Extinction threshold in the spatial stochastic logistic model: Space homogeneous case

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September 19, 2020

Abstract

We consider the extinction regime in the spatial stochastic logistic model in $\mathbb{R}^d$ (a.k.a. Bolker–Pacala–Dieckmann–Law model of spatial populations) using the first-order perturbation beyond the mean-field equation. In space homogeneous case (i.e. when the density is non-spatial and the covariance is translation invariant), we show that the perturbation converges as time tends to infinity; that yields the first-order approximation for the stationary density. Next, we study the critical mortality—the smallest constant death rate which ensures the extinction of the population—as a function of the mean-field scaling parameter $\varepsilon > 0$. We find the leading term of the asymptotic expansion (as $\varepsilon \to 0$) of the critical mortality which is apparently different for the cases $d \geq 3$, $d = 2$, and $d = 1$.

Keywords: extinction threshold, spatial logistic model, mean-field equation, population density, perturbation, correlation function, asymptotic behaviour

2010 Mathematics Subject Classification: 34E10, 92D25, 34D10, 58C15

1 Introduction

The spatial stochastic logistic model was introduced in 1997 by B. Bolker and S. W. Pacala, [4], and it has had a continual interest since then in both population ecology, e.g. [8, 24–27], and (pure) mathematics, e.g. [1–3, 11, 13, 21, 22, 31]. The model describes spatial branching of individuals in a population with a density dependent death rate. We consider it in the following notations.

We fix $m > 0$, the *mortality* constant, and two functions, a_ε^+ and a_ε^-, the *dispersion* and the *competition* kernels, respectively. Here $\varepsilon > 0$ is an artificial scaling parameter:

$$a_\varepsilon^\pm(x) := \varepsilon^d a^\pm(\varepsilon x), \qquad x \in \mathbb{R}^d, \tag{1.1}$$

where $d \geq 1$ and $a^\pm$ are fixed nonnegative integrable functions on $\mathbb{R}^d$, which are assumed to be non-degenerate:

$$\varkappa^\pm := \int_{\mathbb{R}^d} a^\pm(x)\,dx = \int_{\mathbb{R}^d} a_\varepsilon^\pm(x)\,dx > 0, \qquad \varepsilon > 0. \tag{1.2}$$

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Let $\gamma_{t,\varepsilon} \subset \mathbb{R}^d$ denote a discrete random set of positions of individuals at a moment of time $t \geq 0$. The set may be finite or locally finite (the latter means that it has a finite number of points in each compact set from $\mathbb{R}^d$). For an infinitesimally small $\delta > 0$, there happens, with the probability $1 - o(\delta)$, exactly one out of two possible events within the time-interval $[t, t+\delta)$: either the individual placed at an $x \in \gamma_{t,\varepsilon}$ sends an off-spring to an area $\Lambda \subset \mathbb{R}^d$ with the probability

$$\delta \int_\Lambda a_\varepsilon^+(x-y)\,dy + o(\delta);$$

or the individual placed at an $x \in \gamma_{t,\varepsilon}$ dies with the probability

$$\delta\biggl(m + \sum_{y\in\gamma_{t,\varepsilon}\setminus\{x\}} a_\varepsilon^-(x-y)\biggr) + o(\delta).$$

In population ecology, one of the fundamental questions relates to the persistence of populations, or conversely to the possibility of their extinction. The latter can be defined through the equation

$$\lim_{t\to\infty} k_{t,\varepsilon}(x) = 0, \quad x \in \mathbb{R}^d \tag{1.3}$$

(we write henceforth $x \in \mathbb{R}^d$ instead of 'for a.a. $x \in \mathbb{R}^d$'), where $k_{t,\varepsilon}(x) \geq 0$ denotes the local population density given through the equality

$$\mathbb{E}\bigl[|\gamma_{t,\varepsilon} \cap \Lambda|\bigr] = \int_\Lambda k_{t,\varepsilon}(x)\,dx \tag{1.4}$$

which should hold for each compact $\Lambda \subset \mathbb{R}^d$. Henceforth, $\mathbb{E}[\zeta]$ denotes the expected value of a random variable ζ (with respect to the distribution of $\gamma_{t,\varepsilon}$), and $|\eta|$ denotes number of points in a finite subset $\eta \subset \mathbb{R}^d$.

It can be shown, see Section 2 below for details, that, for $\varepsilon \to 0$,

$$k_{t,\varepsilon}(x) = q_t(\varepsilon x) + o(1), \tag{1.5}$$

where $q_t(x) \geq 0$ solves the so-called *mean-field*, or *kinetic*, nonlinear equation, see (2.9) below. Moreover,

$$\lim_{t\to\infty} q_t(\varepsilon x) = 0, \quad x \in \mathbb{R}^d,\ \varepsilon > 0,$$

if and only if $m \geq \varkappa^+$, cf. (1.2). It is natural to expect, however, that (1.3) may take place for smaller value of m, because of the term $o(1)$ in (1.5) which naturally depends on $x \in \mathbb{R}^d$ and $t \geq 0$. To discuss this, one needs the next term of the expansion (1.5), using the approach considered in [6,27,28]. It yields that, for $\varepsilon \to 0$,

$$k_{t,\varepsilon}(x) = q_t(\varepsilon x) + \varepsilon^d p_t(\varepsilon x) + o(\varepsilon^d), \tag{1.6}$$

where $p_t(x)$ can be obtained from a coupled system of linear nonhomogeneous and nonautonomous equations (2.11)–(2.12) (see Section 2 for details).

In the present paper, we consider the space homogeneous regime, when $k_{t,\varepsilon}, q_t, p_t$ do not depend on the space variable. Then both q_t and p_t satisfy ordinary differential equations (3.6), (3.17), respectively, with

$$\lim_{t\to\infty} q_t = \frac{\varkappa^+ - m}{\varkappa^-} =: q^* > 0,$$

for $m < \varkappa^+$. The limit $p^* := \lim_{t\to\infty} p_t$ is found in Theorem 3.2 below. Assuming that that the last term in (1.6) also has a limit as $t \to \infty$ of the same order of ε, we get that the extinction, in the space homogeneous case, takes place iff

$$q^* + \varepsilon^d p^* + o(\varepsilon^d) = 0. \tag{1.7}$$

Note that the conditions we imposed typically lead to $p^* < 0$, that explains why (1.7) should take place for $m < \varkappa^+$. To formalise this, we replace m by $m(\varepsilon) < \varkappa^+$ and reveal the asymptotics of $m(\varepsilon)$ from (1.7). We show that (Theorems 5.1, 6.1, 7.1),

$$q^*(\varepsilon) := \frac{\varkappa^+ - m(\varepsilon)}{\varkappa^-} = \begin{cases} \lambda_3 \varepsilon^d + o(\varepsilon^d), & d \geq 3, \\ \lambda_2 \varepsilon^2 W(\varepsilon^{-2}) + o\big(\varepsilon^2 W(\varepsilon^{-2})\big), & d = 2, \\ \lambda_1 \varepsilon^{\frac{2}{3}} + o\big(\varepsilon^{\frac{2}{3}}\big), & d = 1, \end{cases} \tag{1.8}$$

where $\lambda_3, \lambda_2, \lambda_1$ are explicit positive constants dependent on a^+ and a^-. Here $W(x)$ denotes the Lambert W function that solves $W(x)e^{W(x)} = x$ for $x \geq 0$; using its known asymptotics we also get that, for the case $d = 2$,

$$\frac{\varkappa^+ - m(\varepsilon)}{\varkappa^-} = -2\lambda_2 \varepsilon^2 \log \varepsilon + o(\varepsilon^2 \log \varepsilon).$$

In other words, we show that the mortality needed to ensure that the population (statistically) will extinct as time tends to infinity is less than $\varkappa^+$, namely,

$$m(\varepsilon) = \varkappa^+ - \varkappa^- q^*(\varepsilon),$$

where $q^*(\varepsilon) > 0$ is given by (1.8).

It is worth noting that the orders of the leading terms in the asymptotics (1.8) coincide, for all $d \geq 1$, with the asymptotics of the critical branching parameter for a lattice contact model considered in [5, 9, 10], where ε was the mesh size of the lattice. We expect to discuss a connection between two models as well as to consider the space non-homogeneous case in forthcoming papers.

The paper is organised as follows. In Section 2, we describe further details about the spatial and stochastic logistic model, and discuss how one can derive equations on q_t and p_t. In Section 3, we explain the specific of the space-homogeneous case and prove the existence of the limit $p^* = \lim_{t\to\infty} p_t$. In Section 4, we introduce $m(\varepsilon)$ and discuss the limits of $q^*(\varepsilon)$ and $p^*(\varepsilon)$ depending on the dimension d. Finally, in Sections 5–7 we find the asymptotics (1.8) of $q^*(\varepsilon)$ (and hence of $m_{\mathrm{cr}}(\varepsilon)$) for $d \geq 3$, $d = 2$, and $d = 1$, respectively.

2 Spatial and stochastic logistic model

We consider dynamics of a system consisting of indistinguishable individuals. Each individual is fully characterized by its position $x \in \mathbb{R}^d$, $d \geq 1$. We will always assume that there are not two or more individuals at the same position.

Let $\mathcal{B}_{\mathrm{c}}(\mathbb{R}^d)$ denote the set of all Borel subsets of $\mathbb{R}^d$ with compact closure. We will consider discrete systems only, finite or locally finite. The latter means that, if $\gamma_{t,\varepsilon} = \{x\}$ is a system of individuals at some moment of time $t \geq 0$, then we assume that $|\gamma_{t,\varepsilon} \cap \Lambda| < \infty$ for all $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$. In particular, of course, a finite $\gamma_{t,\varepsilon}$ is also locally finite. We will call such $\gamma_{t,\varepsilon}$ a (finite or locally finite) *configuration*.

The individuals of a configuration are *random*, hence we will speak about random configurations $\gamma_{t,\varepsilon}$ with respect to (w.r.t. henceforth) a probability distribution. Let Γ denote the space of locally finite configurations. We fix the σ-algebra $\mathcal{B}(\Gamma)$ on Γ generated by all mapppings $\Gamma \ni \gamma \mapsto |\gamma \cap \Lambda| \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\lambda \in \mathcal{B}_c(\mathbb{R}^d)$.

