Triangulations and a discrete Brunn-Minkowski inequality in the plane

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1 Introduction

In this paper we write A, B to denote finite subsets of $\mathbb{R}^d$, and $|\cdot|$ stands for their cardinality. We say that $A \subset \mathbb{R}^d$ is *d–dimensional* if it is not contained in any affine hyperplane of $\mathbb{R}^d$. Equivalently, the real affine span of A is $\mathbb{R}^d$. For objects $X_1, \ldots, X_k$ in $\mathbb{R}^2$, $[X_1, \ldots, X_k]$ denotes their convex hull. The *lattice generated by* A is the additive subgroup $\Lambda = \Lambda(A) \subset \mathbb{R}^d$ generated by $A - A$, and A is called *saturated* if it satisfies $A = [A] \cap \Lambda(A)$.

Our starting point are two classical results. The first one is from the 1950's, due to Kemperman [10], and popularized by Freiman [4]: if A and B

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are finite nonempty subsets of $\mathbb{R}$, then

$$|A+B| \geq |A|+|B|-1,$$

with equality if and only if A and B are arithmetic progressions of the same difference. The other result, the Brunn-Minkowski inequality, dates back to the 19th century. It says that if $X, Y \subset \mathbb{R}^d$ are compact nonempty sets then

$$\lambda(X+Y)^{\frac{1}{d}} \geq \lambda(X)^{\frac{1}{d}} + \lambda(Y)^{\frac{1}{d}}$$

where λ stands for the Lebesgue measure. Moreover, provided that $\lambda(X)\lambda(Y) > 0$, equality holds if and only if X and Y are convex homothetic sets.

Various discrete analogues of the Brunn-Minkowski inequality have been established in Bollobás, Leader [1], Gardner, Gronchi [5], Green, Tao [6], González-Merino, Henze [11], Hernández, Iglesias and Yepes [8], Huicochea [9] in any dimension, and Grynkiewicz, Serra [7] in the planar case. Most of these papers use the method of compression, which changes a finite set into a set better suited for sumset estimates, but does not control the convex hull.

Unfortunately the known analogues are not as simple in their form as the original Brunn–Minkowski inequality. For instance, a formula due to Gardner and Gronchi [5] says that, if A is d–dimensional, then

$$|A+B| \geq (d!)^{-\frac{1}{d}}(|A|-d)^{\frac{1}{d}} + |B|^{\frac{1}{d}}. \tag{1}$$

Concerning the case $A = B$, Freiman [4] proved that if the dimension of A is d, then

$$|A+A| \geq (d+1)|A| - \binom{d+1}{2}. \tag{2}$$

Both estimates are optimal. In particular, we can not expect a true discrete analogue of the Brunn–Minkowski inequality if the notion of volume is replaced by cardinality.

We here conjecture and discuss a more direct version of the Brunn–Minkowski inequality where the notion of volume is replaced by the number of full dimensional simplices in a triangulation of the convex hull of the finite set.

For any finite d–dimensional set $A \subset \mathbb{R}^d$ we write T_A to denote some triangulation of A, by which we mean a triangulation of $[A]$ using A as the set of vertices. We denote $|T_A|$ the number of d-dimensional simplices in T_A.

In dimension two the number $|T_A|$ is the same for all triangulations of A, so we denote it $\mathrm{tr}(A)$. More precisely, if Δ_A and Ω_A denote the number of points of A in the boundary $\partial[A]$ and in the interior $\mathrm{int}[A]$, respectively, then

$$\mathrm{tr}(A) = \Delta_A + 2\Omega_A - 2 = 2|A| - \Delta_A - 2. \tag{3}$$

Therefore in dimension two we can formulate the following discrete analogue of the Brunn–Minkowski inequality.

Conjecture 1 *If finite $A, B \subset \mathbb{R}^2$ in the plane are not collinear, then*

$$\operatorname{tr}(A+B)^{\frac{1}{2}} \geq \operatorname{tr}(A)^{\frac{1}{2}} + \operatorname{tr}(B)^{\frac{1}{2}}.$$

One case where Conjecture 1 holds with equality is when A and B are homothetic saturated sets with respect to the same lattice; namely, $A = \Lambda \cap k \cdot P$ and $B = \Lambda \cap m \cdot P$ for a lattice Λ, polygon P and integers $k, m \geq 1$. This follows from the original Brunn-Minkowski equality, since $A + B = \Lambda \cap (k+m) \cdot P$ and the area of any triangle in a suitable triangulation is $\frac{1}{2} \det \Lambda$.

We also note that Conjecture 1, together with the equality (3) and the fact that $\Delta_{A+B} \geq \Delta_A + \Delta_B$, would imply the following inequality of Gardner and Gronchi [5, Theorem 7.2] for sets A and B saturated with respect to the same lattice:

$$|A+B| \geq |A| + |B| + (2|A| - \Delta_A - 2)^{1/2}(2|B| - \Delta_B - 2) - 1.$$

Unfortunately we have not been able to prove Conjecture 1 in full generality. Our main results are the following four cases of it: if $[A] = [B]$ (Theorem 2), in which case we also determine the conditions for equality in Conjecture 1; if A and B differ by one element (Theorem 4); if either $|A| = 3$ or $|B| = 3$ (Theorem 7); and if none of A and B have interior points (Theorem 8). Actually, the last two theorems verify a stronger conjecture (Conjecture 5) discussed below.

We start with the case $[A] = [B]$, which naturally include the case $A = B$.

Theorem 2 *Let $A, B \subset \mathbb{R}^2$ be finite two dimensional sets. If $[A] = [B]$ then Conjecture 1 holds. Moreover equality holds if and only if $A = B$, and*

(a) either A is a saturated set, or

(b) $A = \{z_1, \ldots, z_k\}$ for $k \geq 4$, where $z_1, \ldots, z_{k-3} \in \operatorname{int}[z_{k-2}, z_{k-1}, z_k]$, and $z_1, \ldots, z_{k-2}$ are collinear and equally spaced in this order (see Figure 1).

Let us mention that Theorem 2 (in fact, its particular case $A = B$) gives a simple proof of the following structure theorem of Freiman [4] for a planar set with small doubling. We recall that according to (2), if finite $A \subset \mathbb{R}^2$ is two dimensional, then $|A + A| \geq 3|A| - 3$ and, if the dimension of A is at least 3, then $|A + A| \geq 4|A| - 6$.

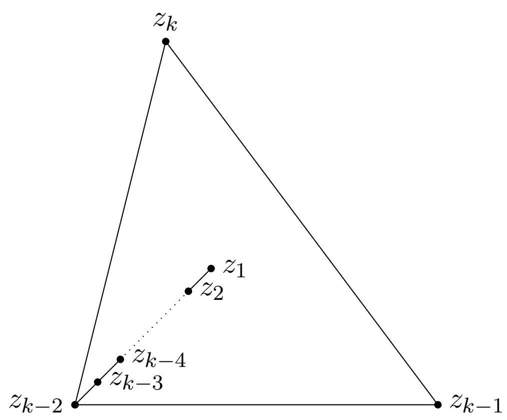

Figure 1: An illustration of case (b) in Theorem 2.

Corollary 3 (Freiman) *Let $A \subset \mathbb{R}^2$ be a fnite two dimensional set and $\varepsilon \in (0,1)$. If $|A| \geq 48/\varepsilon^2$ and*

$$|A+A| \leq (4-\varepsilon)|A|,$$

then there exists a line l such that A is covered by at most

$$\frac{2}{\varepsilon} \cdot (1 + \frac{32}{|A|\varepsilon^2})$$

lines parallel to l.

We note that, for A the grid $\{1, \ldots, k\} \times \{1, \ldots, k^2\}$ and large k,

$$|A+A| \leq (4-\varepsilon)\,|A|, \tag{4}$$

with $\varepsilon = \varepsilon_k = \frac{2}{k}$ and A can not be covered by less than k parallel lines. Therefore the constant 2 in the numerator of $\frac{2}{\varepsilon}$ is asymptotically optimal in Corollary 3.

The next case w address is when A and B differ by one element.

Theorem 4 *Let $A \subset \mathbb{R}^2$ be a finite two dimensional set. If $B = A \cup \{b\}$ for some $b \notin A$ then Conjecture 1 holds.*

For our next results we need the notion of *mixed subdivision* (see De Loera, Rambau, Santos [3] for details). For finite d–dimensional sets $A, B \subset \mathbb{R}^d$ and triangulations T_A and T_B of $[A]$ and $[B]$, we call a polytopal subdivision M of $[A+B]$ a *mixed subdivision* corresponding to T_A and T_B if

(i) every k-cell of M is of the form $F+G$ where F is an i-simplex of T_A and G is a j-simplex of T_B with $i+j=k$;

(i) for any d-simplices F of T_A and G of T_B, there is a unique $b \in B$ and a unique $a \in A$ such that $F+b \in M$ and $a+G \in M$.

We write $\|M\|$ to denote the weighted number of d-polytopes, where $F+G$ has weight $\binom{i+j}{i}$ if F is an i-simplex of T_A, and G is a j-simplex of T_B with $i+j=d$. In particular, all vertices of M are in $A+B$, and the number of d-simplices is $\|M\|$ for any triangulation of M with the same set of vertices (see e.g. [3, Proposition 6.2.11]).

The main goal of this paper is to investigate the following problem: For which triangulations T_A and T_B there exists a corresponding mixed subdivision M for $[A+B]$ such that

$$\|M\|^{\frac{1}{d}} \geq |T_A|^{\frac{1}{d}} + |T_B|^{\frac{1}{d}}. \tag{5}$$

In $\mathbb{R}^2$, we write M_{11} to denote the set of parallelograms in a mixed subdivision M. In this case (5) is equivalent to the following stronger version of Conjecture 1.

Conjecture 5 *For every finite two dimensional sets $A, B \subset \mathbb{R}^2$ there exist triangulations T_A and T_B of $[A]$ and $[B]$ using A and B, respectively, as the set of vertices, and a corresponding mixed subdivision M of $[A+B]$ such that*

$$|M_{11}| \geq \sqrt{|T_A| \cdot |T_B|}. \tag{6}$$

Conjecture 5 offers a geometric and algorithmic approach to prove Conjecture 1.

