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Clique coverings and claw-free graphs

Csilla Bujtás [a,b], Akbar Davoodi [c], Ervin Győri [d,1], Zsolt Tuza [a,b]

[a] Department of Computer Science and Systems Technology, University of Pannonia, Veszprém, Hungary
[b] MTA Rényi Institute, Budapest, Hungary
[c] School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746, Tehran, Iran
[d] MTA Rényi Institute and Central European University, Budapest, Hungary

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ABSTRACT

Let $\mathcal{C}$ be a clique covering for $E(G)$ and let v be a vertex of G. The valency of vertex v (with respect to $\mathcal{C}$), denoted by $val_{\mathcal{C}}(v)$, is the number of cliques in $\mathcal{C}$ containing v. The local clique cover number of G, denoted by $\text{lcc}(G)$, is defined as the smallest integer k, for which there exists a clique covering for $E(G)$ such that $val_{\mathcal{C}}(v)$ is at most k, for every vertex $v \in V(G)$. In this paper, among other results, we prove that if G is a claw-free graph, then $\text{lcc}(G) + \chi(G) \leq n + 1$.

1. Introduction

Throughout the paper, all graphs are simple and undirected. By a *clique* of a graph G, we mean a subset of mutually adjacent vertices of G as well as its corresponding complete subgraph. The *size* of a clique is the number of its vertices. A *clique covering* for $E(G)$ is defined as a family of cliques of G such that every edge of G lies in at least one of the cliques comprising this family.

Let $\mathcal{C}$ be a clique covering for $E(G)$ and let v be a vertex of G. The *valency* of vertex v (with respect to $\mathcal{C}$), denoted by $val_{\mathcal{C}}(v)$, is defined to be the number of cliques in $\mathcal{C}$ containing v. A number of different variants of the clique cover number have been investigated in the literature. The *local clique cover number* of G, denoted by $\text{lcc}(G)$, is defined as the smallest integer k, for which there exists a clique covering for G such that $val_{\mathcal{C}}(v)$ is at most k, for every vertex $v \in V(G)$.

E-mail addresses: bujtas@dcs.uni-pannon.hu (C. Bujtás), davoodi@ipm.ir (A. Davoodi), gyori.ervin@renyi.mta.hu (E. Győri), tuza@dcs.uni-pannon.hu (Z. Tuza).

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This parameter may be interpreted as a variety of different invariants of the graph and the problem relates to some well-known problems such as line graphs of hypergraphs, intersection representation and Kneser representation of graphs. For example, $\mathrm{lcc}(G)$ is the minimum integer k such that G is the line graph of a k-uniform hypergraph. By this interpretation, $\mathrm{lcc}(G) \leq 2$ if and only if G is the line graph of a multigraph.

There is a characterization by a list of seven forbidden induced subgraphs and a polynomial-time algorithm for the recognition that G is the line graph of a multigraph [3,15]. On the other hand, L. Lovász proved in [16] that there is no characterization by a finite list of forbidden induced subgraphs for the graphs which are line graphs of some 3-uniform hypergraphs. Moreover, it was proved that the decision problem whether G is the line graph of a k-uniform hypergraph, for fixed $k \geq 4$, and the problem of recognizing line graphs of 3-uniform hypergraphs without multiple edges are NP-complete [18].

For a vertex $v \in V(G)$, its *(open) neighborhood* $N(v)$ is the set of all neighbors of v in G, while its *closed neighborhood* $N[v]$ is defined as $N[v] := N(v) \cup \{v\}$. Moreover, let $\overline{G}$ stand for the complement of G, and let $\Delta(G)$ and $\delta(G)$ be the maximum degree and the minimum degree of G, respectively. The subgraph induced by a set $Y \subset V(G)$ will be denoted by $G[Y]$. By the notations of $\alpha(G)$, $\omega(G)$, and $\chi(G)$ we mean the independence number, clique number, and chromatic number of G, respectively.

In 1956 E. A. Nordhaus and J. W. Gaddum proved the following theorem for the chromatic number of a graph G and its complement, $\overline{G}$.

Theorem 1 (*[17]*)**.** *Let G be a graph on n vertices. Then $2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1$.*

Later on, similar results for other graph parameters have been found which are known as Nordhaus–Gaddum type theorems. In the literature there are several hundred papers considering inequalities of this type for many other graph invariants. For a survey of Nordhaus–Gaddum type estimates see [1].