The dynamics of configurations in time t is defined through the dynamics of their distributions. Heuristically, the scheme is as follows. We consider, for an $\varepsilon \in (0,1)$, a mapping on measurable functions $F : \Gamma \to \mathbb{R}$ given by

$$(L_\varepsilon F)(\gamma) = \sum_{x\in\gamma}\Bigl(m + \sum_{y\in\gamma\setminus\{x\}} a_\varepsilon^-(x-y)\Bigr)\Bigl(F\bigl(\gamma\setminus\{x\}\bigr) - F\bigl(\gamma\bigr)\Bigr) + \sum_{x\in\gamma}\int_{\mathbb{R}^d} a_\varepsilon^+(x-y)\Bigl(F\bigl(\gamma\cup\{y\}\bigr) - F\bigl(\gamma\bigr)\Bigr)dy. \tag{2.1}$$

Recall that $m > 0$ is a constant and functions $a_\varepsilon^\pm$ are defined through (1.1), where $0 \le a^+, a^- \in L^1(\mathbb{R}^d)$ and (1.2) holds.

Operator (2.1) has two properties: 1) $L_\varepsilon 1 = 0$ and 2) if, for a given function F, a configuration γ^* is such that $F(\gamma^*) \ge F(\gamma)$ for all $\gamma \in \Gamma$ (i.e. if γ^* is a global maximum for F), then $(L_\varepsilon F)(\gamma^*) \le 0$. Hence, formally, L_ε is a *Markov generator*.

The dynamics of $\gamma_{t,\varepsilon}$ if defined then through the differential equation:

$$\frac{d}{dt}\mathbb{E}\bigl[F(\gamma_{t,\varepsilon})\bigr] = \mathbb{E}\bigl[(L_\varepsilon F)(\gamma_{t,\varepsilon})\bigr] \tag{2.2}$$

which should be satisfied for a large class of functions F.

Definition 1. A function $k_{t,\varepsilon} : \mathbb{R}^d \to \mathbb{R}_+ := [0,\infty)$ is said to be *the first order correlation function* (for the distribution of $\gamma_{t,\varepsilon}$), if for any function $g(x) \ge 0$,

$$\mathbb{E}\Bigl[\sum_{x\in\gamma_{t,\varepsilon}} g(x)\Bigr] = \int_{\mathbb{R}^d} g(x)k_{t,\varepsilon}(x)\,dx. \tag{2.3}$$

The function $k_{t,\varepsilon}(x)$ is also called *the density* of individuals of the configuration $\gamma_{t,\varepsilon}$, since, taking $g(x) = 1\!\!1_\Lambda(x)$ for a $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, we get from (2.3) that (1.4) holds.

Definition 2. A symmetric function $k_{t,\varepsilon}^{(2)} : (\mathbb{R}^d)^2 \to \mathbb{R}_+$ is called *the second-order correlation function*, if, for any symmetric function $g_2 : (\mathbb{R}^d)^2 \to \mathbb{R}_+$,

$$\mathbb{E}\Bigl[\sum_{\substack{x\in\gamma_{t,\varepsilon}\\ y\in\gamma_{t,\varepsilon}\\ x\neq y}} g_2(x,y)\Bigr] = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_2(x,y)k_{t,\varepsilon}^{(2)}(x,y)\,dxdy. \tag{2.4}$$

Combining (2.4) with (2.3), we can also write,

$$\mathbb{E}\Bigl[\sum_{\substack{x\in\gamma_{t,\varepsilon}\\ y\in\gamma_{t,\varepsilon}}} g_2(x,y)\Bigr] = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_2(x,y)k_{t,\varepsilon}^{(2)}(x,y)\,dxdy + \int_{\mathbb{R}^d} g_2(x,x)k_{t,\varepsilon}(x)\,dx. \tag{2.5}$$

Substituting to (2.5) the symmetric function

$$g_2(x,y) = \frac{1}{2}\Big(1\!\!1_{\Lambda_1}(x)1\!\!1_{\Lambda_2}(y) + 1\!\!1_{\Lambda_1}(y)1\!\!1_{\Lambda_2}(x)\Big),$$

where $\Lambda_1, \Lambda_2 \in \mathcal{B}_{\mathrm{c}}(\mathbb{R}^d)$, we get

$$\mathbb{E}\big[|\gamma_{t,\varepsilon}\cap\Lambda_1|\,|\gamma_{t,\varepsilon}\cap\Lambda_2|\big] = \int_{\Lambda_1}\int_{\Lambda_2} k^{(2)}_{t,\varepsilon}(x,y)\,dxdy + \int_{\Lambda_1\cap\Lambda_2} k_{t,\varepsilon}(x)\,dx,$$

and hence the *covariance* between random numbers $|\gamma_{t,\varepsilon}\cap\Lambda_1|$ and $|\gamma_{t,\varepsilon}\cap\Lambda_2|$ is given by

$$\begin{aligned}
&\mathbb{E}\Big[\Big(|\gamma_{t,\varepsilon}\cap\Lambda_1| - \mathbb{E}\big[|\gamma_{t,\varepsilon}\cap\Lambda_1|\big]\Big)\Big(|\gamma_{t,\varepsilon}\cap\Lambda_2| - \mathbb{E}\big[|\gamma_{t,\varepsilon}\cap\Lambda_2|\big]\Big)\Big] && (2.6)\\
&= \mathbb{E}\big[|\gamma_{t,\varepsilon}\cap\Lambda_1|\,|\gamma_{t,\varepsilon}\cap\Lambda_2|\big] - \mathbb{E}\big[|\gamma_{t,\varepsilon}\cap\Lambda_1|\big]\,\mathbb{E}\big[|\gamma_{t,\varepsilon}\cap\Lambda_2|\big]\\
&= \int_{\Lambda_1}\int_{\Lambda_2}\Big(k^{(2)}_{t,\varepsilon}(x,y) - k_{t,\varepsilon}(x)\,k_{t,\varepsilon}(y)\Big)\,dxdy + \int_{\Lambda_1\cap\Lambda_2} k_{t,\varepsilon}(x)\,dx. && (2.7)
\end{aligned}$$

Substituting (2.1) into (2.2) and using (2.3) and (2.4), we obtain that $k_{t,\varepsilon}(x)$ satisfies the following equation

$$\begin{aligned}
\frac{\partial}{\partial t}k_{t,\varepsilon}(x) &= \int_{\mathbb{R}^d} a^+_\varepsilon(x-y)k_{t,\varepsilon}(y)\,dy - mk_{t,\varepsilon}(x)\\
&\quad - \int_{\mathbb{R}^d} a^-_\varepsilon(x-y)k^{(2)}_{t,\varepsilon}(x,y)\,dy,
\end{aligned}$$

see e.g. [13, 14] for details. Similarly, the evolution of $k^{(2)}_{t,\varepsilon}(x,y)$ depends on the third order correlation function and so on.

It can be shown, see [12, 14, 21], that then, for $\varepsilon\to 0$,

$$\begin{aligned}
k_{t,\varepsilon}(x) &= q_t(\varepsilon x) + o(1),\\
k^{(2)}_{t,\varepsilon}(x,y) &= q_t(\varepsilon x)q_t(\varepsilon y) + o(1),
\end{aligned} \tag{2.8}$$

where q_t solves the following *mean-field*, or *kinetic*, equation

$$\begin{aligned}
\frac{\partial}{\partial t}q_t(x) &= \int_{\mathbb{R}^d} a^+(x-y)q_t(y)\,dy - mq_t(x)\\
&\quad - q_t(x)\int_{\mathbb{R}^d} a^-(x-y)q_t(y)\,dy.
\end{aligned} \tag{2.9}$$

Note that it was shown using another scaling, which apparently is equivalent to the considered one, see [28] for details. For various properties of solutions to (2.9), see [15–20, 23].

The asymptotics (2.8) however does not describe effectively the covariance (2.6) between random numbers $|\gamma_{t,\varepsilon}\cap\Lambda_1|$ and $|\gamma_{t,\varepsilon}\cap\Lambda_2|$, especially in the case of disjoint $\Lambda_1, \Lambda_2 \in \mathcal{B}_{\mathrm{c}}(\mathbb{R}^d)$, since then, by (2.7) and (2.8),

$$\mathbb{E}\Big[\Big(|\gamma_{t,\varepsilon}\cap\Lambda_1| - \mathbb{E}\big[|\gamma_{t,\varepsilon}\cap\Lambda_1|\big]\Big)\Big(|\gamma_{t,\varepsilon}\cap\Lambda_2| - \mathbb{E}\big[|\gamma_{t,\varepsilon}\cap\Lambda_2|\big]\Big)\Big] = o(1).$$

To partially reveal the covariance above, one needs hence an enhanced asymptotics (2.8). A mathematical approach for this was proposed in [28], justifying the heuristic considerations in the early publication [27]; the approach has been recently generalised in [6]. Namely, it was shown that

$$\begin{aligned} k_{t,\varepsilon}(x) &= q_t(\varepsilon x) + \varepsilon^d p_t(\varepsilon x) + o(\varepsilon^d), \\ k^{(2)}_{t,\varepsilon}(x,y) &= q_t(\varepsilon x) q_t(\varepsilon y) + \varepsilon^d g_t(\varepsilon x, \varepsilon y) + o(\varepsilon^d), \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} \frac{\partial}{\partial t} p_t(x) = &\int_{\mathbb{R}^d} a^+(x-y) p_t(y)\, dy - m p_t(x) - q_t(x) \int_{\mathbb{R}^d} a^-(x-y) p_t(y)\, dy \\ &- p_t(x) \int_{\mathbb{R}^d} a^-(x-y) q_t(y)\, dy - \int_{\mathbb{R}^d} g_t(x,y) a^-(x-y)\, dy; \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} \frac{\partial}{\partial t} g_t(x,y) = &\int_{\mathbb{R}^d} [g_t(x,z) a^+(y-z) + g_t(z,y) a^+(x-z)]\, dz - 2m g_t(x,y) \\ &- g_t(x,y) \int_{\mathbb{R}^d} [a^-(x-z) + a^-(y-z)] q_t(z)\, dz \\ &+ a^+(x-y)[q_t(x) + q_t(y)] - 2a^-(y-x) q_t(y) q_t(x) \\ &- \int_{\mathbb{R}^d} [a^-(x-z) q_t(x) g_t(z,y) + a^-(y-z) q_t(y) g_t(x,z)]\, dz. \end{aligned} \tag{2.12}$$

3 Space-homogeneous case

Let henceforth, for an integrable function f on $\mathbb{R}^d$, $\widehat{f}$ denote its *unitary* Fourier transform given by

$$\widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-2i\pi x \cdot \xi}\, d\xi, \tag{3.1}$$

where $x \cdot \xi$ denotes the standard dot-product in $\mathbb{R}^d$. Note that

$$|\widehat{f}(\xi)| \le \int_{\mathbb{R}^d} |f(x)|\, dx, \qquad \xi \in \mathbb{R}^d. \tag{3.2}$$

We formulate now our basic assumptions on the kernels $a^\pm : \mathbb{R}^d \to [0,\infty)$:

$$\begin{gathered} a^\pm \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d); \qquad \widehat{a}^\pm \in L^1(\mathbb{R}^d) \\ a^\pm(-x) = a^\pm(x), \quad x \in \mathbb{R}^d. \end{gathered} \tag{\textbf{A1}}$$

Note that (**A1**), together with (3.2), imply that $a^\pm, \widehat{a}^\pm \in L^2(\mathbb{R}^d)$. It is also well-known that $\widehat{a}^\pm$ are (uniformly) continuous functions on $\mathbb{R}^d$.