The following example shows that one cannot a priori fix the triangulations T_A and T_B in Conjecture 5:

Proposition 6 *Let*

$$A = \{(0,0), (-1,-2), (2,1)\}.$$

For $k \geq 145$, let

$$B = \{p, q, l_0, \ldots, l_k, r_0, \ldots, r_{k-1}\},$$

where $p = (-1, k+1)$, $q = (k+1, -1)$, $l_i = (i,i)$ for $i = 0, \ldots, k$ and $r_i = (i, i+1)$ for $i = 0, \ldots, k-1$.

Let T_B be the triangulation of B consisting of the triangles

$$[p, l_i, r_i], [q, l_i, r_i],\ i = 0, \ldots, k-1 \text{ and } [p, l_i, r_{i-1}], [q, l_i, r_{i-1}],\ i = 1, \ldots, k.$$

Then, no mixed subdivision of $A+B$ corresponding to T_B and any triangulation T_A of A satisfies (5) for $d=2$.

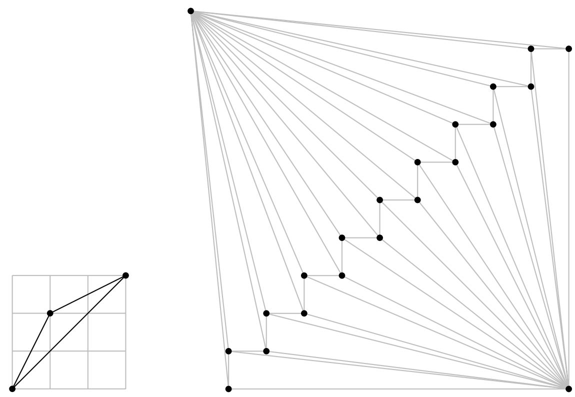

Figure 2: An illustration of the example described in Proposition 6.

Now Conjecture 5 is verified if either A or B has only three elements.

Theorem 7 *If $|B| = 3$, then Conjecture 5 holds for any finite two dimensional set $A \subset \mathbb{R}^2$.*

Remark It follows that if B is the sum of sets of cardinality three, then Conjecture 1 holds for any finite two dimensional set $A \subset \mathbb{R}^2$. For example, if $m \geq 1$ is an integer, and $B = \{(t, s) \in \mathbb{Z}^2 : t, s \geq 0 \text{ and } t + s \leq m\}$, or $B = \{(t, s) \in \mathbb{Z}^2 : |t|, |s| \leq m \text{ and } |t + s| \leq m\}$.

Conjecture 1 was verified by Böröczky, Hoffman [2] if A and B are in convex position; namely, $A \subset \partial[A]$ and $B \subset \partial[B]$. Here we even verify Conjecture 5 under these conditions.

Theorem 8 *Let $A, B \subset \mathbb{R}^2$ be finite two dimensional sets. If $A \subset \partial[A]$ and $B \subset \partial[B]$ then Conjecture 5 holds.*

Part of the reason why we could not verify Conjecture 1 in general is that, except for Theorem 7, our arguments actually prove the inequality $\text{tr}(A + B) \geq 2(\text{tr}(A) + \text{tr}(B))$, which is stronger than Conjecture 1, but which does not hold for all pairs with $A \subset B$. For example, if A are the nonnegative integer points with sum of coordinates at most k and B is the same with sum of coordinates at most l, we have $\text{tr}(A + B) = (k + l)^2$, $\text{tr}(A) = k^2$ and $\text{tr}(B) = l^2$. So we have $\text{tr}(A+B) < 2(\text{tr}(A)+\text{tr}(B))$ if $k \neq l$.

Turning to higher dimensions, we note that if $T_A = T_B$, then a mixed subdivision satisfying (5) does exist.

Theorem 9 *For a finite d–dimensional set $A \subset \mathbb{R}^d$ and for any triangulation T_A of $[A]$ using A as the set of vertices there exists a corresponding mixed subdivision M of $[A+A]$ such that*

$$\|M\| = 2^d|T_A|.$$

Therefore in certain cases, mixed subdivisions point to a higher dimensional generalization of Conjecture 1. This is specially welcome knowing that, if $d \geq 3$, then the order of the number of d-simplices in a triangulation of the convex hull of a finite $A \subset \mathbb{R}^d$ spanning $\mathbb{R}^d$ might be as low as $|A|$ and as high as $|A|^{\lfloor d/2 \rfloor}$ for the same A. In particular, one can not assign the number of d-simplices as a natural notion of discrete volume if $d \geq 3$.

2 Proof of Theorem 2

We will actually prove that

$$\mathrm{tr}(A+B) \geq 2\mathrm{tr}(A) + 2\mathrm{tr}(B), \tag{7}$$

a stronger inequality than Conjecture 1.

For a finite two dimensional set $X \subset \mathbb{R}^2$, we define

$$f_X(z) = \begin{cases} 1 & \text{if } z \in \partial[X] \\ 2 & \text{if } z \in \mathrm{int}\,[X] \end{cases},$$

so that

$$\mathrm{tr}(X) = \left(\sum_{z \in X} f_X(z)\right) - 2.$$

Lemma 10 *Let $A, B \subset \mathbb{R}^2$ satisfy $[A] = [B]$. Then inequality (7) holds. Moreover, equality in (7) yields $A = B$.*

Proof: Let T be a triangulation of $[A] = [B]$ using the points in $A \cap B$ as vertices. One nice thing about inequality (7) is that, since it is linear, it is additive over the triangles of T. Therefore, it suffices to show that, for each triangle t of T, if $A_t = A \cap t$ and $B_t = B \cap t$, then

$$\mathrm{tr}(A_t + B_t) \geq 2\mathrm{tr}(A_t) + 2\mathrm{tr}(B_t), \tag{8}$$

and that equality in (8) implies that $A_t = B_t$ consists of the three vertices of t alone. Moreover, inequality (8) is equivalent to

$$\sum_{p \in A_t + B_t} f_{A_t+B_t}(p) = \left(\sum_{p \in A_t} f_{A_t}(p)\right) + \left(\sum_{p \in B_t} f_{B_t}(p)\right) - 6. \tag{9}$$

Let $A_t \cap B_t = \{v_1, v_2, v_3\}$ be the three vertices of the triangle $t = [A_t] = [B_t]$. We claim that if $i, j \in \{1, 2, 3\}$, $p \in (A_t \cup B_t) \setminus \{v_1, v_2, v_3\}$ and $q \in A_t \cup B_t$, then

$$v_i + p = v_j + q \ \text{ yields } \ v_i = v_j \ \text{ and } \ p = q. \tag{10}$$

We may assume that v_i is the origin and, to get a contradiction, $v_i \neq v_j$. Then the line l passing through v_j and parallel to the side of t opposite to v_j separates t and $v_j + t$, and intersects t only in $v_j \neq p$. Since $v_j + q \in v_j + t$, we get the desired contradiction.

It follows from (10) that the six points $v_i + v_j$, $1 \leq i \leq j \leq 3$, and the points of the form $v_i + p$, $i = 1, 2, 3$ and $p \in (A_t \cup B_t) \setminus \{v_1, v_2, v_3\}$ are all different. Since the six points $v_i + v_j$, $1 \leq i \leq j \leq 3$, belong to $\partial(A_t + B_t)$, we have

$$\left(\sum_{i,j=1,2,3} f_{A_t+B_t}(v_i + v_j) \right) = \left(\sum_{i=1}^{3} f_{A_t}(v_i) \right) + \left(\sum_{j=1}^{3} f_{B_t}(v_j) \right) = 6. \tag{11}$$

On the other hand, we claim that, if $p \in A_t \setminus \{v_1, v_2, v_3\}$ and $q \in B_t \setminus \{v_1, v_2, v_3\}$, then

$$\begin{array}{rcl} \sum_{j=1}^{3} f_{A_t+B_t}(p + v_j) & > & 2 f_{A_t}(p) \\ \sum_{i=1}^{3} f_{A_t+B_t}(v_i + q) & > & 2 f_{B_t}(q). \end{array} \tag{12}$$

Indeed, the inequality readily holds if $p \in \partial[A_t]$ and, if $p \in \operatorname{int}[A_t]$, then $p + v_j \in \operatorname{int}[A_t + B_t]$ for $j = 1, 2, 3$, as well, yielding (12).

By combining (11) and (12) we get (9) and in turn (7). Moreover, (12) shows that if equality holds in (8) then $A_t = B_t$ and, therefore, if equality holds in (7), then $A = B$. $\square$

For a finite two dimensional set $A \subset \mathbb{R}^2$ and a triangulation T of A we denote by A_T the union of A and the set of midpoints of the edges of T (see Figure 3).

Lemma 11 *Let $A \subset \mathbb{R}^2$ be a finite a finite two dimensional set. The equality*

$$\operatorname{tr}(A + A) = 4 \cdot \operatorname{tr}(A)$$

holds if, and only if, for every triangulation T of $[A]$, we have $A_T = \frac{1}{2}(A + A)$.

Proof: Divide each triangle t of T into four triangles using the vertices of t and the midpoints of the sides of t. This way we have obtained a triangulation of $[A] = [A_T]$ using A_T as the vertex set. Therefore

$$\operatorname{tr}(A + A) = \operatorname{tr}(\tfrac{1}{2}(A + A)) \geq \operatorname{tr}(A_T) = 4 \cdot \operatorname{tr}(A).$$

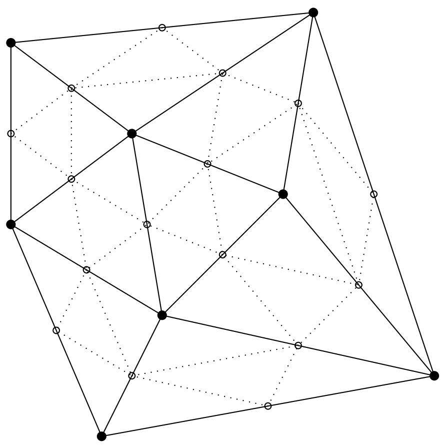

Figure 3: A triangulation and its midpoints.

