frac{h_{\max}}{a}, \\ 0 & \text{if } h \geq \frac{h_{\max}}{a} \end{cases}, \quad (20)$$

190 where

$$191 \quad h_{\min} = \frac{2\sigma \cos(\beta)}{\rho g r_{\max}} \quad h_{\max} = \frac{2\sigma \cos(\beta)}{\rho g r_{\min}}, \quad (21)$$

192 h_{\min} and h_{\max} being the minimum and maximum pressure heads defined by r_{\max} and r_{\min} ,
 193 respectively.

194 Similarly, the main wetting saturation curve can be obtained assuming that the REV is
 195 initially dry and it is flooded with a pressure h . In this case, only the tubes with radius r
 196 smaller than r_h will be fully saturated. Then the main wetting saturation curve S_e^w can be
 197 expressed as:

$$198 \quad S_e^w(h) = \begin{cases} 1 & \text{if } h \leq h_{\min} \\ \frac{h^{D-2} - h_{\max}^{D-2}}{h_{\min}^{D-2} - h_{\max}^{D-2}} & \text{if } h_{\min} \leq h \leq h_{\max} \\ 0 & \text{if } h \geq h_{\max} \end{cases} \quad (22)$$

199 Using the same hypothesis and neglecting film flow on tube surfaces, we can obtain
 200 the main drying and wetting curves for relative hydraulic conductivity. During drainage
 201 only tubes with pore throat radius $a r$ smaller than r_h are fully saturated. Then, the main
 202 contribution to the total volumetric flow through the REV can be obtained by integrating
 203 the individual volumetric flow rates Q_p given by Eq. (5) over the pores that remain fully
 204 saturated ($r_{\min} \leq r \leq r_h/a$):

$$205 \quad Q = \int_{r_{\min}}^{r_h/a} Q_p(r) dN(r). \quad (23)$$

206 Otherwise, according to Buckingham–Darcy’s law (Buckingham 1907), the total volu-
 207 metric flow rate through the REV can be expressed as:

$$208 \quad Q = K_s K_r(h) \frac{\Delta h}{L} \pi R^2, \quad (24)$$

209 where K_r is the relative hydraulic conductivity which is a dimensionless function of h and
 210 varies between 0 and 1.

211 Combining Eqs. (23) and (24), and using Eqs. (5), (8) and (14), we obtain the relative
 212 hydraulic conductivity for the drying process:

$$213 \quad K_r^d = \frac{\left(\frac{r_h}{a}\right)^{4-D} - r_{\min}^{4-D}}{r_{\max}^{4-D} - r_{\min}^{4-D}}. \quad (25)$$

214 Finally, using Eq. (18) we can express Eq. (25) in terms of pressure head:

$$215 \quad K_r^d(h) = \begin{cases} 1 & \text{if } h \leq \frac{h_{\min}}{a} \\ \frac{(ha)^{D-4} - h_{\max}^{D-4}}{h_{\min}^{D-4} - h_{\max}^{D-4}} & \text{if } \frac{h_{\min}}{a} \leq h \leq \frac{h_{\max}}{a} \\ 0 & \text{if } h \geq \frac{h_{\max}}{a} \end{cases} \quad (26)$$

216 Similarly, the main wetting relative hydraulic conductivity curve $K_r^w(h)$ can be derived
217 by integrating Eq. (23) over the range of saturated pores ($r_{\min} \leq r \leq r_h$):

$$218 \quad K_r^w(h) = \begin{cases} 1 & \text{if } h \leq h_{\min} \\ \frac{h^{D-4} - h_{\max}^{D-4}}{h_{\min}^{D-4} - h_{\max}^{D-4}} & \text{if } h_{\min} \leq h \leq h_{\max} \\ 0 & \text{if } h \geq h_{\max} \end{cases} \quad (27)$$

219 Note that saturation and relative hydraulic conductivity expressions for both drying and
220 wetting have analytical closed forms with only four independent parameters with geometrical
221 and physical meaning: a , D , h_{\min} and h_{\max} .

