Distribution of colors in Gallai colorings

András Gyárfás[a], Dömötör Pálvölgyi[b], Balázs Patkós[a], Matthew Wales[c]

[a] Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences
P.O.B. 127, Budapest H-1364, Hungary.
{gyarfas,patkos}@renyi.hu

[b] MTA-ELTE Lendület Combinatorial Geometry Research Group
Institute of Mathematics, Eötvös Loránd University (ELTE), Budapest, Hungary
dom@cs.elte.hu

[c]Department of Pure Mathematics and Mathematical Statistics, University of Cambridge
Wilberforce Road, Cambridge CB3 0WB
mw637@cam.ac.uk

July 29, 2019

Abstract

A Gallai coloring is an edge coloring that avoids triangles colored with three different colors. Given integers $e_1 \geq e_2 \geq \ldots \geq e_k$ with $\sum_{i=1}^k e_i = \binom{n}{2}$ for some n, does there exist a Gallai k-coloring of K_n with e_i edges in color i? In this paper, we give several sufficient conditions and one necessary condition to guarantee a positive answer to the above question. In particular, we prove the existence of a Gallai-coloring if $e_1 - e_k \leq 1$ and $k \leq \lfloor n/2 \rfloor$. We prove that for any integer $k \geq 3$ there is a (unique) integer $g(k)$ with the following property: there exists a Gallai k-coloring of K_n with e_i edges in color i for every $e_1 \leq \ldots \leq e_k$ satisfying $\sum_{i=1}^k e_i = \binom{n}{2}$, if and only if $n \geq g(k)$. We show that $g(3) = 5$, $g(4) = 8$, and $2k - 2 \leq g(k) \leq 8k^2 + 1$ for every $k \geq 3$.

keywords: Gallai colorings, color class sizes, balanced colorings

1 Introduction

Gallai colorings (a term introduced in [4]) of complete graphs are edge colorings that do not contain triangles colored with three different colors. For general properties of Gallai colorings see [1, 2, 3, 4]. Ramsey type properties in Gallai colorings have been studied for example in [5, 6].

Here we are interested in the possible distribution of the number of edges in the color classes of Gallai colorings. For Gallai 2-colorings there is no restriction since any 2-coloring of K_n is a Gallai coloring. In fact, a recent result of Balogh and Li [3] shows that almost all Gallai colorings of K_n are 2-colorings. For more than two colors there are restrictions and it seems not easy to find a good characterization of the sequences that are realizable as color distributions. We give a necessary and several sufficient conditions.

It is well-known that in a Gallai coloring of K_n there exists a monochromatic spanning tree, consequently a color with at least $n-1$ edges. This can be generalized as follows.

Lemma 1. *For positive integers n, ℓ the following is true. In every Gallai coloring of K_n there exists at most ℓ colors such that at least $n-1+n-2+\ldots+n-\ell$ edges are colored with these colors.*

The lower bound of Lemma 1 is tight, we will define the construction showing its tightness in the next subsection. Lemma 1 gives a necessary condition for the distribution of colors in a Gallai coloring of K_n.

Corollary 2. *Assume that $e_1 \geq e_2 \geq \ldots \geq e_k$ and $\sum_{i=1}^{k} e_i = \binom{n}{2}$. If K_n has a Gallai k-coloring with e_i edges of color i, then*

$$\sum_{i=1}^{\ell} e_i \geq n-1+n-2+\ldots+n-\ell$$

for every $1 \leq \ell \leq k$.

The condition of Corollary 2 is not sufficient. For example, for $n=6$ the sequence $e_1 = 7, e_2 = 3, e_3 = e_4 = 2, e_5 = 1$ satisfies the condition, but there is no Gallai coloring of K_6 with the given color distribution. A coloring of K_n is called *balanced* if $e_1 - e_k \leq 1$. Lemma 1 (with $\ell = 1$) implies that for a balanced Gallai k-coloring we have $k \leq \lceil n/2 \rceil$. We prove that this condition is also sufficient.

