# THREE FORMS OF PHYSICAL MEASUREMENT AND THEIR COMPUTABILITY

EDWIN BEGGS, JOSÉ FÉLIX COSTA, AND JOHN V TUCKER

**Abstract.** We have begun a theory of measurement in which an experimenter and his or her experimental procedure are modelled by algorithms that interact with physical equipment through a simple abstract interface. The theory is based upon using models of physical equipment as oracles to Turing machines. This allows us to investigate the computability and computational complexity of measurement processes. We examine eight different experiments that make measurements, and, by introducing the idea of an observable indicator, we identify three distinct forms of measurement process and three types of measurement algorithm. We give axiomatic specifications of three forms of interfaces that enable the three types of experiment to be used as oracles to Turing machines, and lemmas that help certify an experiment satisfies the axiomatic specifications. For experiments that satisfy our axiomatic specifications, we give lower bounds on the computational power of Turing machines in polynomial time using non-uniform complexity classes. These lower bounds break the barrier defined by the Church-Turing Thesis.

**§1. Introduction.** Imagine a simple experiment to measure a physical quantity. An experimenter applies an experimental procedure to some equipment and generates a sequence of values that measures the quantity to some degree of accuracy. The procedure is systematic and precisely defined to enable it to be reproduced by others elsewhere. Due to its systematic and precise nature, the experimental procedure can be thought of as an algorithm that governs each step in the experimental process by instructions that include physical and mathematical operations.

Indeed, in practice, many experiments are controlled, at least partially, by software. The software codes a set of algorithms that can generate data to operate equipment, and collect and analyse mathematically data from observations. The human experimenter is supported by, or is dependent on, software in all sorts of ways. We have begun to develop an algorithmic theory of measurement based on the radical idea that *the human experimenter, the systematic and precise procedure that defines the experiment, and his or her software tools, are all replaced by a computer running a complex program that interacts with physical equipment through an abstract interface.*

The mathematical theory of computability provides a vast range of concepts, methods and theorems to model formally algorithms, programs and machines and to analyse their behaviour, scope and limits. Applying computability theory, we have proposed the following idea:

*The human experimenter and his or her systematic and precise experimental* *procedure can be modelled by a Turing machine; and the interaction with physical equipment can be modelled by using the physical equipment as an oracle that is connected to the Turing machine via an interface and operational protocol for queries.*

The theory developed from this idea enables us to analyse some basic questions about measurement, including:

(i) What is the algorithmic nature of the process of measuring?

(ii) What are the costs in time to perform measurements to a given accuracy?

(iii) What are the limitations to measurements imposed by equipment, experimental procedures and resource constraints?

There are indeed limitations. Elsewhere, in [13], we have begun to answer (iii) by showing that if experiments are controlled by algorithms then not all physical quantities can be measured even in simple classical experiments. The theory focusses on logical properties of measurement and enables us ask new open-ended questions:

(iv) Can measurement processes for obtaining physical data be classified by analysing their logical or algorithmic complexity?

(v) What information can or cannot be computed with the physical data we obtain from measurement processes?

In the matter of (v), we have in mind the processing of data *after* it has been obtained from experiments, instruments, sensors, etc. We have shown – and will show once more here – that a variety of physical equipment used as oracles can boost the power of algorithms beyond the Turing barrier: sets that are algorithmically undecidable can be decided by algorithms enhanced by physical oracles: see the sequence of papers [6, 5, 7, 8, 14, 13, 15].

The first part of the paper addresses question (i) above. We focus on the measurement of a physical quantity by an experimenter following a procedure to operate apparatus. In Section 2, we use Hempel's axiomatic conception of measurement by trial-and-error to introduce a simple taxonomy of three distinct logical types of measurement process. We do this by introducing a function we call an *observable indicator* whose curve illustrates a key characteristic of the measurement. Then, in the following Sections 3-5, we describe eight examples of experiments to illustrate each logical type in the taxonomy. The experiments vary, involving: weighing scales, particle collisions, optical refraction, electrical resistance, scattering in an electric field, and photoelectric effects. The taxonomy and examples lead us to identify three distinct forms of measurement process characterised by three types of algorithm, each with a natural iterative structure; these are defined and compared in Section 6. We call these *signed*, *vanishing* and *threshold measurements*.

The second part of the paper addresses questions (iv) and (v) above. In the following Sections 6-11 we connect the three types of experiments to our theory of measurement based upon physical oracles to Turing machines. In Section 8, we specify axiomatically three abstract interfaces that allows *any* algorithm to operate or access the equipment. The axiomatic specifications of the interfaces to the Turing machines are relatively abstract and so in Section 8 we include *Certification Lemmas*, involving the observable indicators, to help demonstrate that physical experiments actually satisfy the axioms. The axiomatic approach generalises the methods of [18, 16].

To measure a physical quantity we would simply apply one of the three basic measurement algorithms of Section 6 to the axiomatic interface. However, as the quantity is being measured, one can also perform computations with the intermediate results. More radically, the axiomatic specification of the interface allows one to throw *any* algorithm at the equipment. The scope of all possible algorithms applied to the interfaces are mapped by the following theorems.

Each axiomatic specification of an interface defines a very general class $E(\mathcal{G})$ of experiments to measure a physical quantity which can reasonably act as physical oracles to an algorithm. One of the interface's main properties is a set $\mathcal{G}$ of functions on the natural numbers that bound the time taken to make an oracle query, i.e., to initialise and run the equipment, make an observation and return data to the algorithm.

The *certification lemmas* provide sufficient conditions on an experiment for it to belong to the general class $E(\mathcal{G})$ and especially to practically important classes. For each of the three types of experiment, we prove theorems that give a lower bound on complexity of the experiment expressed by its computational power. To do this we use non-uniform computability and complexity notions [2, 3]. Each theorem takes the following form:

THEOREM 1. *Given a decision problem A in the non-uniform complexity class NK, there is a real value $x \in (0,1)$ so that any physical experiment in the class $E(\mathcal{G}_{NK})$ measuring $x$ can be used as an oracle to a Turing machine to decide A in polynomial time.*

The three theorems, stated as Theorem 5 in subsection 10.2, confirm that each form of experiment has the same lower bound on computational power, which is beyond the Turing barrier. We end the paper with some general remarks.

**§2. A theory of measurement.** In developing our ideas we are engaging with the philosophical foundations of measurement, which is a large technical subject, created by problems in the philosophies of physical science and behavioural science. It begins in the nineteenth century in work of Helmholtz in 1887 and acquires a logical axiomatic basis early on, making it mathematical and abstract. A synthesis of approaches by Suppes [31] led to a coherent mathematical representation theory of measurement. The theory explains how numerical representations of qualitative attributes are possible and is laid out in the magnum opus Krantz et al. [28, 32, 29]. We will need a very simple analysis as our guideline.

**2.1. Hempel on measurement.** Consider Carl Hempel's axiomatic analysis of measurement. According to [26], there are three stages to physical measurement:

**Classification.** In which entities are *sorted* according to similarities (e.g., using weight rather than colour, or 'heavy' and 'light').

**Comparison.** In which attributes of entities are *compared* by observations and events (e.g., 'less heavy than').

**Quantification.** In which attributes are *scaled* by assigning numerical values that preserve basic comparisons (e.g., mass is 4 Kilograms).

To *sort* and *compare* attributes, Hempel proposes a *comparative concept*, $(\mathcal{O}|E, L)$ which is a relational structure that consists of a set $\mathcal{O}$ of objects, and two binary relations $E, L$ on $\mathcal{O}$ such that

1. $E$ is an equivalence relation
2. $L$ is transitive
3. $L$ is $E$-connected, i.e. $\neg\ aEb \ \Rightarrow\ (aLb$ or $bLa)$
4. $L$ is $E$-irreflexive, i.e. $aEb \ \Rightarrow\ \neg\ aLb$

A comparative concept is "witnessed" by an *experimental apparatus*, whose physical events define $E$ and $L$.

To *quantify* attributes, Hempel defines a *measurement map* $\mathcal{M} : \mathcal{O} \to \mathbb{R}$ for the comparative concept $(\mathcal{O}|E, L)$, where $\mathbb{R}$ is the set of real numbers. The measurement map is any map obeying the following rules for all $a, b \in \mathcal{O}$:

5. $aEb \ \Rightarrow\ \mathcal{M}(a) = \mathcal{M}(b)$
6. $aLb \ \Rightarrow\ \mathcal{M}(a) < \mathcal{M}(b)$

The framework is most simply illuminated by thinking of the process of measuring mass using a balance scale: imagine an unknown mass $x$ on one pan and placing a test value $a$ on the other to make a comparison say $xLa$. We will return to this model later.

We can use Hempel's abstract framework to investigate the structure of experiments and the limits they frequently impose on what can be measured. For example, in a previous paper [12], we considered the amount of time required to perform measurements to varying degrees of accuracy. We may suppose that getting an exact answer for *equality* of two quantities, $aEb$ is often impractical as it would take an infinite amount of time.[1]

For the moment, we focus on the comparison of two quantities $aLb$, ignore the amount of time taken and assume only that it is finite. Then we can measure such a comparison $aLb$ and get a definite answer in many cases.

**2.2. Three forms of comparison.** As we will show shortly, consideration of several experiments that witness the comparison of two quantities uncovers three distinct and commonly occurring cases. We define them now.

Suppose that $x$ is a physical quantity to be measured and $a$ is a test value generated for comparison in an experiment; we assume that the quantities $x, a \in \mathbb{R}$. We classify experiments into three cases, depending on what comparisons can actually be made:

**1. Signed comparison:** $aLx$ and $xLa$ can be tested separately.

Most of the experiments we have analysed to date have been of this form and we recall some below, e.g., collider machine in which $x$ is the unknown mass and $a$ is the projected particle [13].