222 In the classical models of hysteresis, saturation and relative hydraulic conductivity values
223 are limited by main drying and wetting curves which are obtained for initially fully saturated
224 and dry porous media, respectively. For any intermediate state that does not correspond
225 to a fully saturated or dry medium, scanning curves can be scaled from the main drying
226 and wetting curves for both relationships, $S_e(h)$ and $K_r(h)$. These scanning curves can be
227 generated using different approaches such as play-type (Beliaev and Hassanizadeh 2001) or
228 scaling hysteresis (Parker and Lenhard 1987).

229 Relative hydraulic conductivity K_r can also be expressed in terms of saturation S_e . By
230 combining Eqs. (19) and (25), we obtain the following unique equation for the drying and
231 the wetting:

$$232 \quad K_r = \frac{\left\{ S_e \left[\left(\frac{h_{\min}}{h_{\max}} \right)^{D-2} - 1 \right] + 1 \right\}^{\frac{D-4}{D-2}} - 1}{\left(\frac{h_{\min}}{h_{\max}} \right)^{D-4} - 1}. \quad (28)$$

233 It is interesting to remark that the relationship $K_r(S_e)$ results in a non-hysteretic function
234 across the entire range of saturations and this result is in agreement with a number of exper-
235 imental data (e.g., van Genuchten 1980; Mualem and Klute 1986; Topp and Miller 1966).

236 For $h_{\max} \gg h_{\min}$, Eq. (28) can be reduced to:

$$237 \quad K_r = S_e^{\frac{D-4}{D-2}}, \quad (29)$$

238 which is consistent with the well-known Brooks and Corey model, $K_r = S^{\frac{2+3\lambda}{\lambda}}$, where λ is
239 a dimensionless and empiric parameter related to the pore size distribution. Parameter λ can
240 be related to the fractal dimension D through $\lambda = (D - 2)/(1 - D)$. Considering the range
241 of λ values between 0.21 and 3.02 reported by Assouline (2005) for different type of soils, it
242 yields values of D between 1.249 and 1.826 which are consistent with the admissible range
243 of D values.

244 3 Comparison with Experimental Data

245 In the present section, we test the ability of the proposed model to reproduce available
246 measured data from the research literature. These data sets consist of measured permeability–
247 porosity, relative hydraulic conductivity–saturation, and hysteretic saturation–pressure head
248 values for different soil textures.

Table 1 Values of the fitted parameters (D , C and C_{KC}) and the RMSD corresponding to the proposed model (Eq. 30) and to the KC equation (Eq. 32)

Soil type	D	C (mD)	RMSD	C_{KC} (mD)	RMSD _{KC}
Fluvial and deltaic	1.68	1.336×10^7	1.1386	44.85	1.3856
Timimoun Basin	1.512	3.452×10^5	1.1894	75	1.3942
Fine-grained sandstone	1.498	1.797×10^6	0.5988	4.4×10^3	0.8910
Silty sandstones	1.524	5.1×10^5	0.5478	8.95×10^3	0.7680

3.1 Permeability

In order to test the proposed relationship between permeability and porosity for different types of soils, we selected four data series from Luffel et al. (1991), Hirst et al. (2001) and Chilindar (1964). As it is well known, permeability k and saturated hydraulic conductivity K_s are related through $K_s = \rho g k / \mu$. According to Eq. (16), the proposed permeability model can be expressed as follows:

$$k = C \phi^{\left(\frac{4-D}{2-D}\right)}, \tag{30}$$

where

$$C = \frac{\mu}{\rho g} \alpha f_k f_v^{\left(\frac{2-D}{4-D}\right)}. \tag{31}$$

Equation (30) will be also compared with the Kozeny–Carman equation which reads (Kozeny 1927; Carman 1937):

$$k = C_{KC} \frac{\phi^3}{(1-\phi)^2}, \tag{32}$$

where C_{KC} is a parameter that depends on the specific internal surface area, an empirical geometrical parameter and the tortuosity.

