Toric degenerations of projective varieties with an application to equivariant Hilbert functions

by

Takuya Murata

Bachelor's degree in mathematics, St Mary's University of Minnesota, 2005

Master's degree in mathematics, Hokkaido University (Japan), 2008

Submitted to the Graduate Faculty of

the Dietrich School of Arts and Sciences in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

University of Pittsburgh

2020

UNIVERSITY OF PITTSBURGH

DIETRICH SCHOOL OF ARTS AND SCIENCES

This dissertation was presented

by

Takuya Murata

It was defended on

April 14, 2020

and approved by

Kiumars Kaveh, Department of Mathematics

Bogdan Ion, Department of Mathematics

Gregory M. Constantine, Department of Mathematics

Christopher Manon, University of Kentucky, Mathematics Department

Dissertation Director: Kiumars Kaveh, Department of Mathematics

Toric degenerations of projective varieties with an application to equivariant Hilbert functions

Takuya Murata, PhD

University of Pittsburgh, 2020

A toric degeneration is a flat family over $\mathbb{A}^1$ that is trivial away from the special fiber (fiber over zero) and whose special fiber is a variety acted linearly by a torus with a dense orbit; i.e., the special fiber is a non-normal = not-necessarily-normal toric variety. We introduce a systematic method to construct toric degenerations of a projective variety (embedded up to Veronese embeddings). Part 1 develops the general theory of non-normal toric varieties by generalizing the more conventional theory of toric varieties. A new characterization of non-normal toric varieties as a complex of toric varieties is given. Given a projective variety X of dimension d, the main result of the thesis (Part 2) constructs a finite sequence of flat degenerations with irreducible and reduced special fibers such that the last one is a non-normal toric variety. The degeneration sequence depends on the choice of a full flag of closed subvarieties $X = Y_0 \supset Y_1 \supset \cdots \supset Y_d$ such that each Y_i is a *good divisor* in Y_{i-1}. The notion of a good divisor comes from the asymptotic ideal theory in commutative algebra and the goodness ensures the finite generation of the defining graded ring of the special fiber in each step. This is a generalization of degeneration (or deformation) to normal cone in intersection theory and can be regarded as geometric reinterpretation of the construction of a valuation in [Oko96], the key step in the construction of a Newton–Okounkov body. Part 3 reformulates the main result of [Oko96] in terms of an equivariant Hilbert function; this reformulation may be thought of as a special case of the equivariant Riemann–Roch theorem.

Table of contents

Acknowledgement

The present thesis was written under the supervision of the author's thesis advisor Kiumars Kaveh. The writing of the thesis has partially benefited from inputs from Christopher Manon. The thesis is dedicated to the parents of the author.

0.0 Introduction

Among other things, the present thesis introduces a systematic method to construct a *toric degeneration* of a given projective variety. For our purpose, by a *(flat) degeneration*, we mean a flat morphism

$$\eta : \mathfrak{X} \to \mathbb{A}^1$$

that is trivial away from the special fiber = fiber over 0; i.e., $\mathfrak{X} - \eta^{-1}(0)$ is isomorphic to $X \times (\mathbb{A}^1 - 0)$ over $\mathbb{A}^1 - 0$ for some variety X. All the fibers of η other than the special fiber are isomorphic to one another, namely X; thus, geometrically, η encodes the process of X degenerating to the special fiber $\eta^{-1}(0)$. By a *toric degeneration*, we mean a flat degeneration where we further require that the special fiber $\eta^{-1}(0)$ is *toric* in the sense that $\eta^{-1}(0)$ is a variety[1] that has a $\mathbb{G}_m^r$ = torus action with a dense orbit.[2] If normal, it is then a toric variety. Toric varieties form an important class of algebraic varieties and their geometry has intimate connection with geometry and combinatorics of convex polytopes.

Degeneration techniques have a long history in algebraic geometry; in the literature, it is much more common to consider a degeneration within a fixed ambient projective space; i.e., the case when $\eta : \mathfrak{X} \to \mathbb{A}^1$ factors as

$$\mathfrak{X} \hookrightarrow \mathbb{P}^N \times \mathbb{A}^1 \to \mathbb{A}^1$$

where the first map is a (closed) immersion and the second map is the projection. One key property is that the Hilbert function of the homogeneous coordinate ring of each fiber of η, a closed subvariety of $\mathbb{P}^N$, is independent of the choice of fiber. We call it the Hilbert function attached to $\mathfrak{X}$. In particular, the arithmetic genus is constant along the base $\mathbb{A}^1$. This incidentally means that the special fiber of a toric degeneration is generally a non-normal toric variety, as a normal toric variety has arithmetic genus zero. This motivates us to devote Part 1 of the thesis to the general theory of not-necessarily-normal toric varieties, as no

[1]By convention, a variety is geometrically integral (= geometrically irreducible and geometrically reduced).

[2]In the literature, a *semi-toric degeneration*; i.e., a degeneration whose special fiber is a union of toric varieties is also called a toric degeneration. Here, we do not use that terminology.

commonly used reference exists on the topic.[3] (More substantially, Part 1 is meant to be a special case of the general theory of toric degenerations as a generalization of a toric variety; see Remark 0.0.1 below for more on this.)

In this thesis, given a projective variety $X \subset \mathbb{P}^N$, we will construct a toric degeneration of X up to some Veronese embedding $\mathbb{P}^N \hookrightarrow \mathbb{P}^{N'}$. Thus it will be an embedded toric generation but only going to a Veronese embedding. This is enough for many of the applications we are interested in; the most important being the reframing of [Oko96]. In fact, our construction is a variant of the construction in [Oko96] (which is the origin of the theory of Newton–Okounkov bodies). Thus, to explain the application and the construction, we briefly review [Oko96] first.

Let $X \subset \mathbb{P}^N$ be a projective variety over $\mathbb{C}$ and G a connected reductive group acting on the homogeneous coordinate ring R of X in the grade-preserving manner; i.e., for each $n > 0$, there is a finite-dimensional representation

$$\pi_n : G \to GL(R_n).$$

The paper [Oko96] concerns the asymptotic behavior of π_n as $n \to \infty$; more precisely, the number of times each irreducible representation (i.e., multiplicity) appear in each π_n. The reason for considering the asymptotic behaviors, at least for Okounkov, is that, as $n \to \infty$, one expects (and can show) π_n to exhibit some stable behavior; namely, multiplicities behave like volumes and thus satisfy classical geometric inequalities such as the Brunn–Minkowski inequality. As he mentions in the introduction of the paper, Okounkov was motivated by a similar result in symplectic geometry but he establishes his result without tools from symplectic geometry at all (so, in particular, it is valid over an arbitrary base field).

As we will do in §9, the above result of [Oko96] can be formulated in terms of a G-equivariant Hilbert function attached to a toric degeneration. This is already interesting and in fact this observation was the origin of the thesis. But, in hindsight, the importance of the reformulation here is that it is an instance of an application of a toric degeneration to extend some of notions/constructions in sympletic geometry, which takes place over $\mathbb{C}$, to

[3] See also the webpage https://dacox.people.amherst.edu/toric.html for the list of references on non-normal toric varieties. As the references there indicate, a non-normal toric variety is typically thought of as an instance of a scheme over a “generalized ring”.

algebraic geometry over an arbitrary base field. For example, one can define the moment polytope (the image of a moment map) of a toric degeneration in this way; this is because the data specifying a not-necessarily-normal toric variety is free of the base field and thus, without loss of generality, the base field of the special fiber can be assumed to be $\mathbb{C}$. We note that this phenomenon is very analogous to one in algebraic number theory; there one considers a *flat model* of X over a base such that the special fiber is over a finite field and the generic fiber is over a field of characteristic zero. One then for instance considers a lift or deformation of a Frobenius action from the special fiber to the generic fiber. In the setup of a toric degeneration, an analogous procedure is possible because, again, the data defining a non-normal toric variety is combinatorial. (Here, as an analog of the deformation of a Frobenius action, one can consider, for example, a deformation of a Hamiltonian group action via a toric degeneration.)

0.0.1 Remark (flat model)**.** The present thesis has tried but yet not completed introducing the viewpoint that a toric degeneration is a generalization of a toric variety (or of a toric scheme over an affine line); it is something that can be studied on its own as opposed to the means to study a given variety. Indeed, as we will come back to later, we prefer to view a toric degeneration of X as an example of a *flat model* of X and, in algebraic geometry, to study a space (e.g., a variety or analytic space) as well as objects (e.g., sheaves) on the space, it is a common technique to shift the attention from the space to flat models of that space.[4]

As in [An13], our construction of a toric degeneration is based on Okounkov's construction of a valuation for the function field of a variety X. (This is the most general construction as a toric degeneration inside a projective space necessarily comes from a valuation; cf. Proposition 7.2.4.) For that, we briefly recall his construction in the form that slightly differs from the original one. A key piece is:

0.0.2 Lemma (Lemma 7.2.5)**.** *Let $Y \subset X$ be a codimension-one closed subvariety of an algebraic variety X; i.e., Y is a prime Weil divisor. If*

$$\nu' : k(Y)^* \to \mathbb{Z}^{r-1}$$

[4]The term "flat mode" is not common; in arithmetic geometry, a flat model is more commonly called an integral model and in rigid geometry a flat model is called a formal model.

is a valuation whose image is a free abelian group of rank $r-1$, then there exists a not-necessarily-unique valuation

$$\nu : k(X)^* \to \mathbb{Z}^r,$$

such that (1) $\nu(f) = \nu'(f|_Y)$ for each f in $k(X)$ that does not have zero or pole along Y, so that $f|_Y$ is defined and nonzero, and (2) the image of ν is a free abelian group of rank r.

The proof of the lemma is short. The lemma is then applied inductively to a given partial flag of closed subvarieties

$$X = Y_0 \supset Y_1 \supset \cdots \supset Y_r,$$

where each Y_i is a codimension-one closed subvariety of Y_{i-1}; i.e., a prime Weil divisor, to not-uniquely yield the valuation

$$\nu : k(X)^* \to \mathbb{Z}^r.$$

See Remark 7.3.1.

Now, if $X = \operatorname{Proj} R$ is a projective variety and, for simplicity, if R is an integral domain and contains a degree-one element s (e.g., a global section of $\mathcal{O}_X(1)$) that does not vanish on Y_r, then ν extends to each degree-n piece R_n through the open affine chart $\{s \neq 0\} = \operatorname{Spec}(R[s^{-1}]_0)$:

$$\nu_{R_n} : R_n - 0 \overset{f \mapsto f/s^n}{\longrightarrow} R[s^{-1}]_0 - 0 \overset{\nu}{\to} \mathbb{Z}^r,$$

which in turns determines the integral domain (as R is an integral domain and ν is a valuation):

$$\operatorname{gr}_\nu R := \bigoplus_{n=0}^{\infty} \bigoplus_{a \in \mathbb{Z}^r} \{f \in R_n | f = 0 \text{ or } \nu_{R_n}(f) \geq a\} / \{f \in R_n | f = 0 \text{ or } \nu_{R_n}(f) > a\}.$$

If $r = \dim X$; i.e., if the flag is full, then $\operatorname{gr}_\nu R$ is a graded semigroup algebra that is an integral domain; thus, if it is finitely generated, then Proj of $\operatorname{gr}_\nu R$ is a not-necessarily-normal toric variety. By means of a Rees algebra (§5), one can then construct a toric degeneration $X \rightsquigarrow \operatorname{Proj}(\operatorname{gr}_\nu R)$ inside $\mathbb{P}^{N'}$, N' = the number of the generators of $\operatorname{gr}_\nu R$. We stress that, while ν is intrinsically constructed only from X, $\operatorname{gr}_\nu R$, obviously, depends on R; i.e., the question of "finite generation" is a matter of extrinsic geometry of X.

We note that even if $\mathrm{gr}_\nu R$ is not finitely generated, when the flag is full, it is still a graded semigroup algebra and one can attach a convex set to it[5] and the closure of the convex set is then called the Newton–Okounkov body of R relative to the flag $Y_\bullet$; [KK12] and [LM09]. (This convex set is a convex polytope if and only if $\mathrm{gr}_\nu R$ is finitely generated by Proposition 2.4.10; thus, in that case, the Newton–Okounkov body encodes the Hilbert function of the normalization of $\mathrm{gr}_\nu R$, a very useful information for many applications.)

For the application, the key property of the above valuation is that, *as a graded vector space*, R and $\mathrm{gr}_\nu R$ are isomorphic. Hence, we can view the construction as determining a family of graded ring structures on the same underlying graded vector space; put in another way, the graded ring structure of R is a deformation of the graded ring structure of $\mathrm{gr}_\nu R$. In particular, the procedure leaves intact the Hilbert function as the function is agnostic about the ring structure. More significantly to representation theory, when there is some graded-linear group action on R by a reductive group G, the valuation can be constructed to preserve the linear group action; i.e., R and $\mathrm{gr}_\nu R$ are isomorphic as graded G-modules.

Now, to address the question of the finite generation of $\mathrm{gr}_\nu R$, in this thesis, we introduce the notion of a *good flag*; i.e., a flag giving rise to finitely generated $\mathrm{gr}_\nu R$ in the above construction. In fact, we actually reinterpret the above construction of the valuation in a manner more familiar to algebraic geometers (specifically to those with some background in intersection theory). To define a good flag, for simplicity, suppose Y_i is an effective Cartier divisor on Y_{i-1}. We then say that a flag $Y_\bullet$ is a *good flag* if there are homogeneous elements $x_1, \dots, x_n$ in R of positive degree such that, *as sets*,

$$X = Y_0 \supset Y_1 = V(x_1, \dots, x_{n_1}) \supset \cdots \supset Y_d = V(x_1, \dots, x_{n_d})$$

where

$$0 \le n_i - n_{i-1} - 1 \le \dim Y_i = d - i, \quad n_0 = 0$$

(see Example 7.1.2). Geometrically, if R is the section ring of $\mathcal{O}_X(1)$, then each x_i corresponds to a hypersurface in the linear system $|\mathcal{O}_X(q_i)|$, $q_i = \deg(x_i)$, and so the above condition

[5]This convex set is the slice of the cone generated by the defining graded semigroup of $\mathrm{gr}_\nu R$, which may not be closed. The author personally calls it the Newton–Okounkov convex set of R; but that terminology is non-standard.

means that Y_1 is the (set-theoretic) base locus of the sections $x_1, \ldots, x_{n_1}$, Y_2 is the (set-theoretic) base locus of the sections $x_1, \ldots, x_{n_2}$, etc. The notion of a good flag can still be defined without the "Cartier" assumption.

Now, we can state the following summary of Theorem 7.2.1 and Remark 7.3.1:

0.0.3 Theorem. *Given a good flag $Y_\bullet$ of X as above, we can construct a sequence of flat degenerations:*

$$X \rightsquigarrow X_1 \rightsquigarrow \cdots \rightsquigarrow X_i$$

so that

(1) $X_i = \operatorname{Proj}(\operatorname{gr}_{\nu_i} R)$ *where ν_i is the valuation given by Okounkov's construction from the flag $X = Y_0 \supset \cdots \supset Y_i$ and $\operatorname{gr}_{\nu_i} R$ is finitely generated. In particular, $\mathbb{G}_m^i$ acts on X_i linearly with finite stabilizers.*

(2) Each Y_i is the GIT quotient of X_i by the $\mathbb{G}_m^i$-action in (1).

Conversely, if $\operatorname{gr}_\nu R$ is finitely generated, then the flag used to define ν is a good flag.

See §2.7.1 for the definition of a GIT quotient as well as its basic properties. The key component in the proof of the theorem is that a lifting property of a good flag for a GIT quotient; namely, we prove the following (Proposition 7.1.4):

0.0.4 Proposition (lifting of a good flag)**.** *Let $\pi : X^{ss} \to Y$ be a projective GIT quotient by a linear action of a torus $\mathbb{G}_m^r$ with no negative weights (the characters of the torus are totally ordered with respect to the lexicographical ordering on $\mathbb{Z}^r = \operatorname{Hom}(\mathbb{G}_m^r, \mathbb{G}_m)$).*

Then, given a good flag $Y \supset Y'$ of Y, there exists a good flag $X \supset Z$ of X that is a lift of it; i.e., $\pi(Z \cap X^{ss}) = Y'$ (note Z is generally not an effective Cartier divisor.)

Theorem 0.0.3 is then proved by a repeated application of the above proposition. First, the theorem is true for a good flag of length one essentially by definition (or defining characterization of it; Theorem 6.3.1 (ii)). Next, because there is a GIT quotient $X_1^{ss} \to Y_1$ and $Y_1 \supset Y_2$ is a good flag, we get a good flag $X_1 \supset Z$, which, by the length-one case, degenerates X_1 to X_2 and so forth. (The proposition itself is, roughly, a consequence of the compatibility of valuations and a graded Nakayama lemma.)

Okounkov's original construction uses an analogous tool to inductively construct a valuation out of a flag (Lemma 7.2.5); the above proposition is then may be thought of as the geometric version of that.

The construction of a projective GIT quotient crucially relies on the choice (and existence) of an equivariant ample line bundle[6]; in particular, the quasi-projectivity of the variety. Consequently, the construction of Theorem 0.0.3 relies, in particular, on the quasi-projectivity of the variety (see Conjecture below for a possible resolution to this issue).

We stress that the notion of a good flag is relative to a choice of the defining ring R of the projective variety X; or equivalently a choice of an ample line bundle on X. To reinforce this point further, we make the following comment:

0.0.5 Remark (good divisor)**.** Intuitively speaking, a good flag is a flag consisting of *good divisors*. Intrinsically, (for simplicity) a good divisor on a projective variety X is simply an effective Cartier divisor. The difference from an effective Cartier divisor has to do with extrinsic geometry: when X is equipped an ample effective Cartier divisor H, a good divisor is an effective Cartier divisor that is relatively in a good position with H; e.g., H itself. For the purpose of the construction, we assume that each divisor in the flag is geometrically integral but that itself is incidental to the notion of a good divisor. (Currently, a somewhat more general theory of good divisors is being worked out in a sequel to the thesis [Mu2X], that includes in particular Zariski's theorem on finite generation of a section ring.)

To the readers with some background on geometric invariant theory, a good analogy would be that of a stable point. Relative to an ample equivariant line bundle, one can speak of whether a given closed point is a stable point or not. In much the same way, relative to a given ample line bundle, one can speak of whether a given (effective Cartier) divisor is a good divisor or not.

Finally, [An13] considers a similar idea that one should identify some distinguished class of divisors. It is an interesting question to investigate the relationship between Anderson's divisor and a good divisor in the sense defined here.

It is known that there exists a smooth projective curve of genus ≥ 2 embedded in a

[6] An equivariant line bundle is also known as a linearized line bundle.

projective space that does not admit a toric degeneration, without a change of the embedding other than a change through a Veronese embedding (see Corollary 6.3.2 and [KMM20, §3]). This motivates the following:

Problem. *If a projective variety $X \subset \mathbb{P}^N$ contains a copy of $\mathbb{P}^1$ perhaps in some "favorable position", then show that there exists a good flag of the form $X = Y_0 \supset \cdots \supset Y_{d-1} = \mathbb{P}^1 \supset Y_d$.*

Note that it is not enough that X contains a singular rational curve; see [IW20]. By an argument with Bertini's theorem, without any restriction on the variety X, it is not hard to construct a partial good flag (Example 7.2.3); hence, the thesis in particular recovers the main result of [KMM20]. The above problem thus says that this Bertini argument can be modified to construct a full good flag when there is a copy of $\mathbb{P}^1$ in X.

There is also a conjecture that we want to propose, which is an analog of Raynaud's theorem in rigid geometry.[7]

Conjecture. *Given an algebraically closed field k, there is an equivalence of categories:*

$$\boxed{\textit{toric degenerations}} \;/\sim\; \overset{\simeq}{\to} \boxed{\textit{algebraic varieties over } k}$$
$$[\mathfrak{X}] \qquad\qquad \mapsto \textit{ the generic fiber of } \mathfrak{X}$$

where $/\sim$ refers to a localization (i.e., a quotient) of a category so that if $\mathfrak{X} \sim \mathfrak{X}'$, then they have the same generic fiber.

Moreover, in the above, "proper" is respected in the sense that if a general fiber X of $\mathfrak{X}$ is proper (i.e., complete) over the base field, then the special fiber of $\mathfrak{X}$ is also proper.

In the above, $\sim$ is not explicitly specified and working out the generators of $\sim$ is an important problem (our working guess is that $\sim$ is generated by *admissible blow-ups.*)

In a way, Conjecture is a call for a larger program of working out the theory for degenerations together with morphisms between them. It is quite useful and important to consider a degeneration of not just of a single variety. That will be clarifying, for example, a toric degeneration of a closed subvariety of a projective space without changing the ambient projective space; e.g., Gröbner degeneration can be formulated as a degeneration of a closed

[7] Raynaud's theorem in rigid geometry states that the category of (quasi-compact quasi-separated) rigid analytic k-space is equivalent to the localization of admissible formal schemes over a complete discrete valuation ring of k with respect to admissible blow-ups.

immersion $f : X \hookrightarrow \mathbb{P}^N$. Similarly, the toric degenerations constructed in the thesis can be viewed as a degeneration of $f : X \hookrightarrow \mathbb{P}^N$ together with a Veronese embedding $\mathbb{P}^N \hookrightarrow \mathbb{P}^{N'}$.

0.1 Convention and notations

The following terminology, notations and conventions are used throughout the thesis.

- A finite module means a finitely generated module.
- An algebraic variety is a geometrically integral scheme that is separated and is of finite type over a given base field. (Note some authors such as Fulton assume only that an algebraic variety is irreducible instead of geometrically irreducible and that will cause an issue when we use Bertini's theorem for instance.)
- The precise meaning of the often-used phrase "$\operatorname{Proj} R$ is a projective variety" is given in Definition 2.4.7.
- $\mathbb{R}_+$, the set of nonnegative real numbers.
- $\mathbb{N} = \mathbb{R}_+ \cap \mathbb{Z}$.
- $k[S]$, the semigroup algebra of a unital semigroup S.

Except for Part 3 (where the main concern is representation theory), we have stated the results for a base field that is not necessarily algebraically closed field. This is because, as the readers will easily notice, many of the results belong to commutative algebra and in commutative algebra, it is usually unnatural and unnecessarily to require the base field is an algebraically closed field (even infinite). Geometrically-minded readers should simply assume the base field is algebraically closed. Algebraically-minded readers will notice that, for majority of the results in the thesis, it is not even necessary to assume the base ring is a field (but is still some nice regular ring like a discrete valuation ring). Geometrically-speaking, the case when the base ring has higher dimension corresponds to a family-situation; e.g., a family of toric degenerations.

We should, however, note that most of the nontrivial commutative algebra results here will fail for general Noetherian rings or Noetherian integral domains (cf. Remark 5.3.3); it is crucial to limit ourselves to rings that are finitely generated algebras (the reason is closely related to dimension-theoretic difficulties we encounter when we try to develop intersection theory only using Noetherian rings).

1.0 Part 1: Non-normal toric varieties and inverse systems of semigroups

By definition, an affine non-normal toric variety is the Spec of a finitely generated semigroup k-algebra that is an integral domain:

$$X_S = \operatorname{Spec} k[S].$$

We define a non-normal toric variety as a variety obtained by gluing such X_S for some given system of semigroups S satisfying the compatibility conditions. "Non-normal" refers to the fact that X is not necessarily normal. If X is normal, we call X a toric variety. For example, a fan of cones gives rise to such a system of semigroups (by choosing a lattice and then taking the lattice points of the duals of the cones.)

Given an $\mathbb{N}$-graded finitely generated semigroup $S \subset \mathbb{N} \times \mathbb{Z}^d$, the variety

$$\operatorname{Proj} k[S]$$

is called a projective non-normal toric variety. Our concept of a non-normal toric variety generalizes this notion; such an S determines a system of the semigroups S_u so that $\operatorname{Proj}(k[S])$ is obtained by gluing $\operatorname{Spec} k[S_u]$; concretely, $k[S_u]$ are localizations of $k[S]$.

The use of a system of semigroups allows us to develop a theory that naturally extends that of toric varieties developed in the standard texts such as [Ful93]. Theories of non-normal toric varieties similar to ours are also developed in the papers cited at `http://www.cs.amherst.edu/~dac/toric.html`, the webpage for the book "Toric varieties."

The present section mainly concerns with the definition and some basic properties of a non-normal toric scheme. The implication of the presence of the torus action will be considered in the next section.

1.1 A non-normal toric variety defined by a system of semigroups

We shall use the following notion (cf. [Ro13]).

1.1.1 Definition. By *an inverse system (or a projective system) of semigroups indexed by a category* I, we mean a contravariant functor $i \mapsto S_i$ from a category I to the category of unital commutative semigroups. A fancy way is to say that it is a semigroup-valued presheaf on a category I (we will consider the sheaf condition later; namely, Lemma 1.1.6).

This notion generalizes the properties of a fan (see below) and semigroups arising from it, as we explain. If $\sigma \subset \mathbb{R}^d$ is a nonempty subset, we write

$$\sigma^\vee = \{u \in \mathbb{R}^d | \langle u, v \rangle \geq 0, \, v \in \sigma\}, \;\; \sigma^\perp = \{u \in \mathbb{R}^d | \langle u, v \rangle = 0, v \in \sigma\}.$$

A nonempty subset of $\mathbb{R}^d$ is called a *convex cone* if it is convex and is also stable under the multiplication by positive real numbers. (Convex cones are often assumed to be closed, but that assumption is unnecessary in many cases.)

Given convex cones τ, σ in $\mathbb{R}^d$, we say τ is a *face of* σ if $\tau = \sigma$ or $\tau = \sigma \cap u^\perp$ for some $u \in \sigma^\vee$. A maximal proper face is called a *facet*. It can be shown easily (Lemma 2.2.1) that (1) a finite intersection of faces is a face and (2) "is a face of" is a transitive relation.

By a *fan of convex cones* in $\mathbb{R}^d$, we mean a nonempty set I consisting of cones in $\mathbb{R}^d$ such that

(i) if $\sigma \in I$ and τ is a face of σ; i.e., , then $\tau \in I$

(ii) if $\sigma, \tau \in I$, then $\sigma \cap \tau$ is a face of both σ and τ.

Given σ, τ in I, if τ is a face of σ, then we write $\tau \to \sigma$. Since a face inclusion is a transitive relation, this turns I to a category. For each $\sigma \in I$, let $S_\sigma = \sigma^\vee \cap \mathbb{Z}^d$. Then $\sigma \mapsto S_\sigma$ forms a projective system of semigroups with $S_\sigma \to S_\tau$ induced by the inclusions of faces $\tau \hookrightarrow \sigma$.

Intuitively speaking, we can think S_σ consists of "functions" on σ (hence, the use of dual) and the map $S_\sigma \to S_\tau$ restricts the functions on σ to τ.

In practice, one commonly assumes the cones to be (strongly convex[1]) rational polyhedral cones so that S_σ is finitely generated by Gordan's lemma ([Ful93] Proposition 1).[2]

Given a field k, we fix the standard algebraic torus

$$\mathbb{G}_m^d = \operatorname{Spec} k[\mathbb{Z}^d] = (\operatorname{Spec} k[t, t^{-1}])^{\times d}$$

(more general tori and their actions are considered in the next section.)

Note that $\mathbb{G}_m^d(k) = \operatorname{Mor}(\operatorname{Spec}(k), \mathbb{G}_m^d) = (k^*)^d$, canonically as groups (cf. an example at Definition A.0.1).

Our first goal in this section is to show a projective system of semigroups determines a non-normal toric variety and, conversely, such a variety arises that fashion. This will generalize a fact for toric varieties that says that a fan determines a toric variety and conversely. But there is a key difficulty. Sumihiro's theorem says that a normal $\mathbb{G}_m^d$-variety admits a $\mathbb{G}_m^d$-invariant open affine cover. In the theory of toric varieties, this theorem is used to show that every normal toric variety (i.e., a normal $\mathbb{G}_m^d$-variety with an open dense orbit) arises from a fan. The theorem is not valid without the normality assumption (cf. Example 1.1.12 below). We shall avoid this difficultly by simply limiting ourselves to those satisfying the conclusion of Sumihiro's theorem.

1.1.2 Definition. A *non-normal toric variety over a field* k is a $\mathbb{G}_m^d$-variety over k that admits an open dense $\mathbb{G}_m^d$-orbit as well as a $\mathbb{G}_m^d$-invariant open affine cover. (The latter condition is automatic if X is normal by Sumihiro's theorem.)

If a non-normal toric variety is a normal variety, it is called a *toric variety.*

1.1.3 Example. The torus itself $\mathbb{G}_m^r$ is a toric variety; in fact, intuitively speaking, we tend to view a non-normal toric variety as a *partially compactified torus.* Also, given an algebraic variety X with an (algebraic) action of $\mathbb{G}_m^r$, each $\mathbb{G}_m^r$-orbit closure on X is a non-normal toric variety, assuming the conclusion of Sumihiro's theorem holds for it.

Here is some other (important) way to think about the notion.

[1]A convex cone σ is said to be *strongly convex* if $\sigma \cap (-\sigma) = 0$.

[2]In loc. cit., the name is given as "Gordon" but that is a typo.

1.1.4 Remark (Cox's theorem)**.** We assume the definition of a GIT (= geometric-invariant-theory) quotient is known. Let X be a non-normal toric variety and X^{nor} the normalization of it; then X^{nor} is a toric variety. Hence, by [Cox95] Theorem 2.1., X^{nor} can be written as a GIT quotient of an affine space by some torus action; consequently, there is a composition

$$(\mathbb{A}^N)^{ss} \to X^{\text{nor}} = (\mathbb{A}^N)^{ss} // \mathbb{G}_m^N \to X$$

where $(\mathbb{A}^N)^{ss}$ is the semistable locus. In other words, X is a GIT quotient up to a finite map in the canonical way.

The affine case of the definition can be made concrete (see Theorem 1.2.1 for the projective case).

1.1.5 Lemma. *Let X be an affine $\mathbb{G}_m^d$-variety over an algebraically closed field k with an open dense $\mathbb{G}_m^d$-orbit O. Then*

(i) $X = \operatorname{Spec} k[S]$ *for some finitely generated subsemigroup S of $\mathbb{Z}^d$.*

(ii) For S in (i), S generates $\mathbb{Z}^d$ $\Leftrightarrow$ the composition $\mathbb{G}_m^d \to O \hookrightarrow X$ is birational.

Proof. (i) Let A be the coordinate ring of X; then A is a $\mathbb{G}_m^d$-algebra through a left-regular representation.[3] By a basic result in the theory of linear algebraic groups (Proposition A.0.3), we can find a finite-dimensional $\mathbb{G}_m^d$-submodule W generating A. Since the linear actions of elements of $\mathbb{G}_m^d(k) = (k^*)^d$ are simultaneously diagonalizable on W, W admits a weight space decomposition and then, since an element of A is a polynomial in weight vectors, A itself admits a weight space decomposition:

$$A = \bigoplus_{\chi} A_\chi$$

where $\chi : \mathbb{G}_m^d \to \mathbb{G}_m$ are homomorphisms and $A_\chi = \{f \in A | t \cdot f = \chi(t) f\}$. The inclusion $O \hookrightarrow X$ is birational and it follows that $\dim A_\chi \leq 1$ (indeed, if f, g are weight vectors of the same weight, f/g is torus-invariant and $(f/g)|_O$ must be constant). Hence, A is the semigroup algebra of a subsemigroup $S \subset \operatorname{Hom}(\mathbb{G}_m^d, \mathbb{G}_m) = \mathbb{Z}^d$ (cf. §2.1 for the last equality). Also, S is finitely generated as S is generated by finitely many weights of the weight vectors in W.

[3] In [MFK94], a left-regular representation is called a dual action.

Finally, (ii) holds because S generates $\mathbb{Z}^d$ if and only if $k(X) = Q(k[S]) = Q(k[\mathbb{Z}^d]) = k(\mathbb{G}_m^d)$ if and only if $\mathbb{G}_m^d \to X$ is birational. □

Gluing affine non-normal toric varieties gives rise to a non-normal toric variety. This is done by:

1.1.6 Lemma (gluing lemma)**.** *Let I be a category that admits a finite product and we write $i \cap j$ for the product of i, j; the notation is because if i, j are cones, the product is the intersection of i, j.*

Let $i \mapsto S_i$, $i \in I$ be a projective systems of finitely generated subsemigroups of $\mathbb{Z}^d$ such that

- *(i) For any $i \to j$, the induced map $\operatorname{Spec} k[S_i] \to \operatorname{Spec} k[S_j]$ is an open immersion.*
- *(ii) For any $l \to i, l \to j$ in I, S_l is generated by the images of S_i and S_j.*
- *(iii) I is finite; i.e., it has only finitely many objects.*
- *(iv) I has an initial object i_0 such that S_{i_0} is a group $G \simeq \mathbb{Z}^d$.*

We let

$$X = \varinjlim \operatorname{Spec} k[S_i],$$

that is, X is obtained by gluing $\operatorname{Spec} k[S_i]$ and $\operatorname{Spec} k[S_i]$ along $\operatorname{Spec} k[S_{\inf\{i,j\}}]$.[4]

Then X is a non-normal toric variety in the sense of Definition 1.1.2.

Proof. We must show X is an integral scheme that is of finite type and is separated over the base field k.

We note that X is separated over k if and only if S_l is generated by the images of S_i and S_j for any $l \to i, l \to j$. Indeed, since a closed immersion is local, X is separated over $k \Leftrightarrow \operatorname{Spec} k[S_l] \to \operatorname{Spec} k[S_i] \times_k \operatorname{Spec} k[S_j]$ is a closed immersion for any $l \to i, l \to j \Leftrightarrow k[S_i] \otimes_k k[S_j] \to k[S_l]$, $\chi^u \otimes \chi^{u'} \mapsto \chi^{u+u'}$ is surjective $\Leftrightarrow S_l$ is generated by the images of S_i and S_j for any $l \to i, l \to j$.

Hence, X is separated by (ii). Since X is locally of finite type and is quasi-compact (the latter by (iii)), X is of finite type. Finally, X is connected, as a consequence of (iv), and so is irreducible (since local rings are integral domains.) □

[4]the inductive limit exists as a scheme over k ([Hart77] Ch. II, Exercise 2.12.)

Here is a prototypical example in which the conditions of the lemma are satisfied.

1.1.7 Example. Let $S \subset \mathbb{Z}^d$ be an additive subsemigroup. If $u \in S$, then the localization map

$$k[S] \to k[S][\chi^{-u}] = k[S + \mathbb{N}(-u)]$$

corresponds to $S \hookrightarrow S + \mathbb{N}(-u)$. Geometrically, this localization map corresponds to the open immersion $\{\chi^u \neq 0\} = \mathrm{Spec}(k[S + \mathbb{N}(-u)]) \hookrightarrow \mathrm{Spec}\, k[S]$ and so (i) of the lemma is satisfied.

(Explicitly, for the index category I, we can take the category where the objects are $U_u = \mathrm{Spec}\, k[S][\chi^{-u}]$ and then we have the obvious contravariant functor $U_u \mapsto S_u$.)

In the theory of toric varieties, the above assumes the following more convex-geometric form. Let $\sigma \subset \mathbb{R}^d$ be a closed cone and $S_\sigma = \sigma^\vee \cap \mathbb{Z}^d$. If τ is a face of σ, then $\tau = \sigma \cap u^\perp$ for some $u \in S_\sigma$. Since $(\sigma + \mathbb{R}_+(-u))^\vee = \sigma^\vee \cap (\mathbb{R}_+(-u))^\vee = \tau$, we find:

$$S_\tau = S_\sigma + \mathbb{N}(-u).$$

Thus, by the early discussion, (i) is satisfied for $\tau \to \sigma$. As for (ii), let τ be a cone such that τ is the intersection of the closed cones σ, σ'. By the hyperplane separation theorem, we find a hyperplane $u^\perp = \{\langle u, \cdot \rangle = 0\}$ such that $\mathrm{rel.\,int}(\sigma) \subset \{\langle u, \cdot \rangle > 0\}$ and $\mathrm{rel.\,int}(\sigma') \subset \{\langle u, \cdot \rangle < 0\}$. Shrinking $u^\perp$, we can also achieve $\tau = \sigma \cap u^\perp = \sigma' \cap u^\perp$ Then it is straightforward to see that $S_\tau = S_\sigma + S_{\sigma'}$.

The next theorem is a direct generalization of the analogous statement in the normal case.

1.1.8 Theorem. *Assume the base field k is algebraically closed (for simplicity).*

Let $\mathcal{T}^d$ be the category where

- *the objects are pairs (I, S_*)[5] of a category I and a projective system $S_* = \{S_i | i \in I\}$ on I as in Lemma 1.1.6,*
- *a morphism $\phi : (I, S_*) \to (I', S'_*)$ consists of a functor $I \to I'$ and semigroup homomorphisms $S'_{\varphi(i)} \to S_i$ indexed by the objects of I such that, for each $i \to j$, $S'_{\varphi(j)} \to S_j \to S_i$ coincides with $S'_{\varphi(j)} \to S'_{\varphi(i)} \to S_i$ (in short, a natural transformation $S'_{\varphi(*)} \to S_*$).*

[5] Intuitively speaking, we can think of a pair (I, S_*) as a semigroup-ed-space with the structure sheaf S_* on the space I.

Define the functor from $\mathcal{T}^d$ to the category of algebraic varieties by sending a system $\{S_i | i \in I\}$ in $\mathcal{T}^d$ to $\varinjlim_i \operatorname{Spec} k[S_i]$.

Then this functor is well-defined. Moreover, (up to isomorphisms), each non-normal toric variety with torus $\mathbb{G}_m^d$ is in the image of this functor.

Proof. First we describe how the functor maps morphisms. Given a morphism $\phi : S_* \to S'_*$ in $\mathcal{T}^d$, let $U_i = \operatorname{Spec} k[S_i]$ and $U'_j = \operatorname{Spec} k[S'_j]$. Then, for each $i \in I$, ϕ determines a morphism of varieties:

$$\varphi_* : U_i \to U'_{\varphi(i)}$$

given by $k[S'_{\varphi(i)}] \to k[S_i]$. We claim that the above maps glue. Note that $U_{i \cap j} = U_i \cap U_j$ and thus, given i, j in I, we need to verify that $U_{i \cap j} \hookrightarrow U_i \to U'_{\varphi(i)}$ agrees with $U_{i \cap j} \hookrightarrow U_j \to U'_{\varphi(j)}$. But, by assumption, the first one is equal to $U_{i \cap j} \to U'_{\varphi(i \cap j)} \to U'_{\varphi(i)}$ and the similar equality holds for the second; whence, the agreement.