In this paper, we consider the following two conjectures on the local clique cover number proposed by R. Javadi in 2012.

Conjecture 2. *For every graph G on n vertices,*

$$\mathrm{lcc}(G) + \mathrm{lcc}(\overline{G}) \leq n. \tag{1}$$

Note that Conjecture 2 is a Nordhaus–Gaddum type inequality concerning the local clique cover number of G. Also, he suggested the following weakening of Conjecture 2.

Conjecture 3. *For every graph G on n vertices,*

$$\mathrm{lcc}(G) + \chi(G) \leq n + 1. \tag{2}$$

Let G_1 and G_2 be graphs with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$. The disjoint union of G_1 and G_2, denoted by $G_1 \dot{\cup} G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$.

Lemma 4. *Let $\mathcal{G}$ be a family of graphs which is closed under the operation of taking disjoint union with an isolated vertex. If Conjecture 2 is true for every $G \in \mathcal{G}$, then Conjecture 3 is also true for every $G \in \mathcal{G}$.*

Proof. Let $G \in \mathcal{G}$ and consider the disjoint union $H = G \dot{\cup} \{v\}$. Observe that $\mathrm{lcc}(G) = \mathrm{lcc}(H)$. Hence, assuming that each member of $\mathcal{G}$ satisfies Conjecture 2, we have $\mathrm{lcc}(G) + \mathrm{lcc}(\overline{H}) \leq |V(H)|$. Now, fix an optimal (with respect to lcc) clique covering $\mathcal{C}$ for $\overline{H}$. Clearly, $\chi(G) \leq val_{\mathcal{C}}(v) \leq \mathrm{lcc}(\overline{H})$. These two inequalities together imply $\mathrm{lcc}(G) = \mathrm{lcc}(H) \leq |V(H)| - \mathrm{lcc}(\overline{H}) \leq |V(G)| + 1 - \chi(G)$. □

2. Proof of some variants of the conjectures

Let k be an integer and let G be a graph such that $k \leq \deg(x) \leq k + 1$, for every vertex $x \in V(G)$. Then $\mathrm{lcc}(G) \leq k+1$ and $\mathrm{lcc}(\overline{G}) \leq n-1-k$. Thus, inequality (1) holds for G. Also, If G is a triangle-free graph, then for a vertex v which has the maximum degree in G, $N(v)$ can be properly colored by one color. Thus, $\chi(G) \leq n + 1 - \Delta(G)$. Since $\mathrm{lcc}(G) = \Delta(G)$, Conjecture 3 is true for triangle-free graphs. In what follows we prove that not only (2) but also (1) holds if $\overline{G}$ is triangle-free.

Theorem 5. *Let G be a graph on n vertices. If $\alpha(G) = 2$, then $\mathrm{lcc}(G) + \mathrm{lcc}(\overline{G}) \leq n$.*

Proof. Clearly, $\mathrm{lcc}(\overline{G}) = \Delta(\overline{G}) = n - 1 - \delta(G)$. It is enough to show that $\mathrm{lcc}(G) \leq \delta(G) + 1$. Let v be a vertex of minimum degree in G, and let $K \subset V(G)$ be the set of vertices which are not adjacent to v. Since $\alpha(G) = 2$, the induced subgraph on K, $G[K]$, is a clique in G. Now, for every vertex $u_i \in N(v)$, let $C_i := (N(u_i) \cap K) \cup \{u_i\}$ and define $C_{\delta(G)+1} := G[K]$. These cliques along with the collection of those edges which are not covered by the cliques $C_1, \ldots, C_{\delta(G)+1}$ comprise a clique covering for G, say $\mathcal{C}$. It can be easily checked that $val_{\mathcal{C}}(v) = \delta(G)$ and $val_{\mathcal{C}}(x) \leq \delta(G) + 1$, for every vertex $x \in V(G) - v$. □

It is well-known that $\frac{n}{\alpha(G)}$ and $\omega(G)$ are lower bounds for $\chi(G)$, the chromatic number of G. We show that, if we replace $\chi(G)$ with any of these two general lower bounds in Conjecture 3, then the inequality holds.