Theorem 3. *Assume that $k \leq \lceil n/2 \rceil$. Then K_n has a balanced Gallai k-coloring.*

We shall derive Theorem 3 from the following related result.

Theorem 4. *Assume that $p \geq n-1$ and $\binom{n}{2} = k \times p + q$ with $k, q \geq 0$. Then K_n has a Gallai $(k+1)$-coloring with p edges in k color classes and q edges in one color class.*

Note that these theorems are equivalent if $\binom{n}{2} = kp$.

We prove that for any fixed k and large enough n *any distribution* is possible for a Gallai k-coloring. Formally, there exists the function $g(k)$, the smallest integer m such that for any $n \geq m$ and for any k-partition $e_1, \ldots, e_k$ of $\binom{n}{2}$ there is a Gallai k-coloring of K_n with e_i edges in color i. It is not quite obvious that $g(k)$ is well defined, this follows from the monotone property of Gallai colorings, stated as Lemma 7 in Section 3. We have the following result on $g(k)$.

Theorem 5. *For any $k \geq 3$ we have $2k-2 \leq g(k) \leq 8k^2+1$. Moreover, $g(3) = 5$ and $g(4) = 8$ hold.*

We pose as an open problem to determine the exact order of magnitude of $g(k)$, which we expect to be just slightly superlinear.

1.1 General and special Gallai colorings

A fundamental property of Gallai colorings is the following.

Theorem 6. *[Theorem A, [4]] For any Gallai coloring of K_n there exist at most two colors, say $1, 2$, and a decomposition of K_n into $m \geq 2$ vertex disjoint complete graphs K_{n_i} ($1 \leq i \leq m$) so that all edges between $V(K_{n_i})$ and $V(K_{n_j})$ are colored with the same color and that color is either 1 or 2.*

Briefly stated, every Gallai coloring can be obtained by a sequence of substitutions (of Gallai colored complete graphs) into a (non-trivial) 2-colored complete graph. In our proofs we use a subfamily of Gallai colorings that we call *special Gallai colorings* defined as follows.

Let the vertex set of K_n be $V = [0, n-1]$ and let $S(i)$ be the star with center vertex $i \in [1, n-1]$ and with edge set $\{(i,j) : i > j \geq 0\}$ (note that $S(i)$ has i edges). Let $T_1, \ldots, T_k$ be a partition of $V \setminus \{0\}$ into nonempty parts. Then, for $j = 1, 2, \ldots, k$, color class C_j is defined as $\cup_{i \in T_j} S(i)$. Observe that this coloring is a Gallai coloring, because for every triangle abc with $n-1 \geq a > b > c \geq 0$, the edges ab, ac have the same color. Notice that the special coloring defined by the partition $\{n-1\}, \{n-2\}, \ldots, \{1\}$ shows that the lower bound of Lemma 1 is sharp.

As another example, the sequence $8, 3, 3, 1$ can be realized as a color distribution of a Gallai coloring on K_6 but not as a special Gallai coloring. What we are going to prove is that the Gallai colorings claimed in Theorems 3 and 4 can be special Gallai colorings.

2 Proof of Theorems 3, 4

Proof of Lemma 1. We proceed by induction on n. For any n and $\ell = 1$ the lemma follows from the result cited above: Gallai colorings of K_n contain a monochromatic spanning tree. The case $n = 1$ is trivial (for any ℓ). Let G be a Gallai colored K_n and $\ell \geq 2$. Let m and $e_1, e_2, \dots, e_m$ be the numbers obtained from the partition ensured by Theorem 6. Moreover define α as one or two depending on the number of colors used between the $V(K_{n_i})$-s in Theorem 6.