To complete the proof, let X be a non-normal toric variety with $\mathbb{G}_m^d$-invariant open affine cover U_i's. Then, by Lemma 1.1.5, we can write $U_i = \operatorname{Spec} k[S_i]$. With $S_i \to S_j$ induced by $U_j \to U_i$, the S_i's form a projective system of semigroups. □

We record:

1.1.9 Proposition. *The functor in the theorem specializes to the corresponding functor in the theory of toric varieties.*

Proof. Omitted but is clear since the above theorem is obtained by generalizing the toric-variety case. □

Here are some simple examples and remarks.

1.1.10 Example. Let $S_1 = \mathbb{Z}_{\geq 0}, S_2 = \mathbb{Z}_{\leq 0}$. Then $\{\mathbb{Z}, S_1, S_2\}$ is in $\mathcal{T}^d$ with the inclusions $S_i \to \mathbb{Z}, i = 1, 2$. Gluing $\operatorname{Spec} k[S_i], i = 1, 2$ along $\mathbb{G}_m = \operatorname{Spec} k[\mathbb{Z}]$ results in $\mathbb{P}^1$.

1.1.11 Example. Let $S_1 \subset \mathbb{Z}$ be the subsemigroup generated by $2, 3$ and $S_2 = \mathbb{Z}_{\leq 0}$. Then gluing $\operatorname{Spec} k[S_i], i = 1, 2$ along $\operatorname{Spec} k[\mathbb{Z}]$ results in a non-normal toric variety X whose normalization is $\mathbb{P}^1$.

1.1.12 Example ([Oda78], Ch. I, §2)**.** Let X be a rational curve with a node, obtained by identifying $0 = (0 : 1)$ and $\infty = (1 : 0)$ on $\mathbb{P}^1$. Then $\mathbb{G}_m = k^*$ acts on X but the node has no $\mathbb{G}_m$-invariant affine neighborhood. Hence, it is *not* a non-normal toric variety in the sense of Definition 1.1.2.

1.1.13 Remark. For further discussion of failure of Sumihiro's theorem, see also [Br13].

1.2 Projective non-normal toric varieties

We shall next give a theorem describing how a graded semigroup gives rise to a projective non-normal toric variety and, conversely, every equivariantly-projective non-normal toric variety is of such a form.

First, we recall the notion of an equivariant line bundle. Let X be an algebraic variety X with an action of the torus $\mathbb{G}_m^d$. By a $\mathbb{G}_m^d$*-equivariant line bundle*, we mean an invertible sheaf L such that the corresponding line bundle $Y_L = \operatorname{Spec}_X(\bigoplus_0^\infty (L^*)^n)$ is a $\mathbb{G}_m^d$-variety having the properties

(1) the projection $p : Y_L \to X$ is $\mathbb{G}_m^d$-equivariant,

(2) the action induces the linear transformation $p^{-1}(x) \to p^{-1}(t^{-1} \cdot x)$ for closed points $t \in \mathbb{G}_m^d$ and $x \in X$.

Or, equivalently, L is a $\mathbb{G}_m^d$-linearized invertible sheaf in the sense of [MFK94].

The next theorem is a projective analog of Lemma 1.1.5.

1.2.1 Theorem. *Let $S \subset \mathbb{N} \times \mathbb{Z}^d$ be a finitely generated subsemigroup and $l : \mathbb{N} \times \mathbb{Z}^d \to \mathbb{N}$ the coordinate projection (called the level function). Then the projective variety*

$$\operatorname{Proj} k[S]$$

is a non-normal toric variety, where Proj is taken with respect to $\mathbb{N}$-grading. Concretely, it is given as:

$$\operatorname{Proj} k[S] = \varinjlim_{l(u)>0} \operatorname{Spec} k[S_u],\ S_u = \{x - nu | n \in \mathbb{N}, x, u \in S, l(x) = nl(u)\}$$

where the colimit runs over all elements u of S with $l(u) > 0$ (a special case of Example 1.1.7.)

Conversely, every complete non-normal toric variety that admits a $\mathbb{G}_m^r$-equivariant ample line bundle (cf. Question 1.2.2) has such a form; i.e., one can write $X = \operatorname{Proj} k[S]$ for some graded S as above.

Proof. By construction, $\operatorname{Proj} k[S]$ has an open affine chart consisting of

$$\operatorname{Spec}(k[S][f^{-1}]_0)$$

where R_0 denotes the zero-th degree piece of a graded ring R and f are various homogeneous elements of $k[S]$ of positive degree. We can take the f's here to be the generators of $k[S]$ and $k[S]$ is generated by $\chi^u, l(u) > 0$. Thus, the first assertion follows from Lemma 1.1.6 (cf. Example 1.1.7).

Next, we prove the converse. By assumption, there is a $\mathbb{G}_m^d$-equivariant ample line bundle L on X; so $X = \operatorname{Proj} R(L)$ where $R(L) = \bigoplus_0^\infty \Gamma(X, L^{\otimes n})$ is the section ring of L. Then $R(L)$ is a $\mathbb{G}_m^d$-algebra. Arguing similarly to the last part of the proof of Lemma 1.1.5, we see $R(L)$ is a semigroup algebra. □

1.2.2 Question (of linearization)**.** The author is unaware of a simple answer to the question: which ample line bundle on a non-normal toric variety with torus $\mathbb{G}_m^r$ is $\mathbb{G}_m^r$-linearizable.[6]

But here is some partial result that is marginally interesting. Let X be a complete non-normal toric variety, and L a line bundle on it. Then, for the open dense orbit O on X, $L|_O$ is trivial. Indeed, since O is a normal variety, writing Cl for the divisor class group, we have:

$$\operatorname{Pic}(O) \subset \operatorname{Cl}(O) \simeq \operatorname{Cl}(\mathbb{G}_m^r) = \operatorname{Cl}(\operatorname{Spec} k[x_1, x_1^{-1}, \dots, x_r, x_r^{-1}]) = 0.$$

Thus, we can write $L = \mathcal{O}_X(D)$ for some Cartier divisor D on X (Proof: write $L = \mathcal{O}_X(E)$ for some Cartier divisor E and then take $D = E - \operatorname{div}(f)$ for some rational function f on X such that $\operatorname{div}(f)|_O = E|_O$.)

[6]This is one of the reasons that, in Part 2, we take a global approach when we construct a toric degeneration (so linearization is automatic), as opposed to a local approach.

We have that Supp(D) is $\mathbb{G}_m^r$-invariant, since each irreducible component of it is an irreducible component of $X - O$. Thus, for example, if D is reduced (or more generally Weil[7]), then D is invariant. Now, according to Ch. 1, §3., Proposition 1.5. of [MFK94], some tensor power $L^{\otimes m}$ is linearizable if and only if some tensor power of L in Pic(X) is $\mathbb{G}_m^d$-fixed. Thus, in that case, some tensor power of L is $\mathbb{G}_m^r$-linearizable. (Note: the assumption on D holds for example if X is Cohen-Macaulay.)

1.3 Saturation of a semigroup

Throughout Part 1, we need the notion of a saturation and some basic facts about it.

1.3.1 Definition. Given a subsemigroup $S \subset \mathbb{Z}^d$, if G is the subgroup of $\mathbb{Z}^d$ generated by S, then the semigroup

$$\widetilde{S} = \left(\bigcup_{n>0} \frac{1}{n} S \right) \cap G$$

is called the *saturation* of S. Equivalently, $\widetilde{S} = \mathbb{R}_+ S \cap G$ (use Carathéodory's theorem). The semigroup S is said to be *saturated* if $\widetilde{S} = S$.

The next proposition gives an algebraic characterization of saturation.

1.3.2 Proposition. *Let $S \subset \mathbb{Z}^d$ be an additive subsemigroup, G the group generated by it and $\widetilde{S} = \mathbb{R}_+ S \cap G$ its saturation. Then*

$$\widetilde{S} = \{u \in G | \chi^u \textit{ is integral over } k[S]\}.$$

Consequently, $k[\widetilde{S}]$ is the integral closure of $k[S]$ in the field of fractions of $k[S]$. (Note: S need not be finitely generated here.)

Proof. Since $k[G]$ and $k[S]$ have the same field of fractions and since $k[G]$, the localization of a polynomial ring, is an integrally closed domain, it suffices to show that the integral closure of $k[S]$ in $k[G]$ is $k[\widetilde{S}]$.

[7] Somehow abusively, we say an effective Cartier divisor is Weil if there is no embedded component.

If $u \in \widetilde{S}$, then $nu \in S$ for some positive integer n. Thus, $\chi^{nu} \in k[S]$ and so χ^u is integral over $k[S]$. Conversely, suppose χ^u is integral over $k[S]$; i.e., we can write

$$\chi^{nu} + g_{n-1}\chi^{(n-1)u} + \cdots + g_0 = 0$$

for some $g_i \in k[S]$. That we can cancel χ^{nu} means we have: for some $i > 0$,

$$nu = w + (n-i)u$$

where $\chi^w, w \in S$ is some monomial appearing in g_{n-i}. Then $iu \in S$. □

1.3.3 Corollary. *If $S \subset \mathbb{N} \times \mathbb{Z}^d$ is a finitely generated subsemigroup, then* $\operatorname{Proj} k[\widetilde{S}]$ *is the normalization of* $\operatorname{Proj} k[S]$.

Conversely, if $\operatorname{Proj} k[S]$ *is normal, then $\widetilde{S} - S$ is a finite set.*

To give another corollary, we recall the following algebraic fact:

1.3.4 Theorem (Noether's finiteness theorem and converse)**.** *Let A be an algebra over a field k such that A is an integral domain. Then A is finitely generated as a k-algebra if and only if the integral closure of A in the field of fractions is finitely generated as a k-algebra.*

Proof. The direction ($\Rightarrow$) is the standard fact in commutative algebra. The converse is a consequence of the Artin-Tate lemma ([Ei04], Exercise 4.32). Since the integral closure is integral and of finite type over A, it is finite over A. Thus, the Artin-Tate lemma says that A is a finitely generated algebra over k. □

1.3.5 Corollary. *In the notations of the proposition, S is finitely generated if and only if $\widetilde{S}$ is finitely generated.*

Proof. Consider $\mathbb{C}[S]$ and use Proposition 1.3.2 and Theorem 1.3.4. □

1.3.6 Remark. Let $X = \varinjlim \operatorname{Spec} k[S_i]$ be a non-normal toric variety with the torus $\mathbb{G}_m^d = \operatorname{Spec} k[\mathbb{Z}^d]$ defined by a system of semigroups S_i's. Then $\widetilde{S}_i = \sigma_i^\vee \cap \mathbb{Z}^d$ for some cone σ_i (namely, σ_i is the dual of $\mathbb{R}_+ S$.)

The Picard group of a torus is trivial. This fact is true more generally. We recall that a reduced Noetherian ring A is said to be *seminormal* if there is no element x in the total ring of fractions of A such that x^2, x^3 are in A but $x \notin A$. We say $\operatorname{Spec} A$ is seminormal if A is a seminormal ring.

1.3.7 Theorem. *Let $X_S = \operatorname{Spec} k[S]$ be an affine non-normal toric variety.*

(i) (Quillen-Suslin) If X_S is seminormal, then every finitely generated projective module over $k[S]$ is free; i.e., every vector bundle on X_S is trivial.

(ii) Conversely, if $\operatorname{Pic}(X_S) = 0$; i.e., every line bundle on X_S is trivial, then X_S is seminormal.

For the proofs of the above as well as the general background on the result, see Chapter 8 of [BG08].

1.4 Some non-finite examples of semigroups

Here are some examples that do not quite fit into the framework developed in this section but can be handled by dropping some convention. For example, in Lemma 1.1.6, the assumption that I is finite is needed only to ensure the resulting scheme is of finite type not just locally of finite type. In fact, in the original definition of toric varieties, Mumford allows for an infinite fan.

1.4.1 Example. Let r_i be a decreasing sequence of positive rational numbers such that $r_1 = 1$ and $r_i \to 0$ as $i \to \infty$. Let $\sigma_i \subset \mathbb{R}^2$ be the ray generated by the vector $(r_i, 1)$ and $\sigma_{i,i+1} \subset \mathbb{R}^2$ the cone generated by σ_i and σ_{i+1}. Then $\mathfrak{F} = \{(0,0), \sigma_i, \sigma_{i,i+1} | i = 1, 2, \dots\}$ is a fan and so we get the variety $X_{\mathfrak{F}}$ by the same way we construct a toric variety; explicitly, let $S_{i,i+1} = \sigma_{i,i+1}^\vee \cap \mathbb{Z}^2$ and thus

$$S_{i,i+1} = \{u \in \mathbb{Z}^2 | r_i u_1 + u_2 \geq 0,\ r_{i+1} u_1 + u_2 \geq 0\}.$$

Then $X_{\mathfrak{F}}$ is obtained by gluing $\operatorname{Spec} k[S_{i,i+1}]$ along $\operatorname{Spec} k[S_i], S_i = \sigma_i^\vee \cap \mathbb{Z}^2 = \{u \in \mathbb{Z}^2 | r_i u_1 + u_2 \geq 0\}$.

1.4.2 Example (cf. [An13] Example 5.10)**.** We consider the graded semigroup

$$S = \{(n,a) \in \mathbb{N} \times \mathbb{Z} | 0 \leq a \leq 3n-1\},$$

which is *not* finitely generated. Let $X = \operatorname{Proj} k[S]$. Explicitly, S is generated by $(1,0),(1,1),(i,3i-1)$ for all integers $i > 0$. By definition, X is obtained by gluing $\operatorname{Spec}(k[S][\chi^{-u_j}]_0)$ for the generators u_j (cf. Example 1.1.7). We have $k[S][\chi^{-(1,0)}]_0 = k[\chi^{(0,1)}]$ and $k[S][\chi^{-(1,1)}]_0 = k[S][\chi^{-(1,2)}] = k[\chi^{(0,1)}, \chi^{-(0,1)}]$. Also, for each $i > 0$, we have:

$$k[S][\chi^{-(i,3i-1)}]_0 = k[\chi^{(0,1)}, \chi^{-(0,1)}],$$

since $(0,1) = (2i, 3(2i)-1) - 2(i,3i-1)$. Note that $k[\chi^{(0,1)}, \chi^{-(0,1)}] = k[\sigma^\vee \cap (0 \times \mathbb{Z})]$ for $\sigma = (0,0)$ and $k[\chi^{(0,1)}] = k[\sigma^\vee \cap (0 \times \mathbb{Z})]$ for $\sigma = \mathbb{R}_+(0,1)$. That is, the defining fan is $\{(0,0), \mathbb{R}_+(0,1)\}$, the acting torus is $\operatorname{Spec}(k[0 \times \mathbb{Z}]) \simeq \mathbb{G}_m$ and $X \simeq \mathbb{A}^1$.

1.5 Further remarks on toric schemes

(This section is meant for the readers who are familiar with toric schemes.)

Throughout §1, we have limited ourselves to the case when the base ring is a field. This restriction is easily relaxed but the more substantial reason for this restriction is because in Part 2, we consider the more general situation; namely, a scheme $X \to S$ such that some fibers of X are non-normal toric varieties. It will be called a *toric degeneration*. If, for example, $X = \operatorname{Spec} R[S]$ for some k-algebra R and semigroup $S \subset \mathbb{Z}^d$, then, since $R[S] = R \otimes_k k[S]$, we have:

$$X = \operatorname{Spec}(k[S]) \times_{\operatorname{Spec}(k)} \operatorname{Spec} R \to \operatorname{Spec} R.$$

Hence, a toric scheme is a *special case* of a toric degeneration.

See also [Ogus06] for the theory of schemes arising from monoids in the context of log geometry (the connection between here and there is an interesting one but is not within the scope of the thesis).

2.0 Torus actions and GIT quotients by them

2.1 Torus-lattice correspondence

Given a lattice N, we write $N^* = \mathrm{Hom}(N, \mathbb{Z})$ for its dual lattice and $T_N = \mathrm{Hom}(N^*, \mathbb{G}_m)$ for the torus corresponding to it. By *the category of lattices*, we mean the category of finite-rank free abelian groups. It is known and is easy to see that the functor

$$N \mapsto T_N,$$

that is, the functor $\mathrm{Hom}(-^*, \mathbb{G}_m)$, is an equivalence from the category of lattices to the category of tori.

Explicitly, if, say, $N = \mathbb{Z}^d$ and $N' = \mathbb{Z}^r$, then we can identify $\mathrm{Hom}(N, N') = \{(u_1, \ldots, u_r) | u_i \in \mathbb{Z}^d\}$ by writing a $\mathbb{Z}$-linear map as a matrix. Then the bijection

$$\mathrm{Hom}(N, N') \to \mathrm{Hom}(T_N, T_{N'})$$

is given by sending the vectors $u_1, \cdots, u_r \in \mathbb{Z}^d$ to the group homomorphism

$$T_N = (k^*)^d \to T_{N'} = (k^*)^r,\ x = (x_1, \ldots, x_d) \mapsto (x^{u_1}, \ldots, x^{u_r})$$

where we write $x^a = x_1^{a_1} \ldots x_d^{a_d}$.

2.1.1 Remark. The above discussion can be used to describe an equivariant map between non-normal toric varieties since such a map is completely determined by the restriction to the open torus orbit.

For example, we can view $\mathbb{P}^r$ as a toric variety with the diagonal torus $\mathbb{G}_m^r$. If X is a non-normal toric variety with torus $T_{\mathbb{Z}^d} = \mathbb{G}_m^d = (k^*)^d$, then any equivariant map

$$X \to \mathbb{P}^r$$

has the form: for $x \in (k^*)^d \subset X$,

$$x = (x_1, \ldots, x_d) \mapsto (x^{u_0} : \cdots : x^{u_r})$$

for some $u_0, \dots, u_r \in \mathbb{Z}^d$. Conversely, every projective non-normal toric variety arises in this fashion: first give a map from a torus T to the standard torus of $\mathbb{P}^r$ and then take the closure of the image of T in $\mathbb{P}^r$. (cf. Proposition 1.2.1).

We can also use the above idea to describe a torus action in general in the following way (cf. [AH06] §11).

2.1.2 Lemma (standard representation)**.** *Let $X \subset \mathbb{A}^N$ be an affine variety and $\mathbb{G}_m^N$ act on $\mathbb{A}^N$ in the usual way. Let a torus $T \simeq \mathbb{G}_m^r$ act on $\mathbb{A}^N$ through some $\varphi : T \to \mathbb{G}_m^N \subset$* $\operatorname{Aut}(\mathbb{A}^N) = GL_N$*. If X is T-invariant, then the action of T is given as*

$$\varphi(t)(x_1, \dots, x_N) = (t^{u_1} x_1, \dots, t^{u_N} x_N)$$

for some $u_i \in \operatorname{Hom}(T, \mathbb{G}_m) \simeq \mathbb{Z}^r$.

Proof. This is a consequence of the preceding discussion. □

Here is an example illustrating the lemma.

2.1.3 Example (cf. Example 11.2. of [AH06])**.** Let $X \subset \mathbb{A}^4$ be the closed subvariety defined by $x_1^3 + x_2^4 + x_3 x_4 = 0$. Let $T = \mathbb{G}_m^2$.

We let $\mathbb{G}_m^4$ act on $\mathbb{A}^4$ in the usual way and consider

$$T \to \mathbb{G}_m^4,\ t = (t_1, t_2) \mapsto (t^{u_1}, \dots, t^{u_4})$$

where u_i are chosen so that X is T-invariant; e.g., $(4, 0), (3, 0), (0, 1), (12, -1)$.

2.2 The correspondence between faces and prime torus-invariant ideals

Given a (possibly non-closed) convex cone $C \subset \mathbb{R}^d$, by a *face* of C, we mean either C or a nonempty intersection $C \cap v^\perp$ for some nonzero v in the dual cone $C^\vee$ of C; the hyperplane $v^\perp = \ker(u \mapsto \langle u, v \rangle)$ is called a *supporting hyperplane* for C.

2.2.1 Lemma. *An intersection of at most countably many faces is a face. Also, "is a face of" is a transitive relation.*

Proof. Let C be a convex cone and u_j in $C^\vee$ such that (without loss of generality) $|u_j| = 1$. For $u = \sum 2^{-j} u_j$, we have: $C \cap u^\perp = \cap_j (C \cap u_j^\perp)$. As for the second, if $C' = C \cap v^\perp$ for some v in $C^\vee$ and $F = C' \cap v'^\perp$ for some v' in $C'^\vee$, then, for some large $a > 0$, $v' + av$ is in $C^\vee$ and $F = C \cap (v' + av)^\perp$. □

The last part of the preceding lemma can be used to give a useful characterization of a face of a convex cone (cf. Proposition 2.2.3).

2.2.2 Lemma. *Let C be a (possibly non-closed) convex cone in $\mathbb{R}^d$ and $A \subset C$ a nonempty subset. Then A is a face of C if and only if A is convex and has the property: $x + y \in A \Leftrightarrow x, y \in A$ for any $x, y \in C$.*

Proof. The direction $\Rightarrow$ is trivial and we prove the converse. Note that the assumption implies A is a convex cone. We shall argue by induction on $\dim(A^\perp)$. If $\dim(A^\perp) = 0$, then A spans the whole space $\mathbb{R}^d$ and so, for each x in C, we can write $x = y - z$ for some y, z in A. That is, $x + z$ is in A and so x is in A by assumption. Hence, $A = C$. Next, suppose $A^\perp \neq 0$. Without loss of generality, assume $C \neq \mathbb{R}^d$. Then we can find a nonzero vector v in $C^\vee \cap A^\perp$; since, otherwise, $\mathbb{R}^d = (C^\vee \cap A^\perp)^\vee = \overline{C} + \overline{A} = \overline{C}$. Note $A \subset C' := C \cap v^\perp$. So, by the inductive hypothesis applied to $A \subset C'$ in the space $v^\perp$, we get that A is a face of C'; thus is a face of C by Lemma 2.2.1. □

In a semigroup algebra $k[S]$, the torus-invariant prime ideals are parametrized by the faces of the cone generate by S, as described in the next proposition.

2.2.3 Proposition. *Let $S \subset (\mathbb{Z}^d, +, 0)$ be a subsemigroup. Then there is a natural one-to-one correspondence between the set of the $\mathbb{G}_m^d$-invariant prime ideals of the semigroup algebra $k[S]$ and the faces of the cone $\mathbb{R}_+ S$. Precisely, given a face F, define $\mathfrak{p}_F$ by*

$$0 \to \mathfrak{p}_F \to k[S] \xrightarrow{\varphi} k[S \cap F] \to 0$$

where φ sends χ^u to χ^u if u is in F and to 0 otherwise. Then φ is a k-algebra homomorphism and, consequently, $\mathfrak{p}_F$ is a prime ideal. Moreover,

$$F \mapsto \mathfrak{p}_F$$

is a bijection from the set of the faces of $\mathbb{R}_+S$ *to the set of the* $\mathbb{G}_m^d$*-invariant prime ideals of* $k[S]$ *that reverses the inclusions.*

Proof. Using Lemma 2.2.2, we see that $\varphi(\chi^u\chi^t) = \varphi(\chi^u)\varphi(\chi^t)$; hence, φ is a ring homomorphism.

Suppose $\mathfrak{p} \subset k[S]$ is a torus-invariant prime ideal; i.e., as a $\mathbb{G}_m^d$-module, we have $\mathfrak{p} = \oplus_{u \in P} k \cdot \chi^u$ for some subset P of S. Then, as a $\mathbb{G}_m^d$-module,

$$k[S] \simeq k[S]/\mathfrak{p} \oplus \mathfrak{p} = (\oplus_{u \in S-P} k \cdot \chi^u) \oplus \mathfrak{p}.$$

Since $\mathfrak{p}$ is a prime ideal, we have $u + t \in S - P \Leftrightarrow u, t \in S - P$.

Now, let $F = \mathbb{R}_+(S - P)$. We easily see F satisfies the condition of Lemma 2.2.2 above and so F is a face of the cone $\mathbb{R}_+S$. Moreover, we have $S \cap F = S - P$. Indeed, the inclusion $\supset$ is trivial. Conversely, if $u \in S$ is in F, then $nu \in S - P$ for some integer number $n > 0$. But then, in $k[S]/\mathfrak{p}$, $\chi^{nu} \neq 0$ and so $\chi^u \neq 0$; i.e., $u \in S - P$. □

2.2.4 Corollary. *There is a natural bijection preserving the inclusions:*

$$\{ \text{ the faces of } \mathbb{R}_+S \ \} \xrightarrow{\sim} \{ \text{ the } \mathbb{G}_m^d\text{-subvarieties of } \operatorname{Spec} k[S] \ \}.$$

(cf. Proposition 2.3.4.)

Here is a simple example illustrating the proposition.

2.2.5 Example (projective space)**.** Let $X = \mathbb{P}^d = \operatorname{Proj} k[x_0, \dots, x_d]$ be a projective space, where $x_i = \chi^{e_i}$. For each subset $I \subset \{0, 1, 2, \dots, d\}$, let

$$\mathfrak{p}_I = (x_i | i \in I)$$

be the ideal, which is $\mathbb{G}_m^d$-invariant and corresponds to the face F_I generated by e_i with $i \notin I$. Note: $k[x_0, \dots, x_d]/\mathfrak{p}_I \simeq k[\mathbb{N}^d \cap F_I]$ as k-algebras.

Let $Z_I = \operatorname{Proj} k[x_0, \dots, x_d]/\mathfrak{p}_I = \operatorname{Proj} k[S_I]$. Then Z_I is a projective space with the homogeneous coordinate $x_i, i \notin I$. Then Z_I is a toric variety with the open dense orbit consisting of the points whose coordinates are all nonzero.

2.3 Orbit-closures

We first note the useful characterization of k-points on non-normal toric varieties (in fact schemes):

2.3.1 Lemma. *Given a subsemigroup $S \subset (\mathbb{Z}^d, +, 0)$ and a field k, let $X_S = \operatorname{Spec}(k[S])$. Then the set of k-points $X_S(k)$ on X_S can be identified with the set of semigroup homomorphisms $S \to (k, *)$. In particular, if S is a group, then $X_S(k)$ consists of the group homomorphisms $S \to k^*$.*

Next, if $G(S)$ is the group generated by S (which is free of finite rank) and $T = \operatorname{Spec} k[G(S)]$ the torus given by it, then the action of $T(k)$ on $X_S(k)$ is given by: $t \in T(k), u \in S$,

$$(t \cdot x)(u) = t(u)x(u).$$

Proof. The first part is easy; we have: $X_S(k) = \operatorname{Mor}_{\operatorname{Spec} k}(\operatorname{Spec} k, X_S) = \operatorname{Hom}_{k\text{-alg}}(k[S], k)$. Then each k-algebra homomorphism $\varphi : k[S] \to k$ determines a unital semigroup homomorphism $\varphi' : S \to k$ by $\varphi'(u) = \varphi(\chi^u)$, and conversely.

For the "next" part, the torus action $T \times X_S \to X_S$ corresponds to the algebra homomorphism

$$\sigma^{\#} : k[S] \to k[G(S)] \otimes_k k[S],\ \chi^u \mapsto \chi^u \otimes \chi^u$$

(see Definition A.0.1). Now, let $\varphi = (t, x) \in T(k) \times X_S(k) = \operatorname{Hom}(k[G(S)] \otimes_k k[S], k)$ be given. By pullback, $\sigma^{\#}$ induces $\sigma^{\#,*} : T(k) \times X_S(k) \to X_S(k)$. Unwinding the formula,

$$\sigma^{\#,*}(\varphi)(\chi^u) = \varphi(\chi^u \otimes \chi^u) = t(\chi^u)x(\chi^u) = t(u)x(u).$$

□

2.3.2 Remark (complex points of X_S)**.** *Regardless of the base field k*, the lemma allows us to speak of the complex points of X_S:

$$X_S(\mathbb{C}) := \operatorname{Hom}(S, \mathbb{C})$$

as well as the action of the complex torus $\mathbb{G}_m^r(\mathbb{C}) = (\mathbb{C}^*)^r$ on it. In Part 3, this fact will be a basis for the use of the tools from symplectic geometry.

When no confusion is possible, we will often write T or X_S for their sets of k-points $T(k)$ or $X_S(k)$.

2.3.3 Lemma. *Assume the base field k is algebraically closed. Let $X = \mathrm{Spec}(A)$ be an affine variety with an action of a torus $T \simeq \mathbb{G}_m^r$. Let $O \subset X$ be an orbit. Then the closure $\overline{O} = \mathrm{Spec}\, k[S]$ in X is a non-normal toric variety that is a T-invariant subvariety of X.*

Proof. We know O is open in its closure (e.g., [Sp98] Lemma 2.3.1. (i)). Hence, it follows from Lemma 1.1.5 (i) that $\overline{O} = \mathrm{Spec}\, k[S]$ for some semigroup S. □

The next proposition explains how to parametrize the orbits in an orbit closure. (For simplicity, we only consider the affine case; for the non-affine case, generalize [Ful93] §3.1.).

2.3.4 Proposition. *Assume the base field k is algebraically closed. Let X be an affine T-variety and $\overline{O} = \mathrm{Spec}\, k[S] \subset X$ an orbit closure. For each face F of $\mathbb{R}_+S$, let*

$$O_F = \mathrm{Spec}\, k[G(S \cap F)]$$

with $G(S \cap F)$ = the group generated by $S \cap F$. Then each O_F is a T-orbit of dimension $\dim F$ such that $\overline{O_F} = \mathrm{Spec}\, k[S \cap F]$ and there is the orbit decomposition:

$$\overline{O} = \bigsqcup_F O_F$$

where F runs over all the faces of $\mathbb{R}_+S$.

Proof. To see O_F is an orbit, using Lemma 2.3.1, define the k-point $x_F \in \overline{O} = \mathrm{Spec}\, k[S]$ by

$$x_F(u) = \begin{cases} 1 & \text{if } u \in S \cap F \\ 0 & \text{else.} \end{cases}$$

Then one can show $O_F = T \cdot x_F$ (to be precise, the equality of k-points). The remaining assertion follows from Proposition 2.2.3. □

2.3.5 Corollary. $O_F \subset \overline{O_{F'}} \Leftrightarrow F$ *is a face of F'.*

2.3.6 Corollary. *The irreducible components of the boundary of $\overline{O}$ correspond to the facets of $\mathbb{R}_+S$.*

2.3.7 Corollary. *If $\overline{O}$ is normal (i.e., is a toric variety), then the orbit closures contained in $\overline{O}$ are also normal.*

2.4 Normal fan and moment polytope

In this thesis, we use the following slightly non-standard notion:

2.4.1 Definition (T-ring)**.** Let A be a ring and $T = \mathbb{G}_m^r$. Then a *T-ring*[1] is, by definition, a $\mathbb{Z}^r$-graded ring.

If the ring A contains a field, then a T-ring is the same thing as a ring together with the action of T as $\mathbb{Z}^r$-grading preserving algebra homomorphisms such that A admits a weight space decomposition. The point of the notion is that even when A does not contain a field, we can treat A *as if* there is a torus action.[2] Thus, for example, we write A^T for the zero-th degree component of A and speak on weights as opposed to multi-degrees.

We note:

2.4.2 Lemma (Reynolds operator)**.** *Let A be a T-ring. Then there is a A^T-linear map $P : A \to A^T$ (i.e., $A^T \hookrightarrow A$ is a split injection). Moreover, P is a ring homomorphism if $A = \bigoplus_{\chi \ge 0} A_\chi$; in particular, in that case, the kernel is an ideal.*

Proof. The first assertion is clear. For the second, given f, g in A, we write $f = P(f) + f_+$ and $g = P(g) + g_+$. Then $P(fg) = P(P(f)P(g) + \cdots) = P(f)P(g)$. □

Each graded T-ring comes with the natural convex set:

2.4.3 Definition. Let R be a graded $T = \mathbb{G}_m^r$-ring. Then the *weight convex-set of* R is the convex hull

$$\triangle(R) = \operatorname{conv}\{u/n | n \in \mathbb{N}, u \in \mathbb{Z}^r, R_{n,u} \neq 0\}$$

in $\mathbb{R}^n$.

If R is generated by some finitely many homogeneous T-weight elements x_i of degree n_i, then $\triangle(R)$ is the convex hull of x_i/n_i; thus is a convex polytope and in that case, $\triangle(R)$ is called the *moment polytope* of R or the *weight polytope* of R. The term "moment" comes from a moment map in symplectic geometry. To be pedantic, $-\triangle(R)$ should be called the

[1] In this thesis, we never consider non-split T-rings.

[2] More officially, we can do this by considering torus action over the ring of integers $\mathbb{Z}$ but that will involve formalisms that we prefer to skip.

moment polytope (since that is the image of a moment map), but we like to ignore this distinction (except when the distinction matters).

Moreover, if $X \subset \mathbb{P}^N$ is a closed subvariety carrying R as a homogeneous coordinate ring, then the moment polytope $\triangle_X$ of X is $\triangle(R)$. Similarly, if L is a T-equivariant ample line bundle on X, then we let $\triangle_X(L) = \triangle_X(R(L))$ where $R(L) = \bigoplus_0^\infty \Gamma(X, L^{\otimes n})$ is the section ring of L.

Each convex polytope gives rise to a fan of convex cones.

2.4.4 Lemma (normal fan; cf. Example 2.4.9)**.** *Let $P \subset \mathbb{R}^d$ be a convex polytope with nonempty interior. For each face Q of P,*[3] *let*

$$\sigma_Q = (P + (-Q))^\vee \subset \mathbb{R}^d.$$

Then $\{\sigma_Q | Q$ is a face of $P\}$ is a fan, called the normal fan *to P. The mapping $Q \mapsto \sigma_Q$ is an order-reversing bijection from the set of faces of P to the fan; in particular, $\sigma_P = 0$ is the origin and the facets = maximal proper faces of P correspond to the rays. Also, it is complete in the sense that $\mathbb{R}^d = \bigcup_Q \sigma_Q$.*

Proof (after [Ful93] §1.5). Since $\{\sigma_Q | Q\}$ is unchanged after translating and rescaling P, without loss of generality, we can assume the interior of P contains the origin.

For each face Q of P, we write $\widetilde{Q}$ for the cone over $1 \times Q$ in $\mathbb{R}^{1+d}$. Then $\widetilde{Q}$ is a proper face of $\widetilde{P}$. By duality ([Ful93] §1.2), $\widetilde{Q} \mapsto \widetilde{P}^\vee \cap \widetilde{Q}^\perp$ is an order-reserving bijection from the set of the faces of the cone $\widetilde{P}$ to the set of faces of $\widetilde{P}^\vee$. Let $\pi : \mathbb{R}^{1+d} \to \mathbb{R}^d$ be the projection. Since 0 is in the interior of P, it is easy to see that $\sigma_Q = \pi(\widetilde{P}^\vee \cap \widetilde{Q}^\perp)$ and the set $\{\sigma_Q | Q\}$ corresponds to the set of faces of $\widetilde{P}^\vee$. This proves the first and second assertion. The last assertion is not hard to see. □

The above lemma applies in particular to the moment polytope. Before applying the lemma, we clarify the algebraic meaning of a projective variety. First, we note some basic properties of Proj:

[3] that is, either $Q = P$ or Q is nonempty and $Q = P \cap \{\langle \cdot, v \rangle = a\}$ for some real number a and a nonzero vector v such that $P \subset \{\langle \cdot, v \rangle \geq a\}$

2.4.5 Lemma. *Let R be a graded ring and, given a non-nilpotent homogeneous element f of R of positive degree, let φ_f be the function from the set of homogeneous ideals of R not containing f to the set of proper ideals in $R[f^{-1}]_0$ given by*

$$\varphi_f(I) := IR[f^{-1}]_0 = IR[f^{-1}] \cap R[f^{-1}]_0.$$

Then

(i) $\varphi_f : \{$*homogeneous prime ideals of R not containing f*$\} \to \operatorname{Spec}(R[f^{-1}]_0)$ *is a bijection with the inverse*

$$\psi_f : \mathfrak{q} \mapsto \bigoplus_{n=0}^{\infty} \{g \in R_n | g^{\deg f}/f^n \in \mathfrak{q}\}.$$

(Note: geometrically, ψ_f amounts to taking the closure of $V(\mathfrak{q})$.)

(ii) φ_f *commutes with primary decomposition away from f in the sense: if $I = \cap_i Q_i$ is a primary decomposition, then $\varphi_f(I)$ is the intersection of primary ideals $\varphi_f(Q_i)$ over all i such that $f \notin \sqrt{Q_i}$ and*

$$\varphi_f : \{Q_i | f \notin \sqrt{Q_i}\} \hookrightarrow \{ \text{ primary ideals of } R[f^{-1}]_0 \ \}$$

is well-defined and injective.

(iii) For each integer $m > 0$, the function $\mathfrak{p} \mapsto \mathfrak{p} \cap R^{[m]}$ is a well-defined bijection from $\operatorname{Proj} R$ *to* $\operatorname{Proj}(R^{[m]})$ *with the inverse $\mathfrak{q} \mapsto \sqrt{\mathfrak{q}R}$.*

Proof. (i) is [Va17] Exercise 4.5.E.

(ii) First recall that $\operatorname{Ass}_R(R/I) \cap \{\mathfrak{p} | f \notin \mathfrak{p}\} \overset{\mathfrak{p} \mapsto \mathfrak{p}[f^{-1}]}{\to} \operatorname{Ass}_{R[f^{-1}]}((R/I)[f^{-1}])$ is a bijection ([Bou, Ch. IV, 1, no. 2, Proposition 5.]). By (i) or by a direct argument, given distinct prime ideals $\mathfrak{p}, \mathfrak{p}'$ of $R[f^{-1}]$, we have that $\mathfrak{p}_0, \mathfrak{p}'_0$ are distinct.