Proposition 6. *Let G be a graph with n vertices. Then $\mathrm{lcc}(G) + \omega(G) \leq n + 1$.*

Proof. Assume that $K \subset V(G)$ is a clique of size ω. For every vertex $v_i \in V(G) - K$, $1 \leq i \leq n - \omega$, define $C_i := (N(v_i) \cap K) \cup \{v_i\}$, and let $C_{n-\omega+1} := G[K]$. Now, let F be the set of all the edges which are not covered by the cliques $C_1, \ldots, C_{n-\omega+1}$. Clearly, the cliques C_i for $1 \leq i \leq n - \omega + 1$ together with F form a clique covering $\mathcal{C}$ for G. If $x \in K$, then $val_{\mathcal{C}}(x) \leq 1 + n - \omega(G)$, and for vertex $v_i \in V(G) - K$, $val_{\mathcal{C}}(v_i) \leq n - \omega(G)$. □

Before proving the other inequality $\mathrm{lcc}(G) + \frac{n}{\alpha(G)} \leq n + 1$, we verify a stronger statement involving local parameters. Let $\alpha_G(v) = \alpha(G[N(v)])$ be the maximum number of independent vertices in the neighborhood of vertex v, and let the *local independence number* of graph G be defined as $\alpha_L(G) = \max_{v \in V(G)} \alpha_G(v)$. Clearly, $\alpha_G(v) \leq \alpha_L(G) \leq \alpha(G)$. Further, $\alpha_G(v) \geq 1$ holds if and only if v has at least one neighbor, while $\alpha_G(v) \leq 1$ is equivalent to that the *closed neighborhood* $N_G[v] = N(v) \cup \{v\}$ induces a clique.

Theorem 7. *For every graph G of order n, there exists a clique covering $\mathcal{C}$ such that for each non-isolated vertex $v \in V(G)$ the inequality $val_{\mathcal{C}}(v) + \frac{n}{\alpha_G(v)} \leq n + 1$ holds.*

Proof. A clique covering will be called *good* if it satisfies the requirement given in the theorem. Since the statement is true for all graphs of order $n \leq 3$, we may proceed by induction on n. Let x and y be two adjacent vertices of G. By the induction hypothesis, there is a good clique covering, $\mathcal{C}'$, for $G' = G - \{x, y\}$. We introduce the notations $N_1 := N(x) - N[y]$, $N_2 := N(y) - N[x]$, and $N_{1,2} := N(x) \cap N(y)$. To obtain a good clique covering $\mathcal{C}$ of G from $\mathcal{C}'$, we perform the following steps.

1. To handle vertices whose neighbors are pairwise adjacent, observe that every vertex u from $N_1 \cup N_2 \cup N_{1,2}$ with $\alpha_G(u) = 1$ and $\deg_{G'}(u) \geq 1$ satisfies $\alpha_{G'}(u) = 1$ and hence it is covered by the clique $N_{G'}[u]$ in the good covering $\mathcal{C}'$. Now, for each such vertex u, $N_{G'}[u]$ is extended by x, by y or by both x and y respectively, if $u \in N_1$, $u \in N_2$ or $u \in N_{1,2}$.
2. If $\alpha_G(x) = 1 < \alpha_G(y)$, take the clique $N_G[x]$; if $\alpha_G(y) = 1 < \alpha_G(x)$, take the clique $N_G[y]$; and if $\alpha_G(x) = \alpha_G(y) = 1$, take the clique $N_G[x] = N_G[y]$ into the covering $\mathcal{C}$ (if they were not included in step (1)).
3. If there still exist some uncovered edges between x and N_1, we consider the set $N_1' = \{v \in N_1 \mid xv \text{ is uncovered}\}$ and partition it into some number of adjacent vertex pairs (inducing independent edges) and at most $\alpha(G(N_1'))$ isolated vertices. Then, we extend each of them with x to a K_3 or K_2, and insert these cliques into the covering $\mathcal{C}$. This way, we get at most $\frac{|N_1'| - \alpha(G(N_1'))}{2} + \alpha(G(N_1'))$ new cliques. Then, we define N_2' and $N_{1,2}'$ analogously, and do the

corresponding partitioning procedure for N_2' and $N_{1,2}'$, extending every part of those partitions with y or with $\{x, y\}$, respectively.

4. If the edge xy remained uncovered, we take it as a clique into the covering $\mathcal{C}$.

It is easy to check that $\mathcal{C}$ is a clique covering in G. We prove that it is good.