Moreover, there is equality if and only if $A_T = \frac{1}{2}(A+A)$. $\square$

We observe that the equation in Lemma 11 is equivalent to Conjecture 1 for the case $A = B$. Therefore all we have left to prove is that $\mathrm{tr}(A+A) = 4 \cdot \mathrm{tr}(A)$ if and only if A is of the form either (a) or (b) in Theorem 2. The if part is simple.

Lemma 12 *Suppose that either (a) or (b) in Theorem 2 hold for the finite set A. Then*
$$A_T = \frac{1}{2}(A+A).$$

Proof: Suppose first that $A = [A] \cap \Lambda$ for a lattice Λ. We may assume $\Lambda = \mathbb{Z}^2$. Then clearly the midpoints of sides of every triangulation T of $[A]$ using A as vertex set are precisely the points of $\frac{1}{2}(A+A)$.

Next, if we have property (b), then there is a unique triangulation T of $[A]$ using A as vertex set. For $1 \le i < j \le k$, $[z_i, z_j]$ is an edge of T, unless $j \le k-2$, an hence we have $A_T = \frac{1}{2}(A+A)$ again. $\square$

The next Lemma shows the reverse direction and concludes the proof of Theorem 2.

Lemma 13 *Let $A \subset \mathbb{R}^2$ be a finite two dimensional set. If for every triangulation T of A it holds that*
$$A_T = \frac{1}{2}(A+A),$$
then either (a) or (b) from Theorem 2 hold.

Proof: We first prove two simple claims. All throughout we assume that $A_T = \frac{1}{2}(A + A)$ for every triangulation T of A.

Claim 14 *Let ℓ be a line intersecting A in at least two points and $A_\ell = A \cap \ell$. If $A_\ell + A_\ell = (A + A) \cap (\ell + \ell)$ then the points in A_ℓ form an arithmetic progression. In particular, the points on each side of the convex hull of A form an arithmetic progression.*

Proof: There is a triangulation T of A which contains the edges defined by consecutive points in A_ℓ. Since there are $|A_\ell| - 1$ midpoints of T on A_ℓ, by the hypothesis of the Lemma and of the Claim, we have

$$|A_\ell + A_\ell| = |(A + A) \cap (\ell + \ell)| = |A_T \cap \ell| = 2|A_\ell| - 1,$$

which implies that A_ℓ consists of an arithmetic progression. □

Call a set of four points of A no three of which collinear an empty quadrangle of A if their convex hull contains no further points of A.

Claim 15 *Let $x_1, x_2, x_3, x_4 \in A$ form an empty quadrangle of A. If they are in convex position then the four points form a parallelogram. That is, assuming they are listed in clockwise order, we have $x_1 + x_3 = x_2 + x_4$.*

Proof: There are two triangulations of A containing the edges of the convex quadrangle, one of them containing the edge x_1x_3 and the other one containing x_2x_4. Since A_T cannot depend on the triangulation, the midpoints of these two edges must coincide and therefore $x_1 + x_3 = x_2 + x_4$. □

The proof of the Lemma is by induction on $k = |A|$. The Lemma clearly holds if $k = 3$.

Suppose $k = 4$. If three of the points are collinear then they are on an edge of the convex hull of A and, by Claim 14, they form an arithmetic progression. With the fourth one they form a saturated set. If no three of the points are collinear then the four points form an empty quadrangle. If they are in convex position then by Claim 15 they form a saturated set, otherwise case (b) holds.

Let $k > 4$. Choose a vertex v of the convex hull of A and let $A' = A \setminus \{v\}$. If all points of A' are collinear then by Claim 14 they are in a progresion and, with v, they form a saturated set. Suppose that A' is not on o a line. For every triangulation T' of A there is a triangulation T of A containing T'. The points in $\frac{1}{2}(A' + A')$ are contained in the convex hull of A' and, by the condition of the Lemma, coincide with $A'_{T'}$. By induction either (a) or (b) hold for A'. We consider the two cases.

Case 1. A' is a saturated set.

Case 1.1. There is a convex empty quadrangle formed by v and three points of A'. Then, by Claim 15, v belongs to the lattice generated by A' as well. Moreover, since A' is convex, A is also convex and case (a) holds.

Case 1.2. There is no convex empty quadrangle involving v and three points of A'. Then it is easily checked that A' has at most one empty convex quadrangle.

If there is none in A' then, up to an affine transformation, A' consists of the point $(0, 1)$ or the two points $(0, \pm 1)$, and the remaining points on the line $y = 0$. Then either (i) v belongs to the same line $y = 0$, which satisfies the condition of Claim 14, and all points on that line in A are in arithmetic progression, so that A is a saturated set, or (ii) A' contains only the point $(0, 1)$ and v is on the line $x = 0$, in which case Claim 14 yields that the three points of A on that line are in arithmetic progression and A is a saturated set again, or (iii) A' contains only the point $(0, 1)$ and v belongs to none of the two lines containing A' and case (b) holds (see Figure 4).

If A' contains one convex empty quadrangle then, up to affinities, A' consists of the four points $(0, 0), (1, 0), (1, 1), (0, 1)$ and the remaining ones are on the line $x = y$. Moreover v must belong to the latter line as well and Claim 14 yields that the points on that line are in arithmetic progression and A is a saturated set (see Figure 4).

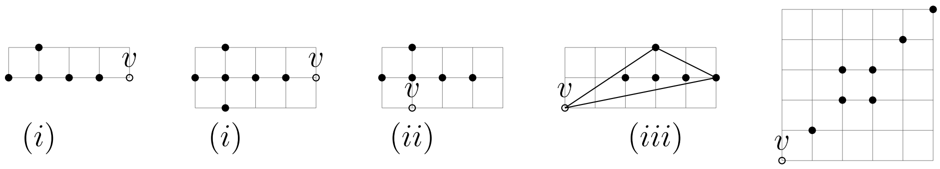

Figure 4: An illustration of Case 1.2.

Case 2. A' is as in (b). We may assume that the progression of points of A' lies on the line $x = 0$. If v is not on this line then it forms a convex empty quadrangle with two extreme points of the progression and one of the vertices w of the triangle. By Claim 15, v must be the point $w + (\pm 1, 0)$, which gives a configuration not satisfying the condition of the Lemma. Therefore v lies on the line $x = 0$ which satisfies the condition of Claim 14, so that v belongs to the progression on that line yielding case (b). □

3 Proof of Theorem 4

The inequality between the quadratic and arithmetic means gives that, if $a, k > 0$, then
$$(4a+2k)^{\frac{1}{2}} > a^{\frac{1}{2}} + (a+k)^{\frac{1}{2}}.$$
Therefore to prove Theorem 4, it is sufficient the verify the following: Let $B = A \cup \{b\}$ for $b \notin A$.

(*) If $\mathrm{tr}(A) = a$ and $\mathrm{tr}(B) = a + k$, then $\mathrm{tr}(A + B) \geq 4a + 2k$.

We fix a triangulation T of A, and let A_T be the union of A and the family of midpoints of the edges of T. It follows by (3) that

$$\Delta_{A_T} + 2\Omega_{A_T} - 2 = \mathrm{tr}(A_T) = 4a.$$

To estimate $\mathrm{tr}(A+B) = \mathrm{tr}(\frac{1}{2}(A+B))$, we isolate certain subset V of A in a way such that
$$A_T \cap (\tfrac{1}{2}(V + \{b\})) = \emptyset. \tag{13}$$
Therefore

$$\begin{aligned} \mathrm{tr}(A+B) \geq\ & 4a + 2|\tfrac{1}{2}(V+\{b\}) \cap \mathrm{int}[B]| + \\ & |\tfrac{1}{2}(V+\{b\}) \cap \partial[B]| + |A_T \cap \partial[A] \cap \mathrm{int}[B]|. \end{aligned} \tag{14}$$

We distinguish two cases depending on how to define V.

Case 1 $b \notin [A]$

We say that $x \in [A]$ is visible if $[b, x] \cap [A] = \{x\}$. In this case $x \in \partial A$. We note that there are exactly two visible points on $\partial[B]$, which are on the two supporting lines to $[A]$ passing through b (see Figure 5). Let $k+1$ be the number of visible points of A, and hence $k \geq 1$. Now $k-1$ visible points of A lie in $\mathrm{int}[B]$, thus (3) yields that $\mathrm{tr}(B) = a + k$. Let V be the set of visible points of A. The condition (13) is satisfied because $[A] \cap (\frac{1}{2}(V + \{b\})) = \emptyset$. We have $|\frac{1}{2}(V + \{b\})| = k + 1$, and $2k - 1$ visible points of A_T lie in $\mathrm{int}[B]$. In particular, $(*)$ follows as (14) yields

$$\mathrm{tr}(A+B) \geq 4a + 2k - 1 + k + 1 = 4a + 3k > 4a + 2k.$$

Case 2 $b \in [A]$

In this case $\mathrm{tr}(B) = a + k$ for $k \leq 2$ by (3), and b is contained in a triangle $T = [p, q, r]$ of T (see Figure 6). We may assume that b is not contained in the sides $[r, p]$ and $[r, q]$ of T. We take $V = \{p, q, r\}$, which satisfies (13). Since $\frac{1}{2}(b + q) \in \mathrm{int}T \subset \mathrm{int}[A]$, (14) yields $\mathrm{tr}(A + B) \geq 4a + 4$. In turn, we

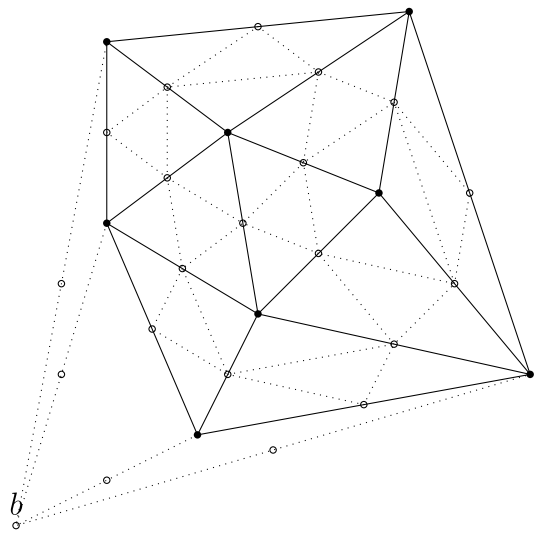

Figure 5: An illustration of Case 1.