(iii) The "well-defined"-ness is clear. Next, given a homogeneous prime ideal $\mathfrak{p}$ in $\operatorname{Proj} R$, let $\mathfrak{q} = \mathfrak{p} \cap R^{[m]}$. We claim: $\mathfrak{p} = \sqrt{\mathfrak{q}R}$. Indeed, the inclusion "$\supset$" is because $\mathfrak{p} \supset \mathfrak{q}R$. Conversely, since $\mathfrak{p}$ is homogeneous, we can choose homogeneous generators x_i's of $\mathfrak{p}$. Then $x_i^m \in R^{[m]}$ and so $x_i^m \in \mathfrak{p} \cap R^{[m]} = \mathfrak{q}$; thus, $x_i \in \sqrt{\mathfrak{q}R}$, proving the claim. The claim implies that $\mathfrak{p}$ is uniquely determined by $\mathfrak{p} \cap R^{[m]}$; i.e., the function in the assertion is injective. The surjectivity follows easily from Lemma 2.5.1 (iii) or, alternatively, from (i) and the equality $R^{[m]}[(f^m)^{-1}]_0 = R[f^{-1}]_0$. □

2.4.6 Proposition. *Let R be a Noetherian graded ring such that $R_0 = k$ is a field, $X = \operatorname{Proj} R$ and $\mathcal{O}_X(l)$ denote the quasi-coherent $\mathcal{O}_X$-module associated to $R(l)$ where $R(l)$ is the $\mathbb{Z}$-graded R-module whose n-th degree piece is R_{l+n}.*

Then the following are equivalent:

- *(i) X is a projective variety over $R_0 = k$; i.e., X is a geometrically reduced and geometrically irreducible and for some $r, m > 0$, $X \hookrightarrow \mathbb{P}^r$ in such a way that $\mathcal{O}_X(m)$ is the restriction of $\mathcal{O}_{\mathbb{P}^r}(1)$.*
- *(ii) There is a graded Noetherian geometrically integral domain S over k[4] and a finite injective ring homomorphism $S \hookrightarrow R$ that is graded of some degree m (i.e., S_l goes to R_{lm}).*
- *(iii) The zero ideal $(0) \otimes_k \overline{k} = (0)$ of $R \otimes_k \overline{k}$[5] is either (1) primary or (2) an intersection of primary ideals Q_1, Q_2 such that $\sqrt{Q_1}$ is the nilradical of R and $\sqrt{Q_2} = R_+ \otimes_k \overline{k}$.*

Assuming that R satisfies the above equivalent conditions, if $R_1 \otimes_k \overline{k}$ does not consist of nilpotent elements (in particular $R_1 \neq 0$) and $(0) \otimes_k \overline{k}$ is primary; i.e., the first case in (iii), then R is geometrically an integral domain.

Proof. Throughout the proof, we assume $k = \overline{k}$. (i) $\Rightarrow$ (ii): Take S to be the homogeneous coordinate ring. The converse (ii) $\Rightarrow$ (i) is valid since $\operatorname{Proj} S \simeq X$.

(ii) $\Rightarrow$ (iii): By Lemma 2.4.5, given a non-nilpotent homogeneous element g of R_+, we have that either (0) is primary or $(0) = Q_1 \cap Q_2$ is a primary decomposition such that $\sqrt{Q_1}$ is the nilradical and $g \in \sqrt{Q_2}$. Since g is arbitrary, we must have either (0) is primary or $(0) = Q_1 \cap Q_2$ where $\sqrt{Q_1}$ is the nilradical of R and $\sqrt{Q_2} = R_+$.

(iii) $\Rightarrow$ (i): Clear.

To see the last assertion, choose some non-nilpotent degree-one element $g \in R$. Then, by (i), $X_g = \operatorname{Spec}(R[g^{-1}]_0)$ is a variety; thus, $R[g^{-1}]_0$ is geometrically an integral domain. Thus, if x is a homogeneous zerodivisor of R of degree n, then, since x is nilpotent as (0) is

[4]A (commutative associative) k-algebra A is a *geometrically integral domain* if $A \otimes_k \overline{k}$ is an integral domain for the algebraic closure $\overline{k}$ of k. The authors is aware that "geometrically" is adverb so the term is grammatically problematic (but not so mathematically).

[5]$\overline{k}$ denotes the algebraic closure.

primary, we have that x/g^n is zero in $R[g^{-1]}]_0$. That is, x is in the kernel of $R \to R[g^{-1]}]$; i.e., $g^m x = 0$. Since (0) is primary and g is not nilpotent, $x = 0$. $\square$

The above proposition prompts us to use the following:

2.4.7 Definition (Proj R is a projective variety)**.** Given a Noetherian graded ring R, we shall say "Proj R is a projective variety" if the equivalent conditions of the above propositions are met.

The next proposition describes the relation between the moment polytope and the fan arising from it.

2.4.8 Proposition. *Let $X = \operatorname{Proj} R$ be a projective variety over an algebraically closed field of dimension d and assume $T = \mathbb{G}_m^d$ acts on R as grade-preserving automorphisms. Assume that the multiplicities are one; i.e., $\dim R_{n,\chi} \leq 1$ for each integer $n > 0$ and a character χ of T.*

Then the normalization X^{nor} of X is the toric variety associated to the normal fan to $\triangle(R)$. Moreover, there is a bijection between the set of the irreducible components of $X^{nor} -$ the open dense T-orbit and the facets of $\triangle(R)$.

Proof. By Theorem 1.2.1, we can write $R = k[S]$ for some finitely generated subsemigroup $S \subset \mathbb{N} \times \mathbb{Z}^d$. Let $P = \triangle(R)$ and $\widetilde{P}$ the cone over it. Then $\widetilde{P} \cap \mathbb{Z}^{1+d}$ is the saturation of S and so Proj of $k[\widetilde{P} \cap \mathbb{Z}^{1+d}]$ is X^{nor}. Finally, it is clear that the toric variety determined by the normal fan to $P = \triangle(R)$ is X^{nor} (see [Ful93] §3.4.) $\square$

The next example is closely related to the Delzant construction (that we will discuss much later in §10).

2.4.9 Example. Let $P = [-1, 1]^2 \subset \mathbb{R}^2$ be the square centered at the origin. Then $\widetilde{P} = (\bigcup_{r \geq 0} r \times [-r, r]^2)$ and $\widetilde{P}^\vee$ is the cone over the convex set

$$\{v \in \mathbb{R}^2 | \langle u, v \rangle \leq 1, u \in P\} = \{v \in \mathbb{R}^2 | \pm v_1 \pm v_2 \leq 1\},$$

called the polar set of P. Thus, the fan looks like the one in [Ful93, page 12] [Ful93] §1.4. page 21, very bottom and, therefore, the toric variety associated to P is $\mathbb{P}^1 \times \mathbb{P}^1$.

By exactly the same reasoning, the toric variety associated to the n-cube $[-1, 1]^n$ is $(\mathbb{P}^1)^n$.

We record the following observation:

2.4.10 Proposition. *Let $S \subset \mathbb{N} \times \mathbb{Z}^d$ be an additive subsemigroup, which we view as a graded semigroup with the grading given by the $\mathbb{N}$-factor; i.e., $S = \bigoplus_0^\infty S_n$ where $S_n = \{x | (n, x) \in S\}$.*

Then S is finitely generated if and only if the weight convex set (Definition 2.4.3) attached to the graded ring $R = \mathbb{C}[S]$ is a polytope.

Proof. By Corollary 1.3.5, we can assume $S = \widetilde{S}$ is saturated and then the assertion is clear. □

2.5 Torus-invariant prime ideals

For later references, we record a few facts on torus-invariant prime ideals. We note the next lemma applies in particular to a graded ring; in fact, it is a generalization of the graded case.

2.5.1 Lemma. *Let $T = \mathbb{G}_m^r$, $r \geq 0$ be a (possibly trivial) torus and A a T-ring.*

(i) A T-ideal I of A is a prime ideal if and only if (1) I is a proper ideal and (2) for each T-weight vectors f, g, $fg \in I \Rightarrow f, g \in I$.

(ii) If $\mathfrak{p}$ is a prime ideal, then $\mathfrak{p}^\# = \bigoplus_\chi \mathfrak{p} \cap A_\chi$ is the smallest prime T-ideal contained in $\mathfrak{p}$. In particular, a prime ideal $\mathfrak{p}$ is a T-ideal if and only if $\mathfrak{p} = \mathfrak{p}^\#$.

(iii) We write $\operatorname{Spec}^T(A)$ for the set of all prime T-ideals of A. If $B \subset A$ is a T-equivariant integral ring extension of T-algebras, then the natural map

$$\operatorname{Spec}^T(A) \to \operatorname{Spec}^T(B),\ \mathfrak{p} \mapsto \mathfrak{p} \cap B$$

is surjective.

(iv) If A is an integral domain that is a finitely generated as a k-algebra, then the map in (iii) preserves heights; i.e., $\operatorname{ht}(\mathfrak{p}) = \operatorname{ht}(\mathfrak{p} \cap B)$.

Proof. (i) and (ii) can be shown easily. For (iii), given a $\mathfrak{q}$ in $\operatorname{Spec}^T B$, since A is integral over B, we can find a prime ideal $\mathfrak{p}$ of A lying over $\mathfrak{q}$; i.e., $\mathfrak{q} = \mathfrak{p} \cap B$. Then $\mathfrak{q} \cap B_\chi \subset \mathfrak{p} \cap A_\chi$ and so $\mathfrak{q} = \mathfrak{q}^\# \subset \mathfrak{p}^\# \cap B$. Since the opposite inclusion is trivial, we conclude $\mathfrak{p}^\# \cap B = \mathfrak{q}$.

(iv) is [HS06] Proposition 4.8.6. More directly, one can show it by Nagata's altitude formula ([Ei04] Exercise 13.12.). □

2.5.2 Lemma. *In the setup of the preceding lemma, let M be a finite T-equivariant A-module. Then each associated prime of M, if any, is a T-ideal; i.e.,* $\operatorname{Ass}(M) \subset \operatorname{Spec}^T A$.

Proof. This is [Ei04] Exercise 3.5. (b) □

2.5.3 Proposition (torus-invariant primary decomposition)**.** *Let I be a T-ideal in a Noetherian T-ring A. Then*

(i) I is an intersection of a finite number of primary T-ideals.

(ii) If I is an intersection of some finite set E of primary T-ideals, then I_χ is the intersection of $\{Q_\chi | Q \in E\}$ where the subscript χ means the weight space of weight χ.

Proof. (i) We repeat the usual proof[6] of the existence of a primary decomposition. Replacing A by A/I, we assume $I = 0$. Since we already know $\operatorname{Ass}_A(A)$ is a finite set, we only need to show the intersection J of all T-primary ideals is zero. Suppose otherwise; then $\operatorname{Ass}_A(J)$ is nonempty and by Lemma 2.5.2, it contains a T-invariant prime ideal $\mathfrak{p}$. Consider the set $\{I | I \text{ a } T\text{-ideal of } A, \mathfrak{p} \notin \operatorname{Ass}_A(I)\}$. Let Q be a maximal element of the set. If Q is not primary, that is, if A/Q has at least two distinct associated primes (both of which are T-invariant), then we can find a T-ideal $I \supset Q$ such that $A/\mathfrak{p}' \simeq I/Q$ for some prime ideal $\mathfrak{p}'$ not $\mathfrak{p}$. Since $\operatorname{Ass}_A(I) \subset \operatorname{Ass}_A(Q) \cup \operatorname{Ass}_A(I/Q)$, we have $\mathfrak{p} \notin \operatorname{Ass}_A(I)$ and so this is a contradiction to the maximality of Q. Hence, Q is a primary ideal and $J \subset Q$. But then $\mathfrak{p} \in \operatorname{Ass}_A(J) \subset \operatorname{Ass}_A(Q)$, a contradiction.

(ii) Clear. □

[6] A proof in Bourbaki's commutative algebra textbook.

2.6 Graded Nakayama lemma

We record the graded Nakayama lemma for later use. Our presentation is a minor extension of Melvin Hochster, Math 711: Lecture Note of September 18 from Math 711, Fall 2006.

2.6.1 Lemma (graded Nakayama lemma)**.** *Let A be a $\mathbb{G}_m$-ring and I a $\mathbb{G}_m$-ideal such that $I \subset A_+ = \bigoplus_{\chi>0} A_\chi$.*

For each $\mathbb{G}_m$-equivariant A-module M such that $M_\chi = 0$ for $\chi \ll 0$, the following hold:

(i) If $IM = M$, then $M = 0$.

(ii) Each finite set of T-weight vectors generating the A/I-module M/IM lift to a set of T-weight vectors generating the A-module M.

In particular, M is finite over A if and only if M/IM is finite over A/I.

Proof. (i) (cf. [Ei04] Exercise 4.6.) Assume $I \neq 0$; otherwise there is nothing to prove. We write $I = I_{\chi_0} \oplus (\oplus_{\chi>\chi_0} I_\chi)$ where $I_{\chi_0} \neq 0$ with $\chi_0 > 0$. We shall show $M_\chi = 0$ for each $\chi \in \mathbb{Z}$. Choose a large enough $n > 0$, depending on χ, such that $M_\mu = 0$ for $\mu \leq \chi - n\chi_0$. Then $M_\chi = (I^n M)_\chi = \oplus_\mu (I^n)_\mu M_{\chi-\mu} = 0$.

(ii) is a standard consequence of (i). $\square$

We say a ring homomorphism $A \to B$ is *finite* if B is finitely generated as an A-module through the homomorphism.

2.6.2 Corollary. *Let A, B be $\mathbb{G}_m$-rings such that $A_\chi = 0$ for $\chi < 0$ and similarly for B. Assume that $\sqrt{B_+}$ is a finitely generated ideal of B (e.g., B is a Noetherian ring).*

Then each $\mathbb{G}_m$-equivariant ring homomorphism $A \to B$ is finite if and only if $A^{\mathbb{G}_m} \to B^{\mathbb{G}_m}$ is finite and $\sqrt{A_+B} = \sqrt{B_+}$, where we write $A_+ = \bigoplus_{\chi>0} A_\chi$ and similarly for B_+.

Proof. ($\Rightarrow$) B/A_+B is finite over $A_0 = A/A_+$; say, $B/A_+B = \sum_1^r A_0\overline{x_i}$. Thus, for each $\mu > \mu_0 := \max_i(\mathrm{wt}(\overline{x_i}))$, we have $(B/A_+B)_\mu = 0$. For $l \gg 0$, we then have $B_+^l \subset \bigoplus_{\mu>\mu_0} B_\mu = \bigoplus_{\mu>\mu_0}(A_+B)_\mu \subset A_+B$. Since $A_+B \subset B_+$ trivially, we get that $\sqrt{B_+} = \sqrt{A_+B}$.

($\Leftarrow$) By assumption, we can choose some large $l > 0$ so that $B_+^l \subset \sqrt{B_+}^l = \sqrt{A_+B}^l \subset A_+B$. Now, choose some $\mathbb{G}_m$-weight vectors x_i that generate B as a $B^{\mathbb{G}_m}$-algebra and are such that $\mathrm{wt}(x_i) \leq \mathrm{wt}(x_1)$. Now, for each $\mu > 0$, if $y \in B_\mu$, we can write $y = \sum_j F_j(x_1, \dots, x_r)$ for some monomials F_j with coefficients in $B^{\mathbb{G}_m}$ such that $\mathrm{wt}(F_j(x_1, \dots, x_r)) = \mu$ for each j. Then

$$\mu = \mathrm{wt}(F_j(x_1, \dots, x_r)) \leq \mathrm{wt}(F_j(x_1, \dots, x_1)) = \deg(F_j)\,\mathrm{wt}(x_1).$$

That is, for $\mu \geq l\,\mathrm{wt}(x_1)$, we have: $(B/A_+B)_\mu = 0$. Now, each degree piece $(B/A_+B)_\chi = B_\chi/(A_+B)_\chi$ is finite over $B^{\mathbb{G}_m}$ as B_χ is finite over $B^{\mathbb{G}_m}$ (since B_χ is spanned by finitely many monomials in x_i's). Hence, B/A_+B is finite over $B^{\mathbb{G}_m}$; thus over $A^{\mathbb{G}_m} = A/A_+$. Hence, B is finite over A by the graded Nakayama lemma. □

2.7 GIT quotients by torus

This subsection includes few materials on geometric-invariant-theory quotients or GIT quotients, that will be referred in §7.

2.7.1 Definition. Let $X = \operatorname{Proj} R$ be a projective variety and a torus $T = \mathbb{G}_m^r$ act on R as graded automorphisms. Let $X^{us} = V(R_+^T R) \subset X$ be the closed subscheme defined by the ideal generated by R_+^T; it is called the *unstable locus*. Then the inclusion $R^T \subset R$ induces the morphism

$$\pi : X^{ss} \to X/\!/T$$

where X^{ss} is the complement $X - X^{us}$ called the *semistable locus*.[7] It is called the *GIT* or *geometric-invariant-theory quotient* of X by T.

Immediately out of the definition, we have:

(i) Since T acts trivially on $X/\!/T$ and π is evidently T-equivariant, π is T-invariant; i.e., $\pi \circ \sigma = \pi \circ p_2$ for the T-action $\sigma : T \times X \to X$ and the projection $p_2 : T \times X \to X$.
In particular, $\pi(O)$ is a point if O is an orbit.

(ii) If $Z \subset X$ is an invariant closed subset, then $Z^{ss} = Z \cap X^{ss}$.

The GIT quotient parametrizes equivalence classes by orbit closures on X^{ss}:

[7] The English language breaks down a bit here: the unstable locus is, more correctly, the non-semi-stable locus.

2.7.2 Lemma. *In the setup of Definition 2.7.1,*

(i) For each invariant closed subvarieties $Z, W \subset X^{ss}$, $\pi(Z)$ is closed in $X//T$ and $\pi(Z \cap W) = \pi(Z) \cap \pi(W)$.

(ii) For each pair of orbits O, O' on X^{ss}, we have: $\overline{O} \cap \overline{O'} \cap X^{ss} \neq \emptyset$ if and only if $\pi(O) = \pi(O')$.

(iii) Each fiber $\pi^{-1}(y)$ is nonempty and contains a unique orbit O that is closed in X^{ss}; in particular, π is surjective.

Proof. (i) Let $\overline{Z}, \overline{W}$ denote the closures of Z, W in X and I, J the defining ideals in R of $\overline{Z}, \overline{W}$. Note $\pi(Z)$ is closed since it is defined by the ideal I^T of R^T. Note that $I + J$ is then the defining ideal of $\overline{Z} \cap \overline{W}$ and $\pi(Z) \cap \pi(W)$ is defined by $I^T + J^T$. On the other hand, since I, J are T-modules, $(I + J)^T = I^T + J^T$.

(ii) By (i), $\overline{O} \cap \overline{O'} \cap X^{ss} \neq \emptyset \Leftrightarrow \pi(\overline{O}^{ss}) \cap \pi(\overline{O'}^{ss}) \neq \emptyset$. Since $\pi(\overline{O}^{ss}) \subset \overline{\pi(O)} =$ a closed point, $\pi(\overline{O}^{ss}) = \pi(O)$ and similarly for O'. Thus, $\pi(\overline{O}^{ss}) \cap \pi(\overline{O'}^{ss}) \neq \emptyset \Leftrightarrow \pi(O) = \pi(O')$.

(iii) For the "moreover" part, if $\mathfrak{q}_y$ is the defining homogeneous prime ideal of y, then $\mathfrak{q}_y = (\mathfrak{q}_y R)^T$ and so π is surjective. The uniqueness of a orbit closed in X^{ss} follows from the first part. □

The next example shows that a GIT quotient may be thought of a generalization of a vector bundle (when the action is not twisted).

2.7.3 Example. Let A be a graded Noetherian integral domain such that $A_0 = k$ is the base field and let $R = A[x]$ with an indeterminate x having degree one. Let $\mathbb{G}_m$ act as grade-preserving automorphisms on R so that A consists of $\mathbb{G}_m$-invariant elements and x has weight one.

Let $X = \operatorname{Proj} R$ and $Y = \operatorname{Proj} A$. On the affine cone level, we have $\operatorname{Spec} R = \operatorname{Spec} A \times \mathbb{A}^1$. We have $A = R^{\mathbb{G}_m}$, $X^{us} = V(A_+ R)$ and $\pi : X^{ss} \to X//\mathbb{G}_m = Y$ is the GIT quotient. Explicitly, if $g \in A$ is a homogeneous element of positive degree, then we have:

$$\pi_g : X_g \to Y_g = X_g//\mathbb{G}_m$$

given by[8] $A_{(g)} = R_{(g)}^{\mathbb{G}_m} \hookrightarrow R_{(g)}$. If g has degree m, $R_{(g)} = A_{(g)}[x^m/g]$ and thus $X_g = Y_g \times \mathbb{A}^1$ and π_g is a projection; i.e., π is a line bundle over Y. In other words, X is a "compactification" of a line bundle over Y where X^{us} amounts to the locus of boundary points.

Next, suppose $\mathbb{G}_m^r$ act on R in such a way x is $\mathbb{G}_m^r$-invariant. Let $T = \mathbb{G}_m^r \times \mathbb{G}_m$. Then, with respect to T, A is spanned by homogeneous weight vectors of weights of the form $(*, \dots, *, 0)$ while x has weight $(0, \dots, 0, 1)$. Also, $X^{us} = V(A_+^T R)$, since $R^T = A^T$. Since $R/A_+^T R \simeq A/A_+^T[x]$, we have: $\dim X^{us} = \dim Y^{us} + 1$, where, if $r = 0$, then Y^{us} is empty and has dimension -1. We also note that Y^{us} is the same as $Y^{us, \mathbb{G}_m^r}$, the unstable locus on Y with respect to $\mathbb{G}_m^r$.

The semistable loci behave compatibly under a finite map:

2.7.4 Proposition. *Let $S \subset R$ be graded rings such that S is a subring and $S \hookrightarrow R$ is graded. Assume $T = \mathbb{G}_m^r$ act on R, S as graded automorphisms in such a way $S \to R$ is T-linear. If R is integral over S, then*

$$\sqrt{S_+^T R} = \sqrt{R_+^T R}.$$

In other words, for $f : X = \operatorname{Proj} R \to Y = \operatorname{Proj} S$, we have: as sets, $f^{-1}(Y^{us}) = X^{us}$.

Proof. Cleary, $\sqrt{S_+^T R} \subset \sqrt{R_+^T R}$ and so we need to show $R_+^T \subset \sqrt{S_+^T R}$. For that end, let f be in R_+^T. Since f is integral over S, we can write

$$f^n + g_1 f^{n-1} + \cdots + g_n = 0$$

for some $g_i \in S_+$. Let $P : R \to R^T$ be the projection (Lemma 2.4.2). Then, since P is R^T-linear,

$$f^n + P(g_1) f^{n-1} + \cdots + P(g_n) = 0.$$

Since $R^T \cap S \subset S^T$, this is to say f^n is in $S_+^T R$. □

We record a few lemmas:

[8] Here we use the notation $-_{(g)} = -[g^{-1}]_0$

2.7.5 Lemma. *Let* $X = \operatorname{Proj} R$ *a projective variety such that a torus* $\mathbb{G}_m^r$ *acts on* R *as graded automorphisms and* $\pi : X^{ss} \to Y = \operatorname{Proj}(R^{\mathbb{G}_m^r})$ *a GIT quotient. Let* $\pi : X^{ss} \to Y$ *be a GIT quotient of a projective variety* $X = \operatorname{Proj} R$ *by a torus as in the previous lemma. Then, for each geometrically connected closed subscheme* $Y' \subset Y$*, the pre-image* $\pi^{-1}(Y')$ *is geometrically connected.*

Proof. Without loss of generality, we assume the base field is algebraically closed. Suppose $\pi^{-1}(Y')$ is not connected; then we can write $\pi^{-1}(Y') = Z_1 \cup Z_2$ for some disjoint closed invariant subsets Z_1, Z_2; namely, one can take Z_1 to be a connected component and Z_2 the union of the rest of the components. Since the Z_i's are invariant, $\pi(Z_i)$ are closed and $\pi(Z_1) \cap \pi(Z_2) = \pi(Z_1 \cap Z_2) = \emptyset$ (Lemma 2.7.2 (i)). Since $Y' = \pi(Z_1) \cup \pi(Z_2)$ and since Y' is connected, we have say $Y' = \pi(Z_1)$, which contradicts the fact that $\pi(Z_1)$ and $\pi(Z_2)$ are disjoint. □

3.0 Preparation from intersection theory

The purpose of this section is to collect the definitions and some fundamental facts in intersection theory for later use; especially in Section 8. In this section, we introduce a group that can be used as a substitute for a Chow group; the former is sheaf-theoretic as opposed to cycle-theoretic and is more convenient for our purpose. The theory of this group is fairly standard; among many references is [Ha09].

3.1 Definition of intersection numbers

Let X be an algebraic scheme (= a scheme that is of finite type and separated over the fixed base field) and $\mathbf{G}(X)$ denote the Grothendieck group of coherent sheaves on X. For each line bundle L on X, let $c_1(L)$ be the endomorphism of the group $\mathbf{G}(X)$ given by: for each class $[F]$,

$$c_1(L)[F] = [F] - [L^{-1} \otimes F].$$

Intuitively speaking, we think the above means we are intersecting F with L. In fact,

3.1.1 Lemma. *Let L be a line bundle on X, $Y \subset X$ a closed subscheme and D an effective Cartier divisor on Y such that $\mathcal{O}_Y(D) \simeq L|_Y$. Then*

$$c_1(L)[\mathcal{O}_Y] = [\mathcal{O}_D].$$

Roughly speaking, D represents the intersection between L and Y.

Proof. Viewing D as a closed subscheme of Y, we have the exact sequence

$$0 \to \mathcal{O}_Y(-D) \to \mathcal{O}_Y \to \mathcal{O}_D \to 0$$

where, by abuse, $\mathcal{O}_D$ above is the pushforward of $\mathcal{O}_D$ along $D \hookrightarrow Y$. From this we get: $[\mathcal{O}_Y] = [\mathcal{O}_Y(-D)] + [\mathcal{O}_D] = [L^{-1} \otimes \mathcal{O}_Y] + [\mathcal{O}_D]$. □

We cite the following fact.

3.1.2 Lemma. *For any line bundle L on X and any $[F]$ in $\mathbf{G}(X)$,*

$$\dim \operatorname{Supp}(c_1(L)F) \leq \dim \operatorname{Supp} F - 1.$$

Proof. This is [Kl05] Lemma B.5. ☐

Now, for any line bundles L and M on X, we have:

$$\begin{aligned} c_1(L)c_1(M)F &= F - L^{-1} \otimes F - M^{-1} \otimes F + L^{-1} \otimes M^{-1} \otimes F \\ &= c_1(L)F + c_1(M)F - c_1(L \otimes M)F. \end{aligned}$$

Note that the expression in the right-hand side is symmetric in L and M; thus, $c_1(L)$ and $c_1(M)$ commute.

3.1.3 Definition (Kleiman et al)**.** If $L_1, \ldots, L_r$ are line bundles on X and F a coherent sheaf on X whose support is proper over the base field k and whose irreducible components have dimension at most r, then their *intersection number* is defined as

$$(L_1 \cdot \ldots \cdot L_r \cdot F) = \chi(X, c_1(L_1) \cdots c_1(L_r)F).$$

Equivalently, it is the coefficient of $\prod n_i$ in the multi-Hilbert polynomial:

$$\chi(X, L_1^{\otimes n_1} \otimes \cdots \otimes L_r^{\otimes n_r} \otimes F).$$

(The Riemann-Roch formula gives the explicit form of this polynomial; see Proposition 3.1.8). If V is a complete subvariety of X of dimension r, then we let

$$(L_1 \cdot \ldots \cdot L_r \cdot V) = (L_1 \cdot \ldots \cdot L_r \cdot \mathcal{O}_V).$$

We have the following notion:

3.1.4 Definition (topological filtration; cf. [Ful98] Example 15.1.5.)**.** We can filter $\mathbf{G}(X)$ by

$$\mathbf{G}(X)_{\leq r} = \{[F] | \dim \operatorname{Supp} F \leq r\}.$$

It is an increasing filtration called the *topological filtration*. We then form the associated graded group:

$$\operatorname{gr}^{\text{top}} \mathbf{G}(X) = \bigoplus_{r=0}^{\infty} \mathbf{G}(X)_{\leq r} / \mathbf{G}(X)_{\leq r-1}.$$

Because, in this paper, we only use the topological filtration, we will usually drop "top" here.

In view of Lemma 3.1.2, $c_1(L)$ is also an endomorphism of $\operatorname{gr} \mathbf{G}(X)$ and the early formula now reads:

$$c_1(L \otimes M) = c_1(L) + c_1(M).$$

The topological filtration is relevant in the following.

3.1.5 Remark (Riemann-Roch theorem in Fulton's *Intersection Theory*)**.** Let $\mathcal{C}$ be the category of algebraic schemes = schemes that are of finite type and separated over the base field. Also, let $\mathcal{C}_{\text{pp}}$ be the subcategory of it that has the same set (or class) of objects as $\mathcal{C}$ does but the morphisms there are proper morphisms.

For each algebraic scheme X, we write $\mathbf{A}(X)$ for the Chow group of X and $\mathbf{A}_{\mathbb{Q}}(X) = \mathbf{A}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. If $K(X)$ denotes the Grothendieck ring of vector bundles on X, the Chern characters determine the $K(X)$-module structure on $\mathbf{A}_{\mathbb{Q}}(X)$: namely, if E is a vector bundle and α is a class in $\mathbf{A}_{\mathbb{Q}}(X)$, then $E \cdot \alpha = \operatorname{ch}(E)\alpha$, where we viewed $\operatorname{ch}(E)$ as an endomorphism. Then $\mathbf{A}_{\mathbb{Q}}$ is a functor from $\mathcal{C}_{\text{pp}}$ to $\mathsf{Mod}_{K(X)}$ = the category of $K(X)$-modules.

Similarly, $\mathbf{G}(X)$ has the structure of a $K(X)$-module given simply by tensor product and $\mathbf{G} : \mathcal{C}_{\text{pp}} \to \mathsf{Mod}_{K(X)}$ is a functor. Then Theorem 18.3. of Fulton's book says that there is a *unique* natural transformation

$$\tau : \mathbf{G} \to \mathbf{A}_{\mathbb{Q}},$$

satisfying the properties generalizing those of the usual Riemann-Roch formula.

The key consequence of the Riemann-Roch theorem in the above remark for us is the following:

3.1.6 Proposition. *For each algebraic scheme X, $V \mapsto \mathcal{O}_V$ induces the isomorphism:*

$$\mathbf{A}_{\mathbb{Q}}(X) \simeq \operatorname{gr} \mathbf{G}_{\mathbb{Q}}(X),$$

where, as before, $\mathbf{G}_{\mathbb{Q}}(X) = \mathbf{G}(X) \otimes \mathbb{Q}$.

Proof. This is the consequence of one of the properties of τ (namely, (5) of Theorem 18.3. of [Ful98].) □

3.1.7 Remark (**G** v.s. gr **G**)**.** As a vector space, $\operatorname{gr} \mathbf{G}_{\mathbb{Q}}(X)$ and $\mathbf{G}_{\mathbb{Q}}(X)$ are isomorphic.

The next proposition shows our definition (Definition 3.1.3) coincides with the usual one (up to the difference between **A** and gr **G**).

3.1.8 Proposition (Example 18.3.6. in [Ful98])**.** *Let X be a complete variety, F a coherent sheaf and $L_1, \ldots, L_r$ line bundles on it. Then*

$$\chi(X, L_1^{\otimes n_1} \otimes \cdots \otimes L_r^{\otimes n_r} \otimes F) = \sum_{j=0}^{\dim X} \frac{1}{j!}(n_1 c_1(L_1) + \cdots + n_r c_1(L_r))^j \tau_X(F)_j$$

where $\tau_X(F)_j$ is the j-th component of $\tau_X(F)$ in $\mathbf{A}_{\mathbb{Q}}(X)$. In particular, it gives the coefficient of $\prod n_i$.

Proof. By the Riemann-Roch theorem discussed in Remark 3.1.5, for $E = L^{\otimes n_1} \otimes \cdots \otimes L^{\otimes n_r}$, we get:

$$\tau_X(E \otimes F) = \operatorname{ch}(E)\tau_X(F).$$

Note that $\operatorname{ch}(E) = e^{n_1 c_1(L_1) + \cdots + n_r c_1(L_r)}$. Taking degree; i.e., the pushforward along the structure map $X \to *$, we get the asserted formula. □

3.2 A few facts on toric varieties

Finally, we record the following that generalizes the corresponding one for toric varieties.

3.2.1 Proposition. *Let $F(X)$ denote either the Chow group of X or* $\operatorname{gr}\mathbf{G}(X)$*. Assume X has a cellular decomposition; i.e., a filtration $X = X_d \supset X_{d-1} \supset \dots X_0 \supset X_{-1} = \emptyset$ by closed schemes such that $X_i - X_{i-1}$ is a disjoint union of U_{ij} isomorphic to affine spaces.*

Then $F(X)$ is generated by the classes of the closures Z_{ij} in X of U_{ij} (or of $\mathcal{O}_{Z_{ij}}$).

Proof. This is [Ful98] Example 1.9.1. □

3.2.2 Corollary. *If X is a non-normal toric variety, then $F(X)$ is generated by the classes of the orbit closures.*

For the use below, we recall:

3.2.3 Lemma. *Let $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$ be closed subvarieties. Suppose there exists a finite surjective morphism $X \to Y$ that is given by some projection from $\mathbb{P}^n$, away from some closed subset, to $\mathbb{P}^m$. Then* $\deg(X) = \deg(Y)$.

Proof. Let R, S be the homogeneous coordinate rings of X, Y, respectively. By assumption, S is a subring of R such that $S \hookrightarrow R$ is grade-preserving and finite. Thus, the S-module R/S is finitely generated and so we can choose some homogeneous element f such that $fR \subset S$. Thus,

$$\dim R_{n-\deg(f)} = \dim(fR)_n \leq \dim S_n \leq \dim R_n.$$

Since $\deg(X)/\dim(X)!$ is the leading coefficient of the Hilbert polynomial of R, the assertion follows. □

The next proposition is a direct generalization of the corresponding fact for a toric variety (this proposition and related results appear in a greater generality in [KK12].) The results of Part 3 can be viewed as a multi-graded or T-equivariant generalization of the proposition.

3.2.4 Proposition. *Let V be a finite-dimensional vector space over an algebraically closed field k and assume the torus $T = \mathbb{G}_m^d$ acts on V as linear automorphisms. Then T acts on the projectivization $\mathbb{P}(V)$ of V. Let $X \subset \mathbb{P}(V)$ be the closure of some T-orbit such that*

$\dim X = d$ *(so that X is a non-normal toric variety). Then the degree of X in $\mathbb{P}(V)$ is given as*

$$\deg_{\mathbb{P}(V)}(X) = d! \operatorname{vol}_d(\triangle_X)$$

where $\triangle_X \subset \mathbb{R}^d$ is the moment polytope of X (Definition 2.4.3) and vol_d *refers to the standard Euclidean volume.*

In other words, the leading term of the Hilbert polynomial $\chi(X, \mathcal{O}_{\mathbb{P}(V)}(n)|_X)$ of X is $\operatorname{vol}_d(\triangle_X)\, n^d$

Proof. By Theorem 1.1.8, we can write $R = k[S]$ for some finitely generated subsemigroup S of $(\mathbb{N} \times \mathbb{Z}^d, +)$, where $\mathbb{N}$ corresponds to the grading on R. We write $S_n = \{x | (n, x) \in S\}$. Then we have $\dim R_n = \#(S_n \cap \mathbb{Z}^d)$.

Now, by Lemma 3.2.3, without loss of generality, we can assume S is saturated; i.e., $S_n = nP \cap \mathbb{Z}^d$ for some convex polytope P. Then, by the Riemann sum approximation,

$$\int_P dx = \lim_{n\to\infty} \sum_{x \in P \cap \frac{1}{n}\mathbb{Z}^d} n^{-d} = \lim_{n\to\infty} n^{-d} \#(nP \cap \mathbb{Z}^d).$$

□

The last part of the above proposition does not generalize to the lower terms of the Hilbert polynomial because X is not necessarily normal. This motivates the construction in the next section.

In the next proposition, it is crucial that X is normal.

3.2.5 Proposition. *Let X be a complete (normal) toric variety and L a line bundle generated by global sections. Then $\mathrm{H}^i(X, L) = 0$ for each $i > 0$.*

4.0 A non-normal toric variety as a toric variety with extra data

The normalization of a variety gives the functor

$$X \mapsto X^{\text{nor}}$$

from the category of non-normal toric varieties with torus $\mathbb{G}_m^d$ to the category of toric varieties with torus $\mathbb{G}_m^d$ (the functoriality is because the morphisms here are dominant; see the proof of Theorem 4.1.6). This functor is not an equivalence, of course. The purpose of this section is to show that the above can be modified to the functor

$$X \mapsto (X^{\text{nor}}, \xi_X)$$

so that it is an equivalence from the category of non-normal projective toric $\mathbb{G}_m^d$-varieties to the category of projective toric $\mathbb{G}_m^d$-varieties together with *extra data* ξ. Concretely, ξ consists of non-normal toric varieties $Y_1, \dots Y_r$ of strictly smaller dimensions together with maps f_i from closed invariant subschemes Y_i' of X to Y.

The construct thus allows one to inductively apply the results in the theory of toric varieties to non-normal toric varieties. As a example of this, we show the Hilbert polynomial of a non-normal toric variety equivariantly embedded into a projective space is some finite sum of Ehrhart polynomials.

4.1 Non-normality data

We introduce the notion of an extra data on a toric variety that can be used to construct a non-normal toric variety from it. Precisely,

4.1.1 Definition. Given a (normal) toric variety X with torus $\mathbb{G}_m^d$, a *non-normality data on* X is a triple (Y', Y, f) consisting of a $\mathbb{G}_m^d$-invariant closed subscheme $Y' \neq X$ of X, a $\mathbb{G}_m^d$-scheme Y and an equivariant map $f : Y' \to Y$ such that (1) the pushout

$$X_\xi = X \cup_f Y$$

is a variety and (2) the natural map $X \to X_\xi$ is finite, equivariant and surjective and (3) $Y \to X_\xi$ is a closed immersion. The data fits into the diagram:

$$\begin{array}{ccc} Y' & \hookrightarrow & X \\ \downarrow{\scriptstyle f} & & \downarrow \\ Y & \hookrightarrow & X_\xi . \end{array}$$

Intuitively, X_ξ is obtained by identifying a closed subscheme of X with Y through f. Usually, f is called the attaching map and X_ξ is said to be the result of attaching X to Y via f.

4.1.2 Remark. Some readers might find the following analogy to a construction in algebraic topology helpful. In algebraic topology, given a space Y, an n-disk D^n (called n-cell) with boundary S^{n-1} and a map $f : S^{n-1} \to Y$, the pushout

$$D^n \cup_f Y$$

is called the space obtained by attaching D^n to Y along f. For example, one can do such attaching finitely times to some finite set of points and the resulting space is called a *spherical complex* ([GH81] Part II. Ch. 19)[1]

As we explained in Example 1.1.3, a torus itself is a toric variety and a toric variety in general is, intuitively speaking, a partially compactified torus (with the defining fan specifying boundary components). Hence, by analogy, we can think of a non-normal toric variety as a complex of toric varieties. We can then compute, for example, homology or cohomology of a non-normal toric variety is the weighted sum of homology or cohomology over the complex; see Proposition 4.1.12 for a trivial instance of this observation.