First note that after performing Step 1, each vertex $v \in V(G) - \{x, y\}$ has the same valency as in $\mathcal{C}'$. Moreover, if two adjacent vertices, say u and x, have $\alpha_G(u) = \alpha_G(x) = 1$, then $N_G[u] = N_G[x]$ must hold. Hence, if $u \in V(G) - \{x, y\}$ and $\alpha_G(u) = 1$, then u is incident with only one clique from $\mathcal{C}$. Thus, $val_{\mathcal{C}}(u) + \frac{n}{\alpha_G(u)} = 1 + n$. If v is a vertex from $V(G) - \{x, y\}$ and $\alpha_G(v) \geq 2$, then the valency of v might increase in Step 2 or 3, but not in both. Therefore, $val_{\mathcal{C}}(v) \leq val_{\mathcal{C}'}(v) + 1$, and clearly $\alpha_{G'}(v) \leq \alpha_G(v)$. Since $\mathcal{C}'$ is assumed to be good, these facts together imply

$$val_{\mathcal{C}}(v) + \frac{n}{\alpha_G(v)} \leq val_{\mathcal{C}'}(v) + 1 + \frac{n-2}{\alpha_{G'}(v)} + \frac{2}{\alpha_G(v)} \leq n + 1.$$

Now, consider the vertex x. If $\alpha_G(x) = 1$, it is covered by only one clique (induced by its closed neighborhood), which was added to $\mathcal{C}$ in Step 1 or 2. In this case $val_{\mathcal{C}}(x) + \frac{n}{\alpha_G(x)} = n + 1$. Also if $\alpha_G(x) \geq \frac{n}{2}$, the trivial bound $val_{\mathcal{C}}(x) \leq \deg(x) \leq n - 1$ implies the desired inequality. Hence, we may suppose $2 \leq \alpha_G(x) < \frac{n}{2}$.

Let us denote by s the number of cliques covering x which were added to $\mathcal{C}$ in Step 1. Choose one vertex u_i with $\alpha_G(u_i) = 1$ from each of these s cliques. The closed neighborhoods $N[u_i]$ are pairwise different cliques. Thus, if S is the set of all u_i's, then S is independent. By the definitions of N_1' and $N_{1,2}'$, there exist no edges between S and $N_1' \cup N_{1,2}'$. Thus, $\alpha(G(N_1')) \leq \alpha_G(x) - s$ and $\alpha(G(N_{1,2}')) \leq \alpha_G(x) - s$. Also, $|N_1'| + |N_{1,2}'| \leq |N_1| + |N_{1,2}| - s = \deg(x) - 1 - s$ follows.

- If $N_{1,2} \neq \emptyset$ and $\alpha_G(y) > 1$, then

$$\begin{aligned} val_{\mathcal{C}}(x) &\leq \frac{|N_1'| - \alpha(G(N_1'))}{2} + \alpha(G(N_1')) \\ &\qquad + \frac{|N_{1,2}'| - \alpha(G(N_{1,2}'))}{2} + \alpha(G(N_{1,2}')) + s \\ &= \frac{|N_1'| + |N_{1,2}'|}{2} + \frac{\alpha(G(N_1')) + \alpha(G(N_{1,2}'))}{2} + s \\ &\leq \frac{\deg(x) - 1 - s}{2} + \frac{2\alpha_G(x) - 2s}{2} + s \leq \frac{n-2}{2} + \alpha_G(x). \end{aligned}$$

  On the other hand, our assumption $2 \leq \alpha_G(x) < \frac{n}{2}$ implies that $\alpha_G(x) + \frac{n}{\alpha_G(x)} \leq 2 + \frac{n}{2}$. Thus,

$$val_{\mathcal{C}}(x) + \frac{n}{\alpha_G(x)} \leq \frac{n-2}{2} + \alpha_G(x) + \frac{n}{\alpha_G(x)} \leq \frac{n-2}{2} + 2 + \frac{n}{2} = n + 1.$$

- If $N_{1,2} \neq \emptyset$ and $\alpha_G(y) = 1$, all edges between $N_{1,2}$ and x are covered by the clique $N_G[y]$, which was added to $\mathcal{C}$ in Step 2 (or maybe earlier, in Step 1). Hence, $N_{1,2}' = \emptyset$ and we have

$$\begin{aligned} val_{\mathcal{C}}(x) &\leq \frac{|N_1'| - \alpha(G(N_1'))}{2} + \alpha(G(N_1')) + 1 + s \\ &= \frac{|N_1'|}{2} + \frac{\alpha(G(N_1'))}{2} + 1 + s \\ &\leq \frac{\deg(x) - 1 - s}{2} + \frac{\alpha_G(x) - s}{2} + 1 + s \leq \frac{n-2}{2} + \alpha_G(x). \end{aligned}$$

  Again, we may conclude $val_{\mathcal{C}}(x) + \frac{n}{\alpha_G(x)} \leq n + 1$.