Let S be the decreasing sequence of positive integers obtained by concatenating the sequences $S_i = e_i - 1, e_i - 2, \dots, 1$ for $i = 1, \dots, m$. For example for

$$n = 14, m = 4, e_1 = 5, e_2 = 4, e_3 = 4, e_4 = 1$$

we get

$$S_1 = 4, 3, 2, 1; S_2 = 3, 2, 1; S_3 = 3, 2, 1; S_4 = \emptyset$$

and

$$S = 4, 3, 3, 3, 2, 2, 2, 1, 1, 1.$$

Let S^* be the subsequence of S defined by the first $\ell - \alpha$ elements of S (if $\ell - \alpha > |S|$, then $S^* = S$). We can partition the elements of S^* into sequences S_i^* of length ℓ_i, $i = 1, \dots, m$ so that $\sum_{i=1}^m \ell_i = \ell - \alpha$ and the elements of S_i^* form an initial segment in S_i for all i. This is not unique, with $\alpha = 2, \ell = 7$ in the example above we have

$$S^* = 4, 3, 3, 3, 2$$

and we can have

$$S_1^* = 4, 3; S_2^* = 3; S_3^* = 3, 2; S_4^* = \emptyset \text{ or } S_1^* = 4, 3, 2; S_2^* = 3; S_3^* = 3; S_4^* = \emptyset$$

.

Now we can apply induction for each Gallai colored K_{e_i} with ℓ_i and together with the α colors from the color set $\{1, 2\}$ we get at most ℓ colors so that the number of edges in these colors is at least

$$t = \sum_{1 \leq i < j \leq m} e_i e_j + \sum_{i=1}^{m} ||S_i^*||$$

where $||S_i^*||$ denotes the sum of elements of the sequence S_i^*. Since the elements of S_i^* form an initial segment in S_i for all i, the number of edges of G not among the t selected edges is at most the sum s of the $(n - m) - (\ell - \alpha)$ elements of S^* that are not in the S_i^*-s. But

$$(n - m) - (\ell - \alpha) \leq n - \ell - 1$$

because if $m = 2$, then $\alpha = 1$ and if $m \geq 3$, then $\alpha \leq 2$. Therefore s is at most the sum of the first $n - \ell - 1$ numbers from the set $\{1, 2, \ldots, n-1\}$ and

$$t \geq \binom{n}{2} - (1 + 2 + \ldots + n - \ell - 1) = n - 1 + n - 2 + \ldots + n - \ell$$

proving the lemma. $\square$

Proof of Theorem 4. We prove by induction on n that the required Gallai coloring can be chosen to be a special Gallai coloring. The proof is built around three primary 'moves' we can make to reduce to a smaller case. When $p \leq 2n - 3$ we form some classes out of a pair of stars $S(i)$ and $S(j), i \neq j$. If $p \geq 2n - 2$, we can't form classes from a pair of stars. We will still place pairs of stars in each class of size p, but then apply induction with the same value of k, and smaller values of p and n. To resolve a final case, we will also remove a star from the class of size q.

We proceed by strong induction on n. The base case $n = 1$ is trival. We can assume $q \leq p - 1$ by finding a partition into one part of size $q \pmod p$, and $k + \lfloor \frac{q}{p} \rfloor$ parts of size p. Observe the hypotheses are still satisfied, and at the end we can merge these newly created parts to form a single size q part. If $q \geq n - 1$, we can find a Gallai coloring of K_{n-1} with one part of size $q - (n-1)$ and k classes of size p. Adding a star on a new vertex in the final color class gives the desired Gallai coloring. We break into 2 primary cases, depending on whether removing a pair of stars can reduce the number of classes. Thus we can assume

$$q \leq n - 2, q \leq p - 1 \tag{1}$$

Case A: $p \leq 2n - 3$

Note that $0 \leq (p - n + 1) < (n - 1)$. We will form some classes out of pairs of stars. If p is odd, we form classes of size p from each of the star pairs $(S(n-1), S(p-n+1)), ..., (S(\frac{p+1}{2}), S(\frac{p-1}{2}))$. What remains is a complete graph on $p - n + 1$ vertices. By induction, since p has not changed (and n decreases), we can construct a Gallai coloring of this graph into some classes of size p, and one class of size q. Adding the stars sequentially gives the desired Gallai coloring.