Equation (2.9) has two constant stationary solutions $q_t(x) = 0$ and $q_t(x) = q^*$, where

$$q^* := \frac{\varkappa^+ - m}{\varkappa^-}. \tag{3.3}$$

We will always assume that

$$\varkappa^+ > m, \tag{\textbf{A2}}$$

i.e. that $q^* > 0$; otherwise, the solution to (2.9) with $q_0(x) \ge 0$, $x \in \mathbb{R}^d$, would uniformly degenerate as $t \to \infty$. We assume also that

$$J^*(x) := a^+(x) - q^* a^-(x) \ge 0, \qquad x \in \mathbb{R}^d. \tag{\textbf{A3}}$$

The reason for this restriction is as follows. By (3.2), assumption (**A3**) yields

$$|\widehat{J}^*(\xi)| \le \int_{\mathbb{R}^d} |J^*(x)|\,dx = \int_{\mathbb{R}^d} J^*(x)\,dx = \varkappa^+ - q^*\varkappa^- = m. \tag{3.4}$$

Next $a^\pm(-x) = a^\pm(x)$ for $x \in \mathbb{R}^d$ implies that $\widehat{a}^\pm(\xi) \in \mathbb{R}$ for $\xi \in \mathbb{R}^d$, and therefore,

$$\varkappa^+ - \widehat{J}^*(\xi) \ge \varkappa^+ - m > 0, \qquad \xi \in \mathbb{R}^d. \tag{3.5}$$

If (3.5) fails, then (under further assumptions) there exists an infinite family of non-constant (in space) stationary solutions to (2.9), see [23].

Under assumption (**A3**), if $q_0(x) = q_0$ for all $x \in \mathbb{R}^d$, then, by [19, Proposition 2.7], the solution to (2.9) is also space homogeneous: $q_t(x) = q_t$, where q_t solves the logistic differential equation

$$\frac{d}{dt}q_t = \varkappa^+ q_t - mq_t - \varkappa^- q_t^2 = \varkappa^- q_t(q^* - q_t). \tag{3.6}$$

It is straightforward to check that then

$$q_t = \frac{q^* q_0}{q_0 + (q^* - q_0)e^{-(\varkappa^+ - m)t}}, \tag{3.7}$$

hence

$$\lim_{t\to\infty} q_t = q^*. \tag{3.8}$$

We will assume henceforth that

$$0 < q_0 < q^*, \tag{3.9}$$

then, by (3.7),

$$0 < q_t < q^*, \qquad t > 0. \tag{3.10}$$

Note that then, by (3.7), $\dfrac{d}{dt}q_t > 0$ for $t > 0$, i.e. q_t is (strictly) increasing.

Equations (2.11) and (2.12) are linear, and it is straightforward to check that, in the space-homogeneous case, when $p_0(x) = p_0$, $g_0(x, y) = g_0(x - y)$ for all $x, y \in \mathbb{R}^d$, this property will be preserved in time, so that (2.10) takes the form

$$\begin{aligned} k_{t,\varepsilon}(x) &= k_{t,\varepsilon} = q_t + \varepsilon^d p_t + o(\varepsilon^d), \\ k_{t,\varepsilon}(x, y) &= k_{t,\varepsilon}(x - y) = q_t q_t + \varepsilon^d g_t(x - y) + o(\varepsilon^d), \end{aligned} \tag{3.11}$$

where, recall q_t solves (3.6) and hence is given by (3.7), and the equations for $p_t, g_t(x)$ take the following form:

$$\frac{d}{dt}p_t = \varkappa^+ p_t - mp_t - 2\varkappa^- q_t p_t - \int_{\mathbb{R}^d} g_t(y)a^-(y)\,dy; \tag{3.12}$$

$$\begin{aligned} \frac{\partial}{\partial t}g_t(x) = 2\int_{\mathbb{R}^d} a^+(x-y)g_t(y)\,dy - 2\varkappa^- q_t g_t(x) - 2mg_t(x) \\ + 2a^+(x)q_t - 2a^-(x)q_t^2 - 2q_t \int_{\mathbb{R}^d} a^-(x-y)g_t(y)\,dy. \end{aligned} \tag{3.13}$$

For $q_0 \in (0, q^*)$, (3.10) holds, and hence

$$J_t(x) := a^+(x) - q_t a^-(x) > J^*(x) \geq 0. \tag{3.14}$$

One can rewrite then (3.13):

$$\frac{\partial}{\partial t} g_t(x) = 2 \int_{\mathbb{R}^d} J_t(x-y) g_t(y)\, dy - 2(\varkappa^- q_t + m) g_t(x) + 2 q_t J_t(x). \tag{3.15}$$

It is straightforward to check that if $g_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, then $g_t \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ for all $t > 0$. One can apply then the Fourier transform to both parts of (3.15) to get

$$\frac{\partial}{\partial t} \widehat{g}_t(\xi) = 2\bigl(\widehat{J}_t(\xi) - \varkappa^- q_t - m\bigr) \widehat{g}_t(\xi) + 2 q_t \widehat{J}_t(\xi). \tag{3.16}$$

By the above, $g_t, a^- \in L^2(\mathbb{R}^d)$, $t \geq 0$, and since we have chosen the unitary Fourier transform (3.1), we can rewrite (3.12), by using the Parseval identity, as follows:

$$\frac{d}{dt} p_t = \varkappa^+ p_t - m p_t - 2\varkappa^- q_t p_t - \int_{\mathbb{R}^d} \widehat{g}_t(\xi) \widehat{a}^-(\xi)\, d\xi. \tag{3.17}$$

We are going to find limits of $\widehat{g}_t$ and p_t as $t \to \infty$. To this end, we prove an abstract lemma which is actually an adaptation of e.g. [29, Theorem 5.8.2] to the case of bounded operators (that apparently requires weaker conditions).

Lemma 3.1. *Let $\bigl(X, \|\cdot\|_X\bigr)$ be a Banach space, and let $\bigl(\mathcal{L}(X), \|\cdot\|\bigr)$ denote the Banach space of linear bounded operators on X. Let $A \in C\bigl([0,\infty) \to \mathcal{L}(X)\bigr)$ be a continuous operator-valued function. Suppose that there exists $c, \nu > 0$ such that, for all $t \geq s \geq 0$, the operator*

$$U(t,s) := \exp\biggl(\int_s^t A(\tau)\, d\tau\biggr) \in \mathcal{L}(X)$$

satisfies

$$\bigl\|U(t,s)\bigr\| \leq c e^{-\nu(t-s)}. \tag{3.18}$$

Let $f \in C\bigl([0,\infty), X\bigr)$ be a continuous X-valued function. Suppose that $f(t)$ converges in X to some $f(\infty) \in X$ and $A(t)$ strongly converges to some $A(\infty) \in \mathcal{L}(X)$ as $t \to \infty$. Finally, suppose that $A(\infty)$ is an invertible operator, i.e. that there exists $A(\infty)^{-1} \in \mathcal{L}(X)$. Then the unique classical solution to the following non-homogeneous Cauchy problem in X:

$$\frac{d}{dt} u(t) = A(t) u(t) + f(t), \qquad u(0) = u_0 \in \mathbb{R}^d, \tag{3.19}$$

converges in X, as $t \to \infty$, to

$$-A(\infty)^{-1} f(\infty) \in X.$$

Proof. Since $A \in C_b\bigl([0,\infty) \to \mathcal{X}\bigr)$, the unique classical solution $u \in C_b\bigl([0,\infty), X\bigr)$ to (3.19) (i.e. such that $u \in C^1\bigl((0,\infty), X\bigr)$) is given by

$$u(t) = U(t,0) u(0) + \int_0^t U(t,s) f(s)\, ds, \tag{3.20}$$

see e.g. [7, Chapter 3] (all integrals are in the sense of Bochner henceforth). By (3.18),

$$\|U(t,0)u(0)\|_X \le e^{-\nu t}\|u(0)\|_X \to 0, \qquad t\to\infty. \tag{3.21}$$

Suppose, firstly, that $f(\infty)=0$. Since then $f(s)\to 0$ in X as $s\to\infty$, one gets that, for any $\varepsilon>0$, there exists $T=T(\varepsilon)>0$ such that $\|f(s)\|\le\varepsilon$ for all $s\ge T$. Since $f\in C\bigl([0,\infty),X\bigr)$, one can define $\|f\|_T := \sup\limits_{t\in[0,T]}\|f(t)\|_X<\infty$. Then

$$\begin{aligned}\left\|\int_0^t U(t,s)f(s)\,ds\right\| &\le \int_0^T \|U(t,s)\|\|f(s)\|_X\,ds + \varepsilon\int_T^t \|U(t,s)\|\,ds\\ &\le c\|f\|_T\int_0^T e^{-\nu(t-s)}\,ds + c\,\varepsilon\int_T^t e^{-\nu(t-s)}\,ds\\ &\le c\|f\|_T\frac{1}{\nu}e^{-\nu(t-T)}+\frac{c\,\varepsilon}{\nu},\end{aligned}$$

and combining this with (3.21), one gets that $u(t)\to 0=u(\infty)$ in X as $t\to\infty$.