- If $N_{1,2} = \emptyset$, the clique xy was added to $\mathcal{C}$ in Step 4, and the same estimation holds as in the previous case.

One can show similarly that $val_{\mathcal{C}}(y) + \frac{n}{\alpha_G(y)} \leq n + 1$. This completes the proof. $\square$

Since for every $v \in V(G)$, $\alpha_G(v) \leq \alpha_L(G) \leq \alpha(G)$, we have the following immediate consequences.

Corollary 8. *Let G be a graph of order n. Then*

(i) $\mathrm{lcc}(G) + \frac{n}{\alpha_L(G)} \le n+1$;
(ii) $\mathrm{lcc}(G) + \frac{n}{\alpha(G)} \le n+1$.

On the other hand, $val_{\mathcal{C}}(v) \ge \alpha_G(v)$, for every arbitrary clique covering $\mathcal{C}$. Hence, $\mathrm{lcc}(G) \ge \alpha_L(G)$. (But $\mathrm{lcc}(G) < \alpha(G)$ may be true.) Also, it is easy to see that $\mathrm{lcc}(G) \ge \frac{\Delta(G)}{\omega-1}$. Next we observe that replacing $\mathrm{lcc}(G)$ with $\alpha(G)$ or $\frac{\Delta(G)}{\omega-1}$ in Conjecture 3, valid inequalities are obtained.

Proposition 9. *If G is a graph on n vertices, then*

1. $\frac{\Delta(G)}{\omega-1} + \chi(G) \le n+1$, *and equality holds if and only if G is the complete graph K_n or the star $K_{1,n-1}$;*
2. $\alpha(G) + \chi(G) \le n+1$, *and equality holds if and only if there exists a vertex $v \in V(G)$ such that $N(v)$ induces a complete graph and $V(G) \setminus N(v)$ is an independent set.*

Proof. To prove (1), first note that it is shown in [10] that there are only two types of graphs G for which $\chi(G) + \chi(\bar{G}) = n+1$,

(a) if $V(G) = K \cup S$ where K is a clique and S is an independent set, sharing a vertex $K \cap S = \{u\}$, or

(b) G is obtained from (a) by substituting C_5, cycle of length 5, into u.

Now, we estimate $\frac{\Delta(G)}{\omega-1} + \chi(G)$ as follows. We write θ for the clique covering number (minimum number of complete subgraphs whose union is the entire vertex set, that is the chromatic number of the complementary graph). Let x be a vertex of degree $\Delta = \Delta(G)$. We have

$$\frac{\Delta}{\omega - 1} \le \theta(G[N(x)]) \le \theta(G) \le n + 1 - \chi(G),$$

where the last inequality is the Nordhaus–Gaddum theorem (Theorem 1). Thus, in order to have $\frac{\Delta}{\omega-1} + \chi = n+1$, it is necessary that G is of type (a) or (b). We shall see that (b) is not good enough, and (a) yields $G = K_n$ or $G = K_{1,n-1}$.

Note that equality does not hold for $G = C_5$ (cycle of length 5), therefore in (b) we have $k = |K - V(C_5)| > 0$. Let $|K-u| = k$ and $|S-u| = s$ in (a). Then after substitution of C_5, we have $n = k+s+5$, $\Delta \le n-1$, $\omega = k+2$ (with $k > 0$), and $\chi = k+3$. Therefore, the most favorable case is $s = 0$, because increasing s by 1 makes $n+1$ increase by 1, while the left-hand side of the inequality increases by at most 1/2. Hence, in the best case we have $n = k+5 \ge 6$, and

$$\frac{\Delta}{\omega - 1} + \chi = \frac{n-1}{n-3} + n - 2 < n+1.$$

Now, we consider case (a). Here, again we have $k > 0$ and $\Delta \le n-1$, moreover now $n = k+s+1$, $\omega = k+1$, and $\chi = k+1$. Thus

$$\frac{\Delta}{\omega - 1} + \chi \le \frac{(k+s)}{k} + k + 1 \le k + s + 2$$

with equality if and only if $s/k = s$, that is $k = 1$ or $s = 0$, where for the case $k = 1$ we also have to ensure $\Delta = s+1$. This completes the proof of (1).