As such, we may assume p is even. We form size p classes from each of the star pairs $(S(n-1), S(p-n+1)), ..., (S(\frac{p}{2}+1), S(\frac{p}{2}-1))$. What remains is a complete graph on $p - n + 1$ vertices, and a star at $\frac{p}{2}$, consisting of $\frac{p}{2}$ edges.

We construct a Gallai Coloring of the K_{p-n+1} into some parts of size $\frac{p}{2}$, and one of size q by induction - recall $\binom{n}{2} = kp + q$, and we only removed classes of size p and a star of size $\frac{p}{2}$. Since $p \leq 2n$ (which implies $\frac{p}{2} \leq p - n$) the induction hypotheses hold. We form a final size $\frac{p}{2}$ class from the leftover star, and pairing off the size $\frac{p}{2}$ classes arbitrarily gives the desired Gallai coloring of K_n.

This resolves Case A.

Case B: $p \geq 2n-2$

In this situation, we try and use the second move to reduce to a smaller case. Observe that $k \leq \frac{n}{4}$ because $\binom{n}{2} \geq kp$. We can therefore form pairs of stars $S(n-i), S(n-2k-1+i)$, with a combined $2n-2k-1$ edges. These are all distinct stars for $1 \leq i \leq k$. With this in mind, let $p' = p-2n+2k+1$, and $n' = n-2k$. We use these for the induction.

If $p' \geq n'-1$, then by induction we can find a Gallai coloring of $K_{n'}$ with k classes of size p' and one class of size q. We then add the star pairs to the size p' classes to obtain the desired Gallai coloring. So, assume this is not the case, and therefore $p \leq 3n-4k-3$. Since also $p \geq 2n-2$, we deduce $4k \leq n-1$. Let $4k = n-\delta$, with $\delta \in \mathbb{Z}^{\geq 1}$. Since by (1) $q \leq n-2$,

$$\binom{n}{2} \leq k(3n-4k-3) + (n-2).$$

Substituting $4k = n-\delta$, we obtain

$$n\delta + \delta(\delta-3) \leq 3n-4.$$

This implies that $\delta < 3$, and so either $\delta = 1$ or $\delta = 2$. Since $2n-2 \leq p \leq 2n+\delta-3$, there are very few remaining possibilities. We check these individually.

If $\delta = 1$, $n = 4k+1$ and $p = 2n-2 = 8k$. Since $\binom{n}{2} = kp+q$, we need $q = 2k$. Remove the k pairs of stars as earlier from the size p classes. It remains to find a Gallai coloring of $K_{n'}(n' = 2k+1)$ with k classes of size $p' = 2k-1$ and one class of size $2k$. We remove a star in the size q class, and now need to Gallai color K_{2k} with k classes of size $2k-1$. But we can do this by induction, and adding back stars gives the desired result. In fact, this case can be settled directly too, by taking one triple of stars $S(4k), S(4k-1), S(1)$ and $k-1$ quadruples of stars $S(4k-2i), S(4k-2i-1), S(2i), S(2i+1)$ (for $i = 1, \ldots, k-1$). The only unused star $S(2k)$ gives the part with q edges.

Thus $\delta = 2$, and so $n = 4k+2$ and either $p = 8k+2$ or $p = 8k+3$. If $p = 8k+2$, we find $q = 4k+1 = n-1$. But this contradicts that $q \leq n-2$. Therefore, $p = 8k+3$, and hence $q = 3k+1$. But in this case, we remove a pair of stars, as at the start of Case B, from each size p class. It'ls now sufficient to find a coloring of $K_{n'}, n' = n-2k = 2k+2$, with k classes of size $p' = 2k$ and 1 class of size $q = 3k+1$. We remove a star of size $n'-1$ from the size q class, and we then need a Gallai coloring of K_{2k+1} with k classes of size $2k$ and 1 class of size k. But this exists by induction. Adding the stars gives the desired coloring, and completes Case B. $\square$