For a general $f(\infty)\in X$, denote $v(t):=u(t)-u(\infty)$. Since $A(\infty)$ is invertible, one can define

$$u(\infty):=-A(\infty)^{-1}f(\infty)\in X.$$

Then

$$\frac{d}{dt}v(t)=\frac{d}{dt}u(t)=A(t)u(t)+f(t)=A(t)v(t)+f(t)+A(t)u(\infty).$$

We set $g(t):=f(t)+A(t)u(\infty)$, $t\ge 0$. By the assumptions on f and A, $g\in C\bigl([0,\infty),X\bigr)$ and

$$\lim_{t\to\infty}g(t)=f(\infty)+A(\infty)\bigl(-A(\infty)^{-1}f(\infty)\bigr)=0,$$

where the limit is in X. Then, by the proved above, $v(t)\to 0$ in X, and hence $u(t)\to u(\infty)$ in X. □

Theorem 3.2. *Let (**A1**)–(**A3**) hold. Let q_0 satisfies (3.9) and $g_0,\widehat{g}_0\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$. Then there exist limits*

$$\widehat{g}^*(\xi) := \lim_{t\to\infty}\widehat{g}_t(\xi)=\frac{q^*\widehat{J}^*(\xi)}{\varkappa^+-\widehat{J}^*(\xi)}\le\frac{m}{\varkappa^-}, \qquad \xi\in\mathbb{R}^d, \tag{3.22}$$

$$p^* := \lim_{t\to\infty}p_t=-\frac{1}{\varkappa^-}\int_{\mathbb{R}^d}\frac{\widehat{J}^*(\xi)}{\varkappa^+-\widehat{J}^*(\xi)}\widehat{a}^-(\xi)\,d\xi\in\mathbb{R}. \tag{3.23}$$

Moreover, the convergence in (3.22) takes place in the norms of both $L^1(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$. As a result, g_t converges, as $t\to\infty$, in $L^\infty(\mathbb{R}^d)$ to g^*, the inverse Fourier transform of $\widehat{g}^*$.

Remark 3.3. We will actually use in the proof a part of the estimate (3.4) only, rather than the more strict assumption (**A3**). More precisely, it is easy to check that all arguments of the proof remain correct if the assume, instead of (**A3**), that, for some $\alpha\in(0,\varkappa^+)$,

$$\varkappa^+-\widehat{J}^*(\xi)\ge\alpha, \qquad \xi\in\mathbb{R}^d;$$

the uppear bound in (3.22) will be then replaced by $q^*\frac{\varkappa^+-\alpha}{\alpha}$.

Proof of Theorem 3.2. We denote

$$j(\xi,t) := \widehat{J}_t(\xi) - \varkappa^- q_t - m, \qquad \xi\in\mathbb{R}^d,\ t\geq 0, \tag{3.24}$$

and apply Lemma 3.1 to equation (3.16), where $X=L^1(\mathbb{R}^d)$ or $X=L^\infty(\mathbb{R}^d)$, $A(t)$ is the multiplication operator by the function $2j(\xi,t)$, and $f(t,\xi)=2q_t\widehat{J}_t(\xi)$. Note that, for any $t\geq s\geq 0$, $\xi\in\mathbb{R}^d$, we have, by (**A1**), (3.2), (3.10),

$$\begin{aligned}
&\widehat{J}_t(\cdot), j(\cdot,t) \in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d),\\
&|j(\xi,t)-j(\xi,s)| \leq 2|\widehat{J}_t(\xi)-\widehat{J}_s(\xi)| + 2\varkappa^-|q_t-q_s| \leq 4\varkappa^-|q_t-q_s|,\\
&|f(t,\xi)-f(s,\xi)| \leq 2|q_t-q_s||\widehat{J}_t(\xi)| + 2q_s|\widehat{J}_t(\xi)-\widehat{J}_s(\xi)|\\
&\qquad\qquad \leq 2\bigl(|\widehat{a}^+(\xi)| + q^*(q^*+1)|\widehat{a}^-(\xi)|\bigr)|q_t-q_s|.
\end{aligned} \tag{3.25}$$

Therefore, $A\in C\bigl([0,\infty)\to\mathcal{L}(X)\bigr)$ and $f\in C\bigl([0,\infty),X\bigr)$ for both $X=L^1(\mathbb{R}^d)$ and $X=L^\infty(\mathbb{R}^d)$. Note that, by (3.20),

$$\widehat{g}_t(\xi) = \exp\biggl(2\int_0^t j(\xi,\tau)\,d\tau\biggr)\widehat{g}_0(\xi) + \int_0^t \exp\biggl(2\int_s^t j(\xi,\tau)\,d\tau\biggr) q_s\widehat{J}_s(\xi)\,ds,$$

and then, by the Riemann–Lebesgue lemma, $\widehat{g}_t(\cdot)\in C(\mathbb{R}^d)$, $t\geq 0$.

By (3.25), we also have that $f(t,\xi)$ converges, in the norm of either of X to $2q^*\widehat{J}^*(\xi)$; and also $A(t)$ strongly converges to the operator $A(\infty)$ of the multiplication by

$$2\lim_{t\to\infty}\widehat{J}_t(\xi) = 2(\widehat{J}^*(\xi)-\varkappa^- q^* - m) = 2(\widehat{J}^*(\xi)-\varkappa^+).$$

By (3.5), operator $A(\infty)$ is invertible.

Next, for all $t>s\geq 0$, $\int_s^t A(\tau)\,d\tau$ is the operator of multiplication by $2\int_s^t j(\cdot,\tau)\,d\tau$. We have

$$\int_s^t j(\xi,\tau)\,d\tau = \bigl(\widehat{a}^+(\xi)-m\bigr)(t-s) - \bigl(\widehat{a}^-(\xi)+\varkappa^-\bigr)\int_s^t q_\tau\,d\tau. \tag{3.26}$$

Since, by (3.6),

$$\frac{d}{dt}\log q_t = \frac{1}{q_t}\frac{d}{dt}q_t = \varkappa^+ - m - \varkappa^- q_t,$$

we get

$$\log q_t - \log q_s = \int_s^t \frac{d}{d\tau}\log q_\tau d\tau = (\varkappa^+-m)(t-s) - \varkappa^-\int_s^t q_\tau d\tau,$$

and hence

$$\int_s^t q_\tau d\tau = q^*(t-s) - \frac{1}{\varkappa^-}\log\frac{q_t}{q_s}. \tag{3.27}$$

Substituting (3.27) into (3.26), and using (3.5) and that q_t is increasing and $|\widehat{a}^-(\xi)|\leq\varkappa^-$ holds, we get

$$\int_s^t j(\xi,\tau)\,d\tau = -\bigl(\varkappa^+-\widehat{J}^*(\xi)\bigr)(t-s) + \frac{\widehat{a}^-(\xi)+\varkappa^-}{\varkappa^-}\log\frac{q_t}{q_s}. \tag{3.28}$$

Therefore, cosnidering a multiplication operator $U(t,s)=\exp\Bigl(\int_s^t A(\tau)\,d\tau\Bigr)$ and using (3.5) and that

$$|\widehat{a}^-(\xi)|\leq \varkappa^-, \qquad 0<q_0\leq q_s<q_t<q^*, \quad t>s>0,$$

we get from (3.28) that, in either of spaces X,

$$\|U(t,s)\|=\sup_{\xi\in\mathbb{R}^d}\exp\biggl(2\int_s^t j(\xi,\tau)\,d\tau\biggr)\leq\biggl(\frac{q^*}{q_0}\biggr)^4 e^{-2(\varkappa^+-m)(t-s)}. \tag{3.29}$$

Therefore, by Lemma 3.1,

$$\widehat{g}_t(\xi)\to -A(\infty)^{-1}f(\infty)=-\frac{1}{2(\widehat{J}^*(\xi)-\varkappa^+)}2q^*\widehat{J}^*(\xi)=\frac{q^*\widehat{J}^*(\xi)}{\varkappa^+-\widehat{J}^*(\xi))}=:\widehat{g}^*(\xi)$$

in the sense of norm in both $L^1(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$ (and, in particular, pointwise). We also have, by (**A1**), that

$$|\widehat{g}^*(\xi)|\leq q^*\frac{m}{\varkappa^+-m}=\frac{m}{\varkappa^-}, \qquad \xi\in\mathbb{R}^d,$$

that finishes the proof of (3.22).

Since $\widehat{g}_t\in L^1(\mathbb{R}^d)$, its inverse Fourier transform coinsides with g_t a.e.; in particular, they coincide as elements of $L^\infty(\mathbb{R}^d)$. Hence if g^* is the inverse Fourier transform of $\widehat{g}^*\in L^1(\mathbb{R}^d)$, then, by (3.2),

$$\|g_t-g^*\|_{L^\infty(\mathbb{R}^d)}\leq\|\widehat{g}_t-\widehat{g}^*\|_{L^1(\mathbb{R}^d)}\to 0, \qquad t\to\infty,$$

that proves the last statement of Theorem 3.2.

We are going to apply now Lemma 3.1 to equation (3.17), with $X=\mathbb{R}$. Since $\widehat{g}_t\to\widehat{g}^*$ in $L^\infty(\mathbb{R}^d)$ and $\widehat{g}_t$ is a classical solution to (3.16) in $L^\infty(\mathbb{R}^d)$ (i.e. a continuous mapping from $[0,\infty)$ to $L^\infty(\mathbb{R}^d)$), function $\widehat{g}_t(\xi)$ is globally bounded in $t\geq 0$ and $\xi\in\mathbb{R}^d$. Then

$$-\int_{\mathbb{R}^d}\widehat{g}_t(\xi)\widehat{a}^-(\xi)\,d\xi\to-\int_{\mathbb{R}^d}\widehat{g}^*(\xi)\widehat{a}^-(\xi)\,d\xi, \qquad t\to\infty,$$

by the dominated convergence theorem. $A(t)$ is given now through the multiplication by $c_t:=\varkappa^+-m-2\varkappa^-q_t$, and, by (3.27),

$$\begin{aligned}\int_s^t c_\tau d\tau&=(\varkappa^+-m)(t-s)-2\varkappa^-q^*(t-s)+2\log\frac{q_t}{q_s}\\&=-(\varkappa^+-m)(t-s)+2\log\frac{q_t}{q_s}\leq-(\varkappa^+-m)(t-s)+2\log\frac{q^*}{q_0}.\end{aligned}$$

Hence, by the same arguments as before, we can apply Lemma 3.1: since, by (3.22),

$$\lim_{t\to\infty}c_t=\varkappa^+-m-2\varkappa^-q^*=-\varkappa^-q^*,$$

we get

$$p_t\to-\frac{1}{\varkappa^-q^*}\int_{\mathbb{R}^d}\widehat{g}^*(\xi)\widehat{a}^-(\xi)\,d\xi,$$

that implies (3.23). □

4 Critical mortality

We are going to discuss now the extinction regime. Recall that $o(\varepsilon^d)$ in (3.11) depends on t, so we have

$$k_{t,\varepsilon} = q_t + \varepsilon^d p_t + o_t(\varepsilon^d),$$

where, for each $t > 0$,

$$\lim_{\varepsilon\to 0} \frac{o_t(\varepsilon^d)}{\varepsilon^d} = 0.$$

We will assume that

$$\lim_{t\to\infty} o_t(\varepsilon^d) = o(\varepsilon^d).$$

Then, the extinction (1.3) takes place if and only if (1.7) holds.