[1]Imposing some conditions on it gives the better-known notion of a CW complex.

The next two simple examples illustrate Definition 4.1.1; in particular they explain what we mean by "f identifies an invariant closed subscheme of X with Y". (The general case goes essentially the same way.)

4.1.3 Example. Let $B = k[t^2, t^3] \subset A = k[t]$ be the rings. Note $B = k[x, y]/(y^2 - x^3)$ via $k[x, y] \to k[t], x \mapsto t^3, y \mapsto t^2$. Since A is integrally closed, A is the integral closure of B. Let $X = \operatorname{Spec} A$. Let I be the annihilator of the B-module A/B; note it is both an ideal of A and an ideal of B (I is usually called a conductor). Let $Y' = \operatorname{Spec} A/I$ and $Y = \operatorname{Spec} B/I$ and $f : Y' \to Y$ given by $A/I \hookrightarrow B/I$. By Lemma 4.1.5 below, $X_\xi = \operatorname{Spec} B$ is the pushout $X \cup_f Y$. Then (Y', Y, f) is a non-normality data on X.

Now, one can see that I is generated by t^2, t^3 as an ideal of B and by t^2 as an ideal of A. Thus, $Y' = \operatorname{Spec} k[t]/t^2$ is the closed subscheme of X defined by the ideal (t^2); i.e., it is a double point on $X = \mathbb{A}^1$. Similarly, $Y = \operatorname{Spec} k$ is the origin on the curve $x^3 = y^2$. The attaching map $f : Y' \to Y$ collapses the double point to a (reduced) point. That is, X_ξ is obtained from $\mathbb{A}^1$ by identifying the double point at the origin.

4.1.4 Example. Let $S \subset \mathbb{Z}^2$ be the subsemigroup generated by $(1, 0), (1, 1), (1, 3)$. Note that $k[S] = k[x, y, z]/(y^2 z - x^3)$; cf. Example 6.2.7. Let $X_\xi = \operatorname{Spec}(k[S])$. The saturation of S is $\widetilde{S} = \sigma^\vee \cap \mathbb{Z}^2$ where $\sigma \subset \mathbb{R}^2$ is the dual of the cone generated by $(1, 0), (1, 3)$. Hence, the normalization X of X_ξ is $X = \operatorname{Spec}(k[\widetilde{S}]) = \operatorname{Spec} k[x, y] \simeq \mathbb{A}^2$ where x, y have $\mathbb{G}_m^2$-weights $(1, 0)$, $(1, 3)$.

Now, it is easy to see that $\widetilde{S}$ is the disjoint union of S and $(1, 2) + F \cap \mathbb{Z}^2$ where $F = \mathbb{R}_+(1, 3)$ is a face of $\sigma^\vee$ (cf. Remark 4.1.9). By Proposition 2.2.3, F corresponds to the prime ideal $\mathfrak{p}_F = \ker(k[\widetilde{S}] \to k[\widetilde{S} \cap F])$. Let I be the annihilator of the $k[S]$-module $k[\widetilde{S}]/k[S]$. Then I is generated by $\chi^{(1,0)}$ and $\chi^{(1,1)}$ as an ideal of $k[S]$. As an ideal of $k[\widetilde{S}]$, $\sqrt{I} = \mathfrak{p}_F$ but $I \neq \mathfrak{p}_F$, since $(\chi^{(1,2)})^2 = \chi^{(1,1)}\chi^{(1,3)}$ but $\chi^{(1,2)} \notin I \subset k[S]$. Note $k[S] \hookrightarrow k[\widetilde{S}]$ induces:

$$k[S]/I \to k[\widetilde{S}]/I$$

and then $X \to X_\xi$ restricts to $Y' = \operatorname{Spec}(k[\widetilde{S}]/I) \to Y = \operatorname{Spec}(k[S]/I)$.

The next lemma is [Ei04] Exercise 11.16.

4.1.5 Lemma. *Let $B \subset A$ be rings and I the annihilator of the B-module A/B. Then B is obtained from A, B/I and A/I as the fiber product*

$$B = A \times_{A/I} B/I$$

where $A, B/I$ are given the structures of A/I-modules through the natural maps.

Note: geometrically, the above equation says that $\operatorname{Spec} B$ *is the pushout of* $\operatorname{Spec} A$ *and* $\operatorname{Spec} B/I$ *along* $\operatorname{Spec} A/I$.

Proof. Let $\varphi : B \to A \times_{A/I} B/I$ be given by $b \mapsto (b, b \mod I)$. It is easy to verify that φ is bijective. □

We now prove the main result of this section.

4.1.6 Theorem. *Let $\mathcal{C}$ be the category where*

- *the objects are pairs (X, ξ) consisting of (normal) toric $\mathbb{G}_m^d$-varieties with non-normality data on them.*
- *a morphism $(X_1, \xi_1 = (Y_1', Y_1, f_1)) \to (X_2, \xi_2)$ consists of a morphism $\varphi : X_1 \to X_2$ as well as a morphism $Y_1 \to Y_2$ such that $\varphi(Y_1') \subset Y_2'$ and $Y_1' \xrightarrow{f_1} Y_1 \to Y_2$ equals $Y_1' \to Y_2' \xrightarrow{f_2} Y_2$.*

Then there is an equivalence of categories

$$X \mapsto (X^{nor}, \xi_X)$$

from the category of non-normal toric varieties with torus $\mathbb{G}_m^d$ to $\mathcal{C}$.

Moreover, let X be a complete non-normal toric variety over an algebraically closed field with torus $\mathbb{G}_m^d$ and L a $\mathbb{G}_m^d$-equivariant line bundle whose pullback to the normalization of X is a line bundle generated by global sections. Let $\triangle_X(L)$ denote the moment polytope of L and $\xi = (Y', Y, f)$ the non-normality data. If $Y_1, \ldots, Y_r$ are the irreducible components of Y with the reduced structures, then there exist some integers $l_1, \ldots, l_r$ such that, for each integer $n \geq 0$,

$$\chi(X, L^{\otimes n}) = \#(n\triangle_X(L) \cap \mathbb{Z}^d) - \sum_{i=1}^{r} \chi(Y_i, L|_{Y_i}{}^{\otimes(n+l_i)}).$$

(Note that the formula can be applied recursively.)

Proof. We first construct the functor $X \mapsto (X^{\mathrm{nor}}, \xi_X)$. Given a non-normal toric variety X, let $\mathcal{I}$ denote the conductor ideal sheaf of $\mathcal{O}_X$, the dual of $\pi_* \mathcal{O}_X$ for the normalization $\pi : \widetilde{X} \to X$. We note that $\mathcal{I}$ is also an ideal sheaf of $\mathcal{O}_{\widetilde{X}}$. Let $Y = \operatorname{Spec}_X(\mathcal{O}_X/\mathcal{I})$ and $Y' = \operatorname{Spec}_X(\mathcal{O}_{\widetilde{X}}/\mathcal{I})$. Then they constitute a non-normality data on the toric variety $\widetilde{X}$ in the sense of Definition 4.1.1; indeed, we can assume X, Y are affine and then this follows from Lemma 4.1.5. Also, $X \to Z$ induces a morphism $(X^{\mathrm{nor}}, \xi_X) \to (Z^{\mathrm{nor}}, \xi_Z)$; indeed, the universal property of normalization says that any dominant morphism from a normal variety W to a variety X factors through the normalization $X^{\mathrm{nor}} \to X$.

To see the functor is an equivalence, we construct a quasi-inverse (= inverse up to natural isomorphisms). Given (X, ξ) in $\mathcal{C}$, we get the non-normal toric variety by forming the pushout $X \cup_f Y$ of X and Y along Y'. Moreover, each $(X_1, \xi_1 = (Y_1' \to Y_1)) \to (X_2, \xi_2)$ induces a morphism $X_1 \cup_{f_1} Y_1 \to X_2 \cup_{f_2} Y_2$ as follows. By assumption, we have the commutative diagram

$$\begin{array}{ccccc}
 & & Y_1' & \hookrightarrow & X_1 \\
 & \diagup\!\!\diagup & \downarrow & & \downarrow \\
Y_1' & \longrightarrow & Y_2' & \hookrightarrow & X_2 \\
\downarrow & & \downarrow & & \downarrow \\
Y_1 & \longrightarrow & Y_2 & \longrightarrow & X_2 \cup_f Y_2.
\end{array}$$

Then, by the universal property of pushout, we get the morphism $X_1 \cup_f Y_1 \to X_2 \cup_f Y_2$. Since the two constructions are clearly inverses of each other, this proves the first assertion of the theorem.

Next, given X, L in the statement of the theorem, for $R = \bigoplus_0^\infty \Gamma(X, L^{\otimes n})$, we can write $R = k[S]$ for some semigroup $S \subset \mathbb{N} \times \mathbb{Z}^d$. Let $\widetilde{S}$ be the saturation of S. Let $M = k[\widetilde{S}]/k[S]$ be the quotient of $k[S]$-modules. If $M \neq 0$, then, by Lemma 2.5.2, we choose an $\mathbb{N} \times \mathbb{Z}^d$-homogeneous prime ideal $\mathfrak{p}$ of $k[S]$ that is an associated prime of M; then we have $k[S]/\mathfrak{p} \hookrightarrow M$ with image say M_1. Then apply the same argument to M/M_1 and repeat the argument. The ascending chain condition ensures the process stops in a finite steps. In the end, we get the filtration (which is non-unique like any composition series):

$$M = M_r \supset M_{r-1} \supset \cdots \supset M_1 \supset M_0 = \{0\}$$

with $M_i/M_{i-1} \simeq (k[S]/\mathfrak{p}_i)(-l_i)$ as $k[S]$-modules, where we used the notion $R(l)_d = R_{l+d}$ for an $\mathbb{N} \times \mathbb{Z}^d$-graded ring R.

Let $Y_i = V(\mathfrak{p}_i) \subset X$ be the subvarieties. Then, for each $n \gg 0$, we have:

$$\dim k[\widetilde{S}]_n = \dim k[S]_n + \dim M_n,$$

while

$$\dim M_n = \sum_{\mathfrak{p} \in \xi_X} \dim(k[S]/\mathfrak{p})_n.$$

Putting these together we see the asserted identity holds for large $n \gg 0$. Now, let $\pi : X^{nor} \to X$ be the normalization of X. By Proposition 3.2.5 and by the assumption on $\pi^* L$, we have $\dim k[\widetilde{S}]_n = \chi(X^{nor}, \pi^* L^{\otimes n})$; thus, $\dim k[\widetilde{S}]_n$ is a polynomial in n. It follows that the asserted identity holds for every integer $n \geq 0$. □

4.1.7 Corollary. *A non-normality data on a toric variety is independent of the base field (it is not geometric in nature).*

4.1.8 Corollary. *Every projective non-normal toric variety can be constructed from a convex polytope together with a non-normality data on the toric variety associated to the polytope.*

4.1.9 Remark. In the notation of the above proof, we have the disjoint union

$$\widetilde{S} - S = \bigsqcup_{i=1}^{r} (g_i + S \cap F_i)$$

where F_i are the faces of the cone $\mathbb{R}_+ S$ that correspond to the prime ideals appearing in the filtration.

In particular, ignoring a choice in the construction of the filtration, ξ amounts to the set of the pairs $(g_i,\, S \cap F_i)$.

4.1.10 Question (of Kiumars Kaveh)**.** Is there a proof of the fact in the above remark that does not use commutative algebra at all. (Probably yes?)

We give an example illustrating (the affine analog of) the above remark.

4.1.11 Example. Let $\sigma \subset \mathbb{R}^2$ be the cone generated by the two vectors $(1,2),(2,1)$. Let $S_\sigma = \sigma^\vee \cap \mathbb{Z}^2 = \{u \in \mathbb{Z}^2 | u_1 + 2u_2 \geq 0, 2u_1 + u_2 \geq 0\}$. We note that $S_\sigma = \langle (2,-1), (-1,2), (1,0), (0,1) \rangle$ as a semigroup. Let

$$S = \langle (2,-1), (-1,2), (0,1) \rangle$$

be the subsemigroup of S_σ. Note that S generates $\mathbb{Z}^2$ since $(1,0) = (2,-1) + (-1,2) - (0,1)$. Since $\mathbb{R}_+ S = \sigma^\vee$, we thus have $\widetilde{S} = S_\sigma$ and S is not saturated since $(1,0) \notin S$. Let $F = \mathbb{R}_+(2,-1)$, which is a face of $\mathbb{R}_+ S = \sigma^\vee$. Then $S \cap F = \mathbb{N}(2,-1)$ and

$$\widetilde{S} - S = (1,0) + \mathbb{N}(2,-1).$$

(Indeed, if x is in $\widetilde{S} - S$, then we write $x = a(2,-1) + b(-1,2) + c(1,0) + d(0,1)$ with $a,b,c,d \in \mathbb{N}$. Since $(2,0) = (2,-1) + (0,1) \in S$, we can assume $c = 1$. Similarly, we find $b = d = 0$.)

For the later use as well as independent interests, we note the following, which says that, in the one-dimensional case, a non-normal normality data is essentially a genus.[2]

4.1.12 Proposition. *Let X be a non-normal toric variety of dimension one with the non-normality data $\xi = (Y', Y, f)$. Let $g = 1 - \chi(X, \mathcal{O}_X)$ be the arithmetic genus of X. Then*

$$g = \#\xi$$

where the right-hand side means: if Y has the irreducible components Y_i's with multiplicity n_i, then $\#\xi = \sum_i n_i$.

Proof ([Hart77] Ch. IV, Exercise 1.8.) Let $\pi : X^{nor} \to X$ be the normalization of X and $\mathcal{O}_{Y_i,X}$ the local ring of X at Y_i. By construction, we have:

$$0 \to \mathcal{O}_X \to \pi_* \mathcal{O}_{X^{nor}} \to \bigoplus_1^r \widetilde{\mathcal{O}_{Y_i,X}} / \mathcal{O}_{Y_i,X} \to 0$$

where $\widetilde{\cdot}$ means integral closure. Since $\mathrm{H}^i(X, \pi_* \mathcal{O}_{X^{nor}}) = \mathrm{H}^i(X^{nor}, \mathcal{O}_{X^{nor}})$, this gives us:

$$\chi(X^{nor}, \mathcal{O}_{X^{nor}}) = \chi(X, \mathcal{O}_X) + \sum_i \mathrm{length}(\widetilde{\mathcal{O}_{Y_i,X}} / \mathcal{O}_{Y_i,X}).$$

□

[2] Since $\mathbb{P}^1$ is the only projective toric variety of dimension one, the corresponding result for a toric variety is uninteresting.

A sequel to this thesis [Mu2X] will give a more general version of the above.

4.1.13 Question (on Cox's theorem)**.** Is there a way to combine Remark 1.1.4 and non-normality data? (Presumably yes, but the precise formulation is unclear.)

5.0 Part 2: Degeneration given by an ideal filtration

This section introduces some definitions and constructions related to flat degenerations that are used through the rest of the paper (they are not necessarily original).

5.1 Definition of an ideal filtration

For the purpose of this paper, we use the following definition.

5.1.1 Definition. Given an algebraic variety X, a *flat degeneration of* X is a flat morphism of varieties:

$$f : X' \to \mathbb{A}^1$$

together with an isomorphism over $\mathbb{A}^1 - 0$:

$$\begin{array}{ccc} X' - f^{-1}(0) & \xrightarrow{\ \sim\ } & X \times (\mathbb{A}^1 - 0) \\ & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle p_2} & \\ & \mathbb{A}^1 - 0 & \end{array}$$

(in other words, $X' - f^{-1}(0)$ is a trivial bundle[1] over $\mathbb{A}^1 - 0$).

The variety X' is called *the total space of the degeneration* and $f^{-1}(0)$, which need not be reduced or irreducible, is called the *special fiber* or the *zero fiber*.

5.1.2 Remark. Since f is a regular function on X', each fiber $f^{-1}(t)$ of f is a principal effective divisor on X'.

Throughout the paper, we are mostly interested in the projective case of a flat degeneration:

5.1.3 Remark (projective case of a flat degeneration)**.** In the above setup, suppose $X = \operatorname{Proj} R$ is a projective variety and X' is a projective scheme over $\mathbb{A}^1 = \operatorname{Spec}(k[t])$; i.e., X' is the Proj of a flat graded $k[t]$-algebra R' that is an integral domain such that $R'_0 = k[t]$ and R' is finitely generated as an R'_0-algebra.

[1] So, the more natural notion is something that admits flat degenerations locally; the notion that will be considered in the next paper.

Suppose, furthermore, that the trivialization lifts in the sense that it is induced by some isomorphism $R'[t^{-1}] \simeq R[t, t^{-1}]$. We then have the notion of the initial form given as follows: if $f \in R$, then we view it as an element of $R'[t^{-1}]$ via $R \subset R[t, t^{-1}] \simeq R'[t^{-1}]$ and then $t^n f \in R' - tR'$ for some $n > 0$. The *initial form* f^* of f is then the image of $t^n f$ in R'/tR'.

Now, suppose $S := R'/tR'$ is $\mathbb{Z}^r$-graded in addition to the original grading; in other words, the torus $\mathbb{G}_m^r$ acts on it as graded algebra automorphisms. We put the lexicographical ordering on $\mathbb{Z}^r$. Then the $\mathbb{Z}^r$-grading on S induces a filtration $\{I_a\}_{a \in \mathbb{Z}^r}$ of R by

$$I_a = \{f \in R - 0 | \text{ the least } \mathbb{Z}^r\text{-degree of } f^* \text{ is } \geq a\}.$$

The last part of the above remark motivates the following definition (cf. [Ka07], where the term "idealistic filtration" is used and Definition 2.4.14. of [La04]. [Sch85] simply uses the term "filtration".)

5.1.4 Definition. An *ideal filtration* $\mathfrak{v} = \{I_a\}_{a \in \mathbb{Z}^r}$[2] of a ring A is a family of ideals of A such that

(i) $I_0 = A$,

(ii) $I_a I_b \subset I_{a+b}$ for each $a, b \in \mathbb{Z}^r$,

(iii) $I_b \subset I_a$ for each $b > a$ in $\mathbb{Z}^r$ in the lexicographical ordering.

If A is graded, we also require that I_a are homogeneous. The definition also has an obvious analog for ideal shaves (the ideal-sheaf version will be considered in §8).

5.1.5 Remark. Because a Noetherian ring, by definition, cannot have an infinite ascending chain of ideals, a descending chain is more convenient (and explains why we use a decreasing filtration here). Some authors, who are interested in non-Noetherian rings like the coordinate rings of infinite-dimensional "varieties", do however consider an increasing filtration.

A basic example of an ideal filtration is given by powers of ideals: if I is an ideal, then let $I_n = A, n \leq 0$, $I_n = I^n, n > 1$ and then $\{I_n\}_{n \in \mathbb{Z}}$ is an ideal filtration. We usually denote this ideal filtration by I^∞. Here is a very typical example not given by powers of ideals.

[2] $\mathfrak{v}$ is mathfrak v; the notation is because the concept is equivalent to that of a quasi-valuation.

5.1.6 Example. Let X be a smooth projective variety, L an ample line bundle on X and $D \subset X$ an effective divisor. Setting

$$I_n = \bigoplus_{l=0}^{\infty} \Gamma(X, L^{\otimes l} \otimes \mathcal{O}_X(-nD)),$$

$\{I_n\}_{n\in\mathbb{Z}}$ is an ideal filtration of the section ring $R = \bigoplus_{l=0}^{\infty} \Gamma(X, L^{\otimes l})$; here, I_n are ideals of R through $\mathcal{O}_X(-nD) \subset \mathcal{O}_X$.

5.1.7 Remark (fractional ideal filtration)**.** The preceding example suggests there is the notion of a *fractional ideal filtration*; namely, when A is an integral domain, in Definition 5.1.4, we require I_a to be a fractional ideal of A instead of an ideal. Then, by the same manner in the above example, a not-necessarily-effective divisor gives rise to a fractional ideal filtration. This notion is not considered here as not needed.

5.1.8 Example. The base loci of a linear series, which is naturally a filtration of closed subschemes, gives another example of an ideal filtration; see [La04] Ch. 2, §4.

5.1.9 Example. The irrelevant ideal of a graded semigroup algebra has a natural structure of an ideal filtration. Indeed, let $R = k[S]$ be a graded semigroup algebra, not necessarily finitely generated, for some subsemigroup $S \subset \mathbb{N} \times \mathbb{Z}^r$. Then the irrelevant $I = R_+$ has a natural structure of an ideal filtration; namely,

$$I_a = k[S]_a := \oplus_{l=0}^{\infty} k \cdot \mathbf{1}_S(l, a)\chi^{(l,a)},$$

where $\mathbf{1}_S$ is the indicator function of S.

5.1.10 Remark (a role of an ideal filtration as generalizing a closed subscheme)**.** Given an ring A and an ideal filtration $\mathfrak{v} = \{I_a\}_{a\in\mathbb{Z}}$ on A, it easy to see that

$$\sqrt{I_a} = \sqrt{I_b}$$

for every $a, b > 0$ and the common radical ideal is called the *radical of the ideal filtration* $\mathfrak{v}$. In particular, $V(I_a) \subset X = \mathrm{Spec}(A)$ all have the same underlying set for $a > 0$ and I_a all share the same set of minimal prime ideals over it; i.e., the set of set-theoretic irreducible components of $V(I_a)$; cf Remark 5.3.7.

For each a in $\mathbb{Z}^r$, let $I_{>a} = \cup_{b>a} I_b$, which is an ideal and then let

$$\operatorname{gr}_{\mathfrak{v}} R := \bigoplus_{a \in \mathbb{Z}^r} I_a / I_{>a},$$

which is called the (generalized) *normal-cone ring along* $\mathfrak{v}$ and the Spec of it is the normal cone along $V(I)$. It comes with the natural action of the torus $\mathbb{G}_m^r$.

5.1.11 Remark (ideal filtration = $\mathbb{Z}^r$-graded ring with constant negative terms)**.** Let $\mathfrak{v}$ be an ideal filtration on a ring A and then the ring

$$A' = \bigoplus_{a \in \mathbb{Z}^r} I_a$$

is called the *Rees algebra of* $\mathfrak{v}$; it is a $\mathbb{Z}^r$-graded ring (which is positive in the sense that all negative components are A).

Conversely, suppose we are given a $\mathbb{Z}^r$-graded ring A' with the property $A'_a = A_0$ for every $a \le 0$ ($\mathbb{Z}^r$ is given the lexicographical ordering). Then it can be written like the above; namely, we set $I_a = A_a$ for each a. Then $\{I_a\}_{a \in \mathbb{Z}}$ is an ideal filtration of A.[3] Hence, *to give an ideal filtration is the same as to give a multi-graded ring with constant negative terms.* In fact, since the consideration of a generalized Rees algebra amounts to doing a generalized blow-up (cf. below), giving an ideal filtration is equivalent to doing a blow-up in some generalized sense. This is among the reasons why the notion of an ideal filtration has come to being in the first place (cf. [Ka07]).

Because of the remark below, we can also think of an ideal filtration as an ideal-theoretic generalization of a valuation.

5.1.12 Remark (ideal filtration = quasi-valuation)**.** Let A be an integral domain. A *quasi-valuation* $\mu : A - 0 \to \mathbb{Z}^r$ on A is a function such that for any $0 \neq f, g \in A$,

(i) $\mu(fg) \ge \mu(f) + \mu(g)$.

(ii) $\mu(f + g) \ge \min\{\mu(f), \mu(g)\}$.

[3]Proof: $AI_a = A_0 A_a \subset A_a = I_a$ and so I_a is an A-module. Since $I_a \subset I_a A = I_a A_{-a} \subset A_0 = A$, I_a is an A-submodule of A; i.e., an ideal. Similarly, for $b > a$, $I_b \subset I_b A_{a-b} \subset I_a$ and the multiplicative property is trivial.

Moreover, the quasi-valuation μ is called a *valuation* if the inequality in (i) is the equality.

Each quasi-valuation determines an ideal filtration by setting

$$I_a = \{f \in A | f = 0 \text{ or } \mu(f) \geq a\},$$

that has the property $\bigcap_{a>0} I_a = 0$. Conversely, given an ideal filtration $\mathfrak{v}$ with $\bigcap_{a>0} I_a = 0$, we can define the associated quasi-valuation ν by

$$\mu(f) = \sup\{a \in \mathbb{Z}^r | f \in I_a\}.$$

Among the fundamental questions on an ideal filtration is finiteness.

5.1.13 Definition (finite type)**.** An ideal filtration $\mathfrak{v}$ on a ring A is said to be *of finite type over* A if the Rees algebra $\bigoplus_{a\in\mathbb{Z}^r} I_a$ of it is finitely generated as an A-algebra. For simplicity, "over A" is frequently dropped.

Note for such $\mathfrak{v}$, we have that, with $I_{>a} = \cup_{b>a} I_b$, the normal-cone ring

$$\operatorname{gr}_{\mathfrak{v}} A := \bigoplus_{a\geq 0} I_a/I_{>a}.$$

is finitely generated as an $A/I_{>0}$-algebra.

We note that "of finite type" is equivalent to "Noetherian" in the following sense:

5.1.14 Proposition. *Let A be a Noetherian ring. Then an ideal filtration $\mathfrak{v} = \{I_a\}$ on A is of finite type $\Leftrightarrow \bigoplus_{a\in\mathbb{Z}^r} I_a$ is a Noetherian ring.*

Proof. ($\Rightarrow$) $\bigoplus I_a$ is a finitely generated algebra over the Noetherian ring A; thus is a Noetherian ring.

($\Leftarrow$) Let $A' = \bigoplus I_a t^{-a}$. Since $A' \simeq \bigoplus I_a$ is a Noetherian ring, the ideal $A'_+ = \bigoplus_{a>0} I_a t^{-a}$ of A' is finitely generated; say, by $g_1 t^{-b_1}, \dots, g_m t^{-b_m}$. If $a > \max\{b_1, \dots, b_m\}$ and $f \in A$, we can write $ft^{-a} = \sum u_i g_i t^{-b_i}$ with u_i in A'. Removing the terms that get cancelled, we can assume $u_i = h_i t^{b_i - a}$ with h_i in A. Thus, by induction on $\mathbb{Z}^r$, we see A' is finitely generated in large degrees and thus is finitely generated. □

The next proposition generalizes the key property of an (ordinary) normal cone.

5.1.15 Proposition (cf. [Ful93] Appendix B.6.6.)**.** *Let* $X = \mathrm{Spec}(A)$ *be an affine variety and* $W \subset X$ *a closed subset without a scheme structure. Let* $\mathfrak{v} = \{I_n\}_{n\in\mathbb{N}}$ *be an ideal filtration of* A *of finite type such that, as a set,* $W = V(\mathfrak{v})$ *and we think* $\mathfrak{v}$ *is a sort of generalized scheme structure on* W*. We write* $W_\mathfrak{v}$ *for* W *together with* $\mathfrak{v}$*. Then the normal cone along* $W_\mathfrak{v}$*:*

$$C_{W_\mathfrak{v}/X} = \mathrm{Spec}\left(\bigoplus_{n\geq 0} I_n/I_{n+1}\right)$$

is of pure dimension; i.e., each irreducible component has the same dimension as the others.

Proof. Consider the generalized extended Rees algebra $B = \bigoplus_{n\in\mathbb{Z}} t^{-n} I_n$ where $I_n = A$ if $n < 0$. Then tB is the defining ideal of $C_{W/X}$ in $\mathfrak{X} = \mathrm{Spec}(B)$. Since B is a Noetherian ring, by Krull's principal ideal theorem, each minimal prime ideal over tB has height one. By the ideal correspondence, each minimal prime ideal of B/tB has the same height. □

5.1.16 Remark (normal ideal filtration)**.** In the proof, suppose B is integrally closed; in that case, we call $\mathfrak{v}$ is *normal*. Then the same proof shows $C_{W_\mathfrak{v}/X}$ has no embedded component; i.e., it is of pure dimension in the strong sense.

5.2 Conversion to a one-parameter Rees algebra

For us, the notion of an ideal filtration is relevant because of the following construction (Theorem 5.2.1) due to Rees, P. Caldero, Brion–Alexeev and D. Anderson.

5.2.1 Theorem (existence of a one-parameter Rees algebra)**.** *Let* A *be a Noetherian ring and* $\mathfrak{v}$ *an ideal filtration of finite type. Then there is a finitely generated* A*-algebra* A' *together with a non-zero-divisor* t *in* A' *such that*

(i) $A'/tA' = \mathrm{gr}_\mathfrak{v} A$,

(ii) There is a natural ring isomorphism $A'[t^{-1}] \simeq A[t, t^{-1}]$ *that commutes with* t*.*

Proof. Let $f_1, \dots, f_p$ be the generators of A such that $f_i \in I_{a_i}$ and the classes $f_i^* \in I_{a_i}/I_{>a_i}$ generate $\mathrm{gr}_\mathfrak{v} A$. The key part of the proof is to choose a linear functional $l : \mathbb{Q}^r \to \mathbb{Q}$. Given such an l, for each integer $n \geq 0$, we let

$$A_{\geq n} = \sum_{l(m_1 a_1 + \cdots + m_p a_p) \geq n} A f_1^{m_1} \dots f_p^{m_p}$$

and then form the ring

$$A' = A[t] \oplus t^{-1} A_{\geq 1} \oplus t^{-2} A_{\geq 2} \oplus t^{-3} A_{\geq 3} \oplus \cdots ,$$

called an extended Rees algebra of A. Since the n-th degree piece of tA' is $t^{-n} A_{\geq n+1}$, we have $A'/tA' = \bigoplus_{n=0}^{\infty} A_{\geq n}/A_{\geq n+1}$. Since $A' \subset A[t, t^{-1}]$, A' has the required property (ii) and we will choose l so that A' also has the property (i).

To choose l, let $S = \mathbb{Z}[x_1, \dots, x_p]$ be a polynomial ring equipped with the $\mathbb{Q}^r$-grading such that $\deg(x_i) = a_i$. If $g \in S$, by the *degree* of g, for the purpose of this proof, we shall mean the *least* degree of the homogeneous components of g. We have the surjective homomorphism

$$S \to \operatorname{gr}_{\mathfrak{v}} A$$

given by $x_i \mapsto f_i^*$. Likewise, with the definition of $A_{\geq *}$, we have the surjection: $S \to \oplus_{n=0}^{\infty} A_{\geq n}/A_{\geq n+1}$ given by $x_i \mapsto f_i^* \in A_{l(a_i)}/A_{l(a_i)+1}$. Now, for any choice of $l \in (\mathbb{Q}^r)^*$, we have:

$$\ker(S \to \oplus_{n=0}^{\infty} A_{\geq n}/A_{\geq n+1}) \subset \ker(S \to \operatorname{gr}_{\mathfrak{v}} A).$$

Indeed, if g is in the kernel on the left, then for $a = \deg(g)$, that means $g(f_1, \dots, f_p)$ is in $A_{\geq l(a)+1}$. Then $g(f_1, \dots, f_p) \in I_{>a}$; since otherwise we have $l(g(f_1, \dots, f_p)) = l(a)$.

Now, we want to choose l so that the above inclusion is the equality. For that, choose homogeneous generators $g_1, \dots, g_q$ of the kernel of $S \to \operatorname{gr}_F A$. Then we can find polynomials h_i in S such that $g_i(f_1, \dots, f_p) = h_i(f_1, \dots, f_p)$ and $\deg(g_i) < \deg(h_i)$. We then define the linear functional $l : \mathbb{Q}^r \to \mathbb{Q}$ by

$$l = \sum_{j=1}^{r} e_j^* N^{r-j}$$

for the standard dual basis e_j^* of $(\mathbb{Q}^r)^*$ and some integer N chosen as follows. Since $\deg(h_i) > \deg(g_i)$ in the lexicographical ordering, for large enough N we have $l(\deg(g_i)) < l(\deg(h_i))$. Since g_i is in the kernel of $S \to \oplus_{n=0}^{\infty} A_{\geq n}/A_{\geq n+1}$, we are done. $\square$

It is worth formulating the above theorem in the following form:

5.2.2 Corollary. *Given a projective variety* $X = \operatorname{Proj} R$ *(with a choice of* R*), there is a one-to-one correspondence between the toric degenerations of* X *(in the appropriate sense*[4]*) and the ideal filtrations of* R *whose associated graded rings are finitely generated semigroups.*

5.2.3 Example (A toric scheme as a toric degeneration)**.** A toric scheme, say over some base scheme S, can be thought of as a special case of a toric degeneration over S if, by *toric degeneration*,[5] we mean something quite general: e.g., a family of varieties/schemes over S whose a distinguished fiber is a non-normal toric variety (=not-necessarily-normal toric variety). The present thesis may be thought of as an attempt to working out the very special situation $S = \mathbb{A}^1$ in details.

We have the notion of an integral closure of an ideal filtration; the terminology is justified by Proposition 5.3.1 below.

5.2.4 Definition (integral closure of an ideal filtration)**.** Given an ideal filtration $\mathfrak{v}$ on an integral domain A such that $\bigcap_{a>0} I_a = 0$ and μ associated quasi-valuation (Remark 5.1.12), for each $f \in A - 0$, we let

$$\overline{\mu}(f) = \lim_{n\to\infty} \mu(f^n)/n.$$

Samuel proved (at least when $r = 1$) that the limit exists, possibly as infinity, and so $\overline{\nu}(f)$ is a well-defined point of $\mathbb{R}^r$ unless infinite. Let $\overline{\mathfrak{v}}$ be the associated ideal filtration:

$$\overline{I_a} = \{f \in A | f = 0 \text{ or } \overline{\mu}(f) \geq a\}$$

for each $a \in \mathbb{Z}^r$. It is called the *integral closure of* $\mathfrak{v}$; or perhaps more precisely the *integral part of the integral closure* of $\mathfrak{v}$.

[4]As this corollary is not needed later, we skip giving the precise meaning

[5]There is also a termonolgical advantage that we can avoid an somehow awkward term "not-necessarily-normal toric variety".

5.3 Integral closure of an ideal filtration

The next result relates the integral closure in the usual sense with Definition 5.3.1.

5.3.1 Proposition. *Let A be an integral domain (the proof goes through without "Noetherian"). For each ideal filtration $\mathfrak{v}$ on A, the Rees algebra $\bigoplus_{a\in\mathbb{Z}^r}\overline{I_a}$ along it is the integral closure of $\bigoplus_{a\in\mathbb{Z}^r} I_a$ in $\bigoplus_{a\in\mathbb{Z}^r} A \simeq A[t,t^{-1}]$.*

Proof. (The first part of the proof follows Kawanoue.) Let f be in the a-th component of $\bigoplus_{a\in\mathbb{Z}^r} A$ and suppose it is integral over $\bigoplus I_a$; i.e., we can write

$$f^n + g_1 f^{n-1} + \cdots + g_n = 0,\; g_i \in I_{ia}.$$

We write $\overline{\mu}$ for the quasi-valuation associated to $\overline{\mathfrak{v}}$ (Remark 5.1.12); i.e., $\overline{\mu}(f) = \sup\{a \in \mathbb{Z}^r | f \in \overline{I_a}\}$. Then the above implies that

$$\overline{\mu}(f^n) \geq \min_{1\leq i\leq n}\{\mu(g_i) + (n-i)\mu(f)\} \geq \min_{1\leq i\leq n}\{ia + (n-i)\mu(f)\}.$$

Set $\lambda_j = 1 - \left(\frac{n-1}{n}\right)^j, j \geq 0$. If, inductively, $\mu(f) \geq \lambda_j a$, then

$$\mu(f^n) \geq (1 + (n-1)\lambda_j)a = n\lambda_{j+1}a.$$

It easily follows from the definition that $\overline{\mu}$ is homogeneous; i.e., $\overline{\mu}(f^n) = n\overline{\mu}(f)$ and so we get $\overline{\mu}(f) \geq \lambda_{j+1}a$. Since $\lambda_j \to 1$ as $j \to \infty$, $\overline{\mu}(f) \geq a$; i.e., $f \in \overline{I_a}$. Since the integral closure C of $\bigoplus I_a$ is a graded subring of $\bigoplus_{a\in\mathbb{Z}^r} A$, we conclude that $\bigoplus_{a\in\mathbb{Z}^r}\overline{I_a}$ contains C.

For the opposite inclusion, fix $a \in \mathbb{Z}^r$ and let $A' = A[t] \oplus I_a t^{-1} \oplus I_{2a}t^{-2} \oplus \cdots$. Then $I_{na} = t^n A' \cap A$ and it is enough to show that $\overline{I_a}$ is contained in the integral closure of A'. Thus, without loss of generality, we can assume I_a is principal. Now, let f be in $\overline{I_a}$; i.e., $\overline{\mu}(f) \geq a$. Given an integer $k > 0$, we can find n_0 such that for all $n \geq n_0$,

$$\mu(f^n)/n \geq a(1 - 1/k);$$

i.e., f^n is in $I_{na(1-1/k)}$ if $n \geq n_0$ and is divisible by k. Now, we recall the fact that the integral closure C is the intersection $\bigcap_V (V \cap A[t,t^{-1}])$ over all the valuation rings V containing

$\bigoplus I_a$. Fix V and write ν for the corresponding valuation. For each ideal J of A, let $\nu(J) = \inf\{\nu(x) | 0 \neq x \in J\}$. Then we see $\nu(J^m) = m\nu(J)$ for any integer $m > 0$ and we get

$$\nu(f^n) \geq \nu(I_{na(1-1/k)}) = n(1 - 1/k)\nu(I_a).$$

Or $\nu(f) \geq (1 - 1/k)\nu(I_a)$. Letting $k \to \infty$, we see $\nu(f) \geq \nu(I_a)$. Varying ν, we conclude f is in C. □

5.3.2 Corollary. *Assume, in addition, that A is a finitely generated algebra over a field. Then an ideal filtration $\mathfrak{v}$ is of finite type if and only if its integral closure $\overline{\mathfrak{v}}$ is of finite type.*

Proof. This follows from Noether's finiteness theorem and its converse (Theorem 1.3.4). □

5.3.3 Remark ([Kn05] Knutson's balanced normal cone)**.** Let $\mathfrak{v}$ be the ideal filtration given by powers of ideals: $I_n = I^n$. Then Knutson calls the normal cone along $\overline{\mathfrak{v}}$ the *balanced normal cone*. Some of the results of the present and two subsequent sections were motivated by this paper of Knutson.

The above corollary is generally false if A is merely a Noetherian integral domain; see, for example, §5 of [HGN90]. In §1.1. of [Kn05], Knutson cites Theorem 4.21. of [Re] to claim the finite generation of a balanced normal cone for an arbitrary Noetherian ring. This is problematic since the cited theorem only gives a point-wise bound for $\overline{q} - q$. Thus, this corollary takes care of this issue and is the (personal) reason why we wanted to record it here.