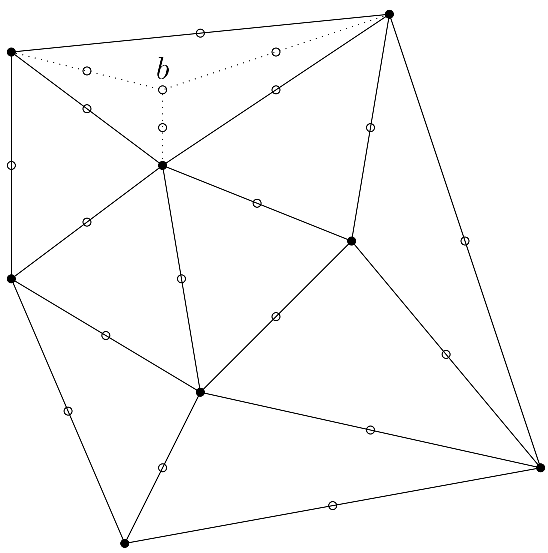

Figure 6: An illustration of Case 2.

conclude Theorem 4.

Remark: The argument does not work if we only asssume that $A \subset B$, because we may have equality in Conjecture 1 in this case.

4 Proof of Theorem 7

Let $A \subset \mathbb{R}^2$ be finite and not contained in any line. By a *path* σ on A we mean a piecewise linear simple path whose vertices are in A, and every point of A in the support of σ is a vertex of the path. We write $|\sigma|$ to denote the number of segments forming σ. We allow the case that σ is a point, and in this case $|\sigma| = 0$. We say that σ is *transversal* to a non-zero vector u if every line parallel to u intersects σ in at most one point. In this case, the segments in σ induce a subdivision of $\sigma + [o, u]$ into $|\sigma|$ parallelograms if $|\sigma| \geq 1$. For the proof of Theorem 7 the idea is to find an appropriate set of paths on A with total length at least $\sqrt{|T_A|}$.

First, we explore the possibilities using only one or two paths. We will see in Remark 16 that one path is not enough, but Proposition 17 shows that using two paths σ_1, σ_2 almost does the job.

Observe that for any given non-zero vector w, the length of the longest path on A transversal to w equals the number of lines parallel to w intersecting A, minus one.

Remark 16 *Given pairwise independent vectors $w_1, \ldots, w_n$ let $f(w_1, \ldots, w_n, s)$ be the minimal number such that, for every finite set $A \subset \mathbb{R}^2$ with $\mathrm{tr}(A) = s$, there is a w_i and a path on A transversal to w_i of length $f(w_1, \ldots, w_n, s)$.*

For $n = 2$, $f(w_1, w_2, s) \geq \sqrt{s/2}$, with equality provided that $k := \sqrt{s/2}$ is an integer. An extremal configuration consists of the points $\{iw_1 + jw_2 : i, j \in \{0, \ldots, k\}\}$.

For $n = 3$, $f(w_1, w_2, w_3, s) \geq \sqrt{2s/3}$ and equality holds provided that $s = 6k^2$. Assuming without loss of generality that $w_1 + w_2 + w_3 = 0$, an extremal configuration is given by the points of the lattice generated by w_1, w_2 in the affine regular hexagon $[\pm kw_1, \pm kw_2, \pm kw_3]$.

Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$, and let σ_1, σ_2 be piecewise linear paths whose vertices are among the vertices of A. We say that the ordered pair (σ_1, σ_2) is a *horizontal-vertical* path if

(i') σ_i is transversal with respect e_{3-i} (possibly a point), $i = 1, 2$;

(ii') the right endpoint a of σ_1 is the upper endpoint of σ_2

(iii') writing $\mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\}$, if $|\sigma_1|, |\sigma_2| > 0$, then

$$((\sigma_1 \backslash \{a\}) + \mathbb{R}_+ e_2) \cap ((\sigma_2 \backslash \{a\}) + \mathbb{R}_+ e_1) = \emptyset.$$

We call σ_1 the horizontal branch, and σ_2 the vertical branch, and a the center.

We observe that if σ_i' is the image of σ_i by reflection through the line $\mathbb{R}(e_1 + e_2)$, then the ordered pair (σ_2', σ_1') is also a horizontal-vertical path.

For any polygon P and non-zero vector u, we write $F(P, u)$ to denote the face of P with exterior normal u. In particular, $F(P, u)$ is either a side or a vertex.

Proposition 17 *For every finite $A \subset \mathbb{R}^2$ not contained in a line, and for every triangulation T of $[A]$ using A as a vertex set, there exists a horizontal-vertical path (σ_1, σ_2) whose vertices belong to A, and satisfies*

$$|\sigma_1| + |\sigma_2| \geq \sqrt{|T| + 1} - \tfrac{1}{2}.$$

Proof: Let us write

$$\begin{aligned} \xi &= |F([A], -e_1) \cap F([A], -e_2)| \leq 1 \\ \Delta_A' &= |(A \cap \partial[A]) \backslash (F([A], -e_1) \cup F([A], -e_2))|. \end{aligned}$$

By the invariance with respect to reflection through the line $\mathbb{R}(e_1 + e_2)$, we may assume that

$$|F([A], -e_2) \cap A| \geq |F([A], -e_1) \cap A|. \tag{15}$$

We set $\{\langle e_1, p \rangle : p \in A\} = \{\alpha_0, \ldots, \alpha_k\}$ with $\alpha_0 < \ldots < \alpha_k$, $k \geq 1$. For $i = 0, \ldots, k$, let $A_i = \{p \in A : \langle e_1, p \rangle = \alpha_i\}$, let $x_i = |A_i|$, and let a_i be the top most point of A_i; namely, $\langle e_2, a_i \rangle$ is maximal. In particular, $x_0 = |F([A], -e_1) \cap A|$. For each $i = 1, \ldots, k$, we consider the horizontal-vertical path $(\sigma_{1i}, \sigma_{2i})$ where

$$\sigma_{1i} = \{[a_0, a_1], \ldots, [a_{i-1}, a_i]\},$$

and the vertex set of σ_{2i} is A_i. In particular, the total length of the horizontal-vertical path is $(\sigma_{1i}, \sigma_{2i})$ is

$$|\sigma_{1i}| + |\sigma_{2i}| = i + x_i - 1.$$

The average length of these paths for $i = 1, \ldots, k$ is

$$\frac{\sum_{i=1}^k (|\sigma_{1i}| + |\sigma_{2i}|)}{k} = \frac{\sum_{i=1}^k (i + x_i - 1)}{k} = \frac{|A| - x_0}{k} + \frac{k}{2} - \frac{1}{2}.$$

We observe that $2|A| = |T| + \Delta_A + 2$, according to (3), and (15) yields

$$2 + \Delta_A - 2x_0 = 2 + \Delta'_A + |F([A], -e_2) \cap A| - \xi - x_0 \geq \Delta'_A + 1.$$

Therefore we deduce from the inequality between the arithmetic and geometric mean that

$$\begin{aligned} \frac{\sum_{i=1}^{k-1}(|\sigma_{1i}| + |\sigma_{2i}|)}{k-1} &= \frac{2|A| - 2x_0}{2k} + \frac{k}{2} - \frac{1}{2} \\ &\geq \frac{1}{2}\left(\frac{|T| + \Delta'_A + 1}{k} + k\right) - \frac{1}{2} \qquad (16) \\ &\geq \sqrt{|T| + \Delta'_A + 1} - \frac{1}{2}. \qquad (17) \end{aligned}$$

Therefore there exists some horizontal-vertical path $(\sigma_{1i}, \sigma_{2i})$ satisfying (17). □

The estimate of Proposition 17 is close to be optimal according to the following example.

Example 18 *Let $k \geq 2$ and $t > 0$. Let A' be the saturated set with $[A']$ having vertices $(0,0), (0,k), (k-1,0)$ and $(k-1,1)$, and let $A = A' \cup \{(k+t, 0)\}$. A triangulation of A has $k^2 + k - 1$ triangles and every horizontal–vertical path (σ_1, σ_2) on A has total length*

$$|\sigma_1| + |\sigma_2| \leq k < \sqrt{|T| + 2} - \tfrac{1}{2}.$$

□

We next proceed to the proof of Theorem 7 by a similar strategy using three paths. Let $B = \{v_1, v_2, v_3\}$ and, for $\{i, j, k\} = \{1, 2, 3\}$ denote by u_i the exterior unit normal to the side $[v_j, v_k]$ of B. A set of three paths $(\sigma_1, \sigma_2, \sigma_3)$ meeting at some point $a \in A$ and using the edges of a triangulation T of A is called a *proper star* if the following conditions hold:

(i) σ_i is transversal with respect $v_j - v_k$ (possibly $\sigma_i = \{a\}$);

(ii) σ_i has an end point $b_i \in \partial[A]$ such that u_i is an exterior unit normal to $[A]$ at b_i, and

$$\langle a, u_i \rangle = \min\{\langle x, u_i \rangle : x \in \sigma_i\};$$

(iii) writing $\mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\}$, if $|\sigma_j|, |\sigma_k| > 0$, then

$$((\sigma_j \backslash \{a\}) + \mathbb{R}_+(v_k - v_i)) \cap ((\sigma_k \backslash \{a\}) + \mathbb{R}_+(v_j - v_i)) = \emptyset.$$

If the semi-open paths $\sigma_i \setminus \{a\}$, $i = 1, 2, 3$, are all non-empty and pairwise disjoint, then (iii) means that they come around a in the same order as the orientation of the triangle $[v_1, v_2, v_3]$ (see Figure 7 for an illustration).