We fix an $m \in (0, \varkappa^+)$ for which (**A3**) holds. We consider a function $m_{\mathrm{cr}} : (0,1) \to (m, \varkappa^+)$ and set, cf. (3.3), for $\varepsilon \in (0,1)$,

$$q^*(\varepsilon) := \frac{\varkappa^+ - m_{\mathrm{cr}}(\varepsilon)}{\varkappa^-} \in \Bigl(0, \frac{\varkappa^+ - m}{\varkappa^-}\Bigr), \tag{4.1}$$

and also, cf. (**A3**),

$$J_\varepsilon(x) := a^+(x) - q^*(\varepsilon) a^-(x) \geq 0, \tag{4.2}$$

because of (**A3**), (4.1). Finally, we set, cf. (3.23), for $\varepsilon \in (0,1)$,

$$p^*(\varepsilon) := -\frac{1}{\varkappa^-} \int_{\mathbb{R}^d} \frac{\widehat{J}_\varepsilon(\xi)}{\varkappa^+ - \widehat{J}_\varepsilon(\xi)} \widehat{a}^-(\xi)\, d\xi. \tag{4.3}$$

Note that, by (3.2), (4.2),

$$|\widehat{J}_\varepsilon(\xi)| \leq \int_{\mathbb{R}^d} \bigl|J_\varepsilon(x)\bigr|\, dx = \int_{\mathbb{R}^d} J_\varepsilon(x)\, dx = m_{\mathrm{cr}}(\varepsilon), \tag{4.4}$$

and hence $p^*(\varepsilon)$ is well-defined: (4.4) and (**A1**) yield

$$|p^*(\varepsilon)| \leq \frac{1}{\varkappa^-} \frac{m_{\mathrm{cr}}(\varepsilon)}{\varkappa^+ - m_{\mathrm{cr}}(\varepsilon)} \int_{\mathbb{R}^d} \bigl|\widehat{a}^-(\xi)\bigr|\, d\xi < \infty, \qquad \varepsilon \in (0,1).$$

In the rest of the paper, our main object of interest will be the following equation, cf. (1.7),

$$q^*(\varepsilon) + \varepsilon^d p^*(\varepsilon) + o(\varepsilon^d) = 0. \tag{4.5}$$

Proposition 4.1. *Let* (**A1**)–(**A3**) *hold. If* (4.5) *holds and there exists* $\lim\limits_{\varepsilon\to 0} m_{\mathrm{cr}}(\varepsilon)$, *then*

$$\lim_{\varepsilon\to 0} m_{\mathrm{cr}}(\varepsilon) = \varkappa^+, \qquad \lim_{\varepsilon\to 0} q^*(\varepsilon) = 0. \tag{4.6}$$

Proof. Clearly, $m_{\mathrm{cr}}(0) := \lim\limits_{\varepsilon\to 0} m_{\mathrm{cr}}(\varepsilon) \leq \varkappa^+$. Suppose that $m_{\mathrm{cr}}(0) < \varkappa^+$. Let $\alpha \in (0,1)$ be such that $\alpha\varkappa^+ > m_{\mathrm{cr}}(0)$. Then there exists $\varepsilon_\alpha \in (0,1)$ such that $m_{\mathrm{cr}}(\varepsilon) < \alpha\varkappa^+$ for all $0 < \varepsilon < \varepsilon_\alpha$. Therefore, by (4.4),

$$|\widehat{J}_\varepsilon(\xi)| < \alpha\varkappa^+, \qquad \xi \in \mathbb{R}^d,\ 0 < \varepsilon < \varepsilon_\alpha.$$

Then, by (4.3), (**A1**),

$$|p^*(\varepsilon)| \le \frac{1}{\varkappa^-}\int_{\mathbb{R}^d} \frac{\alpha\varkappa^+}{\varkappa^+ - \alpha\varkappa^+}\widehat{a}^-(\xi)\,d\xi < \infty,$$

and hence $\varepsilon^d p^*(\varepsilon) \to 0$, $\varepsilon \to 0$. Therefore, by (4.5), we get that $q^*(\varepsilon) \to 0$ and hence, by (4.1), $m_{\mathrm{cr}}(\varepsilon) \to \varkappa^+$ that contradicts the assumption. The statement is proved. □

The behaviour of $p^*(\varepsilon)$ as $\varepsilon \to 0$ depends on the dimension $d \in \mathbb{N}$: the limit (as $\varepsilon \to 0$) of the integrand in (4.3) is equal to, because of (4.6), $\dfrac{\widehat{a}^+(\xi)\widehat{a}^-(\xi)}{\varkappa^+ - \widehat{a}^+(\xi)}$ that has a singularity at the origin, which is, in general, non-integrable for $d < 3$. We discuss this under the following additional assumption:

$$\int_{\mathbb{R}^d} |x|^2 a^+(x)\,dx < \infty. \tag{\textbf{A4}}$$

Since $a^+ \in L^\infty(\mathbb{R}^d)$, the inequality in (**A4**) implies that $\int_{\mathbb{R}^d} |x| a^+(x)\,dx < \infty$, and using that $a^+(-x) = a^+(x)$, $x \in \mathbb{R}^d$, we get

$$\int_{\mathbb{R}^d} x a^+(x)\,dx = 0 \in \mathbb{R}^d.$$

Then, by the Taylor expansion for $\widehat{a}^+(\xi)$ defined by (3.1), we get (cf. [30, Corollary 1.2.7] for another coefficients of the Fourier transform) that

$$\widehat{a}^+(\xi) = \widehat{a}^+(0) - 2\pi^2 \sum_{i,j=1}^{d} a^+_{i,j}\xi_i\xi_j + o(|\xi|^2) = \varkappa^+ - 2\pi^2 A^+\xi\cdot\xi + o(|\xi|^2),$$

for $\xi \to 0$, where

$$a^+_{i,j} := \int_{\mathbb{R}^d} x_i x_j a^+(x)\,dx, \qquad 1 \le i,j \le d, \tag{4.7}$$

and hence $A^+ := \bigl(a_{i,j}\bigr)_{i,j=1}^d$ is a Hermitian (strictly) positive definite matrix. Then there exists a Hermitian (strictly) positive definite matrix B^+ such that $(B^+)^2 = A^+$, and hence

$$\varkappa^+ - \widehat{a}^+(\xi) = 2\pi^2|B^+\xi|^2 + o(|\xi|^2), \qquad \xi \to 0. \tag{4.8}$$

Under assumptions (**A1**)–(**A3**), assumption (**A4**) holds with a^+ replaced by a^- and hence by J_ε or J^*. Let B^-, B_ε, B^* be the Hermitian positive definite matrices corresponding to the functions a^-, J_ε, J^*, respectively. Then, the corresponding analogues of (4.8) hold, with, in particular, $\varkappa^+$ replaced by $\varkappa^- = \widehat{a}^\pm(0)$, $m_{\mathrm{cr}}(\varepsilon) = \widehat{J}_\varepsilon(0)$, $m = \widehat{J}^*(0)$, respectively. It is easy to see also that

$$|B_\varepsilon\xi|^2 = |B^+\xi|^2 - q^*(\varepsilon)|B^-\xi|^2, \qquad \xi \in \mathbb{R}^d,\ \varepsilon \in (0,1). \tag{4.9}$$

Next, for any invertible matrix B,

$$\bigl(\|(B)^{-1}\|\bigr)^{-1}|\xi| \le |B\xi| \le \|B\||\xi|, \qquad \xi \in \mathbb{R}^d, \tag{4.10}$$

Then, for small enough $\delta > 0$,

$$\frac{\pi^2}{\|(B^\pm)^{-1}\|^2}|\xi|^2 \le \varkappa^\pm - \widehat{a}^\pm(\xi) \le 3\pi^2\|B^\pm\|^2|\xi|^2, \qquad |\xi| \le \delta. \tag{4.11}$$

The corresponding double inequalities can be also obtained for J_ε and J^*.

Proposition 4.2. *Let* (**A1**)–(**A4**) *hold. Let also* $m_{\rm cr} : (0,1) \to (m, \varkappa^+)$ *and* $p^*(\varepsilon)$*, defined through* (4.1)–(4.3)*, be such that* (4.6) *holds. Then, for* $d \ge 3$,

$$\lim_{\varepsilon\to 0} p^*(\varepsilon) = -\frac{1}{\varkappa^-}\int_{\mathbb{R}^d} \frac{\widehat{a}^+(\xi)\widehat{a}^-(\xi)}{\varkappa^+ - \widehat{a}^+(\xi)}\, d\xi =: p^*(0) \in \mathbb{R}, \tag{4.12}$$

whereas, for $d \le 2$, $\lim\limits_{\varepsilon\to 0} p^*(\varepsilon) = -\infty$.

Remark 4.3. Note that we do not need to assume (4.5) to get the statement.

Proof. For each $\delta > 0$, one can expand $p^*(\varepsilon)$ as follows

$$\begin{aligned} p^*(\varepsilon) &= p^*_{\le\delta}(\varepsilon) + p^*_{\ge\delta}(\varepsilon) \\ &=: -\frac{1}{\varkappa^-}\int_{|\xi|\le\delta} \frac{\widehat{J}_\varepsilon(\xi)}{\varkappa^+ - \widehat{J}_\varepsilon(\xi)}\widehat{a}^-(\xi)\, d\xi - \frac{1}{\varkappa^-}\int_{|\xi|\ge\delta} \frac{\widehat{J}_\varepsilon(\xi)}{\varkappa^+ - \widehat{J}_\varepsilon(\xi)}\widehat{a}^-(\xi)\, d\xi. \end{aligned}$$

To estimate $p^*_{\ge\delta}(\varepsilon)$, we verify firsly the following inequality:

$$\varkappa^+ - \widehat{J}_\varepsilon(\xi) > m - \widehat{J}^*(\xi) \ge 0, \qquad \xi \in \mathbb{R}^d. \tag{4.13}$$

Namely, by (**A3**), (4.2), the first inequality in (4.13) is equivalent to

$$\varkappa^+ - m > \bigl(q^* - q^*(\varepsilon)\bigr)\widehat{a}^-(\xi), \qquad \xi \in \mathbb{R}^d,$$

that is true since $|\widehat{a}^-(\xi)| \le \varkappa^-$, $\xi \in \mathbb{R}^d$, and $q^* - q^*(\varepsilon) < q^* = \frac{\varkappa^+ - m}{\varkappa^-}$. The second inequality in (4.13) is just (3.4).