5.3.4 Remark (Rees decomposition)**.** One of the main results of [Re] uses the $\mathbb{Q}$-version of integral closure of an ideal filtration to give a decomposition of an ideal filtration (called a quasi-valuation there) in terms of valuations. The interested readers are referred to [Re] Ch 4. (In the thesis, we don't really use valuations; whence, the omission of the discussion).

The significance of having the notion of an ideal filtration is that it allows us to develop a generalization of the classical ideal theory; in particular, [Re] does just that in a systematic way. (The resulting theory has a big implication to multiplicity theory and, as we wish to hint in the present thesis, an implication to intersection theory.)

Among the instances of this generalization is a primary decomposition of integral closures of ideals, as done in Remark 5.3.7 below. We first recall the key features of a primary decomposition of an ideal.

5.3.5 Proposition. *Let A be a ring and I an ideal.*

(i) If $\mathfrak{p}$ is a minimal prime ideal over I, then the pre-image of $IA_{\mathfrak{p}}$ under the localization map $\phi_{\mathfrak{p}} : A \to A_{\mathfrak{p}}$ is the smallest $\mathfrak{p}$-primary ideal containing I.
In particular, if I is $\mathfrak{p}$-primary, then $I = {\phi_{\mathfrak{p}}}^{-1}(IA_{\mathfrak{p}})$.

(ii) Suppose there is a minimal primary decomposition $I = \bigcap_i Q_i$. If $\mathfrak{p} = \sqrt{Q_i}$ is a minimal prime ideal over I, then $Q_i = \phi_{\mathfrak{p}}^{-1}(IA_{\mathfrak{p}})$ and is called the $\mathfrak{p}$-primary component of I.

Proof. (i) Replacing A by A/I we can assume $I = 0$. Let $K = \ker(A \to A_{\mathfrak{p}})$. Since $\mathfrak{p}$ is a minimal prime ideal, $A_{\mathfrak{p}}$ has only one prime ideal; thus, every non-unit there is nilpotent. In particular, every zerodivisor in the subring $A/K \subset A_{\mathfrak{p}}$ is nilpotent and thus K is a primary ideal. Next, let Q be a $\mathfrak{p}$-primary ideal. If $x \in K$, then $sx = 0$ for some $s \in A - \mathfrak{p}$. Then $sx \in Q$ and so $x \in Q$ since $s \notin \mathfrak{p}$. Thus, $K \subset Q$.

(ii) This is essentially the second uniqueness theorem of a primary decomposition ([AM94] Theorem 4.10.). It follows from (i). □

We recall that a ring homomorphism $A \to B$ is said to be a *pure subring* if the natural map $N \to N \otimes_A B$ is an injection for every A-module N. For example, a faithfully flat ring homomorphism is precisely a flat pure subring. Also, if A is a direct summand of B, then $A \to B$ is a pure subring.

5.3.6 Lemma. *Let $A \to R$ be a pure subring (see the above paragraph) and $I \subset A$ an ideal. Then*

$$\overline{IR} \cap A = \overline{I}$$

where the bar $\bar{\cdot}$ means the integral closure of an ideal (recall: an element x is an integral over I if, for some $n > 0$, $x^n \in x^{n-1}I + x^{n-2}I^2 + \cdots + I^n$).

Proof. Let x be in $\overline{IR} \cap A$. For any ideal J of A, since $A \to R$ is a pure subring, we get $A/J \hookrightarrow A/J \otimes_A R = R/JR$; i.e., $J = JR \cap A$. For any ideals $\mathfrak{a}$ and $\mathfrak{b}$ of A, we have $(\mathfrak{a} + \mathfrak{b})R = \mathfrak{a}R + \mathfrak{b}R$. Thus, for some $n > 0$,

$$x^n \in (x^{n-1}IR + \cdots + I^n R) \cap A = (x^{n-1}I + \cdots + I^n)R \cap A = x^{n-1}I + \cdots + I^n.$$

Hence, $\overline{IR} \cap A \subset \overline{I}$. The opposite inclusion is trivial. □

5.3.7 Remark (primary decomposition)**.** Let A be either the coordinate ring of a normal[6] affine variety or a localization of it, $\mathfrak{v} : I_1 \supset I_2 \supset \cdots$ an ideal filtration of A and $A' = A[t] \oplus I_1 t^{-1} \oplus I_2 t^{-2} \oplus \cdots$ the generalized extended Rees algebra associated to $\mathfrak{v}$. Let B be the integral closure of A'.[7] In general, we know that $\mathrm{Ass}(A/x^n A)$ is independent of n for any Noetherian ring A and a nonzerodivisor x.[8] Also, by Serre's criterion for normality, the set $\mathrm{Ass}(B/tB)$ has no non-minimal element (embedded prime). Thus, for each $n > 0$, we have the unique primary decomposition: $t^n B = \bigcap_{\mathfrak{q} \in V_0(tB)} t^n B_{\mathfrak{q}} \cap B$, where $V_0(\mathfrak{a})$ denotes the set of minimal elements of $V(\mathfrak{a})$. Since a nonzero principal ideal in an integrally closed domain is integrally closed, since $t^n B = \overline{I_n B}$ and $\overline{I_n} = t^n B \cap A$ (Lemma 5.3.6), we then get the possibly redundant primary decomposition:

$$\overline{I_n} = \bigcap_{\mathfrak{q} \in V_0(tB)} t^n B_{\mathfrak{q}} \cap A.$$

That each intersection-factor in the right-hand side forms an ideal filtration follows from the basic properties of an ideal filtration recorded in the lemma below.

5.3.8 Lemma (basic properties of an ideal filtration)**.** *Let A be a ring and $\mathfrak{v}$ an ideal filtration on it. Then the following hold:*

(i) For any subring $B \subset A$, $\mathfrak{v} \cap B = \{I_a \cap B\}_{a \in \mathbb{Z}^r}$ is an ideal filration.

(ii) Given a ring homomorphism $\varphi : A \to B$, $\varphi(\mathfrak{v})B = \{\varphi(I_a)B\}_{a \in \mathbb{Z}_r}$ is an ideal filtration on B. In particular, for any multiplicatively closed subset $S \subset A$, $\mathfrak{v}A[S^{-1}]$ is an ideal filtration.

(iii) In the notion of (ii), if $\mathfrak{v}$ is of finite type, then $\varphi(\mathfrak{v})B$ is of finite type.

Proof. (i), (ii) are immediate. (iii) is also straightforward ([Re] Ch 2. §4.). □

The above (i), (ii) can be combined as in:

[6]The "normal" assumption here is to simplify the discussion and can be weakened.
[7]B "can" be called the normalized generalized extended Rees algebra associated to the ideal filtration $\mathfrak{v}$.
[8]Consider $0 \to A/x^n \to A/x^{n+1} \to A/x \to 0$ given in the proof of [Ful98, Lemma A.2.5.].

5.3.9 Example (symbolic power of a prime ideal)**.** Let $\mathfrak{p}^\infty = \{\mathfrak{p}^n\}_{n>0}$ be an ideal filtration given by a power of a prime ideal $\mathfrak{p}$ in a ring A. Then (i), (ii) of the above lemma say that $\mathfrak{p}^{(\infty)} = \{\mathfrak{p}^n A_\mathfrak{p} \cap A\}_{n>0}$ is an ideal filtration, which we sometimes called the *symbolic filtration*; the next section will go deep in the study of this filtration.

5.3.10 Remark (Rees algebra as a base change: an affine blow-up)**.** To visualize Remark 5.3.7, it is useful to have the following view in mind: for an ideal I of a ring A and $X = \mathrm{Spec}(A)$, $V(tA')$ is the fiber product of $V(I) \hookrightarrow X$ and $X' \to X$, where X' is the Spec of $A' = A[t] \oplus It^{-1} \oplus I^2t^{-2} \oplus \cdots$. Since a possibly non-principal ideal (i.e., I) is replaced by a principal ideal (i.e., tA'), this base change is sometimes referred to as an *affine blow-up along I.* Thus, if we used an ideal filtration instead of an ideal, it is a sort of an affine blow-up along a filtration, the key technique used in [Re] throughout.

5.3.11 Lemma (reduction = integral)**.** *Let A be a Noetherian ring and $J \subset I$ ideals. Then the following are equivalent*

- *(i) The blow-up algebra $\bigoplus_0^\infty I^n$ is finite over the *graded subring**[9] $\bigoplus_0^\infty J^n$. *(Note that a Veronese subring is *not* a graded subring).*
- *(ii) For some $m > 0$, $JI^m = I^{m+1}$.*
- *(iii) I is integral over J (cf. Lemma 5.3.6).*

When the above equivalent conditions hold, J is then said to be a reduction of I.

Proof. (i) $\Leftrightarrow$ (ii): Let $b_IA = \bigoplus_0^\infty I^n$ and similarly for b_JA. We recall the following from the standard proof of the Artin–Rees lemma: given a decreasing filtration of an A-module $M = M_0 \supset M_1 \supset \cdots$, we say M_* is an I-stable filtration if $IM_n = M_{n+1}$ for sufficiently large $n > 0$. A key observation was that the filtration M_* is an I-stable filtration if and only if $b_IM = \bigoplus_0^\infty M_n$ is a finite module over b_IA. Then here, with $M_n = I^n$, we have: $b_IA = b_JM$ is finite over b_JA if and only if M_n is J-stable; i.e., $JI^n = I^{n+1}$ for sufficiently large n.

(ii) $\Rightarrow$ (iii): Let x_i be the generators of I. Since x_i is integral over J, we can choose large enough $m > 0$ such that $x_i^m \in x_i^{m-1}J + \cdots + J^m$ for each i. Increasing m, that implies $I^m \subset I^{m-1}J$, which must be the equality.

[9] Here, "graded subring" means that $\bigoplus_0^\infty I^n$ is a graded module over a graded ring $\bigoplus_0^\infty J^n$.

(iii) ⇒ (ii): For each x in I, we have $xI^m \subset JI^m$, which, by the determinant trick, implies that x is integral over J. $\square$

5.3.12 Remark ([HS06] Exercise 8.12)**.** Note that if I is a reduction of J, then $\sqrt{I} = \sqrt{J}$ since $J^{m+1} = IJ^m \subset I \subset J$.

On the other hand, it need not be the case that an ideal I is a reduction of its radical $\sqrt{I}$. For example, if I^2 is a reduction of I, then $I^2 I^m = I^{m+1}$ for some $m > 0$ and that would be a nonsense if I is a radical ideal generated by a nonzerodiviosr.

It is true that $\bigoplus_0^\infty I^n$ is finite over $\bigoplus_0^\infty I^{2n}$ (e.g., by Lemma 5.4.1 (ii) below). But in that case the grading of $\bigoplus_0^\infty I^{2n}$ is such that I^2 has degree two not one.

5.4 Some results from the asymptotic ideal theory

For the remainder of the section, we collect some results in the asymptotic ideal theory for the use in the next two sections. For the sake of self-containedness and the convenience of non-expert readers in this theory, we will often give proofs.

The next lemma is standard.

5.4.1 Lemma. *Let R be a graded ring such that R_0 is a Noetherian ring. Then*

(i) If R is Noetherian, then some Veronese subring

$$R^{[m]} = \bigoplus_{n=0}^{\infty} R_{nm}$$

is generated as an R_0-algebra by a finite number of degree-one elements. Explicitly, if $x_1, \dots, x_r$ are homogeneous generators of R, then one can take $m = rc$ where c is the least common multiple of $\deg(x_i)$'s.

(ii) Assume either (1) R is an integral domain or (2) the degree components of R are decreasing: $R_n \supset R_{n+1}$. If, for some integer $m > 0$, the Veronese subring $R^{[m]}$ is a Noetherian ring, then R is finite over $R^{[m]}$.

Proof. (i) Let $x_1, \dots, x_r$ be the homogeneous generators of R of degrees d_i, c the least common multiple of d_i's and set $m = rc$. For each i, let e_i be a (positive) integer such that $c = e_i d_i$. We note that R_n is spanned by the monomials $x_1^{n_1} \dots x_r^{n_r}$ such that $\sum n_i d_i = n$. Take $n = lm$. Since $lm = \sum n_i d_i \geq m = rc = \sum e_i d_i$, for some i_1, $n_{i_1} \geq e_{i_1}$ and then we factor out $x_{i_1}^{e_{i_1}}$. Note the degree of the remaining monomial is then $lm - c = (lr-1)c$. Iterating this procedure r times, we get:

$$x_1^{n_1} \dots x_r^{n_r} = x_{i_1}^{e_{i_1}} \cdots x_{i_r}^{e_{i_r}} y$$

where y is a monomial of degree $lm - rc = (l-1)m$. Hence, we are done by induction.

We show (ii). First assume the condition (1). We write $R = \bigoplus_{j=0}^{m-1} M_j$ where $M_j = \bigoplus_{n=0}^{\infty} R_{nm+j}$. If $M_j \neq 0$, then it contains a nonzero homogeneous element x. Then the multiplication $x^{m-1} : M_j \to R^{[m]}$ is well-defined and injective since A is an integral domain. It is also a homomorphism of $R^{[m]}$-modules. It follows that R is a finitely generated $R^{[m]}$-module. The proof under the assumption (2) is similar. □

5.4.2 Definition (analytic spread; cf. [Ei04] Exercise 12.5.)**.** Given a Noetherian local ring $(A, \mathfrak{m})$ and an ideal I, consider the following graded ring (called the fiber cone ring):

$$A/\mathfrak{m} \oplus I/\mathfrak{m}I \oplus I^2/\mathfrak{m}I^2 \oplus I^3/\mathfrak{m}I^3 \oplus \cdots .$$

By Nakayama's lemma, for each $n > 0$, the dimension of $I^n/\mathfrak{m}I^n$ as a vector space over $k = A/\mathfrak{m}$ is the minimum number of the generators of I^n. Also, since I is finitely generated, $\dim_k I^n/\mathfrak{m}I^n$ is a polynomial in n large, the Hilbert polynomial.

By definition, the *analytic spread* $\mathsf{an}(I)$ *of* I is one plus the degree of this Hilbert polynomial or, equivalently, the Krull dimension of $\bigoplus_0^\infty I^n/\mathfrak{m}I^n$. See also Lemma 5.4.5 below for another characterization. We note that $\mathsf{an}(I)$ is bounded above by the minimum number of the generators of I.

Note also that $\mathsf{an}(I) = \mathsf{an}(I^r)$ for each $r > 0$ by Lemma 5.4.1 (i).

The next proposition is among the main results in asymptotic theory of prime divisors (recall that a *prime divisor* of an ideal I of a Noetherian ring R is an associated prime of the R-module R/I.)

5.4.3 Proposition. *Let A be either the coordinate ring of an affine variety over a field or a localization of such a ring (in particular, A is an integral domain) and $I, \mathfrak{p}$ an ideal and a prime ideal of A.*

If $\mathfrak{p}$ is a prime divisor of the integral closure $\overline{I^n}$, then it is a prime divisor of $\overline{I^r}$ for every $r \geq n$ as well as a prime divisor of I^r for every $r \gg 0$.

Moreover, the following are equivalent.

(i) *$\mathfrak{p}$ is a prime divisor of $\overline{I^n}$ for $n \gg 0$.*

(ii) *There exists a prime divisor $\mathfrak{p}'$ of $t\overline{A'}$ such that $\mathfrak{p} = \mathfrak{p}' \cap A$, where $\overline{A'}$ denotes the integral closure of $A' = A[t, t^{-1}I]$.*[10]

(iii) *(S. McAdam) $\operatorname{ht}(\mathfrak{p}) = \mathsf{an}(IA_{\mathfrak{p}})$, where an refers to analytic spread (Definition 5.4.2).*

If I is nonzero and principal, then the above conditions are equivalent to

(iv) *$\mathfrak{p}$ has height one.*

Proof. For the equivalence among (i) - (iii), see [Mc83] Proposition 3.18. and Proposition 4.1. The equivalence (i) $\Leftrightarrow$ (iv) is due to [Mc83] Lemma 3.14.

For the first part, from [Mc83, ?], we know $\operatorname{Ass}(A/\overline{I^r})$ is increasing in r, giving us the first item.

For the second item, if $\mathfrak{p}$ is a prime divisor of $\overline{I^n}$, then by the condition (ii), we have $\mathfrak{p} = \mathfrak{q} \cap A$ for some prime divisor $\mathfrak{q}$ of $t\overline{A'}$. By Nagata's altitude formula, $\mathfrak{q} \cap A'$ has height one and thus $\mathfrak{q} \cap A'$ is a prime divisor of tA'; it follows that $\mathfrak{q} \cap A$ is a prime divisor of I^r. $\square$

5.4.4 Corollary ([Ra74] Theorem 2.12.)**.** *If I is generated by a $\operatorname{ht}(I)$ number of elements, then, for every integer $n > 0$, $\overline{I^n}$ is unmixed in height; i.e., every prime divisor of $\overline{I^n}$ has the same height as the others.*

Proof. If $\mathfrak{p}$ is a prime divisor of $\overline{I^n}$, then it is a prime divisor of $\overline{I^r}, r \gg 0$. Since $IA_{\mathfrak{p}}$ is generated by $h = \operatorname{ht}(I)$ elements by assumption, we have: $\operatorname{ht}(\mathfrak{p}) = \mathsf{an}(I^r A_{\mathfrak{p}}) \leq h$. On the other hand, by definition, h is the minimum of the heights of prime ideals containing I; thus, $\operatorname{ht}(\mathfrak{p}) \geq h$. $\square$

[10] We know $\operatorname{Ass}(R/x^n R)$ is independent of n; see Remark 5.3.7. So, the item (ii) here is equivalent if t is replaced by some power of t.

5.4.5 Lemma. *Let $(A, \mathfrak{m})$ be a Noetherian local ring such that $A/\mathfrak{m}$ is infinite and I an ideal. Then the analytic spread of I is the minimal number of elements generating an ideal over which I is integral.*

Proof. Let $q = \operatorname{an}(I)$. Let $R = \bigoplus_0^\infty I^n$ and $B = \bigoplus_0^\infty I^n/\mathfrak{m}I^n$. We have $B = R/\mathfrak{m}R$ since $\mathfrak{m} \subset A = R_0$ consists of degree zero elements and so the n-th degree piece of $R/\mathfrak{m}R$ is $R_n/\mathfrak{m}R_n = I^n/\mathfrak{m}I^n$. Recall that, by definition, the Krull dimension of B is the analytic spread of I. Now, since $B_0 = A/\mathfrak{m}$ is an infinite field, by [Ma70b] Theorem 14.14., B_1 contains elements $x_1^*, \dots, x_q^*$ such that $\sqrt{(x_1^*, \dots, x_q^*)} = B_+$. It is now not terribly hard to show that $R = \bigoplus_0^\infty I^n$ is finite over $\bigoplus_0^\infty (x_1, \dots, x_q)^n$.

Conversely, if $\bigoplus_0^\infty I^n$ is integral over $\bigoplus_0^\infty (x_1, \dots, x_r)^n$, then the former is finite over the latter (since a ring extension is finite if it is integral and is of finite type). Then $\sqrt{(x_1^*, \dots, x_r^*)} = B_+$ and then by Krull's height theorem, we must have $r \ge q$. □

5.4.6 Lemma. *Let A be a Noetherian ring, I, J ideals. Using the notation*

$$(I : J^\infty) = \{f \in A | J^n f \subset I, n \gg 0\},$$

if $I = \bigcap_i Q_i$ is a primary decomposition, then

$$(I : J^\infty) = \textit{the intersection of all } Q_i\textit{'s such that } \sqrt{Q_i} \not\supset J$$

where, by convention, the empty intersection is A.

In other words, $(- : J^\infty)$ removes all primary components whose radicals containing J and leave the rest of components intact.

Proof. It is easy to see: $(I : J^\infty) = \bigcap_i (Q_i : J^\infty)$. Thus, without loss of generality, we assume $I = Q$ is a primary ideal. First, we have:

$$J \subset \sqrt{Q} \Leftrightarrow J^n \subset Q, n \gg 0 \Leftrightarrow (Q : J^\infty) = (1).$$

Next, suppose $J \not\subset \sqrt{Q}$. Trivially, $Q \subset (Q : J^\infty)$. For the opposite inclusion, let f be in $(Q : J^\infty)$. Then $J^m f \subset Q$ for some $m > 0$. Since $\sqrt{Q}$ is a prime ideal, $J^m \not\subset \sqrt{Q}$ and so there is an x in J^m that is not in $\sqrt{Q}$. Then $xf \in Q$ and so $f \in Q$ since Q is primary. Hence, $(Q : J^\infty) = Q$. □

5.4.7 Lemma. *(pure-subring analog of the Eakin–Nagata theorem) Let $A \to B$ be a pure subring (cf. Lemma 5.3.6). If B is a Noetherian ring, then A is a Noetherian ring.*

Proof. Let $I_1 \subset I_2 \subset \cdots$ be an increasing sequence of ideals of A. Then $I_1 B \subset I_2 B \subset \cdots$ stabilizes since B is Noetherian. Since $I_i = I_i B \cap A$ (see the early part of the proof of Lemma 5.3.6), the original sequence stabilizes. □

6.0 Symbolic normal cone

6.1 Definition of a symbolic normal cone

This section introduces a variant of a normal cone defined in terms of a symbolic power of a prime ideal as opposed to an ordinary power.

6.1.1 Remark. Let A be a ring. If $I \subset A$ is an ideal, then we write

$$\operatorname{gr}_I A = \bigoplus_0^\infty I^n/I^{n+1}$$

where, by convention, $I^0 = A$. With the notations $X = \operatorname{Spec}(A)$, $V(I) = \operatorname{Spec}(A/I)$, the scheme

$$\operatorname{Spec}(\operatorname{gr}_I A)$$

is called the normal cone to $V(I)$ in X.

We can consider the following variant of a normal cone. Given a ring A and an ideal I in it, let $S_I = \{x \in A | x \text{ is a non-zerovisior on } A/I\}$. Since S_I is multiplicatively closed, we can form the localization $S_I^{-1}A$. Let $I^{(n)}$ denote the pre-image of the ideal

$$I^n S_I^{-1} A$$

under the localization map $A \to S_I^{-1}A$. In this paper, we call $I^{(n)}$ *the n-th symbolic power of I*. Suppose $I = \bigcap_1^r Q_i$ has a minimal primary decomposition. Then the set $\{\sqrt{Q_i} | 1 \le i \le r\}$ is the same as the set of the associated primes of A/I and the complement of the union of the set in A is precisely S_I defined above. If $I = \mathfrak{p}$ is a prime ideal, then $S_\mathfrak{p} = A - \mathfrak{p}$ and $I^{(n)} = \mathfrak{p}^{(n)}$ is the smallest $\mathfrak{p}$-primary ideal containing $\mathfrak{p}^n$ (Proposition 5.3.5). As $I^{(n)} I^{(m)} \subset I^{(n+m)}$, we can form the associated ring of A

$$\operatorname{gr}_{I^{(*)}} A = \bigoplus_0^\infty I^{(n)}/I^{(n+1)}$$

where, by convention, $I^{(0)} = A$. We then call $\operatorname{Spec}(\operatorname{gr}_{I^{(*)}} A)$ the *symbolic normal cone* to $\operatorname{Spec} A$ along I.

The next proposition collects some properties of symbolic powers for later use. Given a ring A and an ideal $I \subset A$, we call an associated prime of the A-module A/I a *prime divisor* of I; it is called an *embedded prime divisor* if it is not a minimal associated prime.

6.1.2 Proposition. *Let A be a Noetherian ring and I an ideal of A.*

(i) $I^{(1)} = I$.

(ii) *For each prime ideal $\mathfrak{p}$, let $\varphi_\mathfrak{p} : A \to A_\mathfrak{p}$ be the localization map. Then*

$$I^{(n)} = \bigcap_{\mathfrak{p} \in \operatorname{Ass}(A/I)} \varphi_\mathfrak{p}^{-1}(I^n A_\mathfrak{p}).$$

In literature, this is often taken as the definition of a symbolic power.

(iii) $I^{(n)}[f^{-1}] = (I[f^{-1}])^{(n)}$ *for each integer $n > 0$ and each nonzerodivisor $f \in A$.*

Proof. (i) We have $I \subset I^{(1)}$ trivially. Conversely, if $x \in I^{(1)}$, then $sx \in I$ for some $s \in A$ that is a nonzerodivisor modulo I. But then $x \equiv 0$ modulo I; i.e., $x \in I$.

(ii) Clearly, we have "$\subset$". Let S_I be the complement in A of the union of $\operatorname{Ass}(A/I)$. Since each $A_\mathfrak{p}$, $\mathfrak{p}$ a prime divisor of I, is also of the localization $S^{-1}A$, without loss of generality, we replace A by $S^{-1}A$ and then assume that A is a ring whose set of maximal ideals is the set of maximal prime divisors of I = maximal elements of $\operatorname{Ass}(A/I)$. We want to show the injection $I^{(n)} \hookrightarrow$ the right-hand side is a surjection; i.e., the cokernel vanishes. For that, it is enough to show that that is the case at each maximal ideal of A; hence, replacing A by the localization at a maximal ideal, we can assume A is a local ring whose maximal ideal $\mathfrak{m}$ is a maximal prime divisor of I. Then $S_I = A - \mathfrak{m}$ and so $I^{(n)} = \varphi_\mathfrak{m}^{-1}(I^n A_\mathfrak{m})$. On the other hand, $A \to A_\mathfrak{p}$ factors as $A \to A_\mathfrak{m} \to A_\mathfrak{p}$ and thus $\varphi_\mathfrak{m}^{-1}(I^n A_\mathfrak{m}) \subset \varphi_\mathfrak{p}^{-1}(I^n A_\mathfrak{p})$.

(iii) First, let x be in $I^{(n)}[f^{-1}]$. Then $sf^N x \in I^n$ for some $N > 0$ and $s \in S_I$; thus, $sx \in (I[f^{-1}])^n$. Now, if $s \notin S_{I[f^{-1}]}$, then $s \in \mathfrak{p}[f^{-1}]$ for some prime divisor $\mathfrak{p}$ of I not containing f, since

$$\operatorname{Ass}_{A[f^{-1}]}(A[f^{-1}]/I[f^{-1}]) = \{\mathfrak{p}[f^{-1}] | \mathfrak{p} \in \operatorname{Ass}_A(A/I), f \notin \mathfrak{p}\}.$$

That is, $s \notin S_I$, a contradiction. Hence, we conclude x is in $(I[f^{-1}])^{(n)}$.

For the opposite inclusion, let x be in $(I[f^{-1}])^{(n)}$. Then $f^N sx \in I^n$ for some $s \in S_{I[f^{-1}]} \cap A$ and $N \gg 0$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be all (possibly none of) the prime divisors of I containing f.

We write $\mathfrak{p}_i = (I : g_i)$ for some $g_i \notin \mathfrak{p}_i$. Then the ideal $\mathfrak{a} = (s, g_1^n, \dots, g_m^n)$ is not contained in each prime divisor of I. Thus, by prime avoidance, $\mathfrak{a}$ contains an element s' not in any prime divisor of I; i.e., $s' \in S_I$. Then $f^{nN} s' x \in I^n$. Thus, $x \in I^{(n)}[f^{-1}]$. □

The next corollary gives a geometric description of a symbolic power.

6.1.3 Corollary. *Let $X = \operatorname{Spec} A$ be a normal affine variety. Let $I = \cap_1^r Q_i$ be an intersection of primary ideals. Let $\mathfrak{p}_i = \sqrt{Q_i}$ and assume that the $\mathfrak{p}_i$'s all have height one. Let $V_i = V(\mathfrak{p}_i)$. Then $Q_i = \mathfrak{p}_i^{(m_i)}$ for some m_i and*

$$I^{(n)} = \Gamma(X, \mathcal{O}_X(-nm_1V_1 - \cdots - nm_rV_r))$$

where $\mathcal{O}_X(D)$ for a Weil divisor D is defined as

$$\Gamma(X, \mathcal{O}_X(D)) = \left\{ f \in k(X) \,\middle|\, f = 0 \text{ or } \sum_V \operatorname{ord}_V(f) V + D \geq 0 \right\}.$$

Proof. Since A is integrally closed, $A_\mathfrak{p}$ is a one-dimensional integrally closed domain; thus, is a discrete valuation ring. In a discrete valuation ring, every ideal is a power of the maximal ideal. Hence, $Q_i A_{\mathfrak{p}_i} \cap A = \mathfrak{p}_i^{m_i} A_{\mathfrak{p}_i} \cap A = \mathfrak{p}_i^{(m_i)}$ for some integers m_i and then, by (i) of Proposition 5.3.5, $Q_i = \mathfrak{p}_i^{(m_i)}$. The corollary now follows from (ii) of Proposition 6.1.2. □

We use the following standard notation and definition.

6.1.4 Definition. For each $x \in A$, if for some $n \geq 0$, $x \in I^n - I^{n+1}$, we write x^* for the class of x in I^n/I^{n+1}; otherwise, we let $x^* = 0$. Then x^* is called the *initial form of x*.

If I^n is replaced by $I^{(n)}$, we also define the initial form x^* in the same fashion.

The next proposition gives an instance when a symbolic normal cone turns out to be a usual normal cone. We recall that a finite sequence $x_1, \dots, x_r$ in a ring A is called a *regular sequence* if (1) they generate a proper ideal of A and (2) for each $i = 1, \dots, r$, x_i is a nonzerodivisor modulo $(x_1, \dots, x_{i-1})$. The basic fact ([Ful98] Lemma A.6.1.) is: if $x_1, \dots, x_r$ are a regular sequence generating an ideal I, then the surjective homomorphism of graded rings

$$(A/I)[t_1, \dots, t_r] \to \operatorname{gr}_I A, \; t_i \mapsto x_i^*$$

is injective (hence, an isomorphism), where x_i^* is the initial form of x_i. The explicit meaning of the injectivity is this: for each homogeneous polynomial f in $A[t_1, \dots, t_r]$ of degree n, if $f(x_1, \dots, x_r) \in I^{n+1}$, then all the coefficients of f are in I.

We now prove the following:

6.1.5 Proposition. *Let A be a ring and I an ideal generated by a regular sequence $x_1, \dots, x_r$. Then, for every integer $n \geq 1$, $I^{(n)} = I^n$.*

In particular, the symbolic normal cone, the normal cone and the normal bundle to $V(I) \subset \operatorname{Spec} A$ all coincide and

$$\operatorname{gr}_{I^{(*)}} A = \operatorname{gr}_I A = (A/I)[x_1^*, \dots, x_r^*]$$

where x_i^* are the initial forms of x_i's.

Proof. We shall argue by induction on n. The base case $n = 1$ holds by (iii) of Proposition 5.3.5. Assuming the equality is valid for $n - 1$, let y be in $I^{(n)}$. Since $I^{(n)} \subset I^{(n-1)} = I^{n-1}$ by inductive hypothesis, we can write

$$y = \sum_{|\beta|=n-1} z_\beta x^\beta$$

where $\beta = (\beta_1, \dots, \beta_r) \in \mathbb{N}^r$, z_β are in A, $|\beta| = \sum \beta_i$ and $x^\beta = x_1^{\beta_1} \cdots x_r^{\beta_r}$. Since $y \in I^{(n)}$, there is some $s \in A$ such that s is a nonzerodivisor modulo I and $sy \in I^n$. Consider the polynomial:

$$f = \sum_{|\beta|=n-1} s z_\beta t^\beta$$

in $A[t_1, \dots, t_r]$. It is homogeneous of degree $n-1$ and satisfies $f(x_1, \dots, x_r) = sy \in I^n$. Thus, by the fact noted just before the proposition, the coefficients of f are all in I; i.e., $s z_\beta \in I$ for all β. Since s is a nonzerodivisor modulo I, this implies z_β is in I for all β; i.e., y is in I^n.

The last assertion of the proposition holds by the discussion preceding this proposition. □

For the record, we mention the similar results:

6.1.6 Remark. Let A be a Noetherian ring. We have:

(i) If $\mathrm{gr}_I A$ is an integral domain, then $I^{(n)} = I^n$ for every integer $n \geq 1$. (The proof is easy.)

(ii) Corollary 3 of [Sch85] states the following : Let $\mathfrak{p}$ be a prime ideal of A such that $\mathrm{gr}_{\mathfrak{p}A_\mathfrak{p}} A_\mathfrak{p}$ is an integral domain; e.g., $A_\mathfrak{p}$ is a regular local ring. Then

$$(\mathrm{gr}_\mathfrak{p} A)_{\mathrm{red}} \text{ is an integral domain} \Leftrightarrow \mathfrak{p}^{(n)} = \overline{\mathfrak{p}^n} \text{ for all integers } n \geq 1.$$

Here, $-_{\mathrm{red}}$ means the quotient by the nilradical and $\overline{I}$ denotes the integral closure of an ideal I.

The next lemma relates symbolic powers to a valuation.

6.1.7 Lemma. *Let A be a Noetherian ring and $\mathfrak{p}$ a prime ideal. For each $x \in A$, if there is a non-negative integer n such that $x \in \mathfrak{p}^{(n)} - \mathfrak{p}^{(n+1)}$, then let $\mu(x) = n$; if there is no such n, let $\mu(x) = \infty$.*

Then $\mathrm{gr}_{\mathfrak{p}^{(\infty)}} A$ is an integral domain if and only if μ is a valuation; i.e., for any $x, y \in A$, $\mu(xy) = \mu(x) + \mu(y)$ and $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$.

Proof. Note we always have that $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$.

Let $0 \neq x^*, y^*$ denote the classes of $x \in \mathfrak{p}^{(n)}, y \in \mathfrak{p}^{(m)}$ in $\mathfrak{p}^{(n)}/\mathfrak{p}^{(n+1)}$ and $\mathfrak{p}^{(m)}/\mathfrak{p}^{(m+1)}$, where $n = \mu(x), m = \mu(y)$. Then we have:

$$x^*y^* \neq 0 \Leftrightarrow xy \notin \mathfrak{p}^{(n+m+1)} \Leftrightarrow \mu(xy) = n + m.$$

□

A symbolic normal cone can be used to detect regularity, like a usual normal cone.

6.1.8 Proposition. *Let A be a Noetherian integral domain and $\mathfrak{p}$ a prime ideal in it. Then the localization $A_\mathfrak{p}$ is a discrete valuation ring if and only if*

(a) $\mathrm{gr}_{\mathfrak{p}^{(\infty)}} A$ *is an integral domain, and*

(b) $Q^h(\mathrm{gr}_{\mathfrak{p}^{(\infty)}} A)_0 = k(\mathfrak{p})$*, where $Q^h(-)$ means the homogeneous total ring of fractions (i.e., the denominators are homogeneous.*

Proof. (⇒) Let μ be defined as in Lemma 6.1.7 and let ν denote the valuation corresponding to $A_\mathfrak{p}$. Then μ is the restriction of ν to A. Consequently, $\operatorname{gr}_{\mathfrak{p}^{(\infty)}} A$ is an integral domain by Lemma 6.1.7. This proves (a).

For (b), we first note that $(\operatorname{gr}_{\mathfrak{p}^{(\infty)}} A)_0 = A/\mathfrak{p}$. Thus, $k(\mathfrak{p}) \subset Q^h(\operatorname{gr}_{\mathfrak{p}^{(\infty)}} A)_0$. For the opposite inclusion, let x^*/y^* be in $Q^h(\operatorname{gr}_{\mathfrak{p}^{(\infty)}} A)_0$. Note x^*, y^* have the same degree, say, n. Choose $x, y \in A$ that represent x^*, y^*. Then

$$x \in \mathfrak{p}^n A_\mathfrak{p} = y A_\mathfrak{p}$$

and so $sx = ay$ for some $s \in A - \mathfrak{p}$ and $a \in A$. Since $\operatorname{gr}_{\mathfrak{p}^{(\infty)}} A$ is an integral domain, it follows $s^* x^* = a^* y^*$. Hence, $x^*/y^* = a^*/s^* \in Q^h(A/\mathfrak{p})$.

(⇐) We shall show that $\dim_{k(\mathfrak{p})} \mathfrak{p}A_\mathfrak{p}/\mathfrak{p}^2 A_\mathfrak{p} = 1$, which implies, by Nakayama's lemma, that $\mathfrak{p}A_\mathfrak{p}$ is principal and thus $A_\mathfrak{p}$ is a one-dimensional regular local ring; i.e., a discrete valuation ring.

We have $\mathfrak{p}^2 A_\mathfrak{p} \subsetneq \mathfrak{p}A_\mathfrak{p}$ since, otherwise, we get $\mathfrak{p}A_\mathfrak{p} = \mathfrak{p}^2 A_\mathfrak{p} = \mathfrak{p}^3 A_\mathfrak{p} = \cdots$, a violation of Krull's intersection theorem. Thus, we can choose $t \in \mathfrak{p} - \mathfrak{p}^{(2)}$. Let $\overline{x} \in \mathfrak{p}A_\mathfrak{p}/\mathfrak{p}^2 A_\mathfrak{p}$ and x a lift of it in $\mathfrak{p}$ under $A \to A_\mathfrak{p} \to A_\mathfrak{p}/\mathfrak{p}^2 A_\mathfrak{p}$. Since $x \notin \mathfrak{p}^2 A_\mathfrak{p}$, $x \notin \mathfrak{p}^{(2)}$. Thus, the initial form x^* is in $\mathfrak{p}/\mathfrak{p}^{(2)}$. Likewise, t^* is in $\mathfrak{p}/\mathfrak{p}^{(2)}$. Hence, $x^*/t^* \in Q^h(\operatorname{gr}_{\mathfrak{p}^{(\infty)}} A)_0 = k(\mathfrak{p})$ and we can write

$$x^*/t^* = a^*/s^*$$

for some $a, s \in A - \mathfrak{p}$. That is, $s^* x^* = a^* t^*$ and, since $\operatorname{gr}_{\mathfrak{p}^{(\infty)}} A$ is an integral domain, $(sx)^* = (at)^*$. That is to say, $sx - at \in \mathfrak{p}^{(2)}$ and so

$$x - \frac{a}{s} t \in \mathfrak{p}^2 A_\mathfrak{p}.$$

Since a/s is in $k(\mathfrak{p})$, this implies that the images of x and t in $\mathfrak{p}A_\mathfrak{p}/\mathfrak{p}^2 A_\mathfrak{p}$ are proportional over $k(\mathfrak{p})$. □

6.2 Definition of a good prime

Having some preliminaries, we are ready to introduce:

6.2.1 Definition (good prime)**.** Let A be a ring and $\mathfrak{p}$ a geometrically prime idea.[1]

Then we say that $\mathfrak{p}$ is a *good prime* if there is a ring homomorphism $\varphi : A \to O$ from A to some discrete valuation ring $(O, \mathfrak{m}_O)$ such that (1) $\mathfrak{p} = \varphi^{-1}(\mathfrak{m}_O)$ and (2) the generalized Rees algebra $\bigoplus_0^\infty \varphi^{-1}(\mathfrak{m}_O^n)$ is a Noetherian ring.[2]

Note that if $A_\mathfrak{p}$ is a discrete valuation ring and $O = A_\mathfrak{p}$ and φ the localization map, then $\varphi^{-1}(\mathfrak{m}_O^n)$ is the n-th symbolic power of $\mathfrak{p}$. For an integral domain appearing in algebraic geometry, there is a somehow simpler characterization of a good prime.