[1] There are many examples of infinite time in our papers. Even in special cases, there can be problems, such as using quantisation of charge to measure equality of two electrical charges in a finite time involves the need to experimentally verify that all charges come in multiples of a single unit.

**2. Vanishing comparison:** Only the predicate ($aLx$ or $xLa$) can be tested (i.e., only inequality can be tested).

Experiments of this form will be discussed later and include Brewster's angle ($x$ is the angle of incidence of a beam of light), and the heat measurement for Wheatstone bridge ($x$ is the unknown resistance).

**3. Threshold comparison:** Only $aLx$ can be tested.

Experiments of this form will be discussed later and include the broken beam balance, the scattering of charged particles in an Coulombian field and the photoelectric effect.

Before looking at experiments in detail, it will be useful to consider a very general thought experiment to illustrate these three cases and explore the ideas of signed, vanishing and threshold comparisons.

DEFINITION 1. *Consider an experiment in which we generate various test values $a \in \mathbb{R}$ and compare them against a physical quantity $x \in \mathbb{R}$ in an apparatus. Suppose that for each test, and given sufficient finite time, the experiment yields a result that is a measurable physical quantity $F(a) \in \mathbb{R}$, which yields quantitative information on the comparison between $a$ and $x$. Specifically, we suppose that there is a function $F : [0,1] \to \mathbb{R}$ such that if $a = x$ then $F(a) = 0$. The function $F$ we call an* observable indicator *for the experiment.*

We give three graphs in which $F(a)$ is plotted on the vertical axis against $a$ on the horizontal axis with $F(a) = 0$ if $a = x$. The three graphs of Figures 1, 2 and 3 correspond to the three comparison cases mentioned earlier.

The experimental procedure applied to $a$ yields information $F(a)$ from which we can make a comparison with $x$ and proceed to generate further test values.

In the signed case (1), by measuring the sign of $F(a)$ we can find whether $a < x$ or whether $x < a$. If the function $F(a)$ is continuous, we can use bisection to approximate $x$.

In vanishing case (2), a measured value $F(a) > 0$ could correspond to either $a < x$ or $x < a$. In this case it is very obvious that we must make further assumptions on the function $F(a)$ in order to be able to approximate the value $x$. For example, an assumption that $F(a)$ is strictly increasing for $a > x$ and strictly decreasing for $a < x$ will suffice.

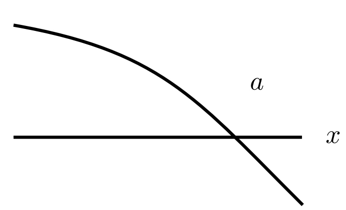


FIGURE 1. **Signed** Measure both $a < x$ and $x < a$. A bisection method can be used to find $x$. Measure both $a < x$ and $x < a$. Assume continuity.

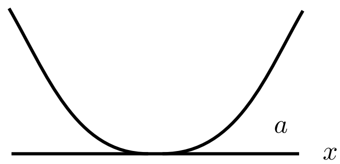


Figure 2. **Vanishing** Can only measure ($a < x$ or $x < a$). A modified bisection method will work. Assume monotonicity on each side of $x$.

In threshold case (3), all values $a \geq x$ give $F(a) = 0$. We can give a sequence of test values tending to $x$ from below. To have any idea of an upper bound for $x$, or how fast the sequence is converging to $x$, we need further information on the the function $F(a)$.

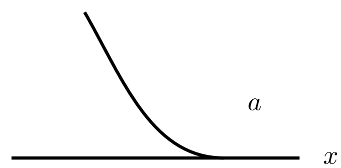


Figure 3. **Threshold** Can only measure $a < x$. Can give a sequence of tests approximating $x$ from below.

Of course, there is another possible case of (3), where we reverse the graph, and have all values $a \leq x$ giving $F(a) = 0$. Instead of having cases (3a) and (3b), we will merely note that a trivial change of variables relates the choices, and continue with the case above.

In all cases, we are dependent on having some information on the observable indicator function $F(a)$ to be able to perform the measurement. All measurement is done within the context of a physical theory, which is the source of such information. Physical theories, and particular fragments of physical theories, are fundamental parameters in our theory of measurement.

**§3. Case 1: Measurements based on signed comparisons.** This is a case we have already met previously when developing our theory (see [13]). It is very common in experimental setups.

**3.1. A balance.** Possibly the simplest is the balance, illustrated in Figure 4. If $x < a$, then the left arm of the balance moves down, and, if $x > a$, then the left arm of the balance moves up. We take the observable indicator $F(a)$ to be the angle of the balance arm with the horizontal; according to the statics, if $a = x$ then $F(a) = 0$.

**3.2. The collider machine.** Next we recall from [13] a simple example, the collider machine, illustrated in Figure 5.

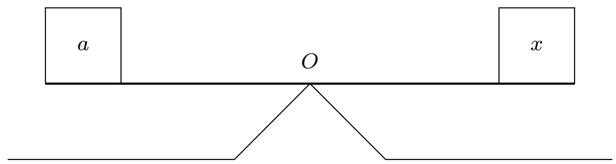


FIGURE 4. Schematic representation of balance.

The initial velocity for the test particle of mass $m$ is approximately 1m/s. After the collision, the test particle is observed to cross the flags $P^{\pm}$ within the time limit, or not (*out of time*). The time $t_{exp}$ taken to reach the flags satisfies (for some constants $A$ and $B$):

$$\frac{A}{|m-\mu|} \leq t_{exp} \leq \frac{B}{|m-\mu|} . \tag{1}$$

If the flag $P^{+}$ is crossed within the time limit, we return $m > \mu$, and if $P^{-}$ is crossed, we return $m < \mu$. We can take the observable indicator $F(m)$ to be the scalar speed of the test particle after the collision.

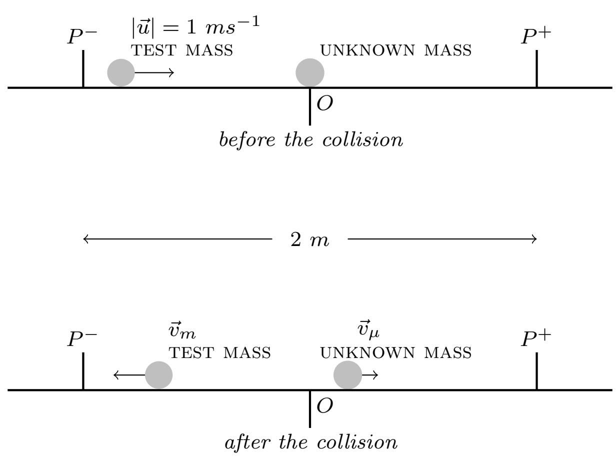


FIGURE 5. Collision experiment.

**3.3. The Wheatstone bridge.** Another example is the Wheatstone bridge experiment to measure an electrical resistance from [14]. The circuit diagram is shown in Figure 6. We measure the fixed resistance $\varrho$ given a user variable resistance $\rho$. If $\rho = \varrho$, then the current $i_0 = 0$.

We measure the sign of $i_0$ by using the current to charge a capacitor – if we wait a long time we can expect that even a small current will build up a measurable charge. We take the observable indicator $F(\rho)$ to be the current $i_0$.

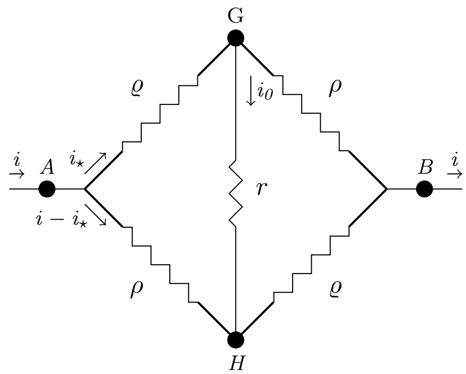


FIGURE 6. The Wheatstone bridge.

**§4. Case 2: Measurements based on vanishing comparisons.** Before introducing a new experiment, we modify the Wheatstone bridge experiment a little to change a signed comparison experiment into a vanishing comparison experiment.

**4.1. Wheatstone revisited.** Let us suppose that we do not have a capacitor to store charge for the Wheatstone bridge experiment in Section 3.3, or indeed any method to directly measure the current. We could look for $i_0 = 0$ by measuring the heating effect due to the current — for example by putting a resistance in a vacuum flask, and measuring the rise in temperature after a certain time. The energy dissipated in the resistance is approximately proportional to $(\varrho - \rho)^2$, so we have no knowledge of the sign of $\varrho - \rho$. We take the observable indicator $F(\rho)$ to be the energy dissipated in the resistance.

This may seem a rather artificial example, as the problem is caused by choosing a limit on the apparatus. But there is another case, where we see no obvious way to find information on the sign — measuring Brewster's angle.

**4.2. Brewster's angle.** This is designed to measure a critical angle for reflecting a light beam from a boundary between two materials of different refractive indexes, labeled mediums (1) and (2) in Figure 7. The construction of this *gedankenexperiment* is based on the principles of classical optics (as main reference we cite [21]). In Figure 7, we indicate the incident, reflected, and the transmitted rays: $\phi$, $\varphi$, and $\psi$ are the angles of incidence, reflexion, and refraction. The plane of incidence is $Oxz$. The directions of propagation of the three waves and the normal to the surface of separation of the two media are coplanar. The electric fields of the three waves are $\vec{E_I} = \langle E_{Ix}, E_{Iy}, E_{Iz}\rangle$ for the incident wave, $\vec{E_R} = \langle E_{Rx}, E_{Ry}, E_{Rz}\rangle$ for the reflected wave, and $\vec{E_T} = \langle E_{Tx}, E_{Ty}, E_{Tz}\rangle$ for the transmitted wave.