To see (2), consider an independent set A of cardinality $\alpha = \alpha(G)$. A proper $(n-\alpha+1)$-coloring always exists as we can assign color 1 to all vertices from A and the further $n-\alpha$ vertices are assigned with pairwise different colors. Hence, $\chi(G) \le n - \alpha + 1$ holds for every graph. Moreover, if the graph induced by $V(G) \setminus A$ is not complete, we can color it properly by using fewer than $n-\alpha$ colors that yields a proper coloring of G with fewer than $n-\alpha+1$ colors. Therefore, $\chi(G) = n-\alpha+1$ may hold only if $V(G) \setminus A$ induces a complete graph. In this case, G is a split graph. Since split graphs are chordal and chordal graphs are perfect [8], $\omega(G) = \chi(G) = n-\alpha+1$. Consequently, if (2) holds with equality, there exists a vertex $v \in A$ which is adjacent to all vertices from $V(G) \setminus A$. This vertex fulfills our conditions as $N(v)$ is a clique and $V(G) \setminus N(v)$ is an independent set.

On the other hand, if a vertex v' with such a property exists in G, then the graph cannot be colored with fewer than $|N(v')| + 1$ colors. This implies $\chi = n - \alpha + 1$ and completes the proof of the second statement. $\square$

3. Claw-free graphs

Several related problems (say, perfect graph conjecture, to mention just the most famous one) are easier for *claw-free graphs*, i.e. for graphs not containing $K_{1,3}$ as an induced subgraph, other problems (say, complexity of finding chromatic number) are not. (For a survey of results on claw-free graphs see e.g. [9].) Concerning local clique cover number, R. Javadi et al. showed in [12] that if G is a claw-free graph then $\mathrm{lcc}(G) \leq c\frac{\Delta(G)}{\log(\Delta(G))}$, for a constant c. In this section, we are going to prove that Conjecture 3 does hold for claw-free graphs.

To prove the main result of this section, we use the following definition and theorem of Balogh et al. [2].

Definition 10 (*[2]*)**.** A graph G is (s, t)-splittable if $V(G)$ can be partitioned into two sets S and T such that $\chi(G[S]) \geq s$ and $\chi(G[T]) \geq t$. For $2 \leq s \leq \chi(G) - 1$, we say that G is s-splittable if G is $(s, \chi(G) - s + 1)$-splittable.

Theorem 11 (*[2]*)**.** *Let $s \geq 2$ be an integer. Let G be a graph with $\alpha(G) = 2$ and $\chi(G) > \max\{\omega, s\}$. Then G is s-splittable.*

Now we prove:

Theorem 12. *Let G be a claw-free graph with n vertices. Then $\mathrm{lcc}(G) + \chi(G) \leq n + 1$. Moreover, for every $n \geq 4$, there exist several claw-free graphs with n vertices such that equality holds.*

Proof. We prove the theorem by induction on n. For small values of n, it is easy to check that a claw-free graph with n vertices satisfies the inequality. Also, the assertion is obvious for $\alpha(G) = 1$.

Let G be a claw-free graph on n vertices. First, we consider the case where $\alpha(G) \geq 3$. Let T be an independent set of size three. By the induction hypothesis, $G - T$ has a clique covering $\mathcal{C}'$ such that every vertex $x \in V(G - T)$ satisfies

$$val_{\mathcal{C}'}(x) \leq (n - 3) + 1 - \chi(G - T) \leq n - 2 - (\chi(G) - 1) = n - 1 - \chi(G). \tag{3}$$

Now, for every vertex $u \in T$, partition $N(u)$ into the $\chi(\overline{G[N(u)]})$ vertex-disjoint cliques. Then, add vertex u to each clique to cover all the edges incident to u. These cliques along with cliques in an optimum clique covering of $G - T$ form a clique covering, say $\mathcal{C}$, for G. Let $u \in T$ and $x \in G - T$. Then we have

$$val_{\mathcal{C}}(u) = \chi(\overline{G[N(u)]}) \leq \chi(\overline{G}) \leq n + 1 - \chi(G),$$
$$val_{\mathcal{C}}(x) \leq val_{\mathcal{C}'}(x) + |N_G(x) \cap T|.$$

Since G is claw-free, $|N_G(x) \cap T| \leq 2$. Thus, by Inequality (3), $\mathrm{lcc}(G) \leq n + 1 - \chi(G)$.