6.2.2 Lemma. *Let A be either the coordinate ring of an affine variety or a localization of such a ring and $\widetilde{A}$ the integral closure of A, $\mathfrak{q}$ a height-one prime ideal of $\widetilde{A}$ such that $\mathfrak{q} \cap A$ is a geometrically prime ideal.*

Then $\mathfrak{p} = \mathfrak{q} \cap A$ is a good prime if and only if there exists an ideal I of A such that for each $n > 0$, $I^n\widetilde{A} \cap A$ is the contraction of the $\mathfrak{q}$-primary component Q of $I^n\widetilde{A}$; i.e., $I^n\widetilde{A} \cap A = Q \cap A$.

In particular, when $A = \widetilde{A}$ is integrally closed, we have that $\mathfrak{p}$ is a good prime if and only if there exists an ideal I such that I^n is $\mathfrak{p}$-primary for each $n > 0$.

Proof. ($\Rightarrow$) By definition, $\bigoplus_0^\infty \mathfrak{q}^{(n)} \cap A$ is a Noetherian ring and so is finitely generated as an algebra over the zero-th degree piece. Hence, there is some $m > 0$ such that $\mathfrak{q}^{(nm)} \cap A = (\mathfrak{q}^{(m)} \cap A)^n$ for each $n > 0$. Also, let $x \neq 0$ in $\mathfrak{p}$ and $l > 0$ such that $x\widetilde{A}_\mathfrak{q} = \mathfrak{q}^l\widetilde{A}_\mathfrak{q}$. Set $m' = lm$ and $I = \mathfrak{q}^{(m')} \cap A$. Then, since $x^m \in I$, clearly, we have:

$$I^n\widetilde{A}_\mathfrak{q} = \mathfrak{q}^{nm'}\widetilde{A}_\mathfrak{q},$$

[1] A prime ideal $\mathfrak{p}$ of a ring R is *geometrically prime* if $R/\mathfrak{p} \otimes_{k(\mathfrak{p})} \overline{k(\mathfrak{p})}$ is an integral domain, where $\overline{k(\mathfrak{p})}$ is the algebraic closure of the residue field $k(\mathfrak{p}) = Q(R/\mathfrak{p})$.

[2] This definition of a good prime suggests that we work out the general theory of pre-image of powers of ideals, of which a symbolic power is a special case. This is beyond the scope of the thesis and here we adopt somehow more ad hoc approach.

which is to say $\mathfrak{q}^{(nm')}$ is the $\mathfrak{q}$-primary component of $I^n\widetilde{A}$. Also, $I^n = I^n\widetilde{A} \cap A$ since I^n is a contraction of an extension of an ideal. Thus, if $I^n\widetilde{A} = \cap Q_i$ is a minimal primary decomposition, then $\cap(Q_j \cap A) = Q_i \cap A$ with $Q_i = \mathfrak{q}^{(nm')}$.

($\Leftarrow$) Let $m > 0$ be an integer such that $I\widetilde{A}_{\mathfrak{q}} = \mathfrak{q}^m\widetilde{A}_{\mathfrak{q}}$. Then $\mathfrak{q}^{(nm)}$ is the $\mathfrak{q}$-primary component of $I^n\widetilde{A}$ and thus, by assumption, $\mathfrak{q}^{(nm)} \cap A \subset I^n\widetilde{A}$. Since $I^n \subset Q \cap A$, we also have $I \subset \mathfrak{p} \subset \mathfrak{q}$. Consider

$$\oplus_0^\infty I^{nm} \subset \oplus_0^\infty \mathfrak{q}^{(nm)} \cap A \subset \oplus_0^\infty I^n\widetilde{A}.$$

We have that $\oplus_0^\infty I^n\widetilde{A}$ is finite over $\oplus_0^\infty I^n$; indeed, we can find a nonzero element $f \in A$ such that $f\widetilde{A} \subset A$ and then f embeds $\oplus_0^\infty I^n\widetilde{A}$ to $\oplus_0^\infty I^n$ as an $\oplus_0^\infty I^n$-module. By Lemma 5.4.1 (ii), $\oplus_0^\infty I^n\widetilde{A}$ is finite (and thus Noetherian) over $\oplus_0^\infty I^{nm}$ and thus $\oplus_0^\infty \mathfrak{q}^{(nm)} \cap A$ is finite over $\oplus_0^\infty I^{nm}$; hence it is a Noetherian ring. By Lemma 5.4.1 (ii), $\oplus_0^\infty \mathfrak{q}^{(n)} \cap A$ is a Noetherian ring. $\square$

We note that the condition on I in the lemma is trivially satisfied if $I^n\widetilde{A}$ itself is primary for each $n > 0$. This observation gives:

6.2.3 Example. Let A be the coordinate ring of an affine variety X and $H = V(x) \subset X$ a geometrically irreducible but possibly non-reduced hypersurface. Assume also that the pre-image of H in the normalization of X is irreducible.[3] Then $\mathfrak{p} = \sqrt{xA}$ is a good prime by Lemma 6.2.2 above with $I = xA$ (because by Serre's criterion, $x^n\widetilde{A}$ has no embedded prime for each $n > 0$.)

Similarly, let R be the homogeneous coordinate ring of a projective variety. Let x be a nonzero homogeneous element of R such that for the integral closure $\widetilde{R}$ of R, $\sqrt{x\widetilde{R}}$ is a prime ideal. Then $\mathfrak{p} = \sqrt{xR}$ is a good prime when it is geometrically prime.

We note that x above is generally *not* a local equation.

6.2.4 Remark. Let $X = \operatorname{Proj} R$ be a normal projective variety. Suppose R is *not* integrally closed (in particular, R is not the section ring of $\mathcal{O}_X(1)$).

[3]One can show that, for $\mathfrak{p} = \sqrt{xA}$, if $A_{\mathfrak{p}}$ is a discrete valuation ring, then this conditions holds; i.e., $\sqrt{x\widetilde{A}}$ is a prime ideal.

Let $Y \subset X$ be the closed subvariety defined by a height-one homogeneous prime ideal $\mathfrak{p}$ of R. Suppose Y is a hypersurface in the sense that $Y = V(xR)$ for some homogeneous element $x \in R$ and suppose $R/\mathfrak{p}$ is integrally closed. Then the equation $xR_{\mathfrak{p}} = \mathfrak{p}R_{\mathfrak{p}}$ would lead to a contradiction. Indeed, assume this equation holds. Let A be the localization of R at R_+. Then x generates the maximal ideal of $A_{\mathfrak{p}}$ (this is not obvious but note that if y is a homogeneous element in $\mathfrak{p}$, then $sy \equiv 0 \bmod (x)$ for some $s \in R - \mathfrak{p}$. Then $s_i y \equiv 0 \bmod (x)$ for each homogeneous component s_i of s.) Since $\sqrt{xA}$ is a prime ideal, the lemma of Hironaka ([Hart77] Ch. III, Lemma 9.12.) then says that $\mathfrak{p} = xA$ and A is integrally closed (thus R is integrally closed), a contradiction.

Given the above Remark 6.2.4, we make the following clarification:

6.2.5 Lemma. *Let $X = \operatorname{Proj} R$ be a projective variety and $Y \subset X$ a closed subvariety defined by a homogeneous geometrically prime ideal $\mathfrak{p}$ of height one. If Y is an effective Cartier divisor on X, then there exists a homogeneous ideal $I \subset R$ such that $\{\mathfrak{p}\} \subset \operatorname{Ass}(R/\overline{I^n}) \subset \{\mathfrak{p}, R_+\}$ for each $n > 0$.*

Let g be a homogeneous element of degree m in R_+ such that $\mathfrak{p}R[g^{-1}]$ is principal and let $S = R^{[m]}$ be the m-th Veronese subring and $\mathfrak{q} = \mathfrak{p} \cap S$. Then $S_{\mathfrak{q}}$ is a discrete valuation ring with the valuation $\nu_{\mathfrak{q}}$ and for each $n > 0$,

$$\mathfrak{p}^{(nm)} \cap S = \mathfrak{q}^{(ne)}$$

where $e = \min\{\nu_{\mathfrak{q}}(x^m) | 0 \neq x \in \mathfrak{p}, x \text{ homogeneous}\}$.

Proof. Let $\pi : \operatorname{Spec}(R) - V(R_+) \to X = \operatorname{Proj}(R)$ be the projection from the affine cone to X. Then $\pi^{-1}(Y)$ is a not-necessarily-reduced effective Cartier divisor on $\operatorname{Spec}(R) - V(R_+)$. Let $I \subset R$ be some homogeneous ideal defining the scheme-theoretic closure of $\pi^{-1}(Y)$ in X. Note that I is locally free except at R_+; thus, by Lemma 2.4.5 (ii) and Corollary 5.4.4, $\operatorname{Ass}(R/\overline{I^n}) \subset \{\mathfrak{p}, R_+\}$.

Let x be a homogeneous element of $\mathfrak{p}$ such that $\nu(x^m) = e$. We note that $\mathfrak{p}R[g^{-1}]_0$. is generated by some element of the form x/g^l where x is a homogeneous element of R and $l > 0$ an integer. Since $\deg(x)$ is divisible by m, we see that x is a homogeneous element of S. Then x generates $\mathfrak{q}S[g^{-1}]$. Indeed, if y is a homogeneous element in $\mathfrak{q}S[g^{-1}]$ of degree n, then y/g^n is in $\mathfrak{p}R[g^{-1}]_0 = x/g^l R[g^{-1}]_0$. As $\deg(x)$ and $\deg(y)$ are both divisible by m, we have $a \in S$; thus, y is in $\mathfrak{q}S[g^{-1}]$. Since $S[g^{-1}] \subset S_{\mathfrak{q}}$, it follows that x generates the maximal ideal of $S_{\mathfrak{q}}$ and so $S_{\mathfrak{q}}$ is a discrete valuation ring.

It remains to verify the equation $\mathfrak{p}^{(nm)} \cap S = \mathfrak{q}^{(ne)}$. First, we have $\mathfrak{q}^e S_{\mathfrak{q}} = x^m S_{\mathfrak{q}} \subset \mathfrak{p}^m S_{\mathfrak{q}} \subset \mathfrak{p}^m R_{\mathfrak{p}}$; it thus follows that $\mathfrak{q}^{(ne)} \subset \mathfrak{p}^{(nm)} \cap S$. For the opposite inclusion, let y be in $\mathfrak{p}^{(nm)} \cap S$. Then $ay \in \mathfrak{p}^{nm}$ for some $a \in R - \mathfrak{p}$. Replacing a by a^m, we can assume $a \in S - \mathfrak{q}$. Now, $\mathfrak{p}^{nm}$ is spanned over R by monomials of the form $\prod_1^{nm} z_i$ with $z_i \in \mathfrak{p}$. Then, since ν is a valuation, we have: $\nu(\prod_1^{nm} z_i^m) = \sum_1^{nm} \nu(z_i^m) \geq \sum_1^{nm} e = nme$. Thus, $\nu(\prod_i^{nm} z_i) \geq ne$. Since $\nu(y) = \nu(ay)$, we have $y \in \mathfrak{q}^{(ne)}$. $\square$

The next standard lemma ([Ei04] Exercise 5.3.) is useful for a concrete computation of a usual normal cone; we omit the (relatively easy) proof.

6.2.6 Lemma. *Let A be a ring and $J \subset I$ ideals. Then*

$$\operatorname{gr}_{I/J}(A/J) = \operatorname{gr}_I(A)/J^*$$

where J^* is the ideal generated by the initial forms of J with respect to I.

The next example is illuminating. In the next section, we will also give a quadric surface example: Example 7.1.5.

6.2.7 Example (elliptic curve)**.** Let $S = k[x, y, z]$, $\mathbb{P}^2 = \operatorname{Proj} S$. Let

$$R = S/(f),\ f = y^2 z - x^3 + xz^2.$$

Then R is the homogeneous coordinate ring of the elliptic curve $X = \operatorname{Proj} R \hookrightarrow \mathbb{P}^2$. Let $\overline{x}, \overline{y}, \overline{z}$ be the images of x, y, z in R.

Let $D = V(\overline{z}) \subset X$ the effective Cartier divisor determined by $\overline{z}$ and $Y = \operatorname{Supp}(D)$. Note $Y = (0 : 1 : 0)$ and is defined by $\mathfrak{p} = \sqrt{(\overline{z})} = (\overline{x}, \overline{z}) \subset R$.

We have $f \equiv 0 \bmod (x, z)$ and $f \equiv y^2 z \bmod (x, z)^2$. That is, the initial form of f is $y^2 z$. Since $\operatorname{gr}_{(x,z)} S = S$, by Lemma 6.2.6, we have:

$$\operatorname{gr}_{\mathfrak{p}} R = S/(y^2 z).$$

Geometrically, as divisors on the affine space $\mathbb{A}^3$, we have:

$$\operatorname{Spec}(\operatorname{gr}_{\mathfrak{p}} R) = 2\{y = 0\} + \{z = 0\}.$$

We shall now identity $\operatorname{gr}_{\mathfrak{p}^{(\infty)}} R = \bigoplus_0^\infty \mathfrak{p}^{(n)}/\mathfrak{p}^{(n+1)}$. We have: $\overline{y}^2\overline{z} = \overline{x}^3 - \overline{xz^2} \in \mathfrak{p}^3$ and so $\overline{z} \in \mathfrak{p}^{(3)}$. Clearly, $\overline{z} \notin \mathfrak{p}^{(4)}$ and so $\deg(\overline{z}) = (1, 3)$. Similarly, we find $\overline{x}, \overline{y}$ have degree $(1, 1)$ and $(1, 0)$ respectively.

Note: geometrically, the calculations correspond to the calculations of order-of-vanishing. For example, we have: $\overline{z} = 0 \Rightarrow \overline{x}^3 = 0$; i.e., $\overline{z}$ vanishes to order 3 at $(0 : 1 : 0)$.

We shall show the classes of $\overline{z}, \overline{x}, \overline{y}$ generate $\operatorname{gr}_{\mathfrak{p}^{(\infty)}} R$. Let $A \subset \operatorname{gr}_{\mathfrak{p}^{(\infty)}} R$ be the subalgebra generated by the classes of $\overline{x}, \overline{y}, \overline{z}$. On the one hand, we have:

$$\begin{aligned}\dim_k A_n &= \#\{n_1(1, 3) + n_2(1, 1) + n_3(1, 0) | n_1 + n_2 + n_3 = n,\ n_i \geq 0\} \\ &= 3n.\end{aligned}$$

On the other hand, by the Riemann-Roch theorem, $\dim_k(\operatorname{gr}_{\mathfrak{p}^{(\infty)}} R)_n = \dim_k R_n = 3n$. Hence, $A = \operatorname{gr}_{\mathfrak{p}^{(\infty)}} R$ and in fact we have:

$$\operatorname{gr}_{\mathfrak{p}^{(\infty)}} R = S/(y^2z - x^3).$$

Geometrically, X degenerates to the cuspidal cubic curve $y^2z = x^3$.

Let $I = (\overline{z})$. Then, by a calculation similar to the one before, we see $\operatorname{gr}_I R = k[x, y, z]/(x^3)$. Then $\operatorname{gr}_I R \to \operatorname{gr}_{\mathfrak{p}^{(\infty)}} R$ is

$$k[x, y, z]/(x^3) \to k[x, y, z]/(y^2z - x^3).$$

Since $R/(\overline{z}) = k[x, y]/(x^3)$ and $R/\mathfrak{p} = k[y]$,

$$\overline{N} := \operatorname{Proj}((R/\mathfrak{p})[z^*]) \simeq \operatorname{Proj}(k[y, z]) \simeq \mathbb{P}^1.$$

We have the finite surjective map

$$\rho : \overline{C} \to \overline{N}$$

induced by $(R/\mathfrak{p})[z^*] \hookrightarrow \mathrm{gr}_{\mathfrak{p}(\infty)} R$. As we have already seen, $\mathrm{gr}_{\mathfrak{p}(\infty)} R$ is generated by x^*, y^*, z^*. Since $\overline{x}^3 = a\overline{z}$ with $a = \overline{xz} - \overline{y}^2 \notin \mathfrak{p}$, we have $\overline{x}^{*3} = \overline{y}^{*2}\overline{z}$ and the surjective map $k[t_0, t_1, t_2] \to \mathrm{gr}_{\mathfrak{p}(\infty)} R,\ t_0, t_1, t_2 \mapsto \overline{x}^*, \overline{y}^*, \overline{z}^*$ has the kernel generated by $t_0^3 - t_1^2 t_3$. Hence,

$$\overline{C} = V(t_0^3 - t_1^2 t_3) \subset \mathbb{P}^2.$$

Thus, the finite map ρ is given by *forgetting* x; i.e., $(t_0 : t_1 : t_2) \mapsto (t_1 : t_2)$. Note ρ is well-defined on X.

We note that we also have the degeneration from X to $\mathbb{P}(N_{D/X} \oplus 1) = \mathrm{Proj}(R/(\overline{z})[\overline{z}^*])$ but the latter is not reduced (since $R/(\overline{z}) = k[x, y]/(x^3)$).

The next theorem (Theorem 6.3.1) gives a useful characterization of a good prime. The equivalence between (i) and (ii) is a mild generalization of the idea found in [Sch88]; whence, the attribution. The equivalence between (i) and (iii) is a result of Kawamata ([Kaw88] Lemma 3.1.) without the assumption on normality.

6.3 Criteria for a good prime

6.3.1 Theorem (good prime criterion)**.** *Let* $X = \mathrm{Proj}\, R$ *be a projective variety over an infinite field*[4] *and* $Y = V(\mathfrak{p}) \subset X$ *a closed subvariety such that* Y *is an effective Cartier divisor on* X*.*

Then the following are equivalent.

- *(i)* $\mathfrak{p}$ *is a good prime.*
- *(ii) (Schenzel)* $\mathfrak{p}$ *is the radical of an ideal generated by at most* $\dim X$ *number of homogeneous elements of positive degree.*
- *(iii) (Kawamata) There exists a projective scheme*

$$g : Z \to \mathrm{Spec}\, R$$

[4]Per usual, we have not investigated the finite field case.

such that (1) g is small; i.e., the exceptional set has codimension ≥ 2.[5] *and (2) $g_*\mathcal{O}_Z(-1)$ is the ideal sheaf of $V(\mathfrak{p}^{(m)}) \subset \operatorname{Spec} R$; i.e., $\mathfrak{p}^{(m)}$ is the space of global sections of $g_*\mathcal{O}_Z(-1)$.*

Before proceeding with the proof, we note that, because $Y \subset X$ is an effective Cartier divisor, after replacing R by a Veronese subring, $R_{\mathfrak{p}}$ is a discrete valuation ring (Lemma 6.2.5). Thus, the meaning of a good prime simplifies: $\mathfrak{p}$ is a good prime $\Leftrightarrow \bigoplus_0^\infty \mathfrak{p}^{(n)}$ is a Noetherian ring.

Proof of 6.3.1. (i) $\Rightarrow$ (ii): Since $\mathfrak{p}$ is a good prime, we can find $m > 0$ such that for every $n > 0$, $\mathfrak{p}^{(nm)} = (\mathfrak{p}^{(m)})^n$. Let $Q = \mathfrak{p}^{(m)}$. Then Q^n is primary for every $n > 0$. Then $\overline{Q^n}$ is primary for every $n > 0$ since, by Proposition 5.4.3, $\operatorname{Ass}(R/\overline{Q^n}) \subset \operatorname{Ass}(R/Q^r) = \{\mathfrak{p}\}$ for $r \gg 0$. Then, by Proposition 5.4.3 and Lemma 5.4.5, for $\mathfrak{r} = R_+$, there exists an ideal $\mathfrak{a}$ in $R_{\mathfrak{r}}$ generated by at most Krull-dim$(R) - 1 = \operatorname{ht}(\mathfrak{r}) - 1$ elements such that $QR_{\mathfrak{r}}$ is integral over $\mathfrak{a}$. Then, by Lemma 5.3.11 and the remark that follows, we have $\mathfrak{p}R_{\mathfrak{r}} = \sqrt{\mathfrak{a}}$. It follows that, for some homogeneous element f of R_+, we have that $\mathfrak{p}R[f^{-1}]$ is the radical of an ideal generated by at most $\operatorname{ht}(\mathfrak{r}) - 1$ elements. Then, by homogenizing the generators, we see that $\mathfrak{p}$ is the radical of an ideal generated by at most $\operatorname{ht}(\mathfrak{r}) - 1$ homogeneous elements.

(ii) $\Rightarrow$ (i): By Lemma 6.2.5, we have a homogeneous ideal I of R such that $\{\mathfrak{p}\} \subset \operatorname{Ass}(R/\overline{I^n}) \subset \{\mathfrak{p}, R_+\}$ for each $n > 0$. Also, by assumption, we can find a homogeneous ideal $\mathfrak{a} \subset R$ generated by Krull-dim$(R) - 1$ elements such that $\mathfrak{p} = \sqrt{\mathfrak{a}}$. Let $A = R_{R_+}$ and $\mathfrak{m} = R_+A$. To simplify the notations, we will write $\mathfrak{p}, \mathfrak{a}$, etc. for $\mathfrak{p}A, \mathfrak{a}A$, etc. Let $m > 0$ be an integer large enough that $\mathfrak{p}^m \subset I$ and $I^m \subset \mathfrak{a}$. By Proposition 5.4.3, for $n \gg 0$, $\mathfrak{m}$ is not a prime divisor of $\overline{\mathfrak{a}^n}$; i.e., taking $(- : \mathfrak{m}^\infty)$ does nothing (cf. Lemma 5.4.6). Thus, for $n \gg 0$,

$$\mathfrak{a}^{nm^2} \subset \mathfrak{p}^{(nm^2)} \subset (\overline{I^{nm}} : \mathfrak{m}^\infty) \subset (\overline{\mathfrak{a}^n} : \mathfrak{m}^\infty) = \overline{\mathfrak{a}^n},$$

which, by an argument similar to one in the proof in Lemma 6.2.2, implies that $\bigoplus_0^\infty \mathfrak{p}^{(nm^2)}$ is a Noetherian ring.

(i) $\Rightarrow$ (iiii): (Sketch) We shall essentially repeat Kawamata's original proof, also reproduced at [Ko08, Exercise 90]). First assume that R is an integrally closed domain. Let Z be the

[5]For the notion of a small morphism, see for example `https://mathoverflow.net/questions/31696/best-strategy-for-small-resolutions`. A small morphism between normal varieties is also called an isomorphism in codimension one.

Proj of $B = R \oplus \mathfrak{p} \oplus \mathfrak{p}^{(2)} \oplus \cdots$ with respect to the grading $B_n = \mathfrak{p}^{(n)}$. Since $\mathfrak{p}$ is a good prime, B is a finitely generated R-algebra and, as a consequence, $\mathcal{O}_Z(m)$ is an ample invertible sheaf for some $m > 0$. Let $g : Z \to \operatorname{Spec} R$ be the structure map. To show that g is small, by way of contradiction, let E be an irreducible component of the exceptional set of g that has codimension 1. Let $\mathcal{I}_E$ denote the ideal sheaf of $E \subset Z$. Since $\mathcal{O}_Z(m)$ is ample, for some integer $l > 0$ divisible by m, we have that $\mathcal{I}_E^*(l) := \mathcal{I}_E^* \otimes \mathcal{O}_Z(l)$ is generated by global sections, where $\mathcal{I}_E^* = \mathcal{H}om_Z(\mathcal{I}_E, \mathcal{O}_Z)$ is the dual module of $\mathcal{I}_E$. Note that $\mathcal{I}_E^*$ admits a nonzero global section; e.g., $\mathcal{I}_E \hookrightarrow \mathcal{O}_Z$ is one and since $\mathcal{I}_E^*$ is torsion-free,[6] we can find an inclusion $\mathcal{O}_Z \hookrightarrow \mathcal{I}_E^*$ that is not the identity morphism. It in turn gives an inclusion $\mathcal{O}_Z(l) \hookrightarrow \mathcal{I}_E^*(l)$. Since g_* is left-exact, that gives $g_*\mathcal{O}_Z(l) \hookrightarrow g_*\mathcal{I}_E^*(l)$. To finish, the proof in literature relies on the normality of $\operatorname{Spec}(R)$.

If R is not necessarily integrally closed, then there exists a good prime $\mathfrak{q}$ of the integral closure $\overline{R}$ lying over $\mathfrak{p}$. Then, by the early part, we get $Z \to \operatorname{Spec}(\overline{R}) \to \operatorname{Spec} R$.

(iii) $\Rightarrow$ (i): (Sketch) Let $\mathcal{B} = \bigoplus_{n=0}^{\infty} g_*\mathcal{O}_Z(-n)$. The assumption says that $\Gamma(Z, \mathcal{O}_Z(-1)) = \Gamma(\operatorname{Spec} R, g_*\mathcal{O}_Z(-1)) = \mathfrak{p}^{(m)}$. Thus, $(\mathfrak{p}^{(m)})^n \subset \Gamma(Z, \mathcal{O}_Z(-1))^n \subset \mathfrak{q}_n := \Gamma(Z, \mathcal{O}_Z(-n))$ and then

$$\mathfrak{p}^{(nm)} \subset \mathfrak{q}_n$$

since the right-hand side is $\mathfrak{p}$-primary. To show the inclusion is the equality, let s be in $\mathfrak{q}_n$. Then we can choose s' in $\mathfrak{p}^{(nm)}$ such that $s - s'$ has support contained in the exceptional set of g, which is a contradiction unless $s - s'$ is identically zero and proves the claim. We thus conclude that $\bigoplus_0^{\infty} \mathfrak{p}^{(nm)} = \Gamma(\operatorname{Spec} R, \mathcal{B})$ is a finitely generated k-algebra (since $\mathcal{B}$ is finitely generated as an $\mathcal{O}_X$-algebra). □

6.3.2 Corollary. *Let $X \subset \mathbb{P}^n$ be a projectively normal smooth projective curve over an algebraically closed field, R the homogeneous coordinate ring of X and $\mathfrak{p} \subset R$ a height-one homogeneous prime ideal. Then $\mathfrak{p}$ is a good prime if and only if $\mathfrak{p}^{(m)}$ is principal for some $m > 0$.*

In other words, $\mathfrak{p}$ is a good prime if and only if the class of $V(\mathfrak{p})$ in the Weil divisor class group of $\operatorname{Spec} R$ *is a torsion element.*

[6] In fact, $\mathcal{I}_E^*$ is a reflexible sheaf in the sense in Hartshorne, R.: Stable reflexive sheaves. Math. Ann.254 (1980), 121?176.

Proof. [7] (⇐) is trivial. (⇒) Let Q be as in Theorem 6.3.1 (ii). Since $R_{\mathfrak{p}}$ is a discrete valuation ring, $Q = \mathfrak{p}^{(m)}$ for some m. Then $\bigoplus_1^\infty Q^n$ is finite (thus integral) over $\bigoplus_1^\infty I^n$ for some nonzero principal ideal I. But, since R is integrally closed, a nonzero principal ideal such as I is integrally closed; i.e., $\mathfrak{p}^{(m)} = I$. □

6.3.3 Corollary (cf. [Kaw88] Lemma 3.2.)**.** *Let $X = \operatorname{Proj} R$ be a projective variety and S a subring of R. Assume that S is a Noetherian graded ring such that $R_0 = S_0 = k$ is the base field and $S \hookrightarrow R$ is graded of some degree and is finite. Let $\mathfrak{q}$ be a height-one homogeneous prime ideal of S such that there is a unique homogeneous prime ideal $\mathfrak{p}$ of R that lies over $\mathfrak{q}$ (i.e., $\mathfrak{p} \cap S = \mathfrak{q}$). Then $\mathfrak{q}$ is a good prime if and only if $\mathfrak{p}$ is a good prime.*

Proof. It is clear that $\mathfrak{p}$ has height-one (cf. Lemma 2.5.1 (iv)).

(⇒) Since $\mathfrak{q}$ is a good prime, we have $g : Z \to \operatorname{Spec} S$ as in (iii). Let $g' : W = Z \times_{\operatorname{Spec} S} \operatorname{Spec} R \to \operatorname{Spec} R$ be the base change along $\operatorname{Spec} R \to \operatorname{Spec} S$. Then g' satisfies the condition of (iii) of the theorem. Hence, $\mathfrak{p}$ is a good prime.

(⇐) Omitted for now (as not needed later). □

6.3.4 Remark (divisor with a growth condition)**.** The early version of the thesis gave a slightly more general version of the above theorem. For the benefits of the interested readers (and for the purpose of historical records), we explain the idea of the general version.

As the readers of the proof would have noticed, the crux of proof is to estimate how *badly* $V(\mathfrak{p})$ fails to be an effective Cartier divisor at the vertex $V(R_+)$. Thus, the idea behind the proof would still work perfectly in the case when $V(\mathfrak{p})$ is the closure of an effective Cartier divisor; we estimate the boundary components of $V(\mathfrak{p})$ (i.e., the components of the non-Cartier locus).

We record the following two simple facts for the use in the next section.

6.3.5 Lemma. *Let A be a $\mathbb{G}_m^r$-ring. If $\mathfrak{p}$ is a $\mathbb{G}_m^r$-invariant prime ideal, then, for each $n > 0$, the symbolic power $\mathfrak{p}^{(n)}$ is a $\mathbb{G}_m^r$-submodule of A; in particular, the symbolic normal cone ring $\operatorname{gr}_{\mathfrak{p}^{(\infty)}} A$ is a $\mathbb{G}_m^r$-algebra.*

[7] cf. `http://www.math.lsa.umich.edu/~hochster/711F07/L09.07.pdf`

Proof. By Proposition 2.5.3 (i), we have a primary decomposition $\mathfrak{p}^n = Q_1 \cap Q_2 \cap \cdots \cap Q_r$ where Q_i are $\mathbb{G}_m^r$-invariant ideals. If, say, $\sqrt{Q_1} = \mathfrak{p}$, then $Q_1 = \mathfrak{p}^{(n)}$. □

6.3.6 Lemma. *Let R be an integral domain and $A \subset R$ a subring. If $\mathfrak{p}$ is a prime ideal of A such that $A_\mathfrak{p}$ is a discrete valuation ring and $\mathfrak{q}$ a prime ideal of R lying over $\mathfrak{p}$, then*

$$\mathfrak{p}^n R_\mathfrak{q} \cap A_\mathfrak{p} = \mathfrak{p}^n A_\mathfrak{p}.$$

Proof. We prove this by induction on n. Let x be a generator of $\mathfrak{p}A_\mathfrak{p}$. The base case $n = 1$ holds since $\mathfrak{p}R_\mathfrak{q} \cap A_\mathfrak{p}$ is a proper ideal of $A_\mathfrak{p}$ containing x. For the inductive step, let y be in the left-hand side. By inductive hypothesis, we can write $y = ax^{n-1}$ for some $a \in A_\mathfrak{p}$. Since $A \to R \to R_\mathfrak{q}$ factors through $A_\mathfrak{p} \to R_\mathfrak{q}$, we have: $\mathfrak{p}^n R_\mathfrak{q} = x^n R_\mathfrak{q}$ and thus $ax^{n-1} \in x^n R_\mathfrak{q}$. By the basic case, $a \in xR_\mathfrak{q} \cap A_\mathfrak{p} = xA_\mathfrak{p}$, establishing the claim. □

Alternative proof. $A_\mathfrak{p} \to R_\mathfrak{q}$ is torsion-free and thus is faithfully flat as $A_\mathfrak{p}$ is a principal ideal domain. In general, for a faithfully flat ring homomorphism $S \to S'$ and an ideal $I \subset S$, we have $IS' \cap S = I$ (cf. the proof of Lemma 5.3.6). Hence, for $I = \mathfrak{p}^n A_\mathfrak{p}$, we have: $IR_\mathfrak{q} \cap A_\mathfrak{p} = I$. □

7.0 Construction of toric degenerations

In the previous section, we saw that, given a projective variety $X = \operatorname{Proj} R$, if $\mathfrak{p}$ is a good prime (Definition 6.2.1), then there is a degeneration from X to another projective variety. We shall iterate this degeneration: each degeneration introduces a nontrivial $\mathbb{G}_m$-action and so, after some finite steps, we reach a projective variety acted by a torus with an open dense orbit; i.e., a non-normal toric variety. We shall also observe that, by unwinding given a toric degeneration, each toric degeneration of a projective variety can be reconstructed in this way.

7.1 Definition and lifts of a good flag

For the purpose of the construction, we use the following notion

7.1.1 Definition (good flag)**.** Let $X = \operatorname{Proj} R$ be a projective variety and

$$Y_\bullet : X = Y_0 \supset Y_1 \supset \cdots \supset Y_r, \operatorname{codim} Y_i = i$$

a partial or complete flag of closed subvarieties of X.

Suppose $Y_i = V(\mathfrak{p}_i)$ for the homogeneous prime ideals $\mathfrak{p}_i \subset R$. Then we say the above flag $Y_\bullet$ is a *good flag with respect to* R if for each i, $\mathfrak{p}_i/\mathfrak{p}_{i-1}$ is a good prime of $R/\mathfrak{p}_{i-1}$ (Definition 6.2.1).

Also, we say a flag is a good flag with respect to an ample line bundle L if it is a good flag with respect to the section ring of L.

Depending on applications, it can be easy to tell whether a flag is good or not, because of the below:

7.1.2 Example (a typical good flag)**.** Let $Y_\bullet$ be a flag of a projective variety $X = \operatorname{Proj} R$ as above and $d = \dim X$. Assume that each Y_i is an effective Cartier divisor on Y_{i-1}. Then, with respect to R, the flag $Y_\bullet$ is a good flag if and only if there is a finite sequence of homogeneous elements $x_1, \dots, x_n$, $0 = n_0 < n_1 < \cdots < n_r = n$ of R such that, as a set, $Y_i = V(x_1, \dots, x_{n_i})$ and $n_i - n_{i-1} - 1 \le d - i$. This immediately follows from Theorem 6.3.1 (ii).

7.1.3 Remark (good primes under a Veronese embedding)**.** A priori, the notion of a good prime depends on R. But the notion of a good prime is stable under a Veronese embedding. In particular, the notion of a good flag is stable under a Veronese embedding.

As mentioned in the beginning of this section, we now want to construction a sequence of degenerations inductively; the construction runs as follows. Given a good flag $Y_1 = V(\mathfrak{p}_1) \supset Y_2 = V(\mathfrak{p}_2) \supset \cdots$, we first degenerate $X = \operatorname{Proj} R$ to

$$X_1 = \operatorname{Proj}(\operatorname{gr}_{\mathfrak{p}_1^{(\infty)}} R),$$

where Proj is with respect to the grading inherited from R. In addition to the inherited grading, $S := \operatorname{gr}_{\mathfrak{p}_1^{(\infty)}} R$ also has the *second grading*, the grading due to the defining direct sum. Thus, X_1 has a $\mathbb{G}_m$-action, which is linear in the sense that it is induced from the $\mathbb{G}_m$-action on the affine cone $\operatorname{Spec} S$. The $\mathbb{G}_m$-invariant subring $S^{\mathbb{G}_m}$ of S is $R/\mathfrak{p}_1$ and so the GIT quotient takes the form:

$$\pi : X_1^{ss} \to Y_1 = \operatorname{Proj}(R/\mathfrak{p}_1).$$

Since $Y_2 \subset Y_1$ is defined by a good prime, we already know we can degenerate Y_1 along Y_2 and we want to *lift* this property to X_1, through the above π.

The next proposition establishes the important lifting property of a GIT quotient map for good primes.

7.1.4 Proposition (lifting property)**.** *Let $X = \operatorname{Proj} R$ be a projective variety and let a torus $T = \mathbb{G}_m^r$ act on R as grade-preserving automorphisms. Let $A = R^T$ be the subring of invariant elements. Assume T acts on R in such a way $R = \bigoplus_{\chi \geq 0} R_\chi$.*

Given a good prime $\mathfrak{p}$ of A, there exists a T-invariant homogeneous geometrically prime[1] *ideal $\mathfrak{q}$ that lies over $\mathfrak{p}$.*

[1] An ideal I of a k-algebra A is *geometrically prime* if $A/\mathfrak{p} \otimes_k \overline{k}$ is an integral domain where $\overline{k}$ is an algebraic closure of k.

Proof. Without loss of generality, we assume R is an integral domain. Let $\widetilde{A}, \widetilde{R}$ be the integral closures of A, R in their respective fields of fractions. Note $\widetilde{A}$ is then a subring of $\widetilde{R}^T$ and $\widetilde{R}^T$ is finite over $\widetilde{A}$.

Let $\widetilde{\mathfrak{p}}$ be a height-one homogeneous prime ideal of $\widetilde{A}$ such that $\bigoplus_0^\infty \widetilde{\mathfrak{p}}^{(n)} \cap A$ is Noetherian (which exists as $\mathfrak{p}$ is a good prime). Choose a height-one homogeneous prime ideal $\widetilde{\mathfrak{q}}$ of $\widetilde{R}$ that lies over $\widetilde{\mathfrak{p}}$. Then Lemma 6.3.6 says:

$$\widetilde{\mathfrak{p}}^n \widetilde{R}_{\widetilde{\mathfrak{q}}} \cap A = \widetilde{\mathfrak{p}}^{(n)} \cap A.$$

Since $\widetilde{R}_{\widetilde{\mathfrak{q}}}$ is a discrete valuation ring, for some $m > 0$, we have: $\widetilde{\mathfrak{q}}^m \widetilde{R}_{\widetilde{\mathfrak{q}}} = \widetilde{\mathfrak{p}} \widetilde{R}_{\widetilde{\mathfrak{q}}}$. Thus,

$$\widetilde{\mathfrak{q}}^{(nm)} \cap A = \widetilde{\mathfrak{p}}^{(n)} \cap A.$$

The proof is now finished by the graded Nakayama lemma (Lemma 2.6.1). Let $I = \bigoplus_{\chi > 0} R_\chi$, which is an ideal (by assumption) and $\varphi : R \to R/I = A$ the projection (cf. Lemma 2.4.2). Consider the surjective ring homomorphism induced by φ:

$$B = \bigoplus_{n=0}^{\infty} (\widetilde{\mathfrak{q}}^{(nm)} \cap R) \to C = \bigoplus_{n=0}^{\infty} (\widetilde{\mathfrak{p}}^{(n)} \cap A).$$

Since A is a direct summand of R, A is a projective R-module; in particular, $\operatorname{Tor}_1^R(-, A) = 0$. Hence, $C = B \otimes_R A = B/IB$. Also, since $\mathfrak{p}$ is a good prime, we have that C is a Noetherian ring. Choose a surjective grade-preserving ring homomorphism $R/I[u_1, \dots, u_q] \to C$, which is a grade-preserving ring homomorphism $R[u_1, \dots, u_q] \to B$ modulo I; here we do not insist the degree of each u_i is one. Let coker be the cokernel of $R[u_1, \dots, u_q] \to B$. Then $\operatorname{coker} \otimes_R A = \operatorname{coker} / I \operatorname{coker}$ is the cokernel of $R/I[u_1, \dots, u_q] \to C$, which is zero. Hence, $\operatorname{coker} = 0$ by the graded Nakayama lemma (Lemma 2.6.1) applied to the grading inherited from R. □

The next example is a very good example illustrating the situation to which the proposition applies.