In Figure 7, are also represented the components of the electrical field in the plane of incidence $E_{IP}$, $E_{RP}$, $E_{TP}$ and in the perpendicular direction $E_{IN}$, $E_{RN}$, $E_{TN}$, which is the direction $Oy$ not appearing in the figure. We consider a component of the reflected or transmitted rays to be positive (or negative) whenever the correspondent ray, when rotated to coincide in direction with the incident ray, gets the same direction (opposite direction).

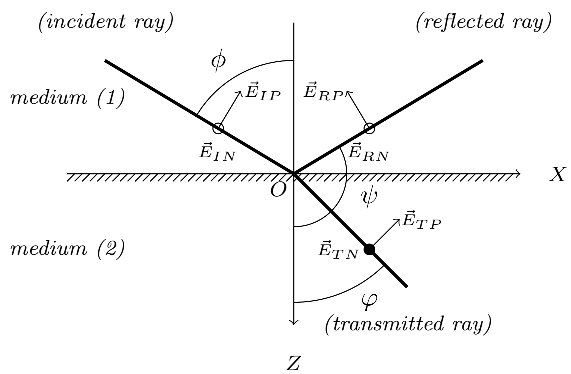


FIGURE 7. Elements of incident, reflection, and transmitted light rays. The indexes of the electrical field $E$ denote the *incident* $I$, *reflected* $R$, and *transmitted* $T$ rays, together with the *normal* $N$ and the *parallel* $P$ components of the field. The black circle denotes the normal component pointing forward and the white circle denotes the normal component pointing backwards.

Optics provide the following subset of relations, called the Fresnel general formulæ for the components of the electrical field:

$$E_{RN} \; = \; -E_{IN}\,\frac{\sin(\phi-\psi)}{\sin(\phi+\psi)} \quad , \quad E_{RP} \; = \; +E_{IP}\,\frac{\tan(\phi-\psi)}{\tan(\phi+\psi)}$$

$$E_{TN} \; = \; +E_{IN}\,\frac{2\sin\psi\cos\phi}{\sin(\phi+\psi)} \quad , \quad E_{TP} \; = \; +E_{IP}\,\frac{2\sin\psi\cos\phi}{\sin(\phi+\psi)\cos(\phi-\psi)}$$

From the Fresnel formulæ, we get immediately the law of Brewster: *For some value of the angle of incidence $\phi_B$, the reflected light is totally polarized in the direction normal to the plane of incidence.* For this particular angle $\phi_B$, we have $E_{RP} = 0$ and $E_{RN} \neq 0$: the reflected light has only the normal component $E_{RN}$. Now, suppose that the incident light wave is polarized in the plane of incidence and the angle of incidence is $\phi_B$: the reflected ray extinguishes.

In finding the Brewster angle in Optics, that the angle of incidence is above or below the critical Brewster angle cannot be known using properties of light. For incident polarized light, when $\phi = \phi_B$, the reflected ray vanishes (see Figure 7). We only know that the light becomes fainter near the critical value to be measured. We measure the intensity by integrating over a given time. (Quantum theory makes this a little easier in practice — we can count photons hitting a detector in a given time.) We take the observable indicator $F(\phi)$ to be the intensity of the reflected ray.

Now the diligent reader will return to the Fresnel formulæ, note that in fact there is a 180° phase change from one side of $\phi_B$ to the other, and ask why we do not measure this phase change. After all, the standard way to measure a 180° phase change is to measure the cancellation effect when combined with a reference wave of *similar* amplitude. But that is the trouble, if the amplitudes are not similar, the effect on phase shift will be small. And here the amplitude of the wave that we are trying to measure is already very small and uncertain. We have not been able to produce any practical measurement process that can measure the phase shift.

## §5. Case 3: Measurements based on threshold comparisons.

**5.1. A broken balance.** The simplest experiment we can have to illustrate this is the broken balance, Figure 8:

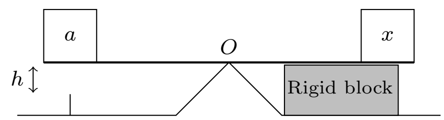


FIGURE 8. Schematic representation of the broken balance.

If weights $x < a$ then the left side of the balance will move down. However if $a < x$ the balance will not move, as the wooden block under the right side prevents it moving down. We take the observable indicator $F(a)$ to be the angle of the balance arm with the horizontal. However, since everyone will think this is too simple to be interesting, we will move to one of the most famous experiments in history.

**5.2. The photoelectric effect.** Figure 9 illustrates the measurement of the photoelectric effect (discovered by Lenard 1902), whose results were explained by Einstein in 1905 in terms of quanta. Electrons are knocked out of the surface by the light. Measure the current given by the electrons between the surface and the grid, as the potential on the grid is changed. Beyond a certain voltage, all the electrons are repelled by the charge on the grid, and no current is observed. We wish to find this critical voltage, and so measure the maximum energy the electrons leave the surface with. We take the observable indicator $F(V)$ to be the current flowing as a function of the grid voltage $V$.

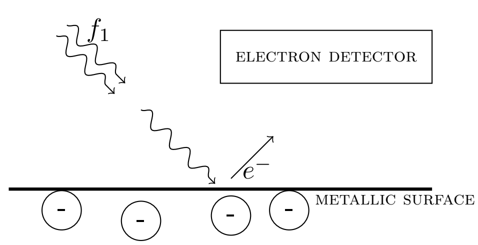


FIGURE 9. Schematic description of the Photoelectric Effect Experiment.

**5.3. Rutherford scattering.** For our third experiment we return to the collider machine for comparing a test mass $m$ with an unknown mass $\mu$, as given in Section 3.2. The reader will recall that there the particles collided head on, and that the collision was perfectly elastic. We shall remove the head on part of the collision (i.e., the initial velocity need not be on the line joining the centres of the particles), and introduce a well understood field, an inverse square repulsion, and see how the experiment differs. This experiment looks like the experiment scattering a beam of alpha particles from a gold foil (Geiger and Marsden 1909). The result of this experiment was that a few alpha particles were scattered back in the direction they had come from. This was explained by Rutherford as scattering of the alpha particles off small but relatively heavy (compared to an alpha particle) atomic nuclei.

We shall take a target consisting of "nuclei" of mass $\mu$, and scatter a beam of test particles of mass $m$ from the target. The interaction is by a repulsive Coulombian field, for example the particles can have the same sign electric charge. We shall study the distribution of scattered angle as the test mass varies. We shall assume that effectively the only forces involved are the repulsive Coulomb interaction, and in particular that any forces holding the target particle fixed are negligible in comparison.

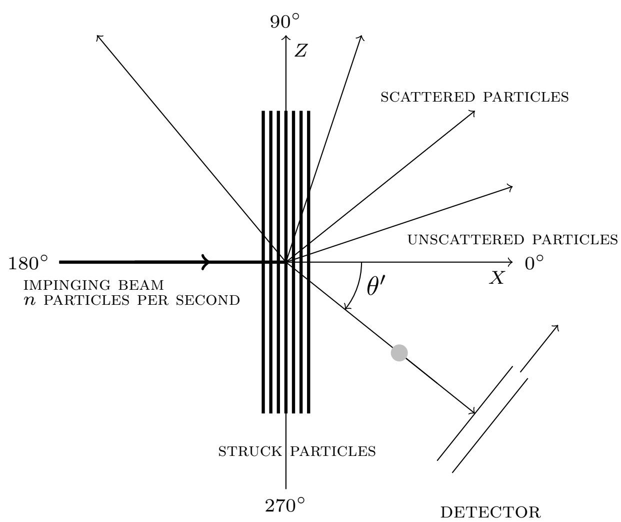


FIGURE 10. Scattering machine experiment. The detector has a non-negligible area and rotates around the block of matter.

The input value we have no effective control over is the *impact parameter* $b$, which is the distance by which the particles would miss each other if there was no interaction. The values of this parameter are so small that we can only deal in probabilities, there is no aiming the experiment for a particular value. A large value of $b$ means that the particles do not come close, and that the scattering angle $\theta'$ is small, whereas a small $b$ gives a large $\theta'$ (see Figure 10).

To describe the results of this experiment, we refer to the usual notation of scattering theory (see [20]). Recalling that the target is so small that the actual line of approach of the incident particle relative to the target (the "impact parameter" mentioned earlier) is random, we measure areas as a way of measuring the probability of the particle being scattered by a given angle $\theta'$ in Figure 10. The area is measured where the particle would hit a plane perpendicular to the direction of the incident beam in the absence of a deflecting force (i.e., in Figure 10 we could use the plane of the front of the target material). The reader may recall that to be deflected by a large angle, the incident particle must pass close to the target particle. The *cross section* $S(\theta)$ is defined to be the area on this plane for which the particle is deflected by an angle of magnitude greater than $\theta$, i.e. where $|\theta'| > \theta$ in Figure 10. The *differential cross section* $\chi(\theta)$ is then just

$$\chi(\theta) \ = \ \left| \frac{\mathrm{d}S}{\mathrm{d}\theta} \right| .$$

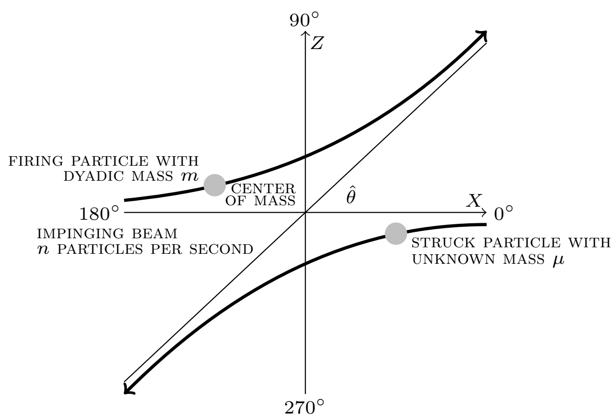


FIGURE 11. Elements of trajectory of an electric particle in a Coulombian field.