For each type of soil, Eqs. (30) and (32) are fitted to measured data by minimizing the root-mean-square deviation (RMSD):

$$\text{RMSD} = \left\{ \frac{1}{n} \sum_{i=1}^n [\log(k_i^{\text{calc}}) - \log(k_i^{\text{dat}})]^2 \right\}^{0.5}, \tag{33}$$

where k^{calc} and k^{dat} correspond to the calculated and measured permeabilities, respectively. A logarithmic scale was considered because of the wide range of variation for the permeability values. The fitted parameters for Eqs. (30) and (32) are listed in Table 1 as well as their respective RMSD. It can be noted that, for all soils, the RMSD of the proposed model is smaller than the ones from the KC equation. Figure 3 shows that the proposed relationship predicts fairly good the observed values over a range of 4–10 orders of magnitude.

3.2 Relative Hydraulic Conductivity

The proposed relative hydraulic conductivity model (Eq. 29) is tested against 8 experimental data series from the Mualem (1974) (see Table 2). These data series have been also used

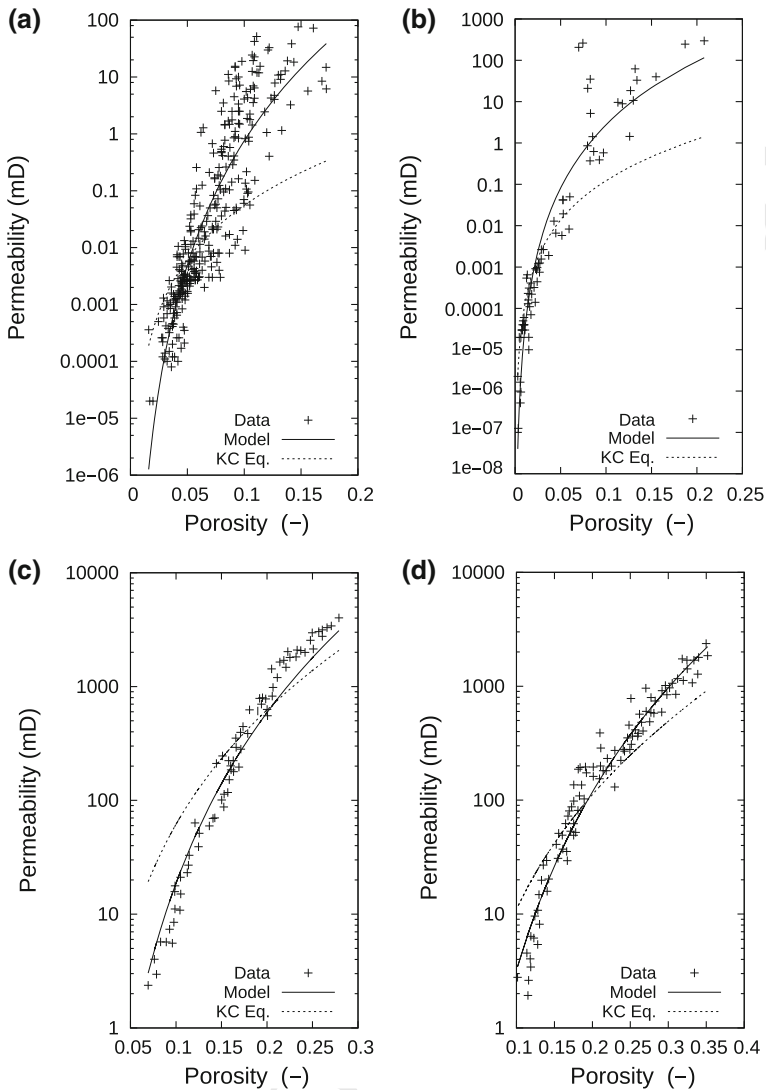


Fig. 3 Comparison between the proposed model, the KC equation and experimental data sets of permeability–porosity. **a** Early Cretaceous Fluvial and Deltaic Channel Sandstones (data from Luffel et al. 1991), **b** Carboniferous and Devonian Timimoun Basin (“tight gas” sandstones) (data from Hirst et al. 2001), **c** fine-grained sandstone and **d** silty sandstones (data from Chilindar 1964)