Next, since the function

$$m - \widehat{J}^*(\xi) \ge m - \widehat{J}^*(0) = 0$$

is continuous in $\xi \in \mathbb{R}^d$, we conclude, cf. (4.3), that, for any $\delta > 0$, there exists $\mu_\delta > 0$, such that

$$m - \widehat{J}^*(\xi) \ge \mu_\delta, \qquad |\xi| \ge \delta,$$

and hence

$$|p^*_{\ge\delta}(\varepsilon)| \le \frac{1}{\varkappa^-}\frac{\varkappa^+}{\mu_\delta}\int_{\mathbb{R}^d} \bigl|\widehat{a}^-(\xi)\bigr|\, d\xi < \infty. \tag{4.14}$$

Next, by an analogue of (4.11) for J_ε, we get, for small enough $\delta > 0$,

$$\varkappa^+ - \widehat{J}_\varepsilon(\xi) \ge m_{\rm cr}(\varepsilon) - \widehat{J}_\varepsilon(\xi) \ge \frac{\pi^2}{\|B_\varepsilon^{-1}\|^2}|\xi|^2,$$

and hence,

$$|p^*_{\le\delta}(\varepsilon)| \le \text{const}\int_{|\xi|\le\delta} \frac{1}{|\xi|^2}\, d\xi < \infty \qquad \text{for } d \ge 3.$$

Combining the latter estimate with (4.14), we get that, for $d \geq 3$, (4.12) holds by (4.6) and the dominated convergence theorem.

Let now $d \leq 2$. By (4.11), one can always choose $\delta_0 > 0$ small enough to ensure that, for $\delta < \delta_0$,

$$\widehat{a}^-(\xi) \geq \varkappa^- - 3\pi^2 \|B^-\|^2 |\xi|^2 > \frac{\varkappa^-}{2} > 0, \qquad |\xi| \leq \delta.$$

Then

$$\widehat{J}_\varepsilon(\xi) = \widehat{a}^+(\xi) - q^*(\varepsilon)\widehat{a}^-(\xi) \geq \widehat{J}^*(\xi), \qquad |\xi| \leq \delta < \delta_0,$$

and, possibly redefining δ_0, we similarly get that

$$\widehat{J}^*(\xi) \geq m - 3\pi^2 \|B^*\|^2 |\xi|^2 > \frac{m}{2} > 0, \qquad |\xi| \leq \delta < \delta_0.$$

Next, by (4.11) applied to $a = J_\varepsilon$, we get

$$\begin{aligned}\varkappa^+ - \widehat{J}_\varepsilon(\xi) &= \varkappa^+ - m_{\mathrm{cr}}(\varepsilon) + m_{\mathrm{cr}}(\varepsilon) - \widehat{J}_\varepsilon(\xi)\\ &\leq \varkappa^+ - m_{\mathrm{cr}}(\varepsilon) + 3\pi^2 \|B_\varepsilon\|^2 |\xi|^2\\ &\leq \varkappa^+ - m_{\mathrm{cr}}(\varepsilon) + 3\pi^2 \|B^+\|^2 |\xi|^2, \qquad |\xi| \leq \delta,\end{aligned}$$

where we used (4.9). Combining the previous inequalities, we get that, for a fixed $\delta < \delta_0$,

$$\begin{aligned}-p^*_{\leq\delta}(\varepsilon) &= \frac{1}{\varkappa^-} \int_{|\xi|\leq\delta} \frac{\widehat{J}_\varepsilon(\xi)}{\varkappa^+ - \widehat{J}_\varepsilon(\xi)} \widehat{a}^-(\xi)\, d\xi\\ &\geq \frac{m}{4} \int_{|\xi|\leq\delta} \frac{1}{\varkappa^+ - m_{\mathrm{cr}}(\varepsilon) + 3\pi^2 \|B^+\|^2 |\xi|^2}\, d\xi.\end{aligned}$$

Therefore, for $d = 2$, we get, by passing to polar coordinates, that

$$-p^*_{\leq\delta}(\varepsilon) \geq c_1 \log\biggl(1 + \frac{c_2}{\varkappa^+ - m_{\mathrm{cr}}(\varepsilon)}\biggr);$$

and, for $d = 1$, we get that

$$-p^*_{\leq\delta}(\varepsilon) \geq \frac{c_3}{\sqrt{\varkappa^+ - m_{\mathrm{cr}}(\varepsilon)}} \arctan \frac{c_4}{\sqrt{\varkappa^+ - m_{\mathrm{cr}}(\varepsilon)}},$$

for certain $c_1, c_2, c_3, c_4 > 0$ (with c_2, c_4 depending on the fixed δ). Since, by (4.6), $\varkappa^+ - m_{\mathrm{cr}}(\varepsilon) \to 0$ as $\varepsilon \to 0$, the statement is proved. □

5 Asymptotics of the critical mortality: $d \geq 3$

We are going to reveal the asymptotic of $m_{\mathrm{cr}}(\varepsilon)$ (or, equivalently, $q^*(\varepsilon)$) assuming that (4.5) does hold. We start with the case $d \geq 3$.

If, additionally to (**A1**)–(**A4**) and (4.5), we assume that the limit $\lim\limits_{\varepsilon\to0} m_{\mathrm{cr}}(\varepsilon)$ exists, then, by Proposition 4.1, (4.6) holds, and hence, by Proposition 4.2, we get (4.12). Then (4.5) implies

$$\lim_{\varepsilon\to0} \frac{q^*(\varepsilon)}{\varepsilon^d} = -\lim_{\varepsilon\to0} p^*(\varepsilon) - \lim_{\varepsilon\to0} \frac{o(\varepsilon^d)}{\varepsilon^d} = -p^*(0) = \frac{1}{\varkappa^-} \int_{\mathbb{R}^d} \frac{\widehat{a}^+(\xi)\widehat{a}^-(\xi)}{\varkappa^+ - \widehat{a}^+(\xi)}\, d\xi.$$

Since $q^*(\varepsilon) > 0$, we will get that $I \geq 0$, where

$$I := \int_{\mathbb{R}^d} \frac{\widehat{a}^+(\xi)}{\varkappa^+ - \widehat{a}^+(\xi)} \widehat{a}^-(\xi)\, d\xi. \tag{5.1}$$

The first statement of the following theorem shows that one can replace the requiremnt about existence of the limit of $m_{\mathrm{cr}}(\varepsilon)$ by the continuity of the function

$$r(\varepsilon) := \frac{o(\varepsilon^d)}{\varepsilon^d} \tag{5.2}$$

in a neighbourhood of 0, where $o(\varepsilon^d)$ is from (4.5). Then, we reveal the next term of the asymptotic under additional smoothness of $r(\varepsilon)$.

Theorem 5.1. *Let $d \geq 3$ and (**A1**)–(**A4**), (4.5) hold. Let r given by (5.2) be continuous for small $\varepsilon > 0$. Let $I \neq 0$, where I is given by (5.1). Then $I > 0$ and*

$$q^*(\varepsilon) = \frac{I}{\varkappa^-}\varepsilon^d + o(\varepsilon^d). \tag{5.3}$$

As a result,

$$m_{\mathrm{cr}}(\varepsilon) = \varkappa^+ - \varepsilon^d I + o(\varepsilon^d). \tag{5.4}$$

If, additionally, $r(\varepsilon)$ is continuously differentiable for small $\varepsilon > 0$ and if $r'(0) := \lim\limits_{\varepsilon\to 0+} r'(\varepsilon) < \infty$, then it determines the next term of the asymptotics, namely, in (5.3), (5.4)

$$o(\varepsilon^d) = -r'(0)\varepsilon^{d+1} + o(\varepsilon^{d+1}). \tag{5.5}$$

Proof. We denote

$$\lambda(\varepsilon) := \varepsilon^{-d} q^*(\varepsilon). \tag{5.6}$$

One can rewrite then (4.5) as follows

$$\lambda(\varepsilon) - \frac{1}{\varkappa^-}\int_{\mathbb{R}^d} \frac{\widehat{a}^+(\xi) - \varepsilon^d\lambda(\varepsilon)\widehat{a}^-(\xi)}{\varkappa^+ - \widehat{a}^+(\xi) + \varepsilon^d\lambda(\varepsilon)\widehat{a}^-(\xi)} \widehat{a}^-(\xi)\, d\xi + r(\varepsilon) = 0. \tag{5.7}$$

Step 1. Note that, since $d \geq 3$, we have $|I| < \infty$, by the arguments above. We set

$$\lambda_3 := \frac{|I|}{\varkappa^-} \in (0,\infty),$$

Let $\delta \in \bigl(0, \min\{\lambda_3, 1\}\bigr)$ be such that, cf. (4.1),

$$(\lambda_3 + \delta)\delta^d < \frac{\varkappa^+ - m}{\varkappa^-}, \tag{5.8}$$

and let also $r(\varepsilon)$, given by (5.2), be continuous on $(0,\delta)$. We set then

$$r(0) := 0 = \lim_{\varepsilon\to 0+} r(\varepsilon), \qquad r(-\varepsilon) := r(\varepsilon), \quad \varepsilon \in (0,\delta). \tag{5.9}$$

For

$$(\lambda,\varepsilon) \in E_\delta := (\lambda_3 - \delta, \lambda_3 + \delta) \times (-\delta,\delta),$$

we consider the function

$$F(\lambda,\varepsilon) := \frac{1}{\varkappa^-}\int_{\mathbb{R}^d} \frac{\widehat{a}^+(\xi) - |\varepsilon|^d\lambda\,\widehat{a}^-(\xi)}{\varkappa^+ - \widehat{a}^+(\xi) + |\varepsilon|^d\lambda\,\widehat{a}^-(\xi)} \widehat{a}^-(\xi)\, d\xi - \lambda\,\mathrm{sgn}(I) - r(\varepsilon).$$

Henceforth $\operatorname{sgn}(I) = 1$ for $I > 0$ and $\operatorname{sgn}(I) = -1$ for $I < 0$.