The book [20] gives the differential cross section for the Coulomb scattering problem for a fixed central charge with potential $q_\mu q_m/r$, where $r$ is the distance between the particles;

$$\chi(\hat{\theta}) \ = \ \frac{\pi (q_\mu q_m)^2}{4\hat{E}^2} \, \frac{\cos(\hat{\theta}/2)}{\sin^3(\hat{\theta}/2)} \ , \tag{2}$$

where $q_\mu$ and $q_m$ are the electric charges of the struck and projected particles, respectively, and $\hat{E}$ is the kinetic energy of each particle of the beam. However this is derived on the assumption that the central charge remains fixed, which may be a reasonable approximation for alpha particles striking gold nuclei, but not where we use similar particles of similar mass. However the mechanics can be considerably simplified by moving to a reference frame with origin the centre of mass of the two particles, as shown in Figure 11.

The simplifying feature about the two body problem is that, when viewed in the centre of mass frame, each particle describes a trajectory as if there were only a single fixed charge at the centre of mass. The cost is that some of the parameters ($\hat{E}$) in the equations have to be changed, and that the angles in the centre of mass frame ($\hat{\theta}$) are not the same as those in the laboratory frame, where the particle of mass $\mu$ is initially stationary.

We wish to measure the particles scattered back along their direction of approach, which means $\theta \in [\frac{\pi}{2}, \pi]$. A quick check shows that in the laboratory frame, the incoming particle is never scattered back if $m > \mu$. If $m \leq \mu$, in the centre of mass frame back scattering corresponds to the interval $\hat{\theta} \in [\arccos(-m/\mu), \pi]$, and integrating Equation (2) gives

$$\text{back scatter cross section} \;=\; \frac{\pi (q_\mu q_m)^2}{4\hat{E}^2}\Big(1 - \frac{1}{\sin^2(\check{\theta}/2)}\Big) \;,$$

where $\check{\theta} = \arccos(-m/\mu)$. Substituting for this angle gives

$$\text{back scatter cross section} \;=\; \frac{\pi (q_\mu q_m)^2}{4\hat{E}^2}\;\frac{m-\mu}{m+\mu} \;.$$

The centre of mass energy $\hat{E}$ of the particle $m$ is, in terms of the laboratory frame energy $E$,

$$\hat{E} \;=\; E\Big(1 + \frac{m}{\mu}\Big)^2 \;.$$

Using the fact that we will be operating in the region where $m$ is approximately equal to $\mu$, we have the approximation for $m \leq \mu$,

$$\text{back scatter cross section} \;\simeq\; \frac{\pi\,(q_\mu q_m)^2\,(m-\mu)}{128\,\mu\,E^2} \;. \tag{3}$$

We take the observable indicator $F(m)$ to be the back scatter cross section, as measured by counting number of particles scattered backwards. If we now conduct an experiment to measure $\mu$ by looking for back scattering of particles of determined mass $m$, we see that we have no back scattering for $m \geq \mu$. We can determine if $m < \mu$, but to do so takes a time inversely proportional to $|m - \mu|$. This is because only the number of particles striking the area given in (3) will be back scattered, and this number is inversely proportional to $|m - \mu|$.

**§6. Algorithmic nature of the three types of measurement.** We now make explicit the general forms of the algorithms that govern the three types of measurement. These will make clear experimental differences due to the restriction in the sort of comparisons allowed, as given in cases (1), (2) and (3).[2]

Suppose we are to determine a value $x$ of a physical quantity, which is a real number in the range $x \in (0, 1)$. Assume there is a pre-set error $\epsilon$ within which we wish to determine $x$. In each case, we use a method of trial and error, comparing the unknown $x$ to a known quantity $a \in [0, 1]$. In each case, the output of the algorithm is an interval $[a_1, a_2]$ containing $x$, where $|a_2 - a_1| < \epsilon$.

For mathematical convenience we take only dyadic rational values of $a$ and, referring back to Figure 3, we suppose that $x$ is the point where the function $F(a)$ vanishes (in case (3), the first point where $F(a)$ vanishes).

[2] Here we are not interested in the total time taken to find $x$ to a given accuracy, nor in any errors inherent in the procedure, but only in the algorithm used. The time taken and dealing with errors can be found in other places [12, 13, 14].

In two of the cases noted below, the Brewster's angle measurement and the Rutherford scattering experiment, the reader may note that the process of particle counting is in fact probabilistic. This is not vital to the point we are making, and we will simply ignore this probabilistic nature, assuming that the particles arrive at 'on average' times. In both cases, given assumptions on the model of physics, we could calculate time limits for each experiment to give a bound for the error due to probabilistic reasons of the whole experiment of, say, $\frac{1}{4}$. From there, repetition of the whole procedure can further reduce the error probability to any desired (strictly positive) amount. To see results which take this sort of probability of error into account, see [13, 16].

**6.1. Case 1: Signed comparisons.** Here the algorithm is a basic linear search method. We assume that the observable indicator $F(a)$ is strictly positive on $[0, x)$ and strictly negative on $(x, 1]$.

**Basic linear search algorithm**
Set $a_1 = 0, a_2 = 1$.
**Loop**:
 Let $a = (a_1 + a_2)/2$.
 If $F(a) > 0$ set $a_1 = a$.
 If $F(a) < 0$ set $a_2 = a$.
 If $F(a) = 0$ set $a_1 = a$ and $a_2 = a$.
 If $|a_2 - a_1| < \epsilon$, then **HALT**.

To see how to implement this for the collider, where $F(a)$ is the speed of test mass $a$ after the interaction, we wait until the test mass passes the flag $P^+$ in Figure 5 (in which case $m > \mu$ and $F(a) > 0$) or it passes the flag $P^-$ (in which case $m < \mu$ and $F(a) < 0$). Of course, if we impose a time limit on the experiment, we may get neither result. However, in this last case, we get an *out of time* result. Thus, this method is quite robust algorithmically — if it goes wrong, it produces an exception or error message. Further, an *out of time* result means that $|m - \mu|$ must be small, so there is useful information to be obtained from it.

**6.2. Case 2: Vanishing comparisons.** Here a simple bisection method will not work. There is no single measurement to determine whether $a > x$ or $a < x$. We assume that the observable indicator $F(a)$ is decreasing on $[0, x]$ and increasing on $[x, 1]$.

**Modified linear search algorithm**
Set $a_1 = 0, a_2 = 1$.
**Loop**:
 Let $a = (3\,a_1 + a_2)/4$ and $b = (a_1 + 3\,a_2)/4$.
 If $F(a) > F(b)$ set $a_1 = a$.
 If $F(a) < F(b)$ set $a_2 = b$.
 If $F(a) = F(b)$ set $a_1 = a$ and $a_2 = b$.
 If $|a_2 - a_1| < \epsilon$, the **HALT**.

To see how to implement this for the Brewster's angle measurement, we count photons for the different angles $a$ and $b$ over a given time. If we cannot decide either $F(a) > F(b)$ or $F(a) < F(b)$ in the given time, we return *out of time*.

However, this is not a very useful result in itself, as it does not imply that $x$ is close to either $a$ or $b$. The values of $a$ and $b$ can be changed a bit, and the experiment repeated.

**6.3. Case 3: Threshold comparisons.** At first sight the algorithm here is very similar to the basic linear search method for Case (1). We assume that the observable indicator $F(a)$ is strictly positive on $[0, x)$ and zero on $[x, 1]$.

**Modified linear search algorithm**
Set $a_1 = 0, a_2 = 1$.
**Loop**:
  Let $a = (a_1 + a_2)/2$.
  If $F(a) > 0$ set $a_1 = a$, otherwise set $a_2 = a$.
  If $|a_2 - a_1| < \epsilon$, then **HALT**.

To see how to implement this for the Rutherford scattering experiment, we count back scattered particles within a certain time limit. If we observe any back scattered particles, then $m < \mu$. And now we come to the difference between this case (3) and case (1) — this method is much less robust. We may not have observed any particles because we did not allow enough time to observe any. In that case we could wrongly assume that $F(a) > 0$ is false, and wrongly set $a_2 = a$ in the algorithm above.

In case (1), the *out of time* result indicated that there was a problem with the experiment. In this case (3), *out of time* is a valid result of the experiment, which can be expected for a large proportion of the time. It can arise either because $a \geq x$ (which is not a problem), or $a < x$ but we did not observe the system for long enough to find this out (which is a problem). To get any reliability, we need to look at the physical theory behind the experiment, in order to see how *fast* the value to be measured $F(a) \to 0$ as $a \to x$ from below.


FIGURE 12. Threshold: A good case (a) and a bad case (b)

In Figure 12 we have a good case and a bad case for our algorithm. In (a) we know how fast the function $F(a)$ decreases as $a \to x$ from below – it is a straight line, though the exact formula is not important. In (b) we will not say anything about how fast the rate of decrease is, and as a result it is very difficult to see where the curve meets the axis. The finite thickness of the lines in the pictures may be taken as illustrating the accuracy of the experiment at a given time limit!

It is however possible to abandon the linear search method entirely, and replace it with a one sided method which will always work. By a finite search procedure, let $a_n$ be the largest dyadic rational in $[0, 1]$ with denominator $2^n$ for which we can show $F(a_n) > 0$ in experimental time $2^n$. Then $a_n$ is an increasing sequence which tends to $x$.

**§7. General algorithmic theory of measurement.** In our computational theory of measurement, we combine experiments and algorithms by using the experiments as oracles. We will describe very briefly the general ideas about physical oracles used to answer questions about measurement in Section 1.

**7.1. Physical oracles.** In the classical model of a Turing machine with an oracle, the oracle is an external device that is consulted by the machine. The oracle may contain non-computable information to boost the computational power of the Turing machine, or computable information provided to speed up computations of the Turing machine. In computability theory, the oracle is a set of strings over an alphabet and the Turing machine accesses the oracle via queries that take one computational step to answer.