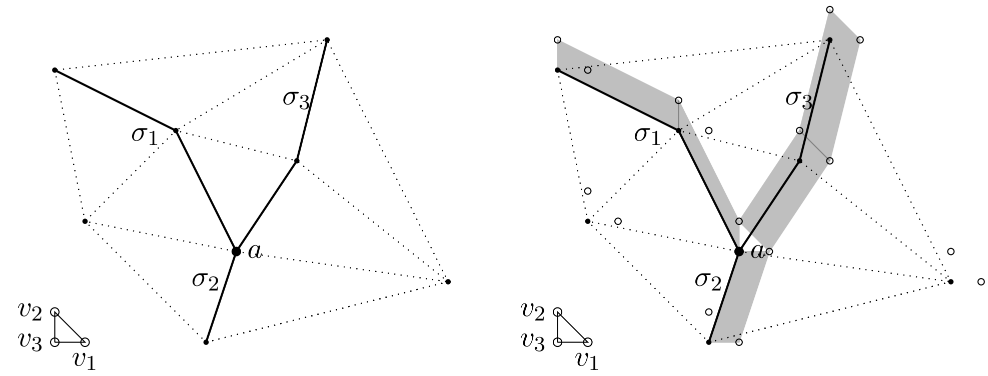

Figure 7: A proper star with respect to v_1, v_2, v_3 centered at a. On the right, paralellograms based on the proper star

The next Lemma shows how to construct an appropriate mixed subdivision of $A + B$ using a proper star.

Lemma 19 *Given a proper star with rays $\sigma_1, \sigma_2, \sigma_3$ such that $|\sigma_1| + |\sigma_2| + |\sigma_3| > 0$, there exists a mixed subdivision M for $A + B$ satisfying*

$$M_{11} = |\sigma_1| + |\sigma_2| + |\sigma_3|.$$

Proof: We may assume that $|\sigma_1| > 0$ and $v_3 = o$. We partition the triangles of T_A into three subsets $\Sigma_1, \Sigma_2, \Sigma_3$ (some of them might be empty). The idea is that if the semi-open paths $\sigma_i \setminus \{a\}$, $i = 1, 2, 3$, are all non-empty and pairwise disjoint and $\{i, j, k\} = \{1, 2, 3\}$, then Σ_i consists of the triangles cut off by $\sigma_j \cup \sigma_k$.

A triangle τ of T_A is in Σ_1 if and only if there exists a $p \in (\text{int}\, \tau) \setminus (a + \mathbb{R} v_1)$ such that

$$|(p - \mathbb{R}_+ v_1) \cap \sigma_2| + |(p - \mathbb{R}_+ v_1) \cap \sigma_3|$$

is finite and odd. Similarly, $\tau \in T_A$ is in Σ_2 if and only if there exists a $p \in \text{int}\, \tau$ such that

$$|(p - \mathbb{R}_+ v_2) \cap \sigma_1| + |(p - \mathbb{R}_+ v_2) \cap \sigma_3|$$

is finite and odd. The rest of the triangles of T_A form Σ_3.

The triangles of the mixed subdivision M are as follows. If $\tau \in \Sigma_i$, then the corresponding triangle in M is $\tau + v_i$. In addition, $[B]+a$ is in M. For the parallelograms, let $\{i,j,k\} = \{1,2,3\}$. If e is an edge of σ_i, then $e + [v_j, v_k]$ is in M. □

For the rest of the section, we fix finite $A \subset \mathbb{R}^2$ and $B = \{v_1, v_2, v_3\} \subset \mathbb{R}^2$ such that both of them spans $\mathbb{R}^2$ affinely, and confirm Conjecture 5 in this case.

The following statement is a simple consequence of the definition of a proper star.

Lemma 20 *Assuming $B = \{v_1, v_2, v_3\}$ with $v_1 = (1,0) = -u_1$, $v_2 = (0,1) = -u_2$ and $v_3 = (0,0)$, and hence $u_3 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, if (σ_1, σ_2) is a horizontal-vertical path for A centered at $a \in A$, then*

- *there exists a proper star $(\sigma_1', \sigma_2', \sigma_3')$ centered at a such that $\sigma_1 \subset \sigma_1'$, $\sigma_2 \subset \sigma_2'$,*
- *if in addition $a \notin F([A], u_3)$, then $|\sigma_3'| \geq 1$.*

Proof of Theorem 7 We may assume that $B = \{v_1, v_2, v_3\}$ with $v_1 = (1,0) = -u_1$, $v_2 = (0,1) = -u_2$ and $v_3 = (0,0)$, and hence $u_3 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. In addition, we may assume that

$$|F([A], -u_2) \cap A| \geq |F([A], -u_1) \cap A|.$$

Using the notation of the proof of (16), we set $\{\langle u_1, p\rangle : p \in A\} = \{\alpha_0, \ldots, \alpha_k\}$ with $\alpha_0 < \ldots < \alpha_k$, and $\Delta_A' = |(A \cap \partial[A]) \backslash (F([A], -u_1) \cup F([A], -u_2))|$. For $i = 0, \ldots, k$, let $A_i = \{p \in A : \langle u_1, p\rangle = \alpha_i\}$, let $x_i = |A_i|$ and let a_i be the top most point of A_i; namely, $\langle u_2, a_i\rangle$ is maximal. According to (16) and (17), we have

$$\frac{\sum_{i=1}^{k}(i + x_i - 1)}{k} \geq \frac{|T_A| + \Delta_A' + 1}{2k} + \frac{k}{2} - \frac{1}{2} \geq \sqrt{|T_A| + 1} - \frac{1}{2}. \tag{18}$$

Let I be the set of all $i \in \{1, \ldots, k\}$ such that

$$i + x_i - 1 \geq \left\lceil \frac{|T_A| + \Delta_A' + 1}{2k} + \frac{k}{2} - \frac{1}{2} \right\rceil = \xi. \tag{19}$$

Since $\xi \geq \sqrt{|T_A| + 1} - \frac{1}{2}$, if strict inequality holds for some i in (19), then we have a required proper star by Lemma 20. Thus we assume that $i + x_i - 1 = \xi$ for $i \in I$.

Let $\theta = |I|$. Since $i + x_i - 1 \leq \xi - 1$ if $i \notin I$, we have

$$\xi - \frac{\sum_{i=1}^{k}(i + x_i - 1)}{k} \geq \frac{k - \theta}{k}.$$

We deduce from (18) that if $i \in I$, then

$$i + x_i - 1 \geq \frac{|T_A| + \Delta'_A + 1}{2k} + \frac{k}{2} - \frac{1}{2} + \frac{k - \theta}{k} = \frac{|T_A| + \Delta'_A + 1}{2k} + \frac{k}{2} + \frac{1}{2} - \frac{\theta}{k}.$$

If $i \in I$ and $a_i \notin F([A], u_3)$, then $\xi \geq \sqrt{|T_A| + 1} - \frac{1}{2}$ and Lemma 20 yields the existence of a required proper star. Therefore we may assume that $a_i \in F([A], u_3)$ for $i \in I$. Since $|F([A], u_3) \cap F([A], -u_2))| \leq 1$, we deduce that

$$\theta \leq \max\{\Delta'_A + 1, k\}. \tag{20}$$

Therefore if $i \in I$, then we conclude using the inequality betwen the arightmetic and the geometric mean at the last inequality that

$$i + x_i - 1 \geq \frac{|T_A| + \theta}{2k} + \frac{k}{2} + \frac{1}{2} - \frac{\theta}{k} \geq \frac{|T_A|}{2k} + \frac{k}{2} + \frac{1}{2} - \frac{\theta}{2k} \geq \sqrt{|T_A|}. \quad \square$$

5 Proof of Theorem 8

We assume in this section that there are no points of A (resp. B) in the interior of $[A]$, (resp. $[B]$).

Recall that Δ_X denotes the number of points of X in the boundary of $[X]$. It is easy to check that Δ_{A+B} has at least as many points as Δ_A and Δ_B together, that is:

$$\Delta_{A+B} \geq \Delta_A + \Delta_B = \mathrm{tr}(A) + \mathrm{tr}(B) + 4$$

As a motivation for the proof, we note that Conjecture 1 follows if the number Ω_{A+B} of points of $A + B$ in $\mathrm{int}([A + B])$ is at least

$$\frac{\mathrm{tr}(A) + \mathrm{tr}(B) - 2}{2} = \frac{\Delta_A + \Delta_B}{2} - 3.$$

Naturally we aim at the stronger Conjecture 5. Given Theorem 7, Theorem 8 follows if A and B being in convex position and $|A|, |B| \geq 4$ yield that there exists a mixed subdivision of $A + B$ satisfying

$$|M_{11}| \geq \frac{\mathrm{tr}(A) + \mathrm{tr}(B)}{2}. \tag{21}$$

Throughout the proof we assume that $[B]$ has at most as many vertices as $[A]$ and v denotes a unit vector (which we assume pointing upwards) not parallel to any side of $[A+B]$. We denote by a_0 and a_1 the leftmost and rightmost vertex of $[A]$ and by b_0 and b_1 the leftmost and rightmost vertex of $[B]$.

To prove (21), we say that A and B form a *strange pair* if $[B]$ is a triangle and the three exterior normals to $[B]$ are also exterior normals of edges of $[A]$.

We will use that, for $t, s \geq 1$,

$$ts \geq t + s - 1. \tag{22}$$

Case 1 A and B are not strange pairs.

We choose a unit vector v as above in the following way: if B is a triangle, then the upper arc of $\partial[B]$ is a side such that $[A]$ has no side with same exterior unit normal; if $[B]$ has at least four sides, then the two supporting lines of $[B]$ parallel to v touch at non-consecutive vertices of $[B]$. For the existence of the latter pair of supporting lines, we note that while continuously rotating $[B]$, the number of upper - lower vertices changes by either zero or two units at a time when a side of $[B]$ is parallel to v, and after rotation by π it changes to its opposite. Hence, at some position that difference is zero or one which implies, since $[B]$ has at least four vertices, that at that position there is at least one upper and one lower vertex, as required.

Claim 21 *One of the two following statements hold:*

$$\begin{aligned} \left|\Big((A+b_0)\cup(a_1+B)\Big)\cap \mathrm{int}[A+B]\right| &\geq \tfrac{\Delta_A+\Delta_B}{2}-3, \ \textit{or} \\ \left|\Big((a_0+B)\cup(A+b_1)\Big)\cap \mathrm{int}[A+B]\right| &\geq \tfrac{\Delta_A+\Delta_B}{2}-3. \end{aligned} \tag{23}$$

Proof: We may assume that $b_1 = a_0 = o$ (see Fig. 8). Observe first that the only repetitions $x+b_0 = a_1+y$ or $x+b_1 = a_0+y$ in these configurations are the points a_1+b_0 and a_0+b_1 (which are interior to $[A+B]$ by our hypothesis). To prove (23), we verify first that

(i) for every $x \in A \setminus \{a_0, a_1\}$ except perhaps two of them, at least one of $x+b_0$ or $x+b_1$ is interior in $A+B$,

(ii) for every $y \in B \setminus \{b_0, b_1\}$ except perhaps two of them, at least one of a_0+y or a_1+y is interior in $A+B$.