Consider now the case $\alpha(G) = 2$. By Proposition 6 we may assume that $\chi(G) > \omega(G)$. Moreover, as the statement clearly holds when $\chi(G) \leq 2$, we may also suppose that $\chi(G) \geq 3$. Then Theorem 11 with $s = 2$ implies that $V(G)$ can be partitioned into two parts, say A and B, such that $\chi(G[A]) \geq 2$ and $\chi(G[B]) \geq \chi(G) - 1$. We assume, without loss of generality, that $A = \{u_1, u_2\}$, where the vertices u_1 and u_2 are adjacent. Then $\chi(G - \{u_1, u_2\}) \geq \chi(G) - 1$.

We will use the notation $N_1 := N(u_1) - N[u_2]$, $N_2 := N(u_2) - N[u_1]$, and $N_{1,2} := N(u_1) \cap N(u_2)$. Since G is claw-free, $N_i \cup \{u_i\}$ induces a clique for $i = 1, 2$. Starting with an optimal clique covering $\mathcal{C}''$ for $G - \{u_1, u_2\}$, we will construct a clique covering $\mathcal{C}$ for G such that $val_{\mathcal{C}}(v) \leq n + 1 - \chi(G)$ holds for every vertex v.

If $N_{1,2} = \emptyset$, then $\mathcal{C} := \mathcal{C}'' \cup \{N_1 \cup \{u_1\}, N_2 \cup \{u_2\}, \{u_1, u_2\}\}$ is a clique covering for G. We observe that $val_{\mathcal{C}}(u_i) \le 2$ holds for $i = 1, 2$ and

$$val_{\mathcal{C}}(v) \le val_{\mathcal{C}''}(v) + 1 \le n - 1 - \chi(\overline{G - \{u_1, u_2\}}) + 1 \le n - \chi(\overline{G})$$

for each vertex v from $V(G - \{u_1, u_2\})$. Hence, $\mathrm{lcc}(G) \le n + 1 - \chi(\overline{G})$.

Otherwise, if $N_{1,2} \neq \emptyset$, partition $N_{1,2}$ into at most $\chi(\overline{G - \{u_1, u_2\}})$ cliques and extend each of them with the vertices u_1 and u_2. These cliques together with $N_1 \cup \{u_1\}$, $N_2 \cup \{u_2\}$, and with the cliques in $\mathcal{C}''$ form a clique covering of G. We show that this clique covering $\mathcal{C}$ satisfies $val_{\mathcal{C}}(x) \le n + 1 - \chi(\overline{G})$ for every vertex $x \in V(G)$. Note that $val_{\mathcal{C}}(u_1) \le \chi(\overline{G - \{u_1, u_2\}}) + 1$, thus the Nordhaus–Gaddum inequality for the chromatic number implies

$$val_{\mathcal{C}}(u_1) \le (n - 2) + 1 - \chi(G - \{u_1, u_2\}) + 1 \le n - \chi(G - \{u_1, u_2\}) \le n + 1 - \chi(G).$$

Similarly, we have $val_{\mathcal{C}}(u_2) \le n + 1 - \chi(G)$. For $v \in V(G - \{u_1, u_2\})$,

$$val_{\mathcal{C}}(v) \le val_{\mathcal{C}''}(v) + 1 \le (n - 2) + 1 - \chi(G - \{u_1, u_2\}) + 1 \le n - \chi(G) + 1.$$

Finally, we note that K_n, $K_n - K_2$, and $K_n - K_{1,2}$ are examples of claw-free graphs with n vertices such that $\mathrm{lcc}(G) + \chi(G) = n + 1$. $\square$

4. A Nordhaus-Gaddum type inequality

A *clique partition* of the edges of a graph G is a family of cliques such that every edge of G lies in exactly one member of the family. The *sigma clique partition number* of G, $\mathrm{scp}(G)$, is the smallest integer k for which there exists a clique partition of $E(G)$ where the sum of the sizes of its cliques is at most k.