Proof of Theorem 3. Observe that it is enough to prove Theorem 3 for the case $n/4 \leq k$. Indeed, otherwise with a suitable positive integer r we have $n/4 \leq rk \leq \lceil n/2 \rceil$ and Theorem 3 provides special balanced Gallai coloring with rk colors. Then, grouping the colors into k parts so that every part consists of r old color classes, we get the required solution. (Note that a special Gallai coloring remains special after merging some color classes.) Therefore we can assume $n = 2k+i$ with $0 \leq i \leq 2k$ and write $\binom{i}{2} = \ell k + m$ where $0 \leq m \leq k-1$. In a balanced distribution, there are $k-m$ color classes with $Z = 2k+2i+\ell-1$ edges and m color classes with $Z' = 2k+2i+\ell$ edges. Assume that ℓ is even. We can create $k - \frac{\ell}{2}$ pairs of stars as follows:

$$(S(2k+i-1), S(i+\ell)), \ldots (S(k+i+\frac{\ell}{2}), S(k+i+\frac{\ell}{2}-1)). \tag{2}$$

If $k - m \geq k - \frac{\ell}{2}$, then the $k - \frac{\ell}{2}$ pairs of centers in (2) cover all numbers in $[i+\ell, n-1]$ exactly once and each pair of stars has total size Z. This implies that $\binom{i+\ell}{2} = (\frac{\ell}{2} - m)Z + mZ'$. Thus, by induction, we can find a balanced special Gallai coloring into $\frac{\ell}{2}$ parts, $\frac{\ell}{2} - m$ color classes of size Z and m color classes with size Z'. Thus together with the pairs in (2) we have the required coloring on K_n. Since $\frac{\ell}{2} \leq \frac{i+\ell}{2}$ and $i + \ell < n = 2k + i$, the induction is justified.

If $k - m < k - \frac{\ell}{2}$, then we have too many pairs of stars with total size Z, thus we need to stop earlier in the pairings at (2). We make the following modification in the pairing.

$$(S(2k+i-1), S(i+\ell)), \ldots, (S(k+i+m), S(k+i-m+\ell-1)) \tag{3}$$

defining $k-m$ pairs with sum Z and continue with pairings with sum Z' as follows.

$$(S(k+i+m-1), S(k+i-m+\ell+1)), \ldots, (S(k+i+\frac{\ell}{2}+1), S(k+i+\frac{\ell}{2}-1)) \tag{4}$$

defining $m - \frac{\ell}{2} - 1$ pairs with sum Z'. Observe that we did not use two numbers as centers of stars from the interval $[i+\ell, n-1]$, namely we skipped $k+i-m+\ell$ and at the end $k+i+\frac{\ell}{2}$ was not used in the pairings. The sum of these is $Z' - (m - \frac{\ell}{2})$. This implies that $\binom{i+\ell}{2} = \frac{\ell}{2}Z' + m - \frac{\ell}{2}$. Since $Z' = 2k+2i+\ell \geq i+\ell-1$, Theorem 4 can be applied with $i+\ell$ in the role of n, $\frac{\ell}{2}$ in the role of k, Z' in the role of p, $m - \frac{\ell}{2}$ in the role of q to get a special Gallai coloring with $\frac{\ell}{2}$ parts of size Z' and one part of size $m - \frac{\ell}{2}$. Adding the star-pairs from (3), (4) together with the two stars unused at (4) we have the required coloring.

Suppose now ℓ is odd. If $k - \frac{\ell+1}{2} \leq m$, then we start with $k - \frac{\ell+1}{2}$ pairs of stars with sum Z' with centers in $[i+l+1, n-1]$.

$$(S(2k+i-1), S(i+\ell+1)), \ldots, (S(k+i+\frac{\ell+1}{2}), S(k+i+\frac{\ell-1}{2})). \tag{5}$$

We need $m - k + \frac{\ell+1}{2}$ further parts of size Z' and $k-m$ parts of size Z with center in $[1, i+\ell]$. This can be done by induction.