In our computational theory of measurement, we model the apparatus for measuring a physical quantity in detail using a precisely defined physical theory. This mathematical model of the behaviour of the apparatus allows us to describe observations in time from given initial states. The algorithm running the Turing machine abstracts the experimental procedure for the measurement chosen by the experimenter: it encodes the sequence of actions that determine the initialisation and observation.

Typically, to measure the value of a physical quantity is to construct a real number $\mu$ bit-by-bit. Real numbers belong in measurement theory as they are an abstract mathematical realisation of the idea of measuring with rationals with unlimited precision (see, e.g., [26]). The Turing machine follows a procedure that generates a sequence of approximate measurements corresponding with oracle consultations. Ideally, the algorithm conducting the experiment should approximate the unknown $\mu$ both from above and from below, in a convergent sequence of experimental values. The algorithm to measure $\mu$ could be one of those in Section 6. Indeed, the signed comparison algorithm has been *proved to be universal* in specific settings (see [13] for one such universal algorithm).

The equipment and the algorithm exchanges data through an interface. We use dyadic rationals, which are trivially denoted by finite binary strings, the data type of the Turing machine. A query initiates a run of the equipment. The query can contain data about test values and precision. We have found that the query time depends upon the size of the query data. Indeed commonly, the time taken to return an observation is exponential in the query size. See [13] for the first complete case study in which the time needed for a single experiment to read the bit $i$ of a mass $\mu$, using the proof particle of mass $m$ of size $i$ (=number of bits) is in the best case exponential in $i$.

Roughly speaking, the model allows us to have a discrete-time description of the scientist's activity and continuous-time analysis of the physical phenomenon being measured (which is described in the physical theory by differential equations, typically). The communication between scientist and apparatus is done via a protocol and has a cost in time that is not an artificial mathematical complexity constraint, but is a consequence of the physical oracle belonging the real world. The cost of the oracle was analysed in previous papers, such as [5, 14, 13], where the computational classes of the overall setting Turing machine and analog device were studied, for different oracle costs. In [17], we addressed the question whether different physical models of the experimental apparatus influences the cost of the oracle.

The basic structure of physical oracle, interface and Turing machine is depicted in Figure 13.

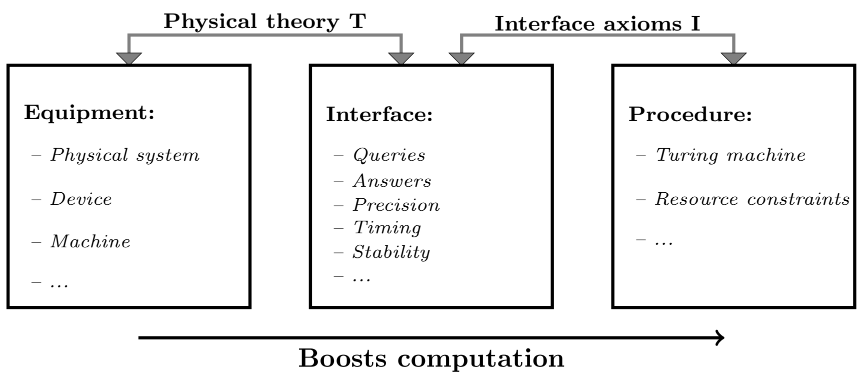


FIGURE 13. An experiment as oracle.

**7.2. Oracles boost computation.** In classical computability theory, Emil Post established the idea that an oracle is a set $A$ and a query is a question of the form is $a \in A$? that is answered after a single time step. If the oracle to a Turing machine is a physical measurement, then the time needed to consult the oracle is a number of time steps that will depend on the size of the query. Thus, a primary difference between classical oracles and physical oracles is the need for a cost function $T$, e.g., a timer that counts the number $T(k)$ of time steps the Turing machine must wait as a function of the size of the query $k$. Provided with such mathematical constructions, the main complexity classes of Turing machines coupled with these measurements, e.g., for the polynomial time case, change and need to be studied (see [5, 8]). Interesting classes emerge, namely those involved in the study of the complexity of hybrid systems and analogue-digital systems such as mirror systems and neural nets (see [22, 30]).

In the physical world, it is not conceivable that an experiment can be initialised with infinite precision. If we consider that precision is not infinite but unbounded, i.e., as big as we need, then it does affect the measurement. Essentially the same complexity classes are defined (see [16]), respectively $P/\log\star$ for the infinite precision case and $BPP//\log\star$ for the unbounded precision case. But suppose that we reject unbounded precision in favour of the most common and realistic *a priori* fixed precision criterion. Then we proved in [5, 13] that, using stochastic methods, we are still able to read the bits of $\mu$. To make our claim rigorous, we say that the lack of precision in measurement, for some types of error distribution, will not constitute an obstacle to the reading of the bits of $\mu$, but *only in the case that the bits of $\mu$ are readable (!)* (see [13]).

The proofs involving lower bounds of time complexity were conducted by encoding advice functions in the unknown quantities to be measured and allowing measurement with errors.

The axiomatic approach generalises the methods of [5, 13] for the signed case. In [16], we proved the following theorems as to how computable and complex are measurements by the three different types of precision in the signed case:

THEOREM 2. *The power of Turing machines equipped with signed comparison oracles with infinite precision, in polynomial time, is $P/\log\star$.*

THEOREM 3. *The power of Turing machines equipped with signed comparison oracles with unbounded or fixed finite precision in polynomial time is $BPP//\log\star$.*

**7.3. Algorithms control experiments.** Finally, in our measurement theory, instead of thinking of the physical oracle boosting the Turing machine, we think of the Turing machine *controlling* the physical oracle. The Turing machine imposes limitations to what is effectively accessible to physical observation. For example, some reasonable definitions of measurable number (as in [13]) imply that not all masses are measurable. This is not due to the limitations in measurements where experimental errors occur (as we address in the subsection above) nor is it because quantum phenomena puts a limit to measurements. It is because of a more essential symbolic and logical limitation of conceiving and modelling physicists and experimental procedures as Turing machines. The mathematics of computation theory does not allow the reading of bits of physical quantities beyond a certain limit: even if infinite precision instruments were available to the algorithmic experimenter, it could not always do the job for physical-mathematical reasons.

The idea of measurable numbers was first considered by Geroch and Hartle in [25], where they introduce the concept in contrast to a computable number. Geroch and Hartle examined at length some desiderata for a theory of measurable numbers. Our theoretical approach provides a mathematical home for several of their speculative ideas. However, the results of our theory do not match with Geroch and Hartle's expectations and speculations: some Geroch and Hartle's measurable numbers become non-measurable in our setting and some of their non-measurable numbers become measurable.

But, either in Geroch and Hartle's setting or in ours, the concept of computable number, introduced by Turing in [33], is not to be confused with the concept of measurable number. Indeed, we showed in [19, 13] that algorithms applied to physical systems can extract non-computable numbers and, vice-versa, some computable numbers cannot be measured; and, moreover, there are limits to measurement in physical systems which arise, not from the limitations of the experimental apparatus, but from the limits of computations performed in algorithmic measurement.

**§8. Axioms and algorithms.** We will now give three axiomatic specifications of the interface between apparatus and algorithm for signed, vanishing and threshold experiments. Each interface defines a very general class $E(\mathcal{G})$ of experiments measuring a physical quantity $x$ that can reasonably act as physical oracles to an algorithm.

For each of the three cases, we first present axioms for an interface. Then we justify the axioms in terms of the behaviour of our hypothetical form of apparatus, set up to make measurements on the physical quantity using the observable indicator $F$. Essentially an experiment with the given behaviour can be used to make a physical oracle: a *certification lemma* provides sufficient conditions for an experiment to satisfy the axioms and hence to act as an oracle. Finally, we give an algorithm which could be used by a Turing machine, equipped with an interface and physical oracle satisfying the axioms, which can be used to find the quantity $x$ within an arbitrarily small error.

The algorithms need only a very abstract interface to operate with the apparatus. For example, while the certification lemmas involve the observable indicators of the experiments, the interfaces do not.

In any experiment, the test value $a$ is generated by a query containing a rational number $y$ of a special form. In general, the equipment could introduce an error so that $a$ is close to $y$ but not identical. This error may be adjustable or fixed. For simplicity, we suppose that there is no error and $a = y$ as our main concern is to illustrate the types of measurement. To see what happens in error prone cases, the reader can look elsewhere [5, 8]. Furthermore, we suppose that there is no error in our observations, that is, if we make a query with number $y$, and the experimentally observed result is $F(y) > 0$, then it really is true that $F(y) > 0$. (However, if $F(y) > 0$ is small, we may not be able to observe that $F(y) > 0$ in a given time.)

One of the interface's main properties is a set $\mathcal{G}$ of functions on the natural numbers that bound the time taken to initialise and run the equipment, make an observation and return data to the algorithm; thus, $\mathcal{G}$ is a class of functions of the form $g : \mathbb{N} \to \mathbb{N}$. Many examples, including those in this paper, have led us to distinguish the following class of time bounds:

$$\mathrm{ExpLin} = \{g : \mathbb{N} \to \mathbb{N} \mid 2^{an+b} \leq g(n) \leq 2^{cn+d}\}.$$

In the time bound for the algorithms, $p(n)$ is a polynomial which is independent of the choice of $g \in \mathcal{G}$. It is a bound on the time used by the Turing machine to execute the algorithm, not counting the waiting time for the physical oracle.

**8.1. Axioms and algorithms for the signed case.**

**8.1.1.** *The axioms for the signed case.*

1. *Real values.* The experiment is designed to find a physical parameter $x \in (0, 1)$.
2. *Queries.* Each query string is interpreted as a binary string of 0s and 1s, $y_1 y_2 \ldots y_k$, giving a dyadic rational $y = 0{\cdot}y_1 y_2 \ldots y_k$.
3. *Finite output.* The result is either $y < x$ or $y > x$ (correctly assigned) or *time out*.
4. *Protocol timer.* There is a $g \in \mathcal{G}$ so that the time taken for the query is $g(k)$, where $k$ is the length of the query.
5. *Sufficiency of the protocol.* If $|x - y| > 2^{-k}$, then the result is either $y < x$ or $y > x$. That is, $k$ determines the error margin in separating $x$ and $y$.