It was conjectured by G. O. H. Katona and T. Tarján, and proved in the papers [4,11,13], that for every graph G on n vertices, $\mathrm{scp}(G) \le \lfloor n^2/2 \rfloor$ holds, with equality if and only if G is the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

Also, this parameter relates to a number of other well-known problems (see [6]). The second author and R. Javadi proved the following Nordhaus–Gaddum type theorem for scp.

Theorem 13 ([5]). *Let G be a graph with n vertices. Then*

$$\frac{31}{50}n^2 + O(n) \le \max\{\mathrm{scp}(G) + \mathrm{scp}(\overline{G})\} \le \frac{9}{10}n^2 + O(n),$$
$$\frac{12}{125}n^4 + O(n^3) < \max\{\mathrm{scp}(G) \cdot \mathrm{scp}(\overline{G})\} < \frac{81}{400}n^4 + O(n^3).$$

In the following result we improve the upper bounds, from 0.9 to less than 0.77 and from 0.2025 to less than 0.15.

Theorem 14. *For every graph G with n vertices,*

$$\mathrm{scp}(G) + \mathrm{scp}(\overline{G}) \le \frac{1203}{1568}n^2 + o(n^2) < 0.76722\, n^2 + o(n^2)$$

and

$$\mathrm{scp}(G) \cdot \mathrm{scp}(\overline{G}) \le \frac{1447209}{9834496}n^4 + o(n^4) < 0.1471564\, n^4 + o(n^4).$$

Proof. Substantially improving on earlier estimates, P. Keevash and B. Sudakov [14] proved via a computer-aided calculation that every edge 2-coloring of K_n contains at least $cn^2 - o(n^2)$ mutually edge-disjoint monochromatic triangles,[2] where

$$c = \frac{13}{196} + \frac{1}{84} - \frac{1}{1568} = \frac{365}{4704}.$$

[2] In the Abstract of [14] the authors announce the lower bound $n^2/13$, and in their Theorem 1.1 they state $n^2/12.89$ (the rounded form of $\frac{9}{116}n^2$, but actually on p. 212 they prove the even better lower bound displayed above).

In our context this means that we can select approximately cn^2 triangles which together cover $3cn^2$ edges in G and $\overline{G}$ at the cost of $3cn^2$. The remaining edges will be viewed as copies of K_2 in the clique partition to be constructed; they are counted with weight 2 in scp. In this way we obtain

$$\mathrm{scp}(G) + \mathrm{scp}(\overline{G}) \leq (1 - 3c)\, n^2 + o(n^2) = \frac{1203}{1568} n^2 + o(n^2)\,.$$

This also implies the upper bound on $\mathrm{scp}(G) \cdot \mathrm{scp}(\overline{G})$. □

Remark 15. The smallest number of cliques in a clique partition of G is called the *clique partition number* of G. As a Nordhaus–Gaddum type inequality for parameter cp, D. de Caen et al. proved in [7] that

$$\mathrm{cp}(G) + \mathrm{cp}(\overline{G}) \leq \frac{13}{30} n^2 - O(n) \approx 0.43333\, n^2 - O(n)\,,$$
$$\mathrm{cp}(G) \cdot \mathrm{cp}(\overline{G}) \leq \frac{169}{3600} n^4 + O(n^3) \approx 0.0469444\, n^2 + O(n^3)\,.$$

Note that if it is possible to select some k edge-disjoint complete subgraphs in G and $\overline{G}$ which together cover m edges, then $\mathrm{cp}(G)+\mathrm{cp}(\overline{G}) \leq \binom{n}{2}+k-m$. As observed within the proof of Theorem 14, the choices $k = \frac{365}{4704} n^2 - o(n^2)$ and $m = 3k$ are feasible for every G on n vertices, thus

$$\mathrm{cp}(G) + \mathrm{cp}(\overline{G}) \leq \left(\frac{1}{2} - \frac{365}{2352}\right) n^2 + o(n^2) = \frac{811}{2352} n^2 + o(n^2)$$
$$< 0.344813\, n^2 + o(n^2)\,,$$

$$\mathrm{cp}(G) \cdot \mathrm{cp}(\overline{G}) \leq \frac{657721}{22127616} n^4 + o(n^4) < 0.029724\, n^4 + o(n^4)\,.$$

These upper bounds improve the results of [7].

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