For $(\lambda, \varepsilon) \in E_\delta$ and δ as in (5.8), we have that $a^+(x) - \lambda|\varepsilon|^d a^-(x) \geq 0$, $x \in \mathbb{R}^d$. Hence one can apply Proposition 4.2 for a (new) function $m_{\mathrm{cr}}(\varepsilon)$ such that $|\varepsilon|^d\lambda = \frac{\varkappa^+ - m_{\mathrm{cr}}(\varepsilon)}{\varkappa^-}$. It yields then

$$\lim_{\varepsilon\to 0} F(\lambda,\varepsilon) = F(\lambda, 0) = \frac{1}{\varkappa^-} I - \lambda \operatorname{sgn}(I), \tag{5.10}$$

and since $|I| \operatorname{sgn}(I) = I$, we get

$$F(\lambda_3, 0) = 0. \tag{5.11}$$

Step 2. By (5.10) and the dominated convergence theorem, F is continuous on E_δ, for small enough $\delta > 0$. Prove that $\frac{\partial}{\partial\lambda}F(\lambda,\varepsilon)$ is continuously differentiable on E_δ, for small enough $\delta > 0$. We have

$$\frac{\partial}{\partial\lambda}F(\lambda,\varepsilon) = -\frac{1}{\varkappa^-}\int_{\mathbb{R}^d} \frac{|\varepsilon|^d\, \widehat{a}^-(\xi)}{\bigl(\varkappa^+ - \widehat{a}^+(\xi) + |\varepsilon|^d\lambda\, \widehat{a}^-(\xi)\bigr)^2} \widehat{a}^-(\xi)\, d\xi - \operatorname{sgn}(I), \tag{5.12}$$

that is continuous for $(\lambda, \varepsilon) \in E_\delta$, $\varepsilon \neq 0$.

For $\varepsilon = 0$, $\lambda \in (\lambda_3 - \delta, \lambda_3 + \delta)$, we have by (5.10),

$$\frac{\partial}{\partial\lambda}F(\lambda, 0) = \lim_{h\to 0}\frac{F(\lambda + h, 0) - F(\lambda, 0)}{h} = -\operatorname{sgn}(I) \neq 0. \tag{5.13}$$

By (4.11), the denominator in (5.12) behaves as $|\xi|^4$ near the origin that is integrable for $d \geq 5$ only. In the latter case, using again (4.11) and the dominated convergence theorem, we get from (5.12) that

$$\lim_{\varepsilon\to 0}\frac{\partial}{\partial\lambda}F(\lambda,\varepsilon) = -\operatorname{sgn}(I), \tag{5.14}$$

hence, by (5.13), $\frac{\partial}{\partial\lambda}F$ is continuous at $(\lambda, 0)$ for all $\lambda \in (\lambda_3 - \delta, \lambda_3 + \delta)$.

Let now $d = 3, 4$. Find $\lim_{\varepsilon\to 0}\frac{\partial}{\partial\lambda}F(\lambda,\varepsilon)$. By the same arguments as for getting (4.14), we obtain that

$$\int_{\mathbb{R}^d\setminus\Delta_\delta} \frac{\bigl(\widehat{a}^-(\xi)\bigr)^2}{\bigl(\varkappa^+ - \widehat{a}^+(\xi) + |\varepsilon|^d\lambda\, \widehat{a}^-(\xi)\bigr)^2}\, d\xi < \infty,$$

for any neighbourhood Δ_δ of the origin, $0 \in \Delta_\delta \subset \mathbb{R}^d$, of a positive Lebesgue measure. Therefore, for any such Δ_δ with small enough $\delta > 0$,

$$\lim_{\varepsilon\to 0}\frac{\partial}{\partial\lambda}F(\lambda,\varepsilon) = -\frac{1}{\varkappa^-}\lim_{\varepsilon\to 0}|\varepsilon|^d h(\varepsilon,\delta) - \operatorname{sgn}(I),$$

where

$$h(\varepsilon,\delta) := \int_{\Delta_\delta} \frac{\bigl(\widehat{a}^-(\xi)\bigr)^2}{\bigl(\varkappa^+ - \widehat{a}^+(\xi) + |\varepsilon|^d\lambda\, \widehat{a}^-(\xi)\bigr)^2}\, d\xi.$$

We define now

$$\Delta_\delta := \{\xi \in \mathbb{R}^d : |B^+\xi| \leq \delta\}, \tag{5.15}$$

that is just an image of the ball $\{|\xi| \leq \delta\}$ under the mapping generated by a Hermitian positive definite matrix $D := (B^+)^{-1}$. Note that $\det(D) > 0$. Since

D generates a bounded continuous linear mapping on $\mathbb{R}^d$, Δ_δ is a bounded neighbourhood of the origin. Then

$$h(\varepsilon,\delta)=\int_{\{|\xi|\le\delta\}}\frac{\bigl(\widehat{a}^-(D\xi)\bigr)^2}{\bigl(\varkappa^+-\widehat{a}^+(D\xi)+|\varepsilon|^d\lambda\,\widehat{a}^-(D\xi)\bigr)^2}\,\det(D)d\xi\ge 0.$$

The inequality (4.10), applied for $B=D$, implies that $o(|D\xi|^2)=o(|\xi|^2)$ for $|\xi|\to 0$. Then, by (4.8), for small enough $\delta>0$ and $|\xi|\le\delta$, there exist $c_1,c_2>0$

$$\begin{aligned}c_1|\xi|^2\le\varkappa^+-\widehat{a}^+(D\xi)=2\pi^2|\xi|^2+o(|\xi|^2)\le c_2|\xi|^2,\\ \frac{\varkappa^-}{2}\le\widehat{a}^-(D\xi)=\varkappa^-+o(1)\le\varkappa^-.\end{aligned}\tag{5.16}$$

Then, there exist $C_1,C_2,C_3,C_4>0$, such that

$$\begin{aligned}h(\varepsilon,\delta)&\le\int_{\{|\xi|\le\delta\}}\frac{C_1}{\bigl(|\xi|^2+|\varepsilon|^d\lambda C_2\bigr)^2}\,d\xi\\ &\le C_3\int_0^\delta\frac{r^{d-1}}{\bigl(r^2+|\varepsilon|^d\lambda C_2\bigr)^2}\,dr\le C_3\delta^{d-3}\int_0^\delta\frac{r^2}{\bigl(r^2+|\varepsilon|^d\lambda C_2\bigr)^2}\,dr\\ &=C_4\Bigl(\frac{1}{\sqrt{|\varepsilon|^d\lambda C_2}}\arctan\Bigl(\frac{\delta}{\sqrt{|\varepsilon|^d\lambda C_2}}\Bigr)-\frac{\delta}{(|\varepsilon|^d\lambda C_2+\delta^2)}\Bigr).\end{aligned}$$

As a result, $|\varepsilon|^d h(\varepsilon,\delta)\le C_5|\varepsilon|^{\frac{d}{2}}$ for some $C_5>0$, and hence (5.14) holds. Therefore, $\frac{\partial}{\partial\lambda}F(\lambda,\varepsilon)$ is continuous at $(\lambda,0)$ as well.

Step 3. As a result, $F(\lambda,\varepsilon)$ is continuous on E_δ and continuously differentiable in λ for small enough $\delta>0$. Since also (5.11) holds, we conclude, by the implicit function theorem, that there exists (possibly smaller) $\delta>0$ and a unique function $\lambda=\lambda(\varepsilon)$, $\varepsilon\in(-\delta,\delta)$, such that $\lambda(0)=\lambda_3$ and

$$F(\lambda(\varepsilon),\varepsilon)=0,\qquad \varepsilon\in(-\delta,\delta);\tag{5.17}$$

moreover, $\lambda(\varepsilon)$ is continuous in $\varepsilon\in(-\delta,\delta)$. The latter implies

$$\lambda(\varepsilon)=\lambda_3+o(1),\quad \varepsilon\to 0.$$

Since (5.17) implies (5.7), we get that

$$q^*(\varepsilon)=\frac{|I|}{\varkappa^-}\varepsilon^d+o(\varepsilon^d).$$

In particular, (4.6) holds, and then, by (4.12), we get from (4.5) that $|I|=I$, i.e. $I>0$. Thus, one has (5.3) and, by (4.1), we get also (5.4).

Step 4. Assume now, additionally, that $r(\varepsilon)$ is continuously differentiable for $\varepsilon\in(0,\delta)$ and that $r'(0):=\lim_{\varepsilon\to0+}r'(\varepsilon)<\infty$. Then (5.9) extends this to $\varepsilon\in(-\delta,\delta)$. We have then that

$$\frac{\partial}{\partial\varepsilon}F(\lambda,\varepsilon)=-\frac{d}{\varkappa^-}\int_{\mathbb{R}^d}\frac{\varepsilon|\varepsilon|^{d-2}\,\lambda\,\widehat{a}^-(\xi)}{\bigl(\varkappa^+-\widehat{a}^+(\xi)+|\varepsilon|^d\lambda\,\widehat{a}^-(\xi)\bigr)^2}\widehat{a}^-(\xi)\,d\xi-r'(\varepsilon),$$

is continuous for $(\lambda,\varepsilon)\in E_\delta$, $\varepsilon\ne0$. The same arguments as above show that, for $d\ge5$, $\frac{\partial}{\partial\varepsilon}F$ is continuous at $(\lambda,0)$, $\lambda\in(\lambda_3-\delta,\lambda_3+\delta)$ with

$$\frac{\partial}{\partial\varepsilon}F(\lambda,0)=-r'(0).$$

For $d=3,4$, we have, by the same arguments as the above,

$$\lim_{\varepsilon\to 0}\frac{\partial}{\partial\varepsilon}F(\lambda,\varepsilon)=C_6\lim_{\varepsilon\to 0}\varepsilon|\varepsilon|^{d-2}O\bigl(|\varepsilon|^{-\frac{d}{2}}\bigr)-r'(0)=-r'(0). \tag{5.18}$$

as $d-1>\frac{d}{2}$ for $d\geq 3$. Next,

$$\frac{\partial}{\partial\varepsilon}F(\lambda,0)=\lim_{\varepsilon\to 0}\frac{F(\lambda,\varepsilon)-F(\lambda,0)}{\varepsilon}=\frac{1}{\varkappa^-}\lim_{\varepsilon\to 0}f(\lambda,\varepsilon)-r'(0),$$

where

$$\begin{aligned}f(\lambda,\varepsilon):&=\frac{1}{\varepsilon}\int_{\mathbb{R}^d}\biggl(\frac{\widehat{a}^+(\xi)-|\varepsilon|^d\lambda\,\widehat{a}^-(\xi)}{\varkappa^+-\widehat{a}^+(\xi)+|\varepsilon|^d\lambda\,\widehat{a}^-(\xi)}-\frac{\widehat{a}^+(\xi)}{\varkappa^+-\widehat{a}^+(\xi)}\biggr)\widehat{a}^-(\xi)\,d\xi\\&=-\frac{\varkappa^+\lambda|\varepsilon|^d}{\varepsilon}\int_{\mathbb{R}^d}b(\lambda,\varepsilon,\xi)\,d\xi,\end{aligned}$$

and

$$b(\lambda,\varepsilon,\xi):=\frac{\widehat{a}^-(\xi)}{\bigl(\varkappa^+-\widehat{a}^+(\xi)+|\varepsilon|^d\lambda\,\widehat{a}^-(\xi)\bigr)\bigl(\varkappa^+-\widehat{a}^+(\xi)\bigr)}.$$