7.1.5 Example. Let $X = V(sy - tx) \subset \mathbb{P}^3 = \operatorname{Proj} k[s,t,x,y]$ be the quadric surface (over $k = \mathbb{C}$). It is the image of the Segre embedding

$$\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\sim} X \subset \mathbb{P}^3,\ ((u_0 : u_1), (v_0 : v_1)) \mapsto (u_0v_0 : u_0v_1 : u_1v_0 : u_1v_1).$$

Let R be the homogeneous coordinate ring of X and write $\overline{x}, \overline{y}$, etc for the images of x, y, etc in R. We shall use the flag: as sets,

$$X = Y_0 \supset Y_1 = V(\overline{x}, \overline{y}) \supset Y_2 = V(\overline{x}, \overline{y}, \overline{s}),$$

which is a good flag by Example 7.1.2. First, let $\mathfrak{p} = (\overline{x}, \overline{y}) \subset R$ be the prime ideal defining Y_1. Then, by the same computation as in Example 6.2.7, the initial form of the defining equation $sy - tx$ with respect to $(x, y) \subset k[x, y, s, t]$ is $sy - tx$ itself. Hence, $R \simeq \operatorname{gr}_{\mathfrak{p}} R$; in particular, the latter ring is an integral domain and so $\mathfrak{p}^{(n)} = \mathfrak{p}^n$ for every n (Remark 6.1.6 (i)) and $\mathfrak{p}$ is a good prime. Note that the isomorphism says in particular that $A = R/\mathfrak{p} = k[s,t]$ is a subring of R.

Next, let $\mathfrak{p}_2 = (s) \subset R/\mathfrak{p}$. Note that the extension $\mathfrak{p}_2 R$ is not a prime ideal since $R/\mathfrak{p}_2 R \simeq k[t, x, y]/(tx)$. In fact, $\mathfrak{p}_2 R = (s, \overline{x}) \cap J$ where $J := (s,t)R = A_+ R$ is the defining ideal of the unstable locus X^{us}. Note $V(s, \overline{x}) = \operatorname{Proj}(k[t, y]) \simeq \mathbb{P}^1$ and so $V(\mathfrak{p}_2 R) = \mathbb{P}^1 \cup X^{us}$. Let $\mathfrak{q} = (s, \overline{x})$. Then it lies over $\mathfrak{p}_2$ and is a good prime.

7.2 Constructing a good degeneration sequence

We are now ready to give the precise version of the construction described early.

7.2.1 Theorem (good degeneration sequence)**.** *Let $X = \operatorname{Proj} R$ be a projective variety of dimension d over an infinite field. Given a good flag*

$$X = Y_0 \supset Y_1 \supset \cdots \supset Y_r$$

with respect to R, for each $1 \le i \le r$, we can inductively construct a sequence of flat degenerations

$$X \rightsquigarrow X_1 \rightsquigarrow \cdots \rightsquigarrow X_i = \operatorname{Proj}(\operatorname{gr}^i R)$$

having the following properties:

(i) The torus $\mathbb{G}_m^i$ acts on $\operatorname{gr}^i R$ as grade-preserving automorphisms and Y_i is the GIT quotient of X_i; i.e., $Y_i = X_i//\mathbb{G}_m^i$.

(ii) The quotient map $\pi : X_i^{ss} \to Y_i$ admits a section; i.e., there is a closed embedding $\iota : Y_i \hookrightarrow X_i^{ss}$ such that $\pi \circ \iota$ is the identity.

(iii) The moment polytope of X_i with respect to $\operatorname{gr}^i R$ has dimension i (which implies a generic stabilizer is finite),

Proof. By Proposition 7.1.4, we can carry out the construction discussed at beginning of the present section, assuming each resulting degeneration has the asserted properties (i) - (iii). Thus, here, we only verify those properties.

In the notions of the theorem, the grade-preserving surjection $S \to R/\mathfrak{p}_i$ gives

$$Y_i \hookrightarrow X_i.$$

Now, for each homogeneous $g \in (R/\mathfrak{p})_+$, the above restricts to $Y_{i,g} \hookrightarrow X_{i,g}$ (where the subscript g means the locus where g doesn't vanish). Since $X_i^{ss} = \bigcup_{g\in(R/\mathfrak{p})_+ \text{ homog}} X_{i,g}$, the above inclusion is in fact:

$$Y_i \hookrightarrow X_i^{ss}.$$

This proves (ii). (iii) is not hard to check by examining the generators of $\operatorname{gr}^i R$.

□

The next example shows that the stable loci of the GIT quotient in the theorem are usually empty without a twisting.

7.2.2 Example. Let $X = \mathbb{P}(V)$ where V is a two-dimensional vector space. With respect to the action of a trivial group with trivial linearlization, we have $X = X^{ss} = X^s$.

Let $R = k[V]$ be the homogeneous coordinate ring of $X = \mathbb{P}(V)$ and $\mathfrak{p} = (x)$ the prime ideal generated by some homogeneous element x of degree one. Since $\mathfrak{p}$ is homogeneous, $\operatorname{gr}_{\mathfrak{p}^{(\infty)}} R$ has a grading inherited from R and then, as a graded k-algebra (we drop $\mathfrak{p}^{(\infty)}$),

$$\operatorname{gr} R = R/(x) \oplus (x)/(x)^2 \oplus \cdots \simeq R.$$

In addition to the inherited grading, $\operatorname{gr} R$ has another $\mathbb{N}$-grading given by the defining direct sum and denoted as $\operatorname{gr} R_*$; i.e., $\operatorname{gr} R_{*,n} = (x)^n/(x)^{n+1}$. One *can* consider the $\mathbb{G}_m$-action on

$X = \mathrm{Proj}(\mathrm{gr}\, R)$ corresponds to this $\mathbb{N}$-grading; in terms of homogeneous coordinates, the action is $t \cdot (x : y) = (t^{-1}x : y)$. Since $\mathrm{gr}\, R_+^{\mathbb{G}_m} = (R/(x))_+ \simeq yR$, we get:

$$X^{ss} = \mathbb{G}_m \cdot (1:1) \cup (0:1).$$

Now, if X^s is nonempty, then, since it is open and invariant, it must be that either $X^{ss} = X^s$ or $X^s = \mathbb{G}_m \cdot (1:1)$. The former is not possible since $\mathbb{G}_m$ fixes $(0:1)$. The latter is not possible either since $\mathbb{G}_m \cdot (1:1)$ is not closed in X^{ss}. Hence, X^s is empty.

Thankfully, there is another natural $\mathbb{G}_m$-action that we can use and is induced as follows. Since $\mathrm{gr}\, R$ is generated as $\mathrm{gr}\, R_{*,0} = R/(x)$-algebra by x and $\mathrm{gr}\, R_{*,0}$ is generated as k-algebra by some degree-one element, say, y, we have an isomorphism of k-algebras:

$$k[u_1, u_2] \to \mathrm{gr}\, R,\ \ u_1 \mapsto x,\ u_2 \mapsto y.$$

Now, SL_2 acts on $k[u_1, u_2]$ in the usual way and so does its diagonal torus $\mathbb{G}_m$. In terms of homogeneous coordinates, this $\mathbb{G}_m$-action is given (up to inverse) by:

$$t \cdot (x : y) = (tx : t^{-1}y).$$

Note, ignoring linearization, this is the same as $t \cdot (x : y) = (t^2 x : y)$; thus, as far as the orbit structure is concerned, this action is the same as the early one. On the other hand, it is easy to see:

$$X^{ss} = X^s = \mathbb{G}_m \cdot (1:1).$$

The next example works out the special case of a good flag that is a flag of set-theoretic hypersurfaces. The example relies on the following Bertini's theorem:

Théorème 6.3 (4) of [Jou83]: given a morphism φ of finite type from a geometrically irreducible scheme to $\mathbb{P}^n$ over an infinite field, if the image of φ has dimension ≥ 2, then $\varphi^{-1}(H)$ is geometrically irreducible for general hyperplanes H on $\mathbb{P}^n$.

(In loc. cit., the theorem is stated for $\mathbb{A}^n$ but can be shown to be valid for $\mathbb{P}^n$ as well.)

7.2.3 Example. In the setup of Theorem 7.2.1, suppose the good flag $Y_i = V(\mathfrak{p}_i)$ has the form $\mathfrak{p}_i = \sqrt{(f_1, \dots, f_i)}$ for some homogeneous elements $f_i \in R$, $1 \le i \le r$. We also assume the base field k is infinite[2] and that each Y_{i+1} is in a general position in Y_i (see below for the precise meaning). Note that, for each i, there is the natural ring homomorphism $R \to R/\mathfrak{p}_i \hookrightarrow \operatorname{gr}^i R$. We shall inductively show:

(i) There is an injective $\mathbb{G}_m^i$-equivariant graded finite ring homomorphism:

$$(R/\mathfrak{p}_i)[u_1, \dots, u_i] \hookrightarrow \operatorname{gr}^i R$$

where u_i's are $\mathbb{G}_m^i$-weight vectors of nonzero weights.

(ii) $\dim X_i^{us} = i - 1$.

(iii) A lift of $\mathfrak{p}_{i+1}$ to $\operatorname{gr}^i R$ is given as $\mathfrak{q}_{i+1} = \sqrt{f_{i+1} \cdot \operatorname{gr}^i R}$.

Let $\mathfrak{q}_2 = \sqrt{f_2 \cdot \operatorname{gr}^1 R} \subset \operatorname{gr}^1 R$ be the lift of $\mathfrak{p}_2 \subset R$ given by (iii). Then we form $\operatorname{gr}^2 R = \operatorname{gr}_{\mathfrak{q}_2^{(*)}} \operatorname{gr}^1 R$. It comes with an injective finite homomorphism $(\operatorname{gr}^1 R/\mathfrak{q}_2)[u_2] \hookrightarrow \operatorname{gr}^2 R$. Since $\mathfrak{q}_2$ lies over $\mathfrak{p}_2/\mathfrak{p}_1[u_1]$ in $R/\mathfrak{p}_1[u_1]$ as easily seen, $R/\mathfrak{p}_1[u_1] \hookrightarrow \operatorname{gr}^1 R$ induces $R/\mathfrak{p}_2[u_1] = \frac{R/\mathfrak{p}_1}{\mathfrak{p}_2/\mathfrak{p}_1}[u_1] \hookrightarrow \operatorname{gr}^1 R/\mathfrak{q}_2$. Composing them we get:

$$(R/\mathfrak{p}_2)[u_1, u_2] \hookrightarrow (\operatorname{gr}^1 R/\mathfrak{q}_2)[u_2] \hookrightarrow \operatorname{gr}^2 R.$$

Thus, on the i-th step, we will have (i). The item (ii) is a consequence of (i). Indeed, let $S = (R/\mathfrak{p}_i)[u_1, \dots, u_i]$ and let $g : X_i = \operatorname{Proj}(\operatorname{gr}^i R) \to X_i' := \operatorname{Proj} S$, which is well-defined and is finite by Corollary 2.6.2. Then it is known (see the end of [MFK94] Ch. I., §5) that $g^{-1}(X_i'^{ss}) = X_i^{ss}$. Hence, $\dim X_i'^{us} = \dim X_i^{us}$. Now, since $S/S_+^{\mathbb{G}_m^i} S = k[u_1, \dots, u_i]$ as in the proof of Proposition 7.1.4, $X_i'^{us} = \mathbb{P}^{i-1}$; in particular, $\dim X_i^{us} = \dim \mathbb{P}^{i-1} = i - 1$.

It remains to show (iii). Let $\pi : X_i^{ss} \to Y_i$ be the GIT quotient. If Y_{i+1} is a general set-theoretic hypersurface on Y_i,[3] then $\pi^{-1}(Y_{i+1})$ is irreducible of codimension one on X_i^{ss} by Bertini's theorem quoted above. Now, note that the set-theoretic hypersurface $H = V(\sqrt{f_{i+1} \cdot \operatorname{gr}^i R}) \subset X_i$ is such that $\operatorname{Supp}(H \cap X_i^{ss}) = \operatorname{Supp}(\pi^{-1}(Y_{i+1}))$. By Krull's principal ideal theorem, H consists of the irreducible components of codimension one in X_i. Since $\operatorname{codim} X_i^{us} \ge \dim X - i + 1 \ge 2$ by (ii), H cannot have an irreducible component disjoint from

[2]We have not investigated the finite field case.

[3]By "set-theoretic hypersurface", we mean a set-theoretic complete intersection of codimension one.

X_i^{ss}. Since $H \cap X_i^{ss}$ is irreducible of codimension one, $H = \overline{H \cap X_i^{ss}}$ and H is irreducible; i.e., $\mathfrak{q}_{i+1} = \sqrt{f_{i+1} \cdot \operatorname{gr}^i R}$ is a prime ideal, which is a lift of $\mathfrak{p}_{i+1}$.

7.2.4 Proposition. *Let X be a projective variety of dimension d over an infinite field k.*

Suppose we are given some flat degeneration $X \rightsquigarrow Z$ to a toric variety Z with the torus $\mathbb{G}_m^d$, where the degeneration takes place inside some projective space. Then we can find some good flag that gives rise to a sequence of degenerations $X \rightsquigarrow \cdots \rightsquigarrow X_d$ with $X_d = Z$.

Proof. Replacing R by some Veronese subring, we assume R is an integral domain. Then, by assumption, there is a valuation $\nu : R - 0 \to \mathbb{Z}^d$ such that $Z = \operatorname{Proj}(\operatorname{gr}_\nu R)$. The valuation ν determines a flag as follows. Without loss of generality (why?), $\nu(R-0) \subset \mathbb{N}^d$. Let $p_i : \mathbb{Z}^d \to \mathbb{Z}$ be the projection onto the i-th component. Then, for each $1 \le j \le i$, let $\mathfrak{p}_i$ be generated by $\{f \in R - 0 \text{ homogeneous } | (p_1 \circ v)(f) = \cdots = (p_{j-1} \circ v)(f) = 0 < (p_j \circ v)(f), 1 \le j \le i\}$. Then clearly $0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_d$ is a chain of prime ideals. We check $\mathfrak{p}_i/\mathfrak{p}_{i-1}$ is a good prime in $R/\mathfrak{p}_{i-1}$. Consider $S = \bigoplus_{a \in \mathbb{N}^d} \{f \in R - 0 | \nu(f) \ge a\}$, which is Noetherian. Then $\bigoplus_{n=0}^{\infty} \mathfrak{p}_i^{(n)}$ is a direct summand of S and thus is a Noetherian ring by Lemma 5.4.7; i.e., $\mathfrak{p}_i$ is a good prime. □

Our construction of a sequence of degenerations from a good flag (Theorem 7.2.1) may be thought of as an extension of the following construction. Okounkov's original construction was in an equivariant setup and will be revisited in §9. The key piece of the construction is the following:

7.2.5 Lemma. *Let $Y \subset X$ be a codimension-one closed subvariety of an algebraic variety X; i.e., Y is a prime Weil divisor. If*

$$\nu' : k(Y)^* \to \mathbb{Z}^{r-1}$$

is a valuation whose image is a free abelian group of rank $r-1$, then there exists a not-necessarily-unique valuation

$$\nu : k(X)^* \to \mathbb{Z}^r,$$

such that (1) $\nu(f) = \nu'(f|_Y)$ for each f in $k(X)$ that does not have zero or pole along Y, so that $f|_Y$ is defined and nonzero, and (2) the image of ν is a free abelian group of rank r.

Proof. Let $\pi : \widetilde{X} \to X$ be the normalization. Choose an irreducible component Z of $\pi^{-1}(Y)$ as well as a generator u of the maximal ideal of the local ring $\mathcal{O}_{\widetilde{X},Z} \subset k(X)$. Let $\mathrm{N} = \mathrm{N}_{Z/Y} : k(Z)^* \to k(Y)^*$ be the norm map. Define $\nu : k(X)^* \to \mathbb{Z}^r$ by

$$\nu(f) = (\mathrm{ord}_Z(f), (\nu' \circ N)(f'))$$

where $f' = (u^{-\mathrm{ord}_Z(f)} f)|_Z$. □

7.3 A good degeneration sequence as the Parshin-Okounkov construction

7.3.1 Remark (Parshin-Okounkov construction in [Oko96])**.** Let X be an algebraic variety together with a flag of closed subvarieties:

$$Y_\bullet : X = Y_0 \supset Y_1 \supset \cdots \supset Y_r.$$

Applying Lemma 7.2.5 inductively then gives a valuation

$$\nu : k(X)^* \to \mathbb{Z}^r.$$

Namely, first we get a valuation $\nu_r : k(Y_r)^* \to \mathbb{Z}$ and from that we get $\nu_{r-1} : k(Y_{r-1})^* \to \mathbb{Z}^2$ and so forth.

If $X \subset \mathbb{P}^N$ is a projective variety, then the above construction also applies to the homogeneous coordinate ring R of X. Indeed, choose a degree-one homogeneous element f of R such that f that does not vanish on Y_r and then, for each $n > 0$, define $\nu : R_n - 0 \to \mathbb{Z}^r$ by $\nu(g) = \nu(g/f^n)$. (Clearly, this valuation is independent of a choice of f).

We note that the above construction depends only the intrinsic geometry of X; i.e., it is independent of the way X is embedded into a projective space or an affine space. In contrast, the valuations constructed in the previous section depends on the extrinsic geometry of X.[4]

[4]The observation that a sequence of degenerations in Theorem 7.2.1 can be composed into a single degeneration is originally due to Christopher Manon.

7.3.2 Proposition. *Let us be in the setup of Theorem 7.2.1 and write $Y_i = V(\mathfrak{p}_i)$ for each $1 \le i \le r$. Replacing R by a Veronese subring, assume R is an integral domain.*

Then there exists a valuation $\nu : R - 0 \to \mathbb{Z}^r$ such that (1) $\operatorname{gr}^r R = \operatorname{gr}_\nu R$ and (2) the valuation

$$k(X) - 0 \to \mathbb{Z}^r,\ f/g \mapsto \nu(f)/\nu(g),$$

where f, g are homogeneous elements of R of the same degree, coincides with the valuation constructed by Remark 7.3.1, up to taking a Veronese subring of R.

Proof. Let $\mathfrak{p} = \mathfrak{p}_1$. Taking a Veronese subring of R, without loss of generality, we can assume $R_\mathfrak{p}$ is a discrete valuation ring; let $\nu_\mathfrak{p}$ be the associated valuation and π a homogeneous element of R that is a generator of $\mathfrak{p}R_\mathfrak{p}$ (it is easy to check that such a π exists). We can also assume that $R - \mathfrak{p}$ contains a degree-one element s. For each homogeneous element f of R, we write $f_s = f/s^{\deg(f)}$. We note that π_s generates the maximal ideal $\mathfrak{m}$ of the local ring $\mathcal{O}_{Y_1,X}$ at Y_1; indeed, suppose f, g are homogeneous elements of R of the same degree such that f/g is in $\mathfrak{m}$. We write $f = a\pi^{\nu_\mathfrak{p}(f)}$ and $g = b\pi^{\nu_\mathfrak{p}(g)}$ for some homogeneous $a, b \in R - \mathfrak{p}$. Clearly, $\nu_\mathfrak{p}(f) > \nu_\mathfrak{p}(g)$ and so $f/g = a/b\pi^{\nu_\mathfrak{p}(f) - \nu_{\mathfrak{p}_1}(g)} \in \pi R_\mathfrak{p}$ and thus $f/g \in \pi_s \mathcal{O}_{Y_1,X}$. In particular, $\nu_{\mathfrak{p}_1}(f_s)$ is the same as the order-of-vanishing of f_s along Y_1.

The proposition is now essentially clear. Let a nonzero homogeneous element f be given. Let $a_1 = \operatorname{ord}_{Y_1}(f_s)$. Then, with g_1 in $R - \mathfrak{p}_1$, we can write $f_s = g_{1,s}u_1^{a_1}$ or $f = g_1 u_1^{a_1}$. Let f_1 be the image of g_1 in $R/\mathfrak{p}_1$, which is a nonzero element. Let $\mathfrak{q}_2$ be a lift of $\mathfrak{p}_2$ to $\operatorname{gr}^1 R$ and $Z_2 := V(\mathfrak{q}_2) \subset X_2 = \operatorname{Proj}(\operatorname{gr}^2 R)$. Then, as above, $\nu_{\mathfrak{q}_2}$ and ord_{Y_2} agree after taking a Veronese subring of R if necessarily. The rest of the proof is now clear.

□

We did not have enough time to verify the following example (so we state it as a question).

7.3.3 Question (Bott-Samelson varieties; cf. [An13] §6.4.)**.** Let $B \subset G = \mathrm{GL}_3(\mathbb{C})$ be the Borel subgroup that consists of the upper-triangular matrices. Let $\alpha = (1, -1, 0), \beta = (0, 1, -1)$ be the two simple roots, P_α, P_β the corresponding parabolic subgroups. Set $\underline{w} = (s_\alpha, s_\beta)$ and let $Z_{\underline{w}}$ be the quotient $P_\alpha \times^B P_\beta \times^B P_\alpha/B$. Let $\varphi : Z_{\underline{w}} \to \mathrm{GL}_3(\mathbb{C})/B$ be the birational morphism induced by the multiplications. Let $L = L(\rho)$ be the very ample line bundle on $\mathrm{GL}_3(\mathbb{C})/B$ corresponding to ρ. We have the flag

$$X = Y_0 \supset Y_1 = \{x = 0\} \supset \cdots \supset Y_3 = \{x = y = z = 0\}$$

Let $M = \varphi^* L \otimes \mathrm{pr}_1 \mathcal{O}(1)$, where $\mathrm{pr}_1 : X \to P_\alpha/B \simeq \mathbb{P}^1$ and R the section ring of M. Then is $Y_\bullet$ a good flag?

8.0 Specialization of ampleness

The style of this section is mostly expository. No particularly new ideas or methods will be introduced; instead, we try to explain the construction in the previous section in the context of intersection theory. With no additional efforts, this will allow us to (1) give a characterization of a good flag in terms of a degeneration of an ample line bundle and (2) extend the intersection-number formula for non-normal toric varieties in §3 to toric degenertaions.

This section is also preparatory for Part 3 in which we consider equivariant analogs of these matters.

8.1 A normal cone in intersection theory

We begin with recalling the use of a normal cone in intersection theory from [Ful93]. First fix a smooth algebraic variety X, which is thought of as an ambient space. Suppose we wish to define the *intersection product* of closed subschemes $V, W \subset X$:

$$V \cdot_X W.$$

This would be easy if X is a vector bundle over W (and then $W \hookrightarrow X$ is the zero-section embedding). The idea in [Ful98] is to reduce the general case to this special case by a degeneration: (for simplicity) say $W \hookrightarrow X$ is a regular embedding and then the normal cone X_0 to it is a vector bundle $X_0 \to W$ and then the degeneration $X \rightsquigarrow X_0$ along W fixes W; in particular, at least as a set, $V \cap W$ remains intact under the degeneration. The degeneration does deform V but it does so in such a way the induced degeneration $V \rightsquigarrow V_0$ is a rational equivalence; i.e., $V \sim V_0$ are rationally equivalent. Hence,

$$V \cdot_X W := V_0 \cdot_{X_0} W$$

is a well-defined product, up to a rational equivalence. This approach is quite similar to the conventional approach that uses the moving lemma, the key difference being: here, the ambient space X is changed to X_0 but the intersection $V \cap W$ itself is kept throughout.

We add a remark that, as already observed by [Kn05], a normal cone can be replaced by a generalization of it such as a balanced normal cone of Knutson or a symbolic normal cone of the present thesis. Indeed, the discussion of an ideal filtration in §5 generalizes in an obvious way to ideal sheaves, as already noted in Definition 5.1.4. Let X be an algebraic variety and $\mathfrak{v} = \{\mathcal{I}_a\}_{a \in \mathbb{N}^r}$ an ideal filtration. The ideal filtration $\mathfrak{v}$ determines a closed subscheme with the filtration $V(\mathcal{I}_a)$ or a *filtered closed subscheme*. For example, if $r = 1$ and $\mathcal{I}_n = \mathcal{I}^n$, the $V(\mathcal{I}^n)$ is the n-th infinitesimal neighborhood of $V(\mathcal{I})$. Also, if X is normal[1] and if $D \subset X$ is a reduced Weil divisor,[2] then $D \subset 2D \subset 3D \subset \cdots$, where $nD = V(\mathcal{O}_X(-nD))$, is another example of a filtered closed subscheme.

8.1.1 Remark (affine blow-up)**.** As before, we say an ideal filtration sheaf $\mathfrak{v} = \{\mathcal{I}_\alpha | \alpha \in \mathbb{N}^r\}$ on X is of finite type if the associated (sheaf version) Rees algebra $\bigoplus_{\alpha \geq 0} \mathcal{I}_\alpha$ is finitely generated as an $\mathcal{O}_X$-algebra. For such an ideal filtration sheaf, because of finiteness, the obvious sheaf analog of Theorem 5.2.1 is valid: namely, we have

$$\mathcal{R}_{\mathfrak{v}} = \mathcal{O}_X[t] \oplus \mathfrak{v}_1 t^{-1} \oplus \mathfrak{v}_2 t^{-2} \oplus \cdots$$

such that $\mathcal{R}_{\mathfrak{v}}/t\mathcal{R}_{\mathfrak{v}} = \operatorname{gr}_{\mathfrak{v}} \mathcal{O}_X \overset{\text{def}}{=} \bigoplus_{\alpha \geq 0} \mathfrak{v}_\alpha / \cup_{\beta > \alpha} \mathfrak{v}_\beta$ and then it defines the flat morphism:

$$\pi_{\mathfrak{v}} : \operatorname{Spec}_X(\mathcal{R}_{\mathfrak{v}}) \to \mathbb{A}^1,$$

which is an affine analog of a generalized extended blow-up. We usually call it a (generalized extended) *affine blow-up*. (We ignore the non-uniqueness of the construction of $\mathcal{R}_{\mathfrak{v}}$.) By definition, the special fiber $\pi_{\mathfrak{v}}^{-1}(0)$ is the generalized normal cone to the filtered closed subscheme $Y_{\mathfrak{v}} = V(\mathfrak{v})$.

[1] "normal" is needed to define $\mathcal{O}_X(-nD)$.

[2] A *reduced Weil divisor* is a pure-codimension-one closed subset, which we can view as the Weil divisor that is the formal sum of the irreducible components of the set.

We note that the definition of a symbolic normal cone in §6 extends to the non-affine case in a straightforward way: given a closed subscheme $Y \subset X$ with an ideal sheaf $\mathcal{I}$, define $\mathcal{I}^{(n)}$ by $\Gamma(U, \mathcal{I}^{(n)}) = \Gamma(U, \mathcal{I})^{(n)}$ for each open affine subset $U \subset X$. Then

8.1.2 Lemma. *In the preceding notations, the ideal filtration $\mathcal{I}^{(\infty)} = \{\mathcal{I}^{(n)} | n \geq 0\}$ is well-defined. It is of finite type, at least, in either of the following two cases (1) Y is an effective Cartier divisor or (2) X is a quasi-projective variety and $Y = X \cap V(\mathfrak{p})$ is defined by a good prime $\mathfrak{p}$ of some graded Noetherian ring whose* Proj *is the closure of X.*

Proof. The first assertion is clear in view of Proposition 6.1.2 (iii). The rest of the assertions are also clear. □

8.2 Quasi-projectivity of a degeneration and a good flag

It is natural to ask whether or when an affine blow-up "quasi-projective-ness". It is answered partially by the following:

8.2.1 Definition (quasi-projective)**.** A flat degeneration $\pi : X' \to \mathbb{A}^1$ is said to be quasi-projective if (1) $X' \subset \mathbb{P}^N \times \mathbb{A}^1$ as a closed subscheme and (2) π is the restriction of the projection.

8.2.2 Proposition. *If X is a quasi-projective variety and if $\mathcal{R}_{\mathfrak{v}}$ is of finite type, then the affine blow-up* $\operatorname{Spec}_X(\mathcal{R}_{\mathfrak{v}})$ *is quasi-projective in the sense of Definition 8.2.1.*

Proof. In view of Theorem 5.2.1, we can assume ideal filtration $\mathfrak{v}$ is indexed by $\mathbb{N}$. Let L be an ample line bundle on X, $R = \bigoplus \Gamma(X, L^{\otimes n})$ its section ring and then set $\overline{X} = \operatorname{Proj} R$. For each $\alpha \in \mathbb{N}$, let $I_\alpha = \bigoplus_{n=0}^{\infty} \Gamma(X, L^{\otimes n} \otimes \mathcal{I}_\alpha)$, which is a homogeneous ideal of R. Let $R' = R[t] \oplus It^{-1} \oplus I_2 t^{-2} \oplus \cdots$. We claim

$$\operatorname{Spec}_X(\mathcal{R}_{\mathfrak{v}}) \subset \operatorname{Proj}(R')$$

in such a way the left-hand side is an open subset of the right-hand side. The claim is seen by unwinding the construction. □

8.2.3 Remark (affine blow-up as the semistable locus)**.** Assume for simplicity $X = \operatorname{Proj} R$ is projective. Then the proof of the preceding proposition in fact says the following: $\operatorname{Spec}_X(\mathcal{R}_{\mathfrak{v}})$ is the semistable locus of $X' = \operatorname{Proj}(R')$ with respect to the natural $\mathbb{G}_m^r$-action on X'.

We also need the next general lemma in the below:

8.2.4 Lemma (constancy of Hilbert polynomial)**.** *Let S be an integral Noetherian scheme and $\mathbb{P}^n_S = \mathbb{P} \times S$ the projective space over it. Then, for each flat closed subscheme $X \subset \mathbb{P}^n_S$, the Hilbert polynomial of X is constant on fibers.*

Proof. This is standard; e.g., [Hart77] Ch. III, Theorem 9.9. □

Having established the preliminary materials on generalized normal cones, we are ready to state the main results of this section. We recall that the substitute Chow functor $\operatorname{gr} \mathbf{G}$ as intersection number was defined in §3. We now consider them in presence of flat degenerations.

Thus, suppose we are given some degeneration $X \rightsquigarrow X_0$ with the total space X' and the trivialization $X' - X_0 \simeq X \times (\mathbb{A}^1 - 0)$ (Definition 5.1.1). Assume that X, X_0 are complete varieties. Then we define the *specialization homomorphism*

$$\alpha : \operatorname{gr} \mathbf{G}(X) \to \operatorname{gr} \mathbf{G}(X_0)$$

as the composition:

$$\operatorname{gr} \mathbf{G}(X) \to \operatorname{gr} \mathbf{G}(X \times (\mathbb{A}^1 - 0)) \simeq \operatorname{gr} \mathbf{G}(X' - X_0) \stackrel{\eta}{\to} \operatorname{gr} \mathbf{G}(X')$$
$$\stackrel{F \mapsto F|_{X_0}}{\to} \operatorname{gr} \mathbf{G}(X_0)$$

where the first map is the induced by the inclusion, the second by the trivialization and η is given by extending coherent sheaves by means of [Hart77, Ch. II, Exercise 5.15.]. That this is well-defined, we will confirm that in Proposition 8.2.5 below.

We will usually write $F_0 = \alpha(F)$ and call F_0 the *specialization of F*. The next proposition collects the key properties of the specialization homomorphism. We note the following terminology.

8.2.5 Proposition. *Given the specialization homomorphism and the trivialization* $\varphi : X \times (\mathbb{A}^1 - 0) \xrightarrow{\sim} X' - X_0$ *as above, the following hold:*

(i) The specialization map defined above

$$\operatorname{gr}\mathbf{G}(X) \to \operatorname{gr}\mathbf{G}(X_0)$$

is a well-defined group homomorphism.

(ii) If $[F] = [\mathcal{O}_V]$*, then* $[F_0] = [\mathcal{O}_{V_0}]$ *where* V_0 *is the scheme-theoretic intersection with* X_0 *of the scheme-theoretic closure* $\overline{\varphi(V \times (\mathbb{A}^1 - 0))}$ *within* X'*.*

(iii) If π *is quasi-projective in the sense of Definition 8.2.1, then, for each coherent sheaf* F *on* X*,*

$$\chi(X, F) = \chi(X_0, F_0).$$

Moreover, if $F = \mathcal{O}_V$ *and* $F = \mathcal{O}_{V_0}$*, then* V_0 *is connected.*

(iv) For each coherent sheaves F, G *on* X*,*

$$[\mathcal{H}om_{\mathcal{O}_X}(F, G)_0] = [\mathcal{H}om_{\mathcal{O}_{\mathcal{X}'}}(F_0, G_0)].$$

Proof. (i) The argument here was adopted from [FL85, Ch. VI. §3 Appendix.] As we noted earlier, [Hart77, in Ch. II. Exercise 5.15.], given a coherent sheaf F on $U = X' - X_0$, we can extend it to a coherent sheaf F' on X'. First we show this determines a well-defined function $\operatorname{gr}\mathbf{G}(X' - X_0) \xrightarrow{\eta} \operatorname{gr}\mathbf{G}(X')$.

Consider the exact sequence $0 \to F_1 \to F_2 \to F_3 \to 0$ of coherent sheaves on X. By (d) of the same exercise in [Hart77], we can find coherent sheaves $G_1 \subset G_2$ such that $G_i|_U = F_i$. Set $G_3 = G_2/G_1$. Then $G_3|_U = F_3$ and we have $[G_2] = [G_1] + [G_3]$. Hence, this gives the well-defined $\mathbf{G}(X' - X_0) \to \mathbf{G}(X')$. Now, since U is dense in X', $\dim \operatorname{Supp}(F) = \dim \operatorname{Supp}(G)$ when G is an extension of F. Hence, the map determines $\eta : \operatorname{gr}\mathbf{G}(X' - X_0) \to \operatorname{gr}\mathbf{G}(X')$. It is a group homomorphism: if G, G' are extensions of F, F', then $(G \oplus G')|_U = F \oplus F'$, which implies

$$[G] + [G'] = \eta([F] + [F']),$$

while $[G] = \eta([F])$ and $[G'] = \eta([F'])$.

We check $F \mapsto F|_{X_0} : \operatorname{gr} \mathbf{G}(X') \to \operatorname{gr} \mathbf{G}(X_0)$ is well-defined. For $i : X_0 \hookrightarrow X$, we recall that, by definition ([Hart77, Ch. II, §5]),

$$F|_{X_0} = i^* F = i^{-1} F \otimes_{i^{-1}\mathcal{O}_X} \mathcal{O}_{X_0}.$$

Thus, given an exact sequence $0 \to F \to G \to H \to 0$ of coherent sheaves on X', we have:

$$0 \to K \to F_0 \to G_0 \to H_0 \to 0,$$

where the subscript 0 refers to the restriction to X_0 and $K = \ker(F_0 \to G_0)$. Then $[F_0] + [K] = [G_0] + [H_0]$. Note that $F_0 \to G_0$ is injective on some open dense subset of X_0 and thus the kernel K has strictly smaller support than that of F_0. Hence, writing $\operatorname{gr} -$ for the class of - in $\operatorname{gr} \mathbf{G}(X_0)$, we then have: $\operatorname{gr}[F_0] = \operatorname{gr}[G_0] + \operatorname{gr}[H_0]$. (Note that this last part is an analog of the fact that if a cycle $\alpha \sim 0$ is rationally equivalent to 0, then its restriction to X_0 is also rationally equivalent to 0.)

(ii) Let V' denote the closure of $\varphi(V \times (\mathbb{A}^1 - 0))$ in X'. Then $\mathcal{O}_{V'}|_U = \mathcal{O}_{V' \cap U}$ and thus $[\mathcal{O}_{V'}]$ is the same as the image of $\operatorname{gr} \mathbf{G}(X) \to \operatorname{gr} \mathbf{G}(X')$. Now, $\mathcal{O}_{V'}|_{X_0}$ is the same as $\mathcal{O}_{V' \cap X_0}$. Hence, the assertion is valid.

(iii) Since F is coherent (i.e., a finite $\mathcal{O}_X$-module), we can find a filtration $F = G_0 \supset G_1 \supset \cdots \supset G_r = 0$ such that G_i/G_{i+1} is of the form $\mathcal{O}_{V_i}$ with $V_i \subset X$ a closed subvariety. Since $\chi(X, F) = \sum_i \chi(X, G_i/G_{i+1})$, without loss of generality, we can assume $F = \mathcal{O}_V$. Now, by Lemma 8.2.4, V, V_0 have the same Hilbert polynomial; i.e., $\chi(X, \mathcal{O}_V(n)) = \chi(X_0, \mathcal{O}_{V_0}(n))$ for each integer $n > 0$; in particular, that is the case for $n = 1$.

For the connnectedness of V_0, as a consequence of the theorem on formal functions, we know that $V' \to \mathbb{A}^1$ has connected fibers (see [Hart77] Exercise ?), where V' is as in the proof of (ii). In particular, V_0, the fiber over 0, is connected.

(iv) For the extensions $\eta(F), \eta(G)$ of F, G from $U = X' - X_0$ to X', since the sheaf Hom commutes with restriction, we have:

$$\mathcal{H}om_{\mathcal{O}_{X'}}(\eta(F), \eta(G))|_U = \mathcal{H}om_{\mathcal{O}_U}(\eta(F)|_U, \eta(G)|_U) = \mathcal{H}om(F, G),$$

which is to say $\mathcal{H}om_{\mathcal{O}_{X'}}(\eta(F), \eta(G)) = \eta(\mathcal{H}om_{\mathcal{O}_U}(F, G))$. □

8.2.6 Remark. The proof of (iii) above also shows that $\operatorname{gr}\mathbf{G}(X)$ is generated by the classes $[\mathcal{O}_V]$. Explicit, if $[F]$ is a class in $\operatorname{gr}\mathbf{G}(X)$, then

$$[F] = \sum \operatorname{length}_{\mathcal{O}_V}(F)[\mathcal{O}_V]$$

where the sum runs over all subvarieties $V \subset X$.