For (i), we note that if both $x+b_0$ or $x+b_1$ are in $\partial[A+B]$, then they are the end points of a segment translated from $[b_0, b_1]$ and only two such

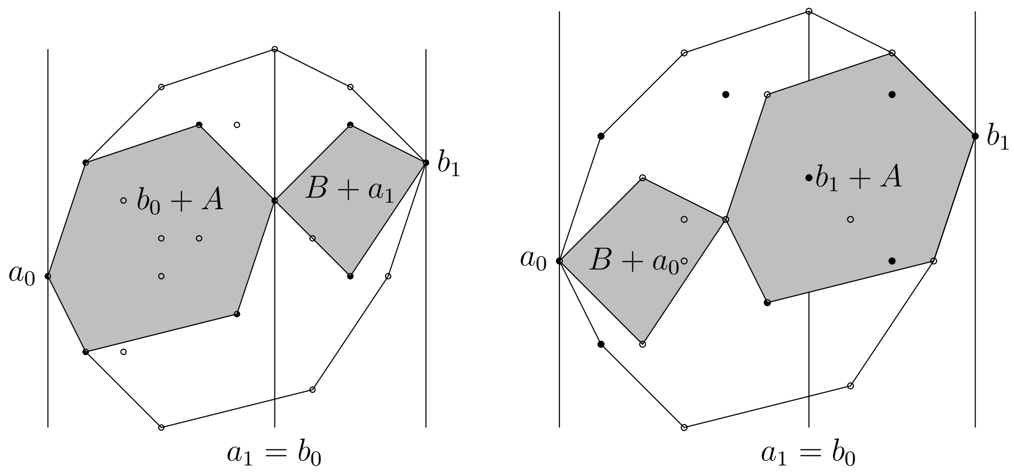

Figure 8: An illustration of the proof of Claim 23.

translations have their end-points in $\partial[A + B]$ because A and B are not a strange pair. The argument for (ii) is similar.

Now (i) and (ii) say that counting the interior points of $(A+b_0)\cup(a_1+B)$ and $(a_0+B)\cup(A+b_1)$ except a_0+b_1 and a_1+b_0 we have altogether at least $|\Delta_A|+|\Delta_B|-8$ of them. Including the latter we have at least $|\Delta_A|+|\Delta_B|-6$ of them and at least half of these in either $(A+b_0)\cup(a_1+B)$ or $(a_0+B)\cup(A+b_1)$, which yields (23). □

Let us construct the suitable mixed triangulation of $[A + B]$. For every path σ in ∂A, we assume that every point of A in σ is a vertex of σ. According to (23), we may assume that

$$|(A \cup B) \cap \mathrm{int}[A + B]| \geq \frac{\Delta_A + \Delta_B}{2} - 3 \tag{24}$$

Let a_{upp} (a_{low}) be the neighboring vertex of $[A]$ to o on the upper (lower) arc of ∂A, and let b_{upp} (b_{low}) be the neighboring vertex of $[B]$ to o on the upper (lower) arc of ∂B. We write ω^A_{upp} and ω^A_{low} to denote the paths determined by $[o, a_{\mathrm{upp}}]$ and $[o, a_{\mathrm{low}}]$ and ω^B_{upp} and ω^B_{low} to denote the paths determined by $[o, b_{\mathrm{upp}}]$ and $[o, b_{\mathrm{low}}]$. Next let σ^A_{upp} (σ^A_{low}) be the longest path on the upper (lower) arc of $\partial[A]$ starting from o such that every segment s of σ^A_{upp} (σ^A_{low}) satisfies that $s+[o, b_{\mathrm{upp}}]$ ($s+[o, b_{\mathrm{low}}]$) is a parallelogram that does not intersect $\mathrm{int}[A]$. Similarly, let σ^B_{upp} (σ^B_{low}) be the longest path on the upper (lower) arc of $\partial[B]$ starting from o such that every segment s of σ^B_{upp} (σ^B_{low}) satisfies that $s + [o, a_{\mathrm{upp}}]$ ($s + [o, a_{\mathrm{low}}]$) is a parallelogram that does not intersect $\mathrm{int}[B]$.

Since $a_1 = b_0 = o$ is a common point of $\sigma^A_{\rm upp}$, $\sigma^A_{\rm low}$, $\sigma^B_{\rm upp}$, $\sigma^B_{\rm low}$, we deduce from (24) that

$$1 + (|\sigma^A_{\rm upp}| - 1) + (|\sigma^A_{\rm low}| - 1) + (|\sigma^B_{\rm upp}| - 1) + (|\sigma^B_{\rm low}| - 1) \geq \frac{\Delta_A + \Delta_B}{2} - 3,$$

equivalently,

$$|\sigma^A_{\rm upp}| + |\sigma^A_{\rm low}| + |\sigma^B_{\rm upp}| + |\sigma^B_{\rm low}| \geq \frac{\Delta_A + \Delta_B}{2}. \tag{25}$$

We construct the mixed subdivision by considering the subdivisions into suitable paralleograms of $\sigma^A_{\rm upp} + \omega^B_{\rm upp}$ and $\sigma^B_{\rm upp} + \omega^A_{\rm upp}$ that have $\omega^A_{\rm upp} + \omega^B_{\rm upp}$ in common, and the subdivisions into suitable parallelograms of $\sigma^A_{\rm low} + \omega^B_{\rm low}$ and $\sigma^B_{\rm low} + \omega^A_{\rm low}$ that have $\omega^A_{\rm low} + \omega^B_{\rm low}$ in common (see Figure 9).

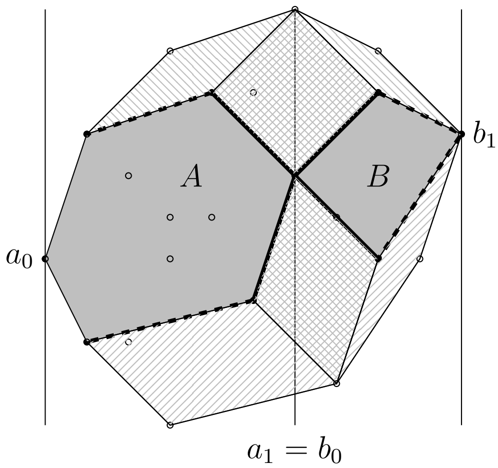

Figure 9: An illustration of the parallelograms of the mixed subdivision in Case 1.

In particular, $|\omega^A_{\rm upp}|, |\omega^B_{\rm upp}| \geq 1$, (22) and (25) yield that

$$\begin{aligned} |M_{11}| &\geq (|\sigma^A_{\rm upp}| - |\omega^A_{\rm upp}|)|\omega^B_{\rm upp}| + (|\sigma^B_{\rm upp}| - |\omega^B_{\rm upp}|)|\omega^A_{\rm upp}| + |\omega^A_{\rm upp}| \cdot |\omega^B_{\rm upp}| + \\ &\quad +(|\sigma^A_{\rm low}| - |\omega^A_{\rm low}|)|\omega^B_{\rm low}| + (|\sigma^B_{\rm low}| - |\omega^B_{\rm low}|)|\omega^A_{\rm low}| + |\omega^A_{\rm low}| \cdot |\omega^B_{\rm low}| \\ &\geq (|\sigma^A_{\rm upp}| - |\omega^A_{\rm upp}|) + (|\sigma^B_{\rm upp}| - |\omega^B_{\rm upp}|) + |\omega^A_{\rm upp}| + |\omega^B_{\rm upp}| - 1 + \\ &\quad +(|\sigma^A_{\rm low}| - |\omega^A_{\rm low}|) + (|\sigma^B_{\rm low}| - |\omega^B_{\rm low}|) + |\omega^A_{\rm low}| + |\omega^B_{\rm low}| - 1 \\ &\geq \frac{\Delta_A + \Delta_B}{2} - 2 = \frac{{\rm tr}(A) + {\rm tr}(B)}{2} \end{aligned}$$

proving (21) in Case 1.

Case 2 A and B form a strange pair with $|A|, |B| \geq 4$, and $[A]$ and $[B]$ are not similar triangles

We write α_{upp} (α_{low}) to denote the number of segments that the points of A divide the upper (lower) arc of $\partial[A]$. We denote by b_2 the third vertex of $[B]$ and by $[x_0, x_1]$ the side of A with $x_1 - x_0 = t(b_1 - b_0)$ for $t > 0$. For $i = 0, 1, 2$, let s_i be the number of segments that the points of B divide the side of $[B]$ opposite to b_i.

Claim 22 *There exists a v such that one of the following holds:*

$$\alpha_{\text{upp}} \geq 2 \textit{ and } \alpha_{\text{upp}} + s_0 + s_1 \geq \frac{1}{2}(\Delta_A + \Delta_B), \textit{ or} \tag{26}$$

$$\alpha_{\text{low}}, s_2 \geq 2 \textit{ and } \alpha_{\text{low}} + s_2 \geq \frac{1}{2}(\Delta_A + \Delta_B). \tag{27}$$

Proof: Since $\alpha_{\text{upp}} + \alpha_{\text{low}} = \Delta_A$ and $s_0 + s_1 + s_2 = \Delta_B$, the claim easily follows if there is a v such that, for each the sets A and B, both the upper arc and the lower arc contain a point of the set strictly between the two supporting lines parallel to v.