275 by Assouline to test his model which predicts K_r from the first two moments of the water
 276 retention (Assouline 2001). For each soil type, the proposed model is fitted to the measured
 277 data by minimizing the RMSD.

278 Figure 4 illustrates the fit of Eq. (29) and Assouline model to 2 sets of experimental data
 279 (Sable de Riviere and Gilat sandy loam) using the parameters given in Table 2. It can be
 280 noticed that the proposed model shows a significant improvement over the one of Assouline
 281 for the Gilat sandy loam (see Fig. 4b). Table 2 lists the resulting best fitted parameters for the

Table 2 Values of the fitted parameters (D and h_{\min}/h_{\max}), the corresponding RMSD and the RMSD of Assouline’s model (2001)

Soil type	D	$\frac{h_{\min}}{h_{\max}}$	RMSD	RMSD (Assouline)
Sable de riviere	1.99	0.101	0.015	0.036
Gilat sandy loam	1.012	1.09×10^{-4}	0.033	0.252
Pouder river sand	1.112	1.09×10^{-4}	0.071	0.076
Amarillo silty clay loam	1.387	0.001	0.009	0.014
Rubicon sandy loam	1.999	0.088	0.021	0.046
Guelph loam	1.918	0.021	0.004	0.037
Weld silty clay loam	1.508	0.061	0.036	0.038
Silt Mont Cerris soils	1.376	1.09×10^{-4}	0.082	0.188

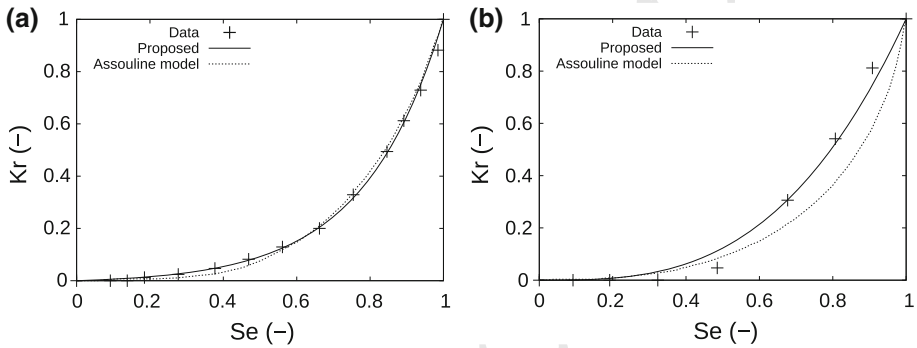


Fig. 4 Comparison between the relative hydraulic conductivity (Eq. 29) and measured data: **a** Sable de Riviere and **b** Gilat sandy loam (data from Mualem 1974). The figure also includes the fit of Assouline model (2001)

282 8 experimental data series, their RMSD and the corresponding RMSD obtained by Assouline
 283 (2001). For all soil types, the RMSD values of our model are smaller than the ones obtained
 284 with the Assouline model. Note that for Gilat sandy loam and Guelph loam soils, the RMSD
 285 is even 1 order of magnitude smaller.

286 **3.3 Saturation Curve Hysteresis**

287 To test the ability of the model to describe the hysteresis phenomena, we compare the main
 288 wetting and drying curves (Eqs. 20, 22) to experimental data from the literature. Two different
 289 soil types from Pham et al. (2003) are used: Beaver Creek sand and a processed silt. The
 290 maximum and minimum values of pressure head (Eq. 21) were determined by try-and-error
 291 method (see Table 3). Then, the fractal dimension D and the radial factor a have been
 292 estimated by minimizing the RMSD between calculated and measured values of both drying
 293 and wetting curves using an exhaustive search method. Table 3 shows the model parameters
 294 and the RMSD values for each soil. Note that even if the model is simple, the hysteretic
 295 behavior of saturation can be fairly fitted by a minimum number of parameters. It is important
 296 to remark that only one set of parameters a and D explains both drying and wetting curves
 297 simultaneously (see Fig. 5).