Otherwise, when $k-\frac{\ell+1}{2}>m$, we can define m pairs of sum Z' and $k-m-\frac{\ell+1}{2}-1$ pairs of sum Z as follows:

$$(S(2k+i-1),S(i+\ell+1)),\ldots,(S(2k+i-m),S(i+\ell+m)), \tag{6}$$

$$(S(2k+i+m-2),S(i+\ell+m+1)),\ldots,(S(k+i+\frac{\ell+1}{2}),S(k+i+\frac{\ell-1}{2}-1)). \tag{7}$$

Note that in (6), (7) we get stars with all centers from $[i+\ell+1,n-1]$ except the ones with centers $2k+i-m-1$ and $k+i+\frac{\ell-1}{2}$. The size of these together is $Z+k-m-1-\frac{\ell+1}{2}$ therefore we have $\binom{i+\ell+1}{2}=(\frac{\ell+1}{2}-1)Z+m-k+1+\frac{\ell+1}{2}$. Since $Z=2k+2i+2\ell-1\geq i+\ell-1$, Theorem 4 can be applied to get a special Gallai coloring on $K_{i+\ell}$ with $\frac{\ell+1}{2}-1$ parts of size Z and one part of size $m-k+1+\frac{\ell+1}{2}$. Adding the pairs at (6), (7) plus the two exceptional stars, we get the desired balanced Gallai k-coloring. □

3 Bounds on $g(k)$

A simple general lower bound is $g(k)\geq 2k-2$, shown by the color distribution $\binom{2k-3}{2}-k+1,1,\ldots,1$ for K_{2k-3}. Indeed, this distribution is impossible for a Gallai k-coloring, since among the edges of color $2,\ldots,k$ there must be two intersecting ones defining a multicolored triangle. This enables us to show that Gallai-colorability is a monotone property in the following sense, implying that $g(k)$ is well defined.

Lemma 7. *Assume that for some n, K_n has a Gallai k-coloring for every distribution $e_1\geq\ldots\geq e_k$ satisfying $\sum_{i=1}^k e_i=\binom{n}{2}$. Then this statement remains true for $n+1$ as well.*

Proof. Assume that K_n has a Gallai k-coloring for every distribution with $\sum_{i=1}^k e_i=\binom{n}{2}$. As shown by the distribution $\binom{n}{2}-k+1,1,\ldots,1$, we must have $n\geq 2k-2$, i.e. $\frac{n+2}{2}\geq k$. Suppose that there is some distribution S, $e_1\geq\ldots\geq e_k$ with $\sum_{i=1}^k e_i=\binom{n+1}{2}$ for which there is no Gallai k-coloring of K_{n+1}. We have $e_1\geq n$ otherwise

$$\sum_{i=1}^k e_i\leq k(n-1)\leq\frac{n+2}{2}(n-1)<\binom{n+1}{2},$$

a contradiction. Replacing e_1 with $e=e_1-n$ and keeping the other e_i-s, we have a distribution realizable as a Gallai coloring on K_n. Adding a star to K_n with n

edges in the color of e, we get a Gallai k-coloring on K_{n+1} with distribution S, a contradiction. $\square$

Proof of Theorem 5. First we prove the general upper bound $g(k) \leq 8k^2 + 1$. We proceed by induction on k with the base case given by the $k = 3$ part of the theorem.

Let $n = 8k^2 + 1$. Because of Lemma 7 it is enough to prove that for any $\sum_{i=1}^{k} e_i = \binom{n}{2}$ there is a Gallai k-coloring of K_n. We will use $e_1 \geq \binom{n}{2}/k \geq 32k^3$ to eliminate color k and reduce the problem to $k-1$ colors.