Otherwise, choose a v such that the side $[b_0, b_1]$ of $[B]$ contains at least 3 points of B (this is possible since $|B| \geq 4$). Then $[x_0, x_1]$ has no other point of A than x_0, x_1 and the other side of $[A]$ at x_i, $i = 0, 1$ is parallel to $[b_i, b_2]$. As $[A]$ and $[B]$ are not similar triangles , $[A]$ has some more sides, which in turn yields that $[b_i, b_2] \cap B = \{b_i, b_2\}$ for $i = 0, 1$. In summary, we have $\alpha_{\text{upp}} = s_0 = s_1 = 1$ and $\alpha_{\text{low}}, s_2 \geq 2$. Since $\alpha_{\text{low}} + s_2 > \alpha_{\text{upp}} + s_0 + s_1$, we conclude (27). □

To prove (21) based on (26) and (27), we introduce some further notation. After a linear transformation, we may assume that v is an exterior normal to the side $[b_0, b_1]$ of $[B]$. We say that $p, q \in \partial[A]$ are opposite if there exists a unit vector w such that w is an exterior normal at p and $-w$ is an exterior normal at q. If $p, q \in \partial[A]$ are not opposite, then we write $\overline{pq}$ the arc of $\partial[A]$ connecting p and q and not containing opposite pair of points.

First we assume that (26) holds and $b_2 = o$. Since $[x_0, x_1]$ has exterior normal v and $\alpha_{\text{upp}} \geq 2$, there exists $a \in A \backslash \{x_0, x_1\}$ such that v is an exterior normal to $\partial[A]$ at a. We write l_{upp} and r_{upp} to denote the number of segments the points of A divide the arcs $\overline{ax_0}$ and $\overline{ax_1}$, respectively. To construct a mixed subdivision, we observe that every exterior normal u to a side of $[A]$ in $\overline{ax_0}$ satisfies $\langle u, b_0 \rangle > 0$, and every exterior normal w to a side of $[A]$ in $\overline{ax_1}$ satisfies $\langle w, b_1 \rangle > 0$. We divide $\overline{ax_0} + [o, b_0]$ into suitable $s_1 l_{\text{upp}}$ parallelograms, and $\overline{ax_1}+[o,b_1]$ into suitable $s_0r_{\rm upp}$ parallelograms. It follows from (22) that

$$\begin{aligned}|M_{11}| &= s_1l_{\rm upp}+s_0r_{\rm upp}\geq l_{\rm upp}+r_{\rm upp}+s_0+s_1-2=\alpha_{\rm upp}+s_0+s_1-2\\ &\geq \tfrac{1}{2}(\Delta_A+\Delta_B)-2=\tfrac{1}{2}({\rm tr}(A)+{\rm tr}(B)).\end{aligned}$$

Secondly we assume that (27) holds. Since $s_2\geq 2$, we may assume that $o\in([b_0,b_1]\backslash\{b_0,b_1\})\cap B$. For $i=0,1$, we write s_{2i} to denote the number of segments the points of B divide $[o,b_i]$. Let $\tilde{x}_0$ and $\tilde{x}_1$ be the leftmost and rightmost points of A such that $-v$ is an exterior normal to $\partial[A]$, where possibly $\tilde{x}_0=\tilde{x}_1$. Since $[A]$ has sides parallel to the sides $[b_2,b_0]$ and $[b_2,b_1]$ of $[B]$, we deduce that $\tilde{x}_0\neq x_0$ and $\tilde{x}_1\neq x_1$. To construct a mixed subdivision, we set $m_{\rm low}=0$ if $\tilde{x}_0=\tilde{x}_1$, and $m_{\rm low}$ to be the number of segments the points of A divide $\overline{\tilde{x}_0,\tilde{x}_1}$ if $\tilde{x}_0\neq\tilde{x}_1$. In addition, we write $l_{\rm low}\geq 1$ and $r_{\rm low}\geq 1$ to denote the number of segments the points of A divide the arcs $\overline{\tilde{x}_0x_0}$ and $\overline{\tilde{x}_1x_1}$, respectively. We divide $\overline{\tilde{x}_0x_0}+[o,b_0]$ into suitable $l_{\rm low}s_{20}$ parallelograms, and $\overline{\tilde{x}_1x_1}+[o,b_1]$ into suitable $r_{\rm upp}s_{21}$ parallelograms. In addition, if $\tilde{x}_0\neq\tilde{x}_1$, then we divide $[\tilde{x}_0\tilde{x}_1]+[o,b_2]$ into suitable $m_{\rm low}$ parallelograms. It follows from (22) that

$$\begin{aligned}|M_{11}| &= l_{\rm low}s_{20}+r_{\rm low}s_{21}+m_{\rm low}\geq l_{\rm low}+r_{\rm low}+m_{\rm low}+s_{20}+s_{21}-2\\ &= \alpha_{\rm low}+s_2-2\geq\tfrac{1}{2}(\Delta_A+\Delta_B)-2=\tfrac{1}{2}({\rm tr}(A)+{\rm tr}(B)),\end{aligned}$$

finishing the proof of (21) in Case 2.

Case 3 $[A]$ and $[B]$ are similar triangles and $|A|,|B|\geq 4$

We recall that s_1,s_2 and s_3 denote the number of segments the points of B divide the sides of $[B]$ and let s_1',s_2',s_3' be the number of segments the points of A divide the corresponding sides of $[A]$. We have ${\rm tr}(A)=s_1'+s_2'+s_3'-2$ and ${\rm tr}(B)=s_1+s_2+s_3-2$. We may assume that s_1 is the largest among the six numbers and that $s_2'\geq s_3'$. Readily

$$|M_{11}|\geq\max\{s_1s_2',s_1's_2,s_1's_3\}. \tag{28}$$

If $s_2'\geq 3$, then

$$|M_{11}|\geq 3s_1\geq\tfrac{1}{2}(s_1+s_2+s_3+s_1'+s_2'+s_3')>\tfrac{1}{2}({\rm tr}(A)+{\rm tr}(B)).$$

If $s_2'=2$, then $s_3'\leq 2$ and

$$|M_{11}|\geq 2s_1\geq\tfrac{1}{2}(s_1+s_2+s_3+s_1'+s_2'+s_3'-4)=\tfrac{1}{2}({\rm tr}(A)+{\rm tr}(B)).$$

Therefore we assume that $s'_2 = s'_3 = 1$. In particular, we may also assume that $s_2 \geq s_3$. Since $s'_1 \geq 2$ and $s_2 \geq 1$ we have $s'_1 s_2 \geq s'_1 + 2s_2 - 2$. Therefore,

$$\begin{aligned} |M_{11}| &\geq \max\{s_1, s'_1 s_2\} \\ &\geq \frac{1}{2}((s_1 + s_2 + s_3 + s'_1 - 2) \\ &\geq \frac{1}{2}(s_1 + s_2 + s_3 + s'_1 - 2) \\ &= \frac{1}{2}(\mathrm{tr}(A) + \mathrm{tr}(B)), \end{aligned}$$

and we conclude (21) in Case 3, as well. □

6 Proof of Theorem 9

Let $A = \{a_1, \ldots, a_n\}$. Naturally, $[A + A]$ has a triangulation $\{F + F : F \in T_A\}$, which we subdive in order to obtain M. We define M to be the collection of the sums of the form

$$[a_{i_0}, \ldots, a_{i_m}] + [a_{i_m}, \ldots, a_{i_k}]$$

where $k \geq 0$, $0 \leq m \leq k$, $i_j < i_l$ for $j < l$, and $[a_{i_0}, \ldots, a_{i_k}] \in T_A$.

To show that we obtain a cell decomposition, let

$$F = [a_{i_0}, \ldots, a_{i_k}] \in T_A$$

be a k-simplex with $k > 0$ where $i_j < i_l$ for $j < l$, and hence

$$F + F = \left\{ \sum_{i=0}^{k} \alpha_j a_{i_j} : \sum_{i=0}^{k} \alpha_j = 2 \; \& \; \forall \alpha_j \geq 0 \right\}.$$

We write $\mathrm{relint}\, C$ to denote the relative interior of a compact convex set C. For some $0 \leq m \leq k$, $\alpha_0, \ldots, \alpha_k \geq 0$ with $\sum_{i=0}^{k} \alpha_j = 2$, we have

$$\sum_{i=0}^{k} \alpha_j a_{i_j} \in \mathrm{relint}\; ([a_{i_0}, \ldots, a_{i_m}] + [a_{i_m}, \ldots, a_{i_k}]) \subset F + F$$

if and only if $\sum_{j<m} \alpha_j < 1$ and $\sum_{i=0}^{m} \alpha_j > 1$ where we set $\sum_{j<0} \alpha_j = 0$. We conclude that M forms a cell decomposition of $[A + A]$.

For any d-simplex $F \in T_A$, and for any $m = 0, \ldots, d$, we have constructed one d-cell of M that is the sum of an m-simplex and a $(d - m)$-simplex. Therefore

$$\|M\| = |T_A| \sum_{m=0}^{d} \binom{d}{m} = 2^d |T_A|.$$

7 Proof of Corollary 3

In this section, let $A \subset \mathbb{R}^2$ be finite and not collinear. We prove four auxiliary statements about A. The first is an application of the case $A = B$ of Conjecture 1 (see Theorem 2).

Lemma 23

$$|A + A| \geq 4|A| - \Delta_A - 3$$

Proof: We have readily $\Delta_{A+A} \geq 2\Delta_A$. Thus (3) and Theorem 2 yield

$$|A + A| = \frac{1}{2}\left(\mathrm{tr}(A + A) + \Delta_{A+A} + 2\right) \geq 2\mathrm{tr}(A) + \Delta_A + 1 = 4|A| - \Delta_A - 3. \ \square$$

We note that the estimate of Lemma 23 is optimal, the configuration of Theorem 2 (b) being an extremal set.

Next we provide the well-known elementary estimate for $|A + A|$ only in terms of boundary points.