Author Proof

Table 3 Values of the fitted parameters (D and a) and the corresponding RMSD

Soil type	D	a	h_{\min} (m)	h_{\max} (m)	RMSD
Beaver Creek sand	1.0266	0.4008	0.112	100.0	1.2566×10^{-2}
Processed silt	1.7598	0.4126	0.510	10.20	1.1178×10^{-2}

Parameters h_{\min} and h_{\max} have been fixed before the estimation of D and a

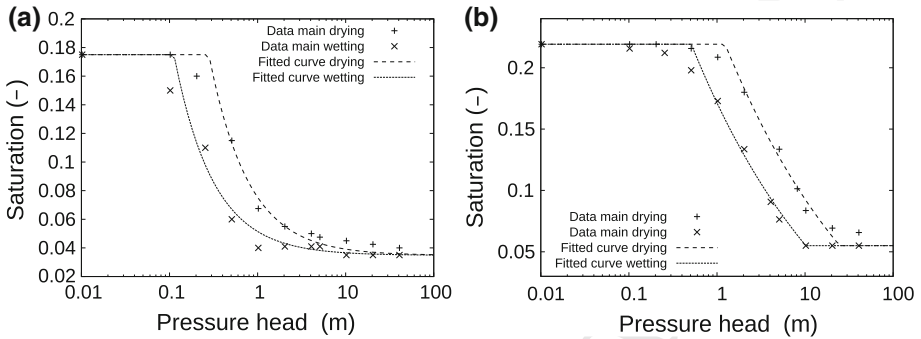


Fig. 5 Comparison of the main drying and wetting saturation curves from the proposed model with experimental data sets: **a** Beaver Creek sand and **b** processed silt (data from Pham et al. 2003)

298 Note that the expression of the drying curve (Eq. 20) depends on parameters a and D .
 299 This enables us to fit these parameters using only experimental data from the main drying
 300 hysteresis loop and then to predict the main wetting curve by using Eq. (22). Following this
 301 alternative fitting procedure, the parameter values of the drying curve are: $a = 0.627$ and
 302 $D = 1.314$ (RMSD 1.877×10^{-2}) for the Beaver Creek sand, and $a = 0.401$ and $D = 1.722$
 303 (RMSD 1.362×10^{-2}) for the processed silt. Note that only the parameters of the processed
 304 silt are similar to the ones listed in Table 3. The prediction of the wetting curve from the
 305 drying curve could be an additional advantage of the proposed model that needs to be verified
 306 with a more exhaustive analysis and additional experimental data.

307 4 Discussion and Conclusion

308 A physically based theoretical model for estimating the hydraulic properties for unsaturated
 309 flow in porous media has been presented. The derivation of the model relies on the assump-
 310 tion that porous media can be represented by a bundle of cylindrical tubes with periodically
 311 throats and a fractal pore size distribution. Based on geometrical properties and physical
 312 laws, analytical closed-form expressions were obtained for the saturation and the relative
 313 hydraulic conductivity as functions of pressure head. These expressions contain four inde-
 314 pendent parameters (a , D , r_{\min} and r_{\max}), all of them with a specific physical or geometrical
 315 meaning. It is worth mentioning that the direct determination of these parameters is a diffi-
 316 cult task due to the need to know in detail the microscopic geometry of the porous media.
 317 Considering the current developments in imaging technology, direct measurements of the
 318 pore structure can be obtained using X-ray tomography (Wildenschild 2002). Lindquist et al.
 319 (2000) applied this technique to measure distributions of channel length, throat size and pore
 320 volume of Fontainebleau sandstones. More recently, Dong and Blunt (2009) developed a max-

321 imal ball algorithm to extract pore networks from X-ray tomography images and computed
 322 distributions of pore and throat size, pore spacing and pore shape factor of Fontainebleau and
 323 Berea sandstones.