We give a procedure to color the edges. Denote the current number of vertices by n' (initially $n' = n$) and the current number of required edges that need color i by e'_i, so that we will always have $\sum_{i=1}^{k} e'_i = \binom{n'}{2}$. If $e'_k \geq n' - 1$, put the star $S(n')$ in color k into the graph, reduce e'_k by n' and reduce n' by one. We repeat this until $e'_k < n' - 1$. As $e_k \leq \binom{n}{2}/k \leq 32k^3 + 4k \leq 5k(8k^2 - 5k + 1) = 5k(n - 5k)$ holds if $k \geq 5$ (in fact if $k \geq 3$), we placed at most $5k$ stars of color k in this first phase.

Since at this point $e'_k < n$, all the further needed edges of color k can be placed within the next set A of $4k+1$ vertices of $K_{n'}$, as $n = 8k^2 + 1 < \binom{4k+1}{2}$. We color all other edges adjacent to A with color 1. There are at most $(4k+1) \cdot 8k^2 - \binom{4k+1}{2} \leq 32k^3 \leq e_1$ such edges, thus this second phase is indeed doable.

Finally we need to check if after removing the vertices of the stars and the vertices in A, we are still left with at least $8(k-1)^2 + 1$ vertices. Or equivalently, we removed less than $16k$ vertices. Indeed, we removed at most $5k + 4k + 1 < 16k$ vertices. Because we eliminated all the required e_k edges of color k, by induction we can give a Gallai $(k-1)$-coloring on the remaining complete graph K according to the sequence $e'_1, e_2, \ldots, e_{k-1}$ where e'_1 is the remaining number of color 1 edges to be placed. The obtained k-coloring is a Gallai coloring because the first two phases give a 2-coloring and every vertex is homogeneously connected in these colors to the coloring of K. $\square$

Next we prove $g(3) = 5$. By Lemma 7 it is enough to consider the case $n = 5$. There are eight possible distributions, six of them with a straightforward special Gallai coloring:

$$(7,2,1) : S_4 \cup S_3, S_2, S_1; (6,3,1) : S_4 \cup S_2, S_3, S_1; (5,4,1) : S_3 \cup S_2, S_4, S_1; \tag{8}$$

$$(5,3,2) : S_4 \cup S_1, S_3, S_2; (4,3,3) : S_4, S_3, S_2 \cup S_1; (4,4,2) : S_4, S_3 \cup S_1, S_2 \tag{9}$$

The distribution $(8,1,1)$ is realized by taking two vertex disjoint edges in colors one and two and color all remaining edges by the third color. The distribution $(6,2,2)$ can be realized by taking a $K_{2,3}$ in color one and the other two colors take care of themselves.

We finish by proving $g(4) = 8$. It is left to the reader to check that no Gallai 4-coloring exists on K_7 with color distribution $(9,4,4,4)$, thus $g(4) \geq 8$.

By Lemma 7, we may assume that $n = 8$. Note that induction on k may work at this point: if some $e_i = 7$, then we can delete e_i and we get a sequence of three numbers whose sum is 21 thus by the $k = 3$ part of the theorem, there is a Gallai 3-coloring on K_7 with the given distribution. Then extending this with a star in the fourth color, we get the required coloring on K_8. This idea can be carried further as follows. If there is no 7 in the sequence but there is a 6, then we can replace e_1 by $e_1 - 7$ and delete the e_i with value 6 and get three numbers whose sum is 15 and by the $k = 3$ part of the theorem there is a Gallai 3-coloring on K_6 with the given distribution. Also, if there is no $7, 6$ but there is a 5 in the sequence, then we can reduce the largest or the two largest numbers by $7 + 6 = 13$ to get a sequence with sum 15. Deleting the e_i with value 5, we can apply the $k = 3$ part of the theorem with K_5 and applying the corresponding extension.

Excluding the values $7, 6, 5$ from the sequence $e_1 \geq e_2 \geq e_3 \geq e_4$ we have one, two or three elements of the sequence larger than 7.