Lemma 24 *Let m_A denote the maximal number of points of A contained in a side of $[A]$. We have,*

$$|A + A| \geq \frac{\Delta_A^2}{4} - \frac{\Delta_A(m_A - 1)}{2}.$$

Proof: We choose a line l not parallel to any side of $[A]$, that we may assume to be a vertical line, and denote by $s_1, \ldots, s_k$ the sides of $[A]$ on the upper chain of $[A]$ in left to right order. Let A_i be the set obtained from $A \cap s_i$ by removing its rightmost point. We may assume that

$$|A_1| + \cdots + |A_k| \geq \frac{\Delta_A}{2}.$$

We observe that, for $1 \leq i < j \leq k$, we have

$$|A_i + A_j| = |A_i| \cdot |A_j| \text{ and } (A_i + A_j) \cap (A_{i'} \cap A_{j'}) = \emptyset \text{ if } \{i, j\} \neq \{i', j'\}.$$

It follows that

$$\begin{aligned} |A + A| &\geq \sum_{1 \leq i < j \leq k} |A_i + A_j| = \sum_{1 \leq i < j \leq k} |A_i| \cdot |A_j| = (\sum_{i=1}^{k} |A_i|)^2 - \sum_{i=1}^{k} |A_i|^2 \\ &\geq \left(\frac{\Delta_A}{2}\right)^2 - (m_A - 1)\frac{\Delta_A}{2}. \ \square \end{aligned}$$

The following Lemma can be found in Freiman [4].

Lemma 25 *Let ℓ be a line intersecting $[A]$ in m points of A. If A is covered by exactly s lines parallel to ℓ, then*

$$|A+A| \geq 2|A| + (s-1)m - s. \tag{29}$$

Moreover,

$$|A+A| \geq (4 - \frac{2}{s})|A| - (2s-1). \tag{30}$$

Proof: We may assume that ℓ is the vertical line through the origin, that $a_1, \ldots, a_s$ are s points of A ordered left to right such that $A = \cup_{i=1}^{s}(A \cap (\ell + a_i))$ and that $|A \cap (\ell + a_1)| = m$. Let $A_i = A \cap (a_i + \ell)$. Then,

$$\begin{aligned}
|A+A| &= |A_1 + A| + |(A \setminus A_1) + A_s| \\
&\geq \sum_{i=1}^{s}(|A_1| + |A_i| - 1) + \sum_{i=2}^{s}(|A_i| + |A_s| - 1) \\
&= 2|A| + (s-1)(|A_1| + |A_s|) - (2s-1),
\end{aligned}$$

from which (29) follows. On the other hand,

$$\begin{aligned}
|A+A| &= \sum_{i=1}^{s}|2A_i| + \sum_{i=1}^{s-1}|A_i + A_{i+1}| \\
&\geq \sum_{i=1}^{s}(2|A_i| - 1) + \sum_{=1}^{s-1}(|A_i| + |A_{i+1}| - 1) \\
&= 4|A| - (|A_1| + |A_s|) - (2s-1).
\end{aligned}$$

If the latter estimate is larger than the former one we obtain (30), otherwise we get the stronger inequality $|A+A| \geq (4 - 2/s^2)|A| - (2s-1)$. □

Proof of Corollary 3 Let $|A+A| \leq (4-\varepsilon)|A|$ where $\varepsilon \in (0,1)$ and $\varepsilon^2|A| \geq 48$. To simply formulae, we set $\Delta = \Delta_A$ and $m = m_A$.

We deduce from Lemma 23 that $\Delta \geq \varepsilon|A| - 3$. Substituting this into Lemma 24 yields

$$\begin{aligned}
(4-\varepsilon)|A| &\geq \frac{\Delta^2}{4} - \frac{\Delta(m-1)}{2} \geq \frac{\Delta(\varepsilon|A| - 3)}{4} - \frac{\Delta(m-1)}{2} \\
&= \frac{\Delta}{2} \cdot (\tfrac{1}{2}\varepsilon|A| - m - \tfrac{1}{2}) \geq \frac{\varepsilon|A| - 3}{2} \cdot (\tfrac{1}{2}\varepsilon|A| - m - \tfrac{1}{2}).
\end{aligned}$$

Therefore

$$\tfrac{1}{2}\varepsilon|A| - (m-1) \leq \frac{8}{\varepsilon}\left(1 - \frac{\varepsilon}{4}\right)\left(1 + \frac{3}{\varepsilon|A| - 3}\right) + \frac{3}{2} < \frac{12}{\varepsilon}$$

as $\varepsilon|A| - 3 \geq \frac{48}{\varepsilon} - 3 > \frac{12}{\varepsilon}$. In particular, $m - 1 > \frac{1}{2}\varepsilon|A| - \frac{12}{\varepsilon}$.

Next let l be the line determined by a side of $[A]$ containing $m = m_A$ point of A, and let s be the number of lines parallel to l intersecting A. According to (29),

$$(4-\varepsilon)|A| \geq 2|A| + (s-1)(m-1) - 1 > 2|A| + (s-1)(\tfrac{1}{2}\varepsilon|A| - \tfrac{12}{\varepsilon}) - 1,$$

thus first rearranging, and then applying $\varepsilon^2|A| \geq 48$ yield

$$2|A| > s \cdot (\tfrac{1}{2}\varepsilon|A| - \tfrac{12}{\varepsilon}) \geq s \cdot \tfrac{1}{4}\varepsilon|A|.$$

Therefore $s < \frac{8}{\varepsilon}$.

We deduce from (30) and $s < \frac{8}{\varepsilon}$ that

$$(4-\varepsilon)|A| > (4 - \tfrac{2}{s})|A| - 2s > (4 - \tfrac{2}{s})|A| - \tfrac{16}{\varepsilon}.$$

Rearranging, and then applying $\varepsilon^2|A| \geq 48$ imply

$$s < \frac{2}{\varepsilon}\left(1 - \frac{16}{\varepsilon^2|A|}\right)^{-1} < \frac{2}{\varepsilon}\left(1 + \frac{32}{\varepsilon^2|A|}\right). \;\square$$

8 Proof of Proposition 6

We call the points of A,

$$a_0 = (0,0), \quad a_1 = (-1,-2), \quad a_2 = (2,1).$$

If $k \geq 2$, then we show that every mixed subdivision M corresponding to T_A and T_B satisfies

$$|M_{11}| \leq 24. \tag{31}$$

We prove (31) in several steps. First we verify

$$[a_1, a_2] + l_i \quad \text{is not an edge of } M \quad \text{for } i = 0, \ldots, k \tag{32}$$
$$[a_1, a_2] + r_i \quad \text{is not an edge of } M \quad \text{for } i = 0, \ldots, k-1. \tag{33}$$

For (32), we observe that $a_1 + l_{i+1}$ if $i \leq k-1$ or $a_1 + l_{i-1}$ if $i \geq 1$ is a point of $A + B$ in $[a_1, a_2] + l_i$ different from the endpoints. Similarly, for (33), we observe that $a_1 + r_{i+1}$ if $i \leq k-2$ or $a_1 + r_{i-1}$ if $i \geq 1$ is a point of $A + B$ in $[a_1, a_2] + r_i$ different from the endpoints.

Next, we have

$$[a_0, a_2] + [l_i, r_i] \quad \text{is not a parallelogram of } M \quad \text{for } i = 0, \ldots, k-1 \tag{34}$$
$$[a_0, a_1] + [r_i, l_{i+1}] \quad \text{is not a parallelogram of } M \quad \text{for } i = 0, \ldots, k-1 \tag{35}$$

as $l_{i+1} \in \text{int}\,[a_0, a_2] + [l_i, r_i]$ and $l_i \in \text{int}\,[a_0, a_1] + [r_i, l_{i+1}]$.

Let us call the edges of T_B of the form either $[l_i, r_i]$ or $[r_i, l_{i+1}]$ for $i = 0, \ldots, k-1$ *small edges*, and the edges of T_B of the form either $[p, l_i]$, $[q, l_i]$ for $i = 0, \ldots, k$, or $[p, r_i]$, $[q, r_i]$ for $i = 0, \ldots, k-1$ *long edges*. In other words, long edges of T_B contain either p or q, while small edges of T_B contain neither.

Concerning long edges, we prove that that the number of parallelograms of M of the form

$$e_A + e_B \text{ for an edge } e_A \text{ of } T_A \text{ and a long edge } e_B \text{ of } T_B \text{ is at most } 12. \quad (36)$$

If e_A is an edge of T_A, then there exist at most two cells of M whose side are $p + e_A$. Since T_A has three edges, there are at most six of parallelograms of M of the form $e_A + e_B$ where e_A is an edge of T_A and e_B is an edge of T_B with $p \in e_B$. Since the same estimate holds if $q \in e_B$, we conclude (36).

Finally, we prove that that the number of parallelograms of M of the form

$$e_A + e_B \text{ for an edge } e_A \text{ of } T_A \text{ and a small edge } e_B \text{ of } T_B \text{ is at most } 12. \quad (37)$$

The argument for (37) is based on the claim that if $e_A + e_B$ is a parallelogram of M for an edge e_A of T_A and a small edge e_B of T_B, then there is a long edge e'_B of T_B such that

$$e_A + e'_B \text{ is a neighboring parallelogram of } M. \quad (38)$$

We have $e_A \neq [a_1, a_2]$ according to (32) and (33). If $e_A = [a_0, a_1]$, then $e_B = [l_i, r_i]$ for some $i \in \{1, \ldots, k-1\}$ according to (35). Now $r_i + e_A$ intersects the interior of $[A + B]$ as $r_i \in \text{int}\,[A]$, thus it is the edge of another cell of M, as well. This other cell is either a translate of $[A]$, which is impossible by (32), (33), and as $r_i \notin p + [A], q + [A]$, or of the form $e_A + e'_B$ for an edge $e'_B \neq e_B$ of T_B containing r_i. However, $e'_B \neq [r_i, l_{i+1}]$ by (35), therefore e'_B is a long edge.

On the other hand, if $e_A = [a_0, a_2]$, then $e_B = [r_i, l_{i+1}]$ for some $i \in \{1, \ldots, k-1\}$ according to (34), and (38) follows as above.

Now if $e_A + e'_B$ is a parallelogram of M for an edge e_A of T_A and a long edge e'_B of T_B, then there is at most one neighboring paralellogram of the form $e_A + e_B$ for a small edge e_B of T_B because $e_A + e_B$ does not intersect $e_A + p$ and $e_A + q$. In turn, (37) follows from (36) and (38). Moreover, we conclude (31) from (36) and (37).

Finally, it follows from (31) that if $k \geq 145$, then

$$|M_{11}| \leq 24 < \sqrt{4k} = \sqrt{|T_A| \cdot |T_B|}. \ \Box$$

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