324 Hysteresis in the saturation and relative hydraulic conductivity curves have been easily
 325 introduced in the model by assuming periodic constrictivities through the radial factor a and
 326 the length factor c (Fig. 1). It is interesting to note that when the relative hydraulic conductivity
 327 is expressed in terms of saturation a unique non-hysteretic relationship is obtained for both
 328 drainage and imbibition tests (Eq. 29). This behavior is consistent with previous studies
 329 and experimental data (Fig. 4; Topp and Miller 1966; van Genuchten 1980; Mualem and
 330 Klute 1986). Several causes have been proposed to justify hysteresis phenomena (e.g., Jury
 331 et al. 1991; Klausner 1991). These results enhance the hypothesis that hysteresis originates
 332 from pore throats or “ink-bottle” effects. Nevertheless, other effects could also explain or
 333 contribute to hysteresis in porous media, such as network effects, contact angle hysteresis
 334 and film flow (e.g., Blunt et al. 2002; Spiteri et al. 2008; Maineult et al. 2017). Note that
 335 when the radial factor $a = 1$ (straight tubes), the hysteresis disappears from the saturation
 336 and relative hydraulic conductivity curves (see Sect. 2.3).

337 The presence of throats in the capillary tubes also modifies the porosity and permeability
 338 through the factors f_v and f_k (Eqs. 10, 14), respectively. Both factors depend on a and c , and
 339 vary between 0 and 1 (Eqs. 3, 6). Nevertheless, the factor f_k that modifies the permeability is
 340 always smaller than the factor f_v that affects the porosity. This allows the model to describe
 341 media with high porosity, low permeability and low specific surface area, which cannot be
 342 properly represented with straight tube models.

343 The fractal dimension D is a geometrical parameter that determines the pore size distribu-
 344 tion of the model. This fractal distribution has been found to be useful to describe groundwater
 345 flow in the literature (e.g., Tyler and Wheatcraft 1990; Yu et al. 2003; Yu and Li 2001; Guar-
 346 racino et al. 2014). The fractal dimension can be related to the pore size distribution index λ
 347 proposed in the Brooks and Corey model (see Sect. 2.3), providing a geometrical meaning
 348 to this empirical parameter.

349 The proposed model also provides a relationship between permeability and porosity
 350 (Eq. 30), which under simplifying assumptions is similar to the KC equation. However,
 351 the proposed model performs better than the KC equation when compared to experimental
 352 permeability data ranging over 4–10 orders of magnitudes (Fig. 3).

353 This study allowed the development of a framework to describe saturation and relative
 354 hydraulic conductivity curves that include hysteresis phenomena. The relative hydraulic
 355 conductivity has been validated using experimental data from different type of soils, showing
 356 better agreements than Assouline model (Fig. 4). The hysteretic saturation curves have also
 357 been successfully tested with experimental data by fitting only 2 model parameters: a and D
 358 (Fig. 5).

359 From a mathematical point of view, all the expressions have analytical closed forms, which
 360 are simple and easy to evaluate. Therefore, their implementation in numerical flow codes is
 361 straightforward and involves little additional computational effort compared to non-hysteretic
 362 simulations.

363 This simple constitutive model can be a starting point to describe other physical phenomena
 364 that require hydraulic description at pore scale, such as generation of streaming potential
 365 (e.g., Jougnot et al. 2012), ionic transport and mixing in capillaries (e.g., Dentz et al. 2011),
 366 geochemical reactions in porous media (e.g., Guarracino et al. 2014) and wave-induced fluid
 367 flow (e.g., Rubino et al. 2013).

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