Case 1. $e_1 \geq 8 > e_2$**.** In this case $3 \leq e_2+e_3+e_4 \leq 12$. We show that the distribution e_2, e_3, e_4 can be realized as a Gallai coloring on the union of vertex disjoint complete graphs. Then the complement of this graph has e_1 edges and form a complete partite graph on eight vertices, providing the required Gallai 4-coloring.

We give the partition according to the sum $S = e_2+e_3+e_4$ and show only the non-trivial part of the partitions (the K_1 parts are omitted). Finding the corresponding Gallai colorings is easy, we leave this to the reader.

- $S = 12$. $(4, 4, 4) \to K_4 \cup K_4$
- $S = 11$. $(4, 4, 3) \to K_5 \cup K_2$
- $S = 10$. $(4, 4, 2), (4, 3, 3) \to K_5$
- $S = 9$. $(4, 4, 1), (4, 3, 2), (3, 3, 3) \to K_4 \cup K_3$
- $S = 8$. $(4, 3, 1), (4, 2, 2), (3, 3, 2) \to K_4 \cup K_2 \cup K_2$
- $S = 7$. $(4, 2, 1), (3, 3, 1), (3, 2, 2) \to K_4 \cup K_2$
- $S = 6$. $(4, 1, 1), (3, 2, 1), (2, 2, 2) \to K_3 \cup K_3$
- $S = 5$. $(3, 1, 1), (2, 2, 1) \to K_3 \cup K_2 \cup K_2$
- $S = 4$. $(2, 1, 1) \to K_3 \cup K_2$
- $S = 3$. $(1, 1, 1) \to K_2 \cup K_2 \cup K_2$

Case 2. $e_1, e_2 \geq 8 > e_3$**.** Replacing e_2 by $e_2 - 7$ and reordering, we have a new sequence with only $e_1 \geq 8$. Now the method of Case 1 can be used with $n = 7$. From easy inspection of e_2 and $e_3 + e_4 \leq 8$ (before the reduction), we get $S = e_2 + e_3 + e_4 \leq 11$. Since only the case $S = 12$ used eight vertices we finish as in Case 1.
Case 3. $e_1 \geq e_2 \geq e_3 \geq 8 > e_4$**.** Replacing e_2, e_3 by $e_2 - 7, e_3 - 6$ and reordering, we have a new sequence with only one element larger than 7. Now $n = 6$ and easy inspection shows that $S = e_2 + e_3 + e_4 \leq 7$, when case 1 uses more than six vertices only for $S = 5$. Thus the only problem is when we end up with a sequence with $e_2 + e_3 + e_4 = 5$. However, this is impossible because then $e_1 = 10$ and from the assumption of the subcase $e_2 \geq e_3 \geq 8$ was true before the reduction, thus $e_4 \leq 0$, contradiction.

Funding: Research supported by the National Research, Development and Innovation Office - NKFIH under the grants SNN 129364 and K 116769, and by the Lendület program of the Hungarian Academy of Sciences (MTA), under grant number LP2017-19/2017.

References

[1] J. de Oliveira Bastos, F. S. Benevides, G. O. Mota, I. Sau, Counting Gallai 3-colorings of complete graphs, *arXiv:1805.06805*

[2] J. de Oliveira Bastos, F. S. Benevides, J. Han, The number of Gallai k-colorings of complete graphs, *arXiv:1812.10465v2*

[3] J. Balogh, L. Li, The typical structure of Gallai colorings and their extremal graphs, *arXiv:1812.07747*

[4] A. Gyárfás, G. Simonyi, Edge colorings of complete graphs without tricolored triangles, *Journal of Graph Theory* **46** (2004) 211-216.

[5] A. Gyárfás, G. N. Sárközy, A. Sebő, S. Selkow, Ramsey-type results for Gallai colorings, *Journal of Graph Theory* **64** (2010) 233-243

[6] C. Magnant, I. Schiermeyer, Gallai-Ramsey number for K_5, *arXiv: 1901.03622*