6. *Repeatability.* Identical queries will result in identical results.

**8.1.2.** *The justification for the signed case axioms.* In this case the observable indicator $F(y) > 0$ for $y < x$ and $F(y) < 0$ for $y > x$ (the other way round would be similar). We have the following result which allows us to implement the axioms, given an experimental apparatus satisfying certain conditions:

LEMMA 1. ***Certification lemma:*** *Suppose that there is an experimental setup and a $g \in \mathcal{G}$ so that:*
*For all $n \geq 1$, if $y \in [0, x - 2^{-n}]$, we can experimentally observe $F(y) > 0$ in time $g(n)$.*
*For all $n \geq 1$, if $y \in [x + 2^{-n}, 1]$, we can experimentally observe $F(y) < 0$ in time $g(n)$.*
*Then we can use the experimental setup to construct a physical oracle satisfying the signed axioms.*

PROOF. Given $y = 0 \cdot y_1 y_2 \ldots y_k$, perform the experiment to measure $F(y)$ for time $g(k)$. If we measure $F(y) > 0$, then output "$y < x$". If we measure $F(y) < 0$, then output "$y > x$". If neither, output *time out.* ⊣

**8.1.3.** *The algorithm for the signed case.*

PROPOSITION 1. *Given a Turing machine with a physical oracle satisfying the axioms for the signed case for $x \in (0,1)$ and $g \in \mathcal{G}$, there is an algorithm to find $a_n \in [0,1)$ for integer $n \geq 0$ whose binary expansion has at most $n+1$ significant figures, and so that $a_n < x \leq a_n + 1/2^n$. The time taken by this algorithm to find $a_n$ is*

$$p(n) + \sum_{i=2}^{n+1} g(i) \ ,$$

*where $p(n)$ is independent of $x$ and $g$, and is the time taken to run the algorithm on the Turing machine, not counting wait times for the physical oracle.*

PROOF. This sequence is constructed recursively via oracle calls in the following manner: Begin with $a_0 = 0$. Now suppose that we have constructed $a_n$. Make one call to the oracle with a length $k = n+2$ string, with $y = a_n + 2^{-n-1}$. There are three outcomes:

**i)** If the query gives time out, set $a_{n+1} = a_n + 2^{-n-2}$.
**ii)** If the query gives $y < x$, set $a_{n+1} = a_n + 2^{-n-1}$.
**iii)** If the query gives $x < y$, set $a_{n+1} = a_n$.

To see that this works, note that by Axiom 5, if we make a query $y$ of length $k = n+2$, and the result is *time out*, then $y + 2^{-n-2} > x > y - 2^{-n-2}$. Cases (ii) and (iii) are self-explanatory once the relevant value of $y$ is plugged in. ⊣

### 8.2. Axioms and algorithms for the vanishing case.

**8.2.1.** *The axioms for the vanishing case.*

1. *Real values.* The experiment is designed to find a physical parameter $x \in (0,1)$.
2. *Queries.* Each query string is interpreted as a binary string of 0s and 1s, $y_1 y_2 \ldots y_k$, giving a dyadic rational $y = 0 \cdot y_1 y_2 \ldots y_k$.

3. *Finite output.* The result is either $y < x$ or $x < y + 2^{-k}$ (correctly assigned) or *time out*.
4. *Protocol timer.* There is a $g \in \mathcal{G}$ so that the time taken for the query is $g(k)$, where $k$ is the length of the query.
5. *Sufficiency of the protocol.* If $x \leq y$, then the result is $x < y + 2^{-k}$, and if $y + 2^{-k} \leq x$ then the result is $y < x$.
6. *Repeatability.* Identical queries will result in identical results.

**8.2.2.** *The justification for the vanishing case axioms.* In this case $F : [0, x] \to \mathbb{R}$ is strictly decreasing, $F : [x, 1] \to \mathbb{R}$ is strictly increasing, and $F(x) = 0$. We have the following result which allows us to implement the axioms, given an experimental apparatus satisfying certain conditions:

LEMMA 2. ***Certification lemma:*** *Suppose that there is an experimental setup and a $g \in \mathcal{G}$ so that:*
*For all $n \geq 1$, if $y \in [0, x - 2^{-n}]$, we can observe $F(y + 2^{-n}) < F(y)$ in time $g(n)$.*
*For all $n \geq 1$, if $y \in [x, 1 - 2^{-n}]$, we can observe $F(y + 2^{-n}) > F(y)$ in time $g(n)$.*
*Then we can use the experimental setup to construct a physical oracle satisfying the vanishing axioms.*

PROOF. Given $y = 0 \cdot y_1 y_2 \ldots y_k$, first check if $y > 1 - 2^{-k}$, in which case output "$x < y + 2^{-k}$". Otherwise perform the experiment to compare $F(y)$ and $F(y + 2^{-k})$ for time $g(k)$. If we measure $F(y + 2^{-k}) > F(y)$, then output "$x < y + 2^{-k}$". If we measure $F(y + 2^{-k}) < F(y)$, then output "$y < x$". If neither, output *time out*.

We have four cases for the value of $y$:

First, $y \in [x, 1 - 2^{-k}]$, in which case we observe $F(y + 2^{-n}) > F(y)$, and correctly output "$x < y + 2^{-k}$". Second, $y \in [0, x - 2^{-k}]$, in which case we observe $F(y + 2^{-k}) < F(y)$, and correctly output "$y < x$". Third, $y \in (x - 2^{-k}, x)$, in which case we can correctly return any output. Fourth, $y > 1 - 2^{-k}$, in which case the output is correctly "$x < y + 2^{-k}$". ⊣

**8.2.3.** *The algorithm for the vanishing case.*

PROPOSITION 2. *Given a Turing machine with a physical oracle satisfying the axioms for the vanishing case for $x \in (0, 1)$ and $g \in \mathcal{G}$, there is an algorithm to find $a_n \in [0, 1)$ for integer $n \geq 0$ whose binary expansion has at most $n + 1$ significant figures, and so that $a_n < x \leq a_n + 1/2^n$. The time taken by this algorithm to find $a_n$ is*

$$p(n) + 2 \sum_{i=2}^{n+1} g(i) \ ,$$

*where $p(n)$ is independent of $x$ and $g$, and is the time taken to run the algorithm on the Turing machine, not counting wait times for the physical oracle.*

PROOF. This sequence is constructed recursively via oracle calls in the following manner: Begin with $a_0 = 0$. Now suppose that we have constructed $a_n$. Make two calls to the oracle with length $k = n + 2$ strings, one with $y = a_n + 2^{-n-2}$ and the other with $y = a_n + 2^{-n-1}$. There are four outcomes, taken in order:

**i)** If either of the queries gives time out, set $a_{n+1} = a_n + 2^{-n-2}$.
**ii)** If the result of the second query is $y < x$, set $a_{n+1} = a_n + 2^{-n-1}$.
**iii)** If the result of the first query is $x < y + 2^{-k}$, set $a_{n+1} = a_n$.
**iv)** Otherwise, set $a_{n+1} = a_n + 2^{-n-2}$.

To see that this works, note that by Axiom 5, if we make a query $y$ of length $k = n+2$, and the result is *time out*, then $y+2^{-n-2} > x > y$. Cases (ii) and (iii) are self-explanatory once the relevant value of $y$ is plugged in. If none of (i-iii) apply, then we must have the results $y < x$ from the first query and $x < y+2^{-k}$ from the second. ⊣

**8.3. Axioms and algorithms for the threshold case.**

**8.3.1.** *The axioms for the threshold case.*

1. *Real values.* The experiment is designed to find a physical parameter $x \in (0,1)$.
2. *Queries.* Each query string is interpreted as a binary string of 0s and 1s, $y_1 y_2 \dots y_k$, giving a dyadic rational $y = 0{\cdot}y_1 y_2 \dots y_k$.
3. *Finite output.* The result is either $y < x$ (correctly assigned) or *time out.*
4. *Protocol timer.* There is a $g \in \mathcal{G}$ so that the time taken for the query is $g(k)$, where $k$ is the length of the query.
5. *Sufficiency of the protocol.* If $y < x - 2^{-k}$, then the result is $y < x$.
6. *Repeatability.* Identical queries will result in identical results.

**8.3.2.** *The justification for the threshold case axioms.* In this case $F(y) > 0$ for $y < x$, and $F(y) = 0$ for $y \geq x$. We have the following result which allows us to implement the axioms, given an experimental apparatus satisfying certain conditions:

Lemma 3. ***Certification lemma:*** *Suppose that there is an experimental setup and a $g \in \mathcal{G}$ so that:*
*For all $n \geq 1$, if $y < x - 2^{-n}$, we can observe $F(y) > 0$ in time $g(n)$.*
*Then we can use the experimental setup to construct a physical oracle satisfying the threshold axioms.*

Proof. Given $y = 0{\cdot}y_1 y_2 \dots y_k$, , perform the experiment to measure $F(y)$ for time $g(k)$. If we measure $F(y) > 0$, then output "$y < x$". Otherwise, output *time out.* ⊣

**8.3.3.** *The algorithm for the threshold.*

Proposition 3. *Given a Turing machine with a physical oracle satisfying the axioms for the threshold case for $x \in (0,1)$ and $g \in \mathcal{G}$, there is an algorithm to find $a_n \in [0,1)$ for integer $n \geq 0$ whose binary expansion has at most $n+1$ significant figures, and so that $a_n < x \leq a_n + 1/2^n$. The time taken by this algorithm to find $a_n$ is*

$$p(n) + 2\sum_{i=2}^{n+1} g(i) \ ,$$

*where $p(n)$ is independent of $x$ and $g$, and is the time taken to run the algorithm on the Turing machine, not counting wait times for the physical oracle.*

PROOF. This sequence is constructed recursively via oracle calls in the following manner: Begin with $a_0 = 0$. Now suppose that we have constructed $a_n$. Make two calls to the oracle with length $k = n + 2$ strings, corresponding to the numbers $y = a_n + 2^{-n-2}$ and $y = a_n + 2^{-n-1}$. We have three cases:

**i)** If the query $a_n + 2^{-n-2}$ gives *time out*, then set $a_{n+1} = a_n$.
**ii)** If the query $a_n + 2^{-n-2}$ gives $a_n + 2^{-n-2} < x$ and the query $a_n + 2^{-n-1}$ gives *time out*, then set $a_{n+1} = a_n + 2^{-n-2}$.
**iii)** If the query $a_n + 2^{-n-2}$ gives $a_n + 2^{-n-2} <$ and the query $a_n + 2^{-n-1}$ gives $a_n + 2^{-n-1} < x$, then set $a_{n+1} = a_n + 2^{-n-1}$.