8.2.7 Corollary. *$(F^*)_0 = (F_0)^*$ where the superscript $*$ refers to the dual sheaf; i.e. $F^* = \mathcal{H}om_{\mathcal{O}_X}(F, \mathcal{O}_X)$.*

Proof. Take $G = \mathcal{O}_X$ and use (iv) of the proposition. □

We note the following result:

8.2.8 Proposition. *Let X be an algebraic variety and $Y_\bullet : X = Y_0 \supset Y_1 \supset \cdots \supset Y_r$ a flag of closed subvarieties such that each Y_{i+1} is an effective Cartier divisor on Y_i.*

Then there exists a flat degeneration $\pi : X' \to \mathbb{A}^1$ of X of finite type such that

$$\pi^{-1}(0) = \operatorname{Spec}_X(\operatorname{Sym}_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2))$$

where $\mathcal{I}$ is the ideal sheaf of $Y_r \hookrightarrow X$. Also, if

$$\nu : k(X) - 0 \to \mathbb{Z}^r$$

is the valuation constructed from $Y_\bullet$ by Remark 7.3.1, then, for each sufficiently small affine open subset $U = \operatorname{Spec} A \subset X$,

$$\Gamma(U, \mathcal{O}_{\pi^{-1}(0)}) = \operatorname{gr}_\nu A.$$

Proof. Since the assertions are all local, without loss of generality, we can assume $X = \operatorname{Spec} A$ is affine and that $\mathcal{I}$ is an ideal of A of height r generated by a regular sequence $x_1, \ldots, x_r$. Then the assertion is clear (it is precisely a theorem of Rees). □

The specialization homomorphism can now be used to state the following characterization of a good flag:

8.2.9 Theorem. *Let X be a projective variety, L an ample line bundle on it and $\pi : X' \to \mathbb{A}^1$ a flat degeneration of X constructed from some flag $Y_\bullet$ by Proposition 8.2.8. Then the following are equivalent:*

(i) X' is quasi-projective in such a way $X' \subset \mathbb{P}^N \times \mathbb{A}^1$ as a closed subscheme so that, for some $m > 0$, $L^{\otimes m}$ is the pullback of $\mathcal{O}_{\mathbb{P}^N}(1)$ under $X \hookrightarrow X'$. and π is the restriction of the projection.

(ii) After replacing L by some tensor power, the specialization L_0 is an ample line bundle.

(iii) $Y_\bullet$ is a good flag with respect to L.

Proof. This is essentially the restatement of the results of §7. □

It is natural to ask:

8.2.10 Question. Given a flag $Y_\bullet$ (with some Cartier-type assumption) on a projective variety X, can we find an ample line bundle L on X with respect to which $Y_\bullet$ is a good flag?

The question is trivially true for a projective curve (since any point on a curve is a hypersurface with respect to some embedding). We plan to address the question in the forthcoming [Mu2X].

8.3 (A very special case of) specialization of intersection numbers

Generally speaking, we expect (and can show that) intersection numbers to be preserved under specialization when the total space of a degeneration is smooth (cf. [Ful93, Corollary 10.1]). However, the degenerations considered in this thesis are almost never smooth and, accordingly, we *do not expect* the intersection numbers to be preserved, generally speaking, in our setup (see also Remark 8.3.3 for the discussion of what is missing). But there is some exception that is perhaps worth recorded. For the sake of transparency of the discussion, we state it for curves but the same approach would work in the higher dimension case.

We recall that, in §3, we defined intersection number in terms of Euler characteristic (see Definition 3.1.3). For example, for a line bundle L and a reducible curve C on some complete variety X, we have $L \cdot C = \chi(\mathcal{O}_C) - \chi(L^{-1} \otimes \mathcal{O}_C)$.

8.3.1 Proposition. *Given a quasi-projective degeneration $\pi : X' \to \mathbb{A}^1$ of an algebraic variety X (Definition 8.2.1), let L be a line bundle on X such that the specialization L_0 is a* *line bundle. Then, for any curve*[3] C *on* X*,*

$$L \cdot C = L_0 \cdot C_0$$

where the left-hand side $L \cdot C$ *is the intersection number of* L *and* C *on* X *and that right-hand side that is on* X_0*.*

Proof. Since $L^{-1} \otimes F = L^* \otimes F = \mathcal{H}om_{\mathcal{O}_X}(L, F)$, we can write:

$$c_1(L)F = F - \mathcal{H}om(L, F).$$

Then, by (iv) of Proposition 8.2.5,

$$(c_1(L)F)_0 = F_0 - \mathcal{H}om_{\mathcal{O}_X}(L_0, F_0) = c_1(L_0)F_0$$

and thus

$$\chi((c_1(L)F)_0) = \chi(c_1(L_0)F_0).$$

Since χ is preserved by the specialization homomorphism (Proposition 8.2.5 (iii)), this implies the assertion. □

The next corollary follows from the second part of Theorem 4.1.6. We recall that we defined the moment polytope $\triangle(L)$ of an ample line bundle at Definition 2.4.3.

8.3.2 Corollary. *In the setup of Proposition 8.3.1 above, assume the special fiber* X_0 *is a complete non-normal toric variety and* L_0 *is a ample line bundle. Let* $C_0 = \bigcup C_{0,i}$ *the decomposition into irreducible components:*

$$L \cdot C = \sum_i m_i(\dots)$$

where L_i *is the pullback of* L *to* $C_{0,i,red}$ *= the reduced structure of* $C_{0,i}$ *and* m_i *the multiplicity of* $C_{0,i,red}$ *in* $C_{0,i}$*.*

In particular, if X *is a surface and* $L = \mathcal{O}_X(C)$*, then* $C^2 = \deg(C) = \mathrm{vol}(\triangle_X(C))$*.*

[3]By a curve on X, we mean a one-dimensional closed subscheme (i.e., a reducible not-neceesarily-reduced curve) of X.

Proof. In light of Proposition 8.3.1, without loss of generality, we can assume that X is a non-normal toric variety and that C is reduced and irreducible. By definition, $L \cdot C = \chi(\mathcal{O}_C) - \chi(L^{-1} \otimes \mathcal{O}_C)$. □

8.3.3 Remark. It is beyond the scope of this thesis but what is needed to weaken the condition on L_0 is to extend the definition of $c_1(L)$ from a line bundle L to some more general coherent sheaf. Or, equivalently, to answer what is a divisor that is not a Cartier divisor? (For a normal variety, we have the notion of a divisorial sheaf and so the equivalent question is to find the definition of a divisorial sheaf for non-normal varieties.)

9.0 Part 3: Equivariant Hilbert functions and their leading terms

In this section, we give the reformulation of the result of [Oko96] in terms of characters of representations that paves the way for a generalization in the next section. This section requires some background in representation theory, which, for convenience of the readers, is recalled in Appendix.

Let G be a connected reductive linear algebraic group over an algebraically closed field k and V a finite-dimensional representation of G. Let $X \subset \mathbb{P}(V)$ be a G-subvariety and R the homogeneous coordinate ring of it. Now, G acts on R as grade-preserving automorphism; i.e., there is a sequence of finite-dimensional representations

$$\pi_n : G \to GL(R_n).$$

We can then take their traces $\operatorname{tr} \pi_n$; i.e., the characters of (π_n, R_n). Since $\operatorname{tr} \pi_n(1) = \dim R_n$, for large n, $\operatorname{tr} \pi_n$ may be thought of an equivariant analog of a Hilbert function and this is what we meany by "equivariant Hilbert function" in the title of the present section.

9.1 Representation theory setup

We shall derive the integral formula for the leading term of $\operatorname{tr} \pi_n$ (as $n \to \infty$) by combining [Oko96] and Weyl's character formula.

First we need some setup. We fix a maximal torus $T \subset G$ of dimension r and then let $\mathfrak{t} = \mathfrak{t}_{\mathbb{C}} = \operatorname{Hom}(\mathbb{G}_m, T) \otimes_{\mathbb{Z}} \mathbb{C}$ where $\operatorname{Hom}(\mathbb{G}_m, T)$ is the group of 1-parameter subgroups of T written additively. Then the dual space $\mathfrak{t}^*$ is $\Lambda_{\mathrm{wt}} \otimes_{\mathbb{Z}} \mathbb{C}$ where $\Lambda_{\mathrm{wt}} = \operatorname{Hom}(T, \mathbb{G}_m)$ is the weight lattice of T, again written additively..[1] We write $\langle \cdot, \cdot \rangle : \mathfrak{t}^* \times \mathfrak{t} \to \mathbb{C}$ for the (usual) pairing given by the composition. Let $C \subset \mathfrak{t}^*$ be some choice of the positive Wely chamber (see A.0.8).

[1]Note that if $k = \mathbb{C}$, then $\mathfrak{t}$ is the Lie algebra of T. In other words, $\mathfrak{t}_{\mathbb{C}}$ is the complex points of the $\mathbb{Z}$-Lie algebra of T but this type of a general point of view is not needed here.

Let (π^λ, V^λ) denote the irreducible representations of G parametrized by those $\lambda \in C \cap \Lambda_{\mathrm{wt}}$ (Theorem A.0.9). For each n, according to Corollary A.0.7, we have the decomposition of R_n:

$$R_n \simeq \bigoplus_{\lambda \in C \cap \Lambda_{\mathrm{wt}}} \mathrm{Hom}_G(V^\lambda, R_n) \otimes V^\lambda.$$

Let

$$\mathrm{mult}_{\pi_n}(\lambda) := \dim \mathrm{Hom}_G(V^\lambda, R_n)$$

be the multiplicity of V^λ in R_n, the number of times V^λ appears in the decomposition.

9.2 Okounkov's result in terms of an equivariant Hilbert function

The next proposition is a variant of the main result of [Oko96]; we will prove it using the same idea as in that paper. By a *relatively open face* of a polyhedral convex cone, we mean the relative interior of a closed face of the cone.

9.2.1 Proposition (cf. [Oko96])**.** *For each relatively open face $F \subset \overline{C}$ of dimension r (defined just above), there exist a convex compact set $\triangle_F \subset \mathbb{R}^{\dim F}$ with nonempty interior as well as a map:*

$$\mu : \triangle_F \to \mathfrak{t}^*$$

such that

(i) μ is the restriction of an $\mathbb{R}$-linear map $\mathbb{R}^{\dim F} \to \mathfrak{t}^*$ and the image of μ is F and coincides with the convex hull of the bounded set

$$\{\lambda/n | n > 0, \lambda \in F \cap \Lambda_{wt}, \mathrm{mult}_{\pi_n}(\lambda) > 0\}.$$

(ii) For each z in $\mathfrak{t}$,

$$\lim_{n\to\infty} n^{-\dim F} \sum_{\lambda \in F} \mathrm{mult}_{\pi_n}(n\lambda) e^{\langle \lambda, z\rangle} = \int_{\triangle_F} e^{\langle \mu(x), z\rangle} dx.$$

Proof. Let B be the Borel subgroup of G corresponding to our choice of positive Weyl chamber and U be the unipotent part of B (T is then the semisimple part of B). We know (see Theorem A.0.9)

$$\mathrm{mult}_{\pi_n}(\lambda) = \dim R^U_{n,\lambda},$$

where $R^U_{n,\lambda}$ denotes the T-weight space of R_n with weight λ.

Let $\nu : R^U - 0 \to \mathbb{Z}^d$ be a valuation having the property that: if $E \subset R^U$ is a finite-dimensional vector subspace, then $\#(\nu(E - 0)) = \dim E$. Such a valuation exists by [Oko96] (and also by Remark 7.3.1).

Let

$$R_F = \bigoplus_{(n,n\lambda)\in\mathbb{N}\times(F\cap\Lambda)} R^U_{n,n\lambda}.$$

It is an algebra over the base field k. Let

$$S_n = \{(n\lambda, \nu(f)) | 0 \neq f \in R^U_{n,n\lambda}, \lambda \in F\},$$

which is a subset of $E \times \mathbb{R}^d$ such that $S_n + S_m \subset S_{n+m}$ (since R_F is an algebra).

Define $\triangle_F$ to be the closure

$$\overline{\bigcup_{n>0} S_n/n}.$$

It is known and is also easy to see that $\triangle_F$ is a compact convex set.

Let $q = \dim F$. As in the proof of the lemma in §2.5 of [Oko96] or by Theorem 1.6. of [KK12], we have: for any open ball U in $\mathbb{R}^r$,

$$\lim_{n\to\infty} n^{-q}\#(S_n \cap nU) = \mathrm{vol}_q(\triangle_F \cap U)$$

where vol_r denotes some suitably normalized volume.

As a matter of notation, if $E \subset \mathbb{R}^r$ is a subset, we write 1_E for a characteristic function on E. If E is a finite set, we view it also as a discrete measure:

$$\sum_{x\in\mathbb{R}^r} 1_E(x)\delta_x$$

where δ_x is the point mass measure at x.

With this notation, the above means that the discrete measure $n^{-q}1_{S_n/n}$ converges weakly to $1_{\triangle_F}$, up to some normalization constant.

For any continuous function f on E, we have:

$$\begin{aligned}\langle 1_{S_n/n}, f\circ\mu\rangle &= \sum_{\lambda\in F}\left(\sum_{a\in\mathbb{R}^d} 1_{S_n}(n\lambda, a)\right) f(\lambda)\\ &= \sum_{\lambda\in F}\left(\#\nu(R^U_{n,n\lambda}-0)\right) f(\lambda)\\ &= \mathrm{mult}_{R_n}(n\lambda) f(\lambda).\end{aligned}$$

For each fixed z in $\mathfrak{t}$, dividing both sides by n^r, taking $f(\lambda) = e^{\langle\mu(\lambda),z\rangle}$ and then letting $n\to\infty$ gives the assertion. □

9.2.2 Remark. The proof suggests that $\triangle_F$ are related to each other, but we omit the discussion on the relations (as not needed).

The above proposition describes the asymptotic behavior of $\mathrm{mult}_{\pi_n}(n\lambda)$ as $n\to\infty$ but nothing about the remainder. There is, however, an a priori estimate that we can and shall give; the more details information, which is the point of this thesis, can be obtained by an application of a toric degeneration as in the next section §10.

First, we note the following restatement of the Hilbert-Serre theorem (see the proof of the lemma for the statement of the theorem).

9.2.3 Lemma. *Let A be a graded Noetherian ring and λ a $\mathbb{C}$-valued additive function on the class of finite A_0-modules; here "additive" means for each exact sequence $0\to M'\to M\to M''\to 0$, we have $\lambda(M) = \lambda(M') + \lambda(M'')$.*

Let M be a finite A_0-module and m the least common multiple of the degrees of some finitely many generators of M. Then, for some integer $m>0$ and ζ an m-root of unity in $\mathbb{C}$,

$$\lambda(M_n) = \sum_{j=0}^{m-1} F_j(n)\zeta^{jn}$$

where each F_j is a polynomial. A function like the right-hand side is called a quasi-polynomial; hence, in short, $\lambda(M_n)$ is a quasi-polynomial in n.

Proof. We recall that the Hilbert-Serre theorem ([AM94] Theorem 11.1.) says that for the Poincare series $P(M,t) = \sum_{n=0}^{\infty} \lambda(M_n) t^n$ and homogeneous generators $x_1, \dots, x_s$ of M, we have: $P(M,t)\prod_{i=1}^{s}(1 - t^{\deg x_i})$ is a polynomial in t with integer coefficients. By partial fraction, we can write:

$$P(M,t) \equiv \sum_{i,j} a_{ij}(1 - \zeta^i t)^{-j-1} \bmod \mathbb{Q}[t].$$

Then, by binomial series, if n is large,

$$\dim M_n = \sum_{i,j} a_{ij} \zeta^{ni} \binom{n+j}{j}.$$

□

We note the following estimate:

9.2.4 Lemma. *For each relatively open face F of the closure of the positive Wely chamber, if f is a function on $\mathfrak{t}^*$, then*

$$\sum_{\lambda \in F} \mathrm{mult}_{\pi_n}(n\lambda) f(\lambda)$$

is a quasi-polynomial in n of degree $\leq \dim \triangle_F$ (for the definition of "quasi-polynomial", see Lemma 9.2.3 just above).

Proof. The proof consists of two steps.

(i) Show $R_F = \bigoplus_{n \geq 0, \lambda \in F} R^U_{n,n\lambda}$ is a finitely generated algebra over the base field k.

(ii) Apply the Hilbert-Serre theorem to the finite $k[u_1, \dots, u_s]$-module R_F to conclude the proof.

By Proposition A.0.10, to do (i), it is enough to show the graded ring $\bigoplus_{n=0}^{\infty} R_{n,n\lambda}$ is a Noetherian ring. But that ring is a direct summand of the Noetherian ring R; thus is a Noetherian ring. □

The irreducible characters can be computed by the Weyl character formula. Following [Kac90] §10.6., we define the *character* of a finite-dimensional G-module (V, π_V) to be the function $\mathrm{ch}_V : \mathfrak{t} \to \mathbb{C}$ given by

$$\mathrm{ch}_V(z) = \sum_{\lambda \in \mathfrak{t}^*} \mathrm{mult}_V(\lambda) e^{\langle \lambda, z \rangle}$$

where, for $\lambda \in C \cap \Lambda_{\mathrm{wt}}$, $\mathrm{mult}_V(\lambda) = \dim \mathrm{Hom}(V^\lambda, V)$; otherwise, $\mathrm{mult}_V(\lambda) = 0$. Note that $\mathrm{ch}_V(z) = \mathrm{tr}\, \pi_V(\exp z)$ if the base field is $k = \mathbb{C}$ and $\exp : \mathfrak{t} \to T$ is the exponential map.

Using this definition of a character, the Weyl character formula says: for any $z \in \mathfrak{t}$,

$$D(z)\, \mathrm{ch}_{V^\lambda}(z) = \sum_{w \in W} (-1)^{l(w)} e^{\langle w(\lambda+\rho)-\rho, z \rangle}$$

where

- $l(w)$ is the length of w; i.e., the minimum number of the elements in the decomposition of w into simple reflections.
- $D(z) = \prod_{\alpha > 0} (1 - e^{-\langle \alpha, z \rangle})$.

We next mix Proposition 9.2.1 and the Weyl character formula. By linearity,

$$\begin{aligned} D(z)\, \mathrm{ch}_{\pi_n}(z) &= \sum_{\lambda} \mathrm{mult}_{\pi_n}(\lambda) D(z)\, \mathrm{ch}_{V^\lambda}(z) \\ &= \sum_{w \in W} (-1)^{l(w)}\, \mathrm{mult}_{\pi_n}(\lambda) e^{\langle w(\lambda+\rho)-\rho, z \rangle} \end{aligned}$$

By making the change of a variable $\lambda \mapsto nw^{-1}\lambda$, we have:

$$D(z/n)\, \mathrm{ch}_{\pi_n}(z/n) = \sum_{\lambda, w} (-1)^{l(w)}\, \mathrm{mult}_{\pi_n}(nw^{-1}\lambda) e^{\langle \lambda, z \rangle + \langle w\rho - \rho, z \rangle / n}.$$

This leads to the main result of this section:

9.2.5 Theorem. *In the notations of Proposition 9.2.1, with* $\triangle = \triangle_C$ *and fixed* $z \in \mathfrak{t}$*, there is an integer* $m > 0$ *such that for each integer* $n \gg 0$ *divisible by* m*,*

$$D(z/n)\,\mathrm{ch}_{\pi_n}(z/n) = \left(\int_{\triangle} e^{\langle \mu(x), z\rangle} dx\right) n^r + \epsilon_z(n)$$

where ϵ_z *is bounded above by a polynomial of degree* $< r = \dim T$ *that depends on* z*.*

Proof. If w is not the identity element, then the W-fixed set is a subcone F of C of strictly smaller dimension and thus, the term involving such w cannot contribute to the leading term. The estimate on the error term $\epsilon_z(n)$ comes from Lemma 9.2.4. □

10.0 The remaining terms in the abelian case

We keep using the notations introduced in the previous section but, for simplicity, we take $G = T$ to be a torus. It is *not* essential to limit ourselves to the abelian case, but the statements of the results will be less transparent. For the same desire for transparency, we also only consider the saturated case.

We deduce our result as a corollary of the result of Khovanskii and Pukhlikov ([KP92], [KP92-b]); or more precise following significant refinement due to Brion and Vergne [BV97] (cf. [Gu97]):

10.0.1 Theorem. *(Euler-Maclaurin formula for rational convex polytopes) Let* $\triangle \subset (\mathbb{R}^d)^*$ *be a convex polytope given as, say,*

$$\triangle = \{x \in (\mathbb{R}^d)^* | \langle x, v_i \rangle + a_i \geq 0,\ 1 \leq i \leq N\}$$

for some $(v_i, a_i) \in \mathbb{Z}^d \times \mathbb{Z}$.

Let $\tau_I(x_1, \ldots, x_N)$ *denote the Todd function associated to* I *as in [BV97]. Then, for each linear functional* l *such that the series below converges, we have:*

$$\sum_{\lambda \in \triangle \cap \mathbb{Z}^d} e^{\langle l, \lambda \rangle} = \sum_I \tau_I \left(\partial / \partial h \right) |_{h=0} \int_{\triangle_h} e^{\langle l, x \rangle} dx$$

where $\triangle_h = \{x \in (\mathbb{R}^d)^* | \langle x, v_i \rangle + a_i + h_i \geq 0,\ 1 \leq i \leq N\}$.

10.0.2 Corollary. *In the setup of the previous section, suppose there is a toric degeneration* $X \rightsquigarrow X_0 = \mathrm{Proj}(k[S])$ *from* X *to a normal toric variety* X_0*; i.e., there is a convex polytope* $\triangle$ *such that* $S_n = n\triangle \cap \mathbb{Z}^d$*. Moreover, assume the degeneration is* T*-equivariant. Then*

$$\mathrm{ch}_{\pi_1}(z) = \sum_I \tau_I \left(\partial / \partial h \right) |_{h=0} \int_{\triangle_h} e^{\langle \mu, z \rangle} dx.$$

Proof. Since

$$\dim \mathbb{C}[S]_{n,\lambda} = \#(n\triangle \cap \mathbb{Z}^d \cap \mu^{-1}(\lambda)),$$

we have:

$$\begin{aligned} \mathrm{ch}_{\pi_n}(z) &= \sum_{\lambda\in\Lambda} \dim R_{n,\lambda} e^{\langle\lambda,z\rangle} = \sum_{\lambda\in\Lambda} \dim \mathbb{C}[S]_{n,\lambda} e^{\langle\lambda,z\rangle} \\ &= \sum_{(\lambda,a)\in n\triangle\cap\mathbb{Z}^d} e^{(\langle\mu(\lambda,a),z\rangle}. \end{aligned}$$

□

Appendix Representation theory

This section collects some basic results and facts from representation theory that are used in Part 3 (and some clarification on group actions that is referred in Part 1). The main reference is [Br05] (especially the notion of a reductive group).

First we recall the notions of a group action in algebraic geometry and linearization of it in the form we use.

A.0.1 Definition (group action)**.** Let G be a linear algebraic group[1] over a field k. Let $k[G]$ be the coordinate ring of G; i.e., $G = \operatorname{Spec} k[G]$. By a *group action of G on a scheme X over k,* we mean a morphism $\sigma : G \times X \to X$ over k that satisfies the two axioms of a group action (associativity and unitality).

Now, suppose $X = \operatorname{Spec} A$ is affine. Then the group action axioms state explicitly that the algebra homomorphism $\sigma^{\#} : A \to k[G] \otimes A$ satisfy:

(1) $A \overset{\sigma^{\#}}{\to} k[G] \otimes A \overset{1 \otimes \epsilon}{\to} k \otimes A \simeq A$ is the identity map, where $\epsilon : k[G] \to k$ corresponds to the identity element of G.

(2) The diagram

$$\begin{array}{ccc} A & \xrightarrow{\sigma^{\#}} & k[G] \otimes A \\ \downarrow{\scriptstyle \sigma^{\#}} & & \downarrow{\scriptstyle \mu \otimes 1_A} \\ k[G] \otimes A & \xrightarrow{1_{k[G]} \otimes \sigma^{\#}} & k[G] \otimes k[G] \otimes A \end{array}$$

commutes, where $\mu : k[G] \to k[G] \otimes k[G]$ corresponds to the multiplication on G.

For example, if $G = \mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}]$, then for each k-algebra R, we have, as groups,

$$R^* \overset{r \mapsto f_r}{=} \mathbb{G}_m(R) := \operatorname{Hom}(k[t, t^{-1}], R)$$

where $f_r(t) = r$ and the binary operation on $\mathbb{G}_m(R)$ is given by the pullback along $t \mapsto t \otimes t$. That is to say, the multiplication on $\mathbb{G}_m$ is given by $\mu : k[t, t^{-1}] \to k[t, t^{-1}] \otimes k[t, t^{-1}]$, $t \mapsto t \otimes t$. Hence, if there is a dominant and equivariant map $\mathbb{G}_m \to X$ (so $k[X]$ is a subring of $k[t, t^{-1}]$

[1] To be definite, an affine group scheme that is of finite type over the base field

and the group action of X extends the multiplication of $\mathbb{G}_m$), then the group action on X is given by $k[X] \to k[t, t^{-1}] \otimes k[X]$, $f \mapsto f \otimes f$.

The above is the general definition of a group action. But at least in the thesis, we are exclusively interested in group actions coming from a linear representation of a group; i.e., when the action is linearizable. The next proposition says that an (algebraic) group action on an affine variety is always linearizable.

We recall that, as usual, a vector space V is given the structure of a scheme as the Spec of the ring of polynomial functions on V (at least when the base field k is infinite).

A.0.2 Proposition. *Let G be a linear algebraic group acting on an affine variety X. Then there exists a finite-dimensional vector space V such that (1) V is a G-module and (2) there is a G-equivariant closed immersion $X \hookrightarrow V$.*

A.0.3 Corollary. *The coordinate ring A of an affine G-variety, viewed as a left regular representation of G, is rational; i.e., for each vector $v \in A$, $G \cdot v$ spans a finite-dimensional vector subspace.*

Here is the projective case of the above.

A.0.4 Proposition. *Let G be a linear algebraic group and R a graded ring such that $R_0 = k$ is the base field and G act on R as grade-preserving automorphisms. Then* $\operatorname{Proj} X$ *is a G-scheme. Moreover, if R is finitely generated as a k-algebra, then there is a G-module V such that $X \hookrightarrow \mathbb{P}(V)$ is a G-equivariant closed immersion.*

Next we recall the notion of a reductive group.

A.0.5 Proposition. *For a linear algebraic group G over an algebraically closed field k of characteristic zero, the following are equivalent:*

(i) G is reductive; i.e., has no nontrivial closed normal unipotent subgroup.

(ii) Every finite-dimensional G-module is semisimple (i.e., a direct sum of simple modules).[2]

(iii) Every G-module is semisimple.

[2]Such a G is called linear reductive.

Moreover, if $k = \mathbb{C}$, then the above conditions are equivalent to saying G contains a Zariski-dense compact subgroup.

Proof. This is well-known; see for example [Br05] §1. Theorem 1.23. □

Because of the last part of the above proposition, to some extent, the representation theory of a reductive group is related to that of a compact group; in particular, we have the theorem below.

A.0.6 Theorem (Peter-Weyl)**.** *Let G be a reductive linear algebraic group with the coordinate ring $k[G]$, k algebraically closed of characteristic zero. Let $G \times G$ act on $k[G]$ by $(g, h) \cdot f(x) = f(g^{-1}xh)$. Then, as a $G \times G$-module, there is the decomposition*

$$k[G] = \bigoplus_{\lambda} \mathrm{End}_G(V^\lambda)$$

where each λ denotes the isomorphism class of a simple $G \times G$-module and V^λ a representative of λ.

Proof. See [Br05] §2. Lemma 2.2. □

Note that $\mathrm{End}_G(V^\lambda) = (V^\lambda)^* \otimes V^\lambda$.

A.0.7 Corollary (isotypic decomposition)**.** *If V is a G-module, then*

$$V = \bigoplus_{\lambda} \mathrm{Hom}_G(V^\lambda, V) \otimes V^\lambda$$

Proof.

$$\begin{aligned} k[G] = V \otimes_{k[G]} k[G] &= \bigoplus_{\lambda} V \otimes_{k[G]} (V^\lambda)^* \otimes_k V^\lambda \\ &= \bigoplus_{\lambda} \mathrm{Hom}_G(V^\lambda, V) \otimes V^\lambda \end{aligned}$$

□

For a finite group, the above theorem reduces to a standard fact on the decomposition of a regular representation. For a connected reductive group, on the other hand, there is a convenient and conventional way to parametrize all the simple G-modules: the theorem of the highest weight. To formulate it, we need some setup.

Let G be as above but now assumed to be connected. Let T be a maximal torus of G. By a *root* (relative to G and T), we mean that a group homomorphism $\alpha : T \to \mathbb{G}_m$ that is a weight of the adjoint action of T on the Lie algebra $\mathfrak{g}$ of G. Let E be the real vector space spanned by the character group of T, the characters written additively; note $r = \dim T = \dim E$. The set Φ of roots then forms a root system on E. Let $E_{\text{reg}} = E - \bigcup_{\alpha \in \varphi} \{\langle \alpha, \cdot \rangle = 0\}$. The connected components of E_{reg} are then called the *Weyl chambers*. The key fact is:

A.0.8 Lemma (Weyl chambers)**.** *There are bijections between the following sets*

- *(i) The set of Weyl chambers.*
- *(ii) The set of choices of simple roots of* Φ*.*
- *(iii) The set of choices of Borel subgroups of* G *containing* T*.*

Fix some Borel subgroup B or equivalently choose a Weyl chamber, which is called the *positive Weyl chamber*. By the *weight lattice* $\Lambda_{\text{wt}} \subset E$, we mean that the character group of T. Let U denote the unipotent radical of B (so that B is the semidirect product of T and U).

A.0.9 Theorem (theorem of the highest weight)**.** *There is the bijection*

$$C \cap \Lambda_{wt} \to \text{ the set of isomorphism classes of simple } G\text{-modules.}$$

Also, for each G*-module* V*,*

$$\dim \operatorname{Hom}_G(V^\lambda, V) = \dim V_\lambda^U$$

where V_λ^U *is the* T*-weight space of weight* λ *in* V^U*.*

Proof. This is well-known. □

The following is the Hilbert's theorem on the finiteness of an invariant ring for a unipotent radical.

A.0.10 Proposition. *Let A be a finitely generated algebra over k. Then A^U is also a finitely generated algebra over k.*

Proof. This is the theorem of Hadziev and Grosshans (see [Br05] Theorem 2.7.) □

Bibliography

[AB04] Alexeev, V.; Brion, M.: Toric degenerations of spherical varieties, Selecta Math. (N.S.) 10 (2004), no. 4, 453–478.

[AH06] Altmann, K.; Hausen, J.: Polyhedral Divisors and Algebraic Torus Actions. Math, Ann. 334, 557-607 (2006).

[An13] Anderson, D.: Okounkov bodies and toric degeneration, Math. Ann. 356 (2013), no. 3, 1183–1202.

[AM94] Atiyah, M.; Macdonald, I.G.: *Introduction to Commutative Algebra*, Addison–Wesley, 1994.

[Bou] Bourbaki, N.: *Algèbre commutative*, Springer-Verlag Berlin Heidelberg, 2006; edition originale publiѐe par Diffusion C.C.L.S., Paris, 1985

[Br05] Brion, M.: Introduction to actions of algebraic groups, `https://www-fourier.ujf-grenoble.fr/~mbrion/notes_luminy.pdf`

[Br13] Brion, M.: On linearization of line bundles, J. Math. Sci. Univ. Tokyo 22 (2015), 1–35. arXiv:13126267

[BV97] Brion, M.; Vergne, M.: Residue formulae, vector partition functions and lattice points in rational polytopes. J. Amer. Math. Soc., 10 (1997), pp. 797–833.

[BG08] Bruns W.; Gubeladze, J.: *Polytopes, rings, and K -theory,* Springer, 2009.

[Cox95] Cox, D. A.: The homogeneous coordinate ring of a toric variety, J. Algebraic Geom. 4 (1995), 17–50.

[Ei04] Eisenbud, D.: *Commutative Algebra with a View Toward Algebraic Geometry*, Springer, 1995.

[FL85] Fulton, W.; Lang S.: *Riemann-Roch Algebra*, Springer, 1985.

[Ful93] Fulton, W.: *Introduction to Toric Varieties*, Princeton, 1993.

[Ful98] Fulton, W.: *Intersection Theory,* second edition, Springer, 1998.

[GH81] Greenberg, M. J.; Harper, J. R.: *Algebraic Topology: A First Course.* Revised. Benjamin/Cummings Publ. Co. 1981.

[Hart77] Hartshorne, R.: *Algebraic geometry*, Springer, 1977.

[Gu97] Guillemin, V.: Riemann-Roch for toric orbifolds, J. Diff. Geom. 45 (1997), 53–73.

[Ha09] Haution, O.: Steenrod operations and quadratic forms, Ph.D. thesis, Université Paris.

[HS06] Huneke, C.; Swanson, I.: *Integral closure of ideals, rings, and modules,* London Mathematical Society Lecture Note Series, 336, Cambridge, UK: Cambridge University Press, 2006.

[HGN90] Herrmann, M.; Goto, S.; Nishida, K.: On the structure of Noetherian symbolic Rees Algebras. Manuscripta mathematica 67.2 (1990): pp. 197–226.

[Jou83] Jouanolou, J.P.: *Théorèmes de Bertini et applications*, Progress in Mathematics, vol. 42, Birkhäuser Boston Inc., Boston, MA, 1983.

[Kac90] Kac, V.: *Infinite Dimensional Lie Algebras,* Cambridge University Press, 1990.

[KK12] Kaveh, K.; Khovanskii, A. G.: Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory. Ann. of Math. (2) 176 (2012), no. 2, 925–978.

[KMM20] Kaveh, K.: Manon, C.; Murata, T.: On degenerations of projective varieties to complexity-one T-varieties, arXiv:1708.02698 [math.AG]

[Kaw88] Kawamata, Y.: Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces, Ann. of Math., 2, 127 (1988): 93–163

[Ka07] Kawanoue, H.: Toward resolution of singularities over a field of positive characteristic. I. Foundation; the language of the ideal filtration, Publ. Res. Inst. Math. Sci. 43, No. 3, 819-909 (2007).

[KP92] Khovanskii, A. G.; Pukhlikov, A. V.: Finitely Additive Measures of Virtual Plytopes, Algebra Anal. 4(2), 161–185 (1992) [St. Petersburg Math. J. 4 (2), 337–356 (1993)].

[KP92-b] Khovanskii, A. G.; Pukhlikov, A. V.: A Riemann-Roch Theorem for Integrals and Sums of Quasipolynomials over Virtual Polytopes, Algebra Anal. 4(4), 188?216 (1992) [St. Petersburg Math. J. 4 (4), 789–812 (1993)].

[Kl05] Kleiman, S. L.: The Picard scheme: Appendix B., *Fundamental algebraic geometry,* Math. Surveys Monogr. 123, Providence, R.I.: American Mathematical Society, arXiv:math/0504020, 2005.

[Kn05] Knutson, A.: Balanced normal cones and Fulton-MacPherson's intersection theory, arXiv:math/0512632 [math.AG], 2005.

[Ma70b] Matsumura, H.: *Commutative Ring Theory*, Cambridge Studies in Advanced Mathematics (2nd ed.), Cambridge University Press, 1989.

[Ko08] Kollár, J.: Exercises in the birational geometry of algebraic varieties. Analytic and algebraic geometry, 495–524, IAS/Park City Math. Ser., 17, Amer. Math. Soc., Providence, RI, 2010.

[Mc83] McAdam, S.: *Asymptotic Prime Divisors*, Lecture Notes in Mathematics, No. 1023, Springer-Verlag, New York (1983)

[MFK94] Mumford, D.; Fogarty, J.; Kirwan, F.: *Geometric invariant theory.* Third edition. Springer-Verlag, Berlin, 1994.

[Mu2X] Murata, T.: Intrinsic toric degeneration of an algebraic variety, in preparation.

[LM09] Lazarsfeld, R.: Mustaţă, M.: Convex bodies associated to linear series, Ann. Sci. Ecole Norm. Sup. 42 (2009), no. 5, 783–835

[La04] Lazarsfeld, R.: *Positivity in Algebraic Geometry I - Classical Setting: Line Bundles and Linear Series,* Springer-Verlag Berlin Heidelberg, pp. 269–270, 2004.

[Oda78] Oda, T.: *Lectures on torus embeddings and applications (Based on joint work K. Miyake)*, Tata Inst. of Fund. Research 58 (1978), Springer-Verlag.

[Ogus06] Ogus, A.: Lectures on Logarithmic Algebraic Geometry, `https://math.berkeley.edu/~ogus/logpage.html`, 2006.

[Oko96] Okounkov, A.: Brunn-Minkowski inequality for multiplicities, Invent. Math. 125 (1996), 405–411.

[Sch85] Schenzel, P.: Symbolic powers of prime ideals and their topology, Proc. Amer. Math. Soc, 93 (1985), 15–20.

[Sch88] Schenzel, P.: Filtrations and Noetherian symbolic blow-up rings, Proc. Amer. Math. Soc. 102 (1988), 817–822.

[Re] Rees. D.: *Lectures on the asymptotic theory of ideals*, London Mathematical Society Lecture Note. Series 113, Cambridge University Press, Cambridge, 1988.

[Ra74] Ratliff, L.: Locally quasi-unmixed Noetherian rings and ideals of the principal class. Pacific J. Math. 52 (1974), no. 1, 185–205.

[Ro13] Rohrer, F.: The geometry of toric schemes. J. Pure Appl. Algebra 217 (2013), 700–708.

[Sp98] Springer, T.A.: *Linear algebraic groups*, second edition, Progress in Mathematics, vol. 9, Birkhäuser, 1998.

[Va17] Vakil, R.: The Rising Sea: Foundations Of Algebraic Geometry Notes, November 18, 2017 version, `http://math.stanford.edu/~vakil/216blog/`.

[IW20] Ilten, N.; Wrobel, M.: Khovanskii-finite valuations, rational curves, and torus actions. J. Comb. Algebra 4 (2020), no. 2, 141–166.