To see that this works, note that by 5, if we make a query $y$ of length $k = n + 2$, and the result is *time out*, then $x \le y + 2^{-n-2}$. Now in case **(i)**, we get $a_n < x \le a_n + 2^{-n-1}$. In case **(ii)**, we get $a_n + 2^{-n-2} < x \le a_n + 2^{-n-1} + 2^{-n-2}$. In case **(iii)**, we get $a_n + 2^{-n-1} < x \le a_n + 2^{-n}$. ⊣

**§9. Experimental times, errors, and the axioms.** From the paper [18] we have the following result:

THEOREM 4. *Suppose we are given an experimental procedure to measure $x \in (0, 1)$ in the following manner. Suppose there are rationals $A, B, E > 0$ and integers $n, q, p \ge 1$ so that*

*(a) Given $y = 0 \cdot y_1 y_2 \ldots y_k$ we can set up the experiment with a test value $y' \in \mathbb{R}$ with $|y - y'| < 2^{-s}$ in time $2^{p \max\{s,k\}} E$.*

*(b) We can run the experiment to determine if $x < y'$ or $x > y'$ in time*

$$\frac{A}{|x - y'|^q} \;\le\; T_{\text{experiment}}(x, y') \;\le\; \frac{B}{|x - y'|^n}\ .$$

*Then there is a physical oracle to a Turing machine using the experiment which obeys the axioms 8.1 for the signed case, with $\mathcal{G} = \text{ExpLin}$.*

The proof of this result, where an error prone experiment ends up giving invariably correct results, depends on hiding the error in the output category *time out*. (The possibility of error could also be written into the axioms, of course, but they would be different axioms.)

However, in this paper our purpose is to explain the indicator function and the different sorts of experiments, so we shall ignore errors in what follows.

We shall however take the experimental time bound in Theorem 4 as typical of 'realistic' thought experiments. In this paper we measure $x$ by observing the indicator function $F(y)$. We shall suppose that determining a difference $F(y_1) > F(y_2)$ takes an experimental time bounded by, for constants $A, B, q, n$,

$$\frac{A}{|F(y_1) - F(y_2)|^q} \;\le\; T_{\text{experiment}}(y_1, y_2) \;\le\; \frac{B}{|F(y_1) - F(y_2)|^n}\ . \tag{4}$$

In fact (4) is too specialised and complicated for our needs. If we take $\delta$ to be the change in $F$ that we need to observe, let $T(\delta)$ be an upper bound on the time needed to observe it. Thus, for example, (4) gives $T(\delta) = B/\delta^n$. Now analysing the various cases requires examining the graph of $F$, so the reader should look at Figure 3 again.

**9.1. The signed case.** Recall Certification Lemma 1. Suppose that there are numbers $\delta_n > 0$ so that

For all $n \geq 1$, if $y \in [0, x - 2^{-n}]$, then $F(y) \geq \delta_n$.
For all $n \geq 1$, if $y \in [x + 2^{-n}, 1]$, then $F(y) \leq -\delta_n$.

To distinguish these numbers $F(y)$ from 0 could be done in time $T(\delta_n)$, so the conditions for certification Lemma 1 are true if $T(\delta_n) \leq g(n)$. The condition on $\delta_n$ can be restated as

$$0 \, < \, \delta_n \, \leq \, \min\{|F(y)| : y \in [0, x - 2^{-n}] \cup [x + 2^{-n}, 1]\}$$

**9.2. The vanishing case.** Recall Certification Lemma 2. Suppose that there are numbers $\delta_n > 0$ so that

For all $n \geq 1$, if $y \in [x, 1 - 2^{-n}]$, then $F(y + 2^{-n}) - F(y) \geq \delta_n$.
For all $n \geq 1$, if $y \in [0, x - 2^{-n}]$, then $F(y) - F(y + 2^{-n}) \geq \delta_n$.

To distinguish these numbers from 0 could be done in time $T(\delta_n)$, so the conditions for certification Lemma 2 are true if $T(\delta_n) \leq g(n)$. The condition on $\delta_n$ can be restated as

$$0 \, < \, \delta_n \, \leq \, \min\{|F(y + 2^{-n}) - F(y)| : y \in [x, 1 - 2^{-n}] \cup [0, x - 2^{-n}]\}$$

**9.3. The threshold case.** Recall Certification Lemma 3. Suppose that there are numbers $\delta_n > 0$ so that

For all $n \geq 1$, if $y < x - 2^{-n}$, then $F(y) \geq \delta_n$.

To distinguish these numbers from 0 could be done in time $T(\delta_n)$, so the conditions for certification Lemma 3 are true if $T(\delta_n) \leq g(n)$. The condition on $\delta_n$ can be restated as

$$0 \, < \, \delta_n \, \leq \, \min\{|F(y)| : y \in [0, x - 2^{-n}]\} \; .$$

**§10. The computational power of the physical oracles.** In this section, we will examine the computational power of a Turing machine using a physical oracle governed by the axioms given in the three cases of Section 8. The computational power is measured by *non-uniform complexity theory*, and as this has been summarised elsewhere (see [2, 4]), we give only a short account.

**10.1. Non-uniform computability and complexity.** Traditionally computability theory is phrased in terms of a word acceptance problem — a word $w$ in a certain alphabet is input to a Turing machine, and the word is either accepted, or the computation does not terminate. The idea is to consider the class of words which are accepted by a certain machine in a certain time, expressed as a function of the word length. Thus, a particular word acceptance problem is deemed to be in the class $P$ if there is a Turing machine $M$ and a polynomial $p$ so that, when given a word $w$ as an input, $M$ accepts $w$ (assuming that $w$ is an acceptable word) in a time $p(|w|)$, where $|w|$ is the length of the word $w$.

In particular, a word recognition problem in $P$ is computable by a Turing machine. The idea of non-uniform complexity is to go beyond such computable problems, but in a controlled manner. This time, we use a function $f$, called an *advice function*, from the natural numbers to words. Given a word $w$, we take the word $\langle w, f(|w|)\rangle$ given by appending $f(|w|)$ to $w$ (in fact we need to be slightly careful about how to do this, so that we can see where $w$ ends and

$f(|w|)$ begins, but that is a technicality). We can now ask if the collection of $\langle w, f(|w|)\rangle$, that is words plus their respective *advice*, is in class $P$.

If the length $|f(n)|$ of the advice word $f(n)$ is bounded by $a+b\ \log_2(n)$, we say that advice is *logarithmic*, and the resulting non-uniform class is termed $P/\log$. If we ask that $f(n)$ is the appropriate length initial segment of a single infinite word, we get the class $P/\log\star$.

**10.2. Lower bounds on power.** Using the same techniques as in [13] (this was the signed case in our current notation), *mutatis mutandis*, the algorithms in Section 8 allow us to prove Theorem 5. Note that the signed case of this theorem is proved in [5], and the threshold case in [9]. We only place a lower bound on the computational power here, an upper bound requires further assumptions and more complicated arguments. However, the lower bound is firmly outside the purely Turing computable domain.

THEOREM 5. *Suppose that we are given a word acceptance problem in the class $P/\log\star$. Then there is a real number $x \in (0,1)$ so that a Turing machine equipped with a physical oracle satisfying any of the axioms for*

*a) the signed case (Subsection 8.1) for $x \in (0,1)$ and $\mathcal{G} = \mathrm{ExpLin}$,*

*b) the vanishing case (Subsection 8.2) for $x \in (0,1)$ and $\mathcal{G} = \mathrm{ExpLin}$,*

*c) the threshold case (Subsection 8.3) for $x \in (0,1)$ and $\mathcal{G} = \mathrm{ExpLin}$,*

*can solve the problem in time polynomial in the word length.*

PROOF. We can give a proof that is uniform for the three cases. The real $x \in (0,1)$ encodes the advice function for the $P/\log\star$ problem. The method of encoding is described in detail in [5]. The alphabet is first coded as binary words, so that without loss of generality we can take the alphabet to be $\{0,1\}$. Next the digits 0 and 1 in the infinite advice word are replaced by binary triplets. The resulting binary word is used as the digits after the binary point in the binary representation of a real $x \in (0,1)$. The triplets are chosen so that $x$ lies in a Cantor set with the following property: For any integer $k$ and any integer $m \geq 1$, $|x - k/2^m| > 1/2^{m+5}$.

Now input a word $w$. Our first problem is to read a number $a\ \log_2(|w|) + b$ of binary digits from the expansion of $x$ (we take $n$ to be the integer value, rounded down, of $a\ \log_2(|w|)+b$). For this, we use the algorithms listed in Subsections 8.1, 8.2 and 8.3 for the respective cases.

These say that there is an algorithm to find an integer $k_{n+5}$ (where $a_{n+5} = k_{n+5}/2^{n+6}$) so that

$$\frac{k_{n+5}}{2^{n+6}} < x \leq \frac{k_{n+5}}{2^{n+6}} + \frac{1}{2^{n+5}}$$

in a time bounded by

$$p(n+5) + 2\sum_{i=2}^{n+6} g(i)\ , \tag{5}$$

where $p$ is a polynomial, and $g \in \mathrm{ExpLin}$. Now $|x - \frac{k_{n+5}}{2^{n+6}}| < 1/2^{n+5}$, so there is no number of the form $k/2^n$, for integer $k$, between $x$ and $\frac{k_{n+5}}{2^{n+6}}$, so the first $n$ digits of $x$ are the same as the first $n$ digits of $\frac{k_{n+5}}{2^{n+6}}$.

If we take a bound $g(i) \leq 2^{ci+d}$ (with $c, d > 0$) for $g \in \mathrm{ExpLin}$, then the time bound from (5) has

$$\begin{aligned} p(n+5) + 2\sum_{i=2}^{n+6} g(i) &\leq p(n+5) + 2\sum_{i=2}^{n+6} 2^{ci+d} \\ &\leq p(n+5) + 2\,(n+5)\,2^{cn+6c+d} \\ &\leq p(n+5) + 2\,(n+5)\,2^{6c+d}\,2^{ac\,\log_2|w|+bc} \\ &\leq p(n+5) + 2\,(n+5)\,2^{6c+d+bc}\,|w|^{ac}\,. \end{aligned}$$

As $n \leq a\,\log_2(|w|) + b$ we get a polynomial bound in the time taken for this stage.

Now we can decode the first $n$ digits of $x$ to get the required advice $f(|w|)$. We append $f(|w|)$ to the word $w$ to get $\big\langle w, f(|w|)\big\rangle$, using a method which allows a distinction to be made between the first and second parts. The most common way of doing this doubles the length and adds two to the number of characters, but this does not affect the result. Now the Turing machine runs the polynomial time algorithm for the word $\big\langle w, f(|w|)\big\rangle$ to get the result. $\dashv$

**§11. The interaction between physical theory and the axioms.** The axiomatic specifications in Section 8 make no mention of physical experiments. The experimenters and their algorithms do not see any apparatus. It is as though the two processes were in separate rooms: in one room is the experimental apparatus, in another the Turing machine. Between them is a channel with an interface defined by the axioms. Thus, the Turing machine sees the physical oracle as a black box — given a query in a certain form, in a certain amount of time given by $g \in \mathcal{G}$, an answer of a certain form will be delivered. To the experimentalists, in the design of their algorithms, the axiomatic specifications are contracts that the apparatus may be expected to satisfy.

So how do the experimentalists know that their contract will be fulfilled? The answer is that they don't know, they cannot guarantee that the experiment will behave exactly as the contract requires. What they do have is some physical theory that predicts that the experiment behaves in such a way. In a quote from [1] which summarises commonly accepted ideas: "There is no measurement without theory. Theory precedes measurement or, more properly, every measurement implies theory."

A look at our examples of experiments shows that to determine the physical value we use a single observable, the *observable indicator*. The experiments are arranged so that the observable indicator vanishes at the given physical value, and we measure the value of the observable indicator at one or two values specified by the input to the physical oracle. It is assumed that the time taken to measure the observable indicator increases as the required accuracy of the observation increases — this is certainly true of the given experiments. Then the type of axiom required, and the class of functions $\mathcal{G}$ required, is determined by the shape of the observable indicator as a function of the input parameter, and the time taken to observe it to a given accuracy. The certification lemmas are intended to bridge the gap.

How would we find that our theory was not capable of adequately modeling the experiment? One way would be if the outputs from the physical oracle were not consistent with the axioms — e.g., for the signed case we might have outputs $x > \frac{1}{2}$ and $x < \frac{1}{4}$. In principle, a Turing machine could carry out some limited checking for such inconsistencies, but the algorithms listed in Section 8 do not perform such checks.

A more subtle problem would be if the axioms did indeed perform as stated for some value of $x$, but that value might not be the 'expected' value. For example, most current theories of physics say that inertial and gravitational mass are the same. In principle, cross referencing between two experiments, one measuring inertial mass and the other gravitational mass, might discover an inconsistency. However a single experiment might quite consistently measure the inertial mass, whereas we expected the value of the gravitational mass, having erroneously assumed that they were the same.

**§12. Reflections.** Finally, here are some comments on the nature of the experiments and their taxonomy.

**12.1. On classifying experiments.** We are creating our theory of measurement by exploring the algorithmic nature of the process of a scientist making a measurement using an experimental procedure and some apparatus, and working within a scientific theory. The logical nature of measurement was analysed by philosophers, using the methods of mathematical logic and was deeply investigated in the last century (see [23, 24, 26, 28, 32, 29])). In our approach, the scientist is modelled as a Turing machine. Thus, the Turing machine represents the discrete-time activity of the scientist; and the oracle, is an analog device belonging to the continuous-time world where the measurement takes place. Details of the model were given in Section 7.

Let us note that anthropic interpretations of the Turing machines originate with Turing's 1936 image of a human following a procedure to calculate with symbols. The Church-Turing Thesis on the limits of computability rests most firmly on Turing's anthropic argument. For us the machine models a scientist measuring the value of a physical quantity. We believe that this idea has tremendous scope.[3]

The axiomatic analysis in Section 2.1 from Measurement Theory encourages us to think that the computational taxonomy of Section 2 is very general. We believe that analysing case studies of experiments is vital to the construction of the theory. However, describing a physical experiment can easily consume too much space, a fact that motivates us to look at simple experiments which are easy to specify and illuminate a property. In this paper, we have reinterpreted and added experiments. For example, threshold measurements are distinct from signed measurements and so are not covered by the theory we developed in previous papers such as [13, 14]. In particular, by including the modified Rutherford experiment of Section 5.3, we wanted to emphasise that complex thought experiments can be of the same type as simple experiments. We conjecture that *the threshold measurements have a canonical example in the broken beam balance.*

[3] In learning theory, the machine can model a human learning from a text, such as in [27].

The most simple experiments, which to a physicist may seem unacceptably simple and unworthy of discussion, are conceptually very rich.

Subsequently, we have analysed further the threshold oracles in [10] and the vanishing oracles in [11].

**12.2. On idealisations.** We commonly encounter two reactions that express caution towards the way we are developing our theory. The experiments we use are simple — we have dealt with this reservation in subsection 12.1. The experiments we use involve idealized apparatus that cannot be built — we will address this now.

Although we have not helped our case by using the term *gedankenexperiment* for many physical oracles, it is rather tiresome to have to point out repeatedly that this reaction is a consequence of a diffuse and naive "philosophy" that considers the Turing machine as an object of a different kind from a mathematical model in physics. It is not. Both the abstract mathematical models of a physical experiment and the Turing machine are entities of the same kind. For example, they are equally non-realizable. For the implementation of the Turing machine an engineer would need either unbounded space and an unlimited physical support structure, or unbounded precision in some finite interval to code for the contents of the tape; each time the size of the written word in the working tape increases by one symbol, the precision needed will increase.[4] The experiment can be set up to some precision in the same way that the Turing machine can be implemented up to some accuracy.

Knowing that both objects, the Turing machine and the measurement device, are of the same ideal nature, we may now wonder what is the purpose of such an experiment from the computational point of view. The physical experiment exhibits the character of an oracle, an external device to the Turing machine. It gives to the concept of an oracle a new epistemology, one close to Turing's description in 1939: the oracle is not any more a purely abstract *mathematical* entity, but an abstract *physical* entity; the oracle is not any more a one step transition of the Turing machine, but a device that needs time to be consulted; the oracle is not any more a relativisation mechanism, but it has physical content: it can only be consulted up to some accuracy; moreover the degrees of accuracy in the consultation of the oracle can be studied. For some, this setting can be seen as that of a computer connected to an analogue device — a kind of hybrid system.

Interestingly, an emergent result is the conclusion that infinite precision and unbounded precision are of the same ontological nature, as the computational process has taught us for decades (see [5]). Different experiments imply a conclusion similar to that of the work on neural nets in the 1990s (see [30]): to compute up to time $t$,[5] only $\mathcal{O}(f(t))$ bits of the unknown are needed, where $f$ is a function depending on the undergoing experiment. This result is more about the nature of numbers and arithmetic than about physics or neurodynamics.

[4]Note that, once the limits of miniaturization are attained, the only way to build unbounded memories is by increasing the volume.

[5]For example, to simulate the physical system up to time $t$.

Indeed, since the Turing machine and the measurement device are of the same ideal nature, the motivations for their study can be exchanged. From the physical sciences point of view, what does the idea of a Turing machine connected with an abstract physical device (that cannot be built) offer? The orthodox motivation for studying Turing machines in computer science applies: the theory is able to describe the logical and algorithmic nature, scope and limits of a *computer*. Equivalently, our theory is able to describe the logical and algorithmic nature, scope and limits of experiments that make measurements. Our results suggest logical and algorithmic aspects of measurement matter.

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COLLEGE OF SCIENCE
SWANSEA UNIVERSITY, SINGLETON PARK, SWANSEA, SA2 8PP
WALES, U.K.
*E-mail*: E.J.Beggs@Swansea.ac.uk

INSTITUTO SUPERIOR TÉCNICO
UNIVERSIDADE DE LISBOA
PORTUGAL
and
CENTRO DE MATEMÁTICA E APLICAÇÕES FUNDAMENTAIS
UNIVERSIDADE DE LISBOA
PORTUGAL
*E-mail*: fgc@math.tecnico.ulisboa.pt

COLLEGE OF SCIENCE
SWANSEA UNIVERSITY, SINGLETON PARK, SWANSEA, SA2 8PP
WALES, U.K.
*E-mail*: J.V.Tucker@swansea.ac.uk