Since $d>1$, by re-choosing $\delta>0$, we get, similarly to the arguments above, that

$$\lim_{\varepsilon\to 0}f(\lambda,\varepsilon)=-\varkappa^+\lambda\lim_{\varepsilon\to 0}\frac{|\varepsilon|^d}{\varepsilon}\int_{\Delta_\delta}b(\lambda,\varepsilon,\xi)\,d\xi,$$

where Δ_δ is given by (5.15). By the change of variables and (5.16), we have, for small enough $\delta>0$, and some constants $c_1,c_2,c_3,c_4>0$,

$$\begin{aligned}&\left|\frac{|\varepsilon|^d}{\varepsilon}\int_{\Delta_\delta}b(\lambda,\varepsilon,\xi)\,d\xi\right|\\&\leq|\varepsilon|^{d-1}\varkappa^-\det(D)\int_{\{|\xi|\leq d\}}\frac{1}{\bigl(c_1|\xi|^2+\frac{\varkappa^-}{2}|\varepsilon|^d\lambda\bigr)\bigl(c_1|\xi|^2\bigr)}\,d\xi\\&\leq|\varepsilon|^{d-1}c_2\int_{\{|\xi|\leq d\}}\frac{1}{\bigl(|\xi|^2+c_3\lambda|\varepsilon|^d\bigr)|\xi|^2}\,d\xi=|\varepsilon|^{d-1}c_4\int_0^\delta\frac{r^{d-1}}{\bigl(r^2+c_3\lambda|\varepsilon|^d\bigr)r^2}\,d\xi\\&\leq|\varepsilon|^{d-1}c_4\delta^{d-3}\int_0^\delta\frac{1}{\bigl(r^2+c_3\lambda|\varepsilon|^d\bigr)}\,d\xi\\&=c_4\delta^{d-3}|\varepsilon|^{d-1}\frac{1}{\sqrt{c_3\lambda|\varepsilon|^d}}\arctan\frac{\delta}{\sqrt{c_3\lambda|\varepsilon|^d}}\to 0,\qquad\varepsilon\to 0.\end{aligned}$$

Therefore, for $d=3,4$, we also have

$$\frac{\partial}{\partial\varepsilon}F(\lambda,0)=-r'(0)$$

and hence $\frac{\partial}{\partial\varepsilon}F$ is continuous at $(\lambda,0)$ for $\lambda\in(\lambda_3-\delta,\lambda_3+\delta)$.

As a result, both partial derivatives $\frac{\partial}{\partial\lambda}F$ and $\frac{\partial}{\partial\varepsilon}F$ are continuous on E_δ, hence F is continuously differentiable on E_δ.

Then, the implicit function theorem ensures that the unique function $\lambda=\lambda(\varepsilon)$, $\varepsilon\in(-\delta,\delta)$, such that $\lambda(0)=\lambda_3$ and (5.17) holds, is continuously differentiable in ε. Differentiating (5.17) in ε, we get

$$\lambda'(\varepsilon)\frac{\partial}{\partial\lambda}F(\lambda(\varepsilon),\varepsilon)+\frac{\partial}{\partial\varepsilon}F(\lambda(\varepsilon),\varepsilon)=0,$$

and hence, by (5.13), (5.18)

$$\lambda'(0) = \frac{\partial}{\partial\varepsilon} F(\lambda(\varepsilon), \varepsilon)\Big|_{\varepsilon=0} = -r'(0).$$

As a result,

$$\lambda(\varepsilon) = \lambda_3 - r'(0)\varepsilon + o(\varepsilon),$$

that, by (5.6) implies (5.5). □

Remark 5.2. Note that $\lambda_3 = \frac{I}{\varkappa^+}$, where I is given by (5.1), does not actually depend on $\varkappa^+$ nor $\varkappa^-$.

6 Asymptotics of the critical mortality: $d = 2$

Let $d = 2$ and (4.5) hold. We start with some heuristic arguments. Let, additionally, (4.6) hold; then, by Proposition 4.2, $\lim_{\varepsilon\to 0} p^*(\varepsilon) = -\infty$. By (4.5), (4.6), we can expect then that

$$q^*(\varepsilon) \approx \varepsilon^2 l(\varepsilon), \tag{6.1}$$

where $l(\varepsilon) \sim p^*(\varepsilon)$, $\varepsilon \to 0$, and therefore,

$$\lim_{\varepsilon\to 0} l(\varepsilon) = \infty, \qquad \lim_{\varepsilon\to 0} \varepsilon^2 l(\varepsilon) = 0. \tag{6.2}$$

Then, by (4.3), the anzatz (6.1) implies

$$\begin{aligned} l(\varepsilon) &\sim \frac{1}{\varkappa^-} \int_{\mathbb{R}^2} \frac{\widehat{a}^+(\xi) - \varepsilon^2 l(\varepsilon)\,\widehat{a}^-(\xi)}{\varkappa^+ - \widehat{a}^+(\xi) + \varepsilon^2 l(\varepsilon)\,\widehat{a}^-(\xi)} \widehat{a}^-(\xi)\,d\xi \\ &= \frac{\varkappa^+}{\varkappa^-} \int_{\mathbb{R}^2} \frac{1}{\varkappa^+ - \big(\widehat{a}^+(\xi) - \varepsilon^2 l(\varepsilon)\widehat{a}^-(\xi)\big)} \widehat{a}^-(\xi)\,d\xi - \frac{1}{\varkappa^-} \int_{\mathbb{R}^2} \widehat{a}^-(\xi)\,d\xi. \end{aligned}$$

By the same arguments as above, the singularity of the latter expression as $\varepsilon \to 0$ is fully determined by the integral

$$\sigma(\delta, \varepsilon) := \frac{\varkappa^+}{\varkappa^-} \int_{\Delta_\delta} \frac{1}{\varkappa^+ - \big(\widehat{a}^+(\xi) - \varepsilon^2 l(\varepsilon)\widehat{a}^-(\xi)\big)} \widehat{a}^-(\xi)\,d\xi$$

for small enough $\delta > 0$, where Δ_δ is given by (5.15). By (6.10) and the change of variables as above,

$$\sigma(\delta, \varepsilon) \sim \text{const} \cdot \int_{|\xi|\leq\delta} \frac{1}{|\xi|^2 + \varkappa^-\varepsilon^2 l(\varepsilon)}\,d\xi = \text{const} \cdot \int_0^\delta \frac{r}{r^2 + \varkappa^-\varepsilon^2 l(\varepsilon)}\,dr.$$

Integrating, we conclude that, heuristically, for some $c_1, c_2 > 0$,

$$l(\varepsilon) \sim c_1 \log\left(1 + \frac{c_2}{\varepsilon^2 l(\varepsilon)}\right) \sim c_1 \log \frac{c_2}{\varepsilon^2 l(\varepsilon)},$$

by (6.2), i.e. $\frac{l(\varepsilon)}{c_1} e^{\frac{l(\varepsilon)}{c_1}} \sim \frac{c_2}{c_1\varepsilon^2}$. Therefore,

$$l(\varepsilon) \sim c_1 W\left(\frac{c_2}{c_1\varepsilon^2}\right),$$

where $W(z)$, $z > 0$, is the unique solution to the equation $ye^y = z$, $y > 0$, (the principal branch of) of the so-called Lambert W function. It is well-known that

$$W(z) = \log z - \log\log z + o(1), \qquad z \to +\infty.$$

Therefore,

$$W\left(\frac{c_2}{c_1\varepsilon^2}\right) = -2\log\varepsilon - \log(-\log\varepsilon) + O(1),$$

in other words, two leading terms of the asymptotics of $q^*(\varepsilon) \approx c'\varepsilon^2 W(c\varepsilon^{-2})$ depend on c' but not on c. This gives a hint to define $\lambda(\varepsilon)$ in the proof of the following theorem.

Theorem 6.1. *Let $d = 2$ and (**A1**)–(**A4**) hold. Let (4.5) hold with $o(\varepsilon^2)$ such that the function*

$$r(\varepsilon) := \frac{o(\varepsilon^2)}{\varepsilon^2 W(\varepsilon^{-2})} \tag{6.3}$$

is continuous for small $\varepsilon > 0$. Then

$$q^*(\varepsilon) = \lambda_2\varepsilon^2 W(\varepsilon^{-2}) + o\bigl(\varepsilon^2 W(\varepsilon^{-2})\bigr), \tag{6.4}$$

where

$$\lambda_2 := \frac{\varkappa^+}{2\pi\sqrt{a_{11}a_{22} - a_{12}^2}}, \tag{6.5}$$

and a_{ij}, $1 \le i, j \le 2$, are given by (4.7).

Remark 6.2. Note that λ_2 does not actually depend on $\varkappa^+$ (cf. Remark 5.2).

Proof. We define $\lambda(\varepsilon) > 0$, $\varepsilon \in (0, 1)$, through the equality

$$q^*(\varepsilon) = \lambda(\varepsilon)\varepsilon^2 W(\varepsilon^{-2}). \tag{6.6}$$

One can rewrite then (4.5) as follows

$$\lambda(\varepsilon) - \frac{1}{\varkappa^- W(\varepsilon^{-2})}\int_{\mathbb{R}^2} \frac{\widehat{a}^+(\xi) - \varepsilon^2 W(\varepsilon^{-2})\lambda(\varepsilon)\,\widehat{a}^-(\xi)}{\varkappa^+ - \widehat{a}^+(\xi) + \varepsilon^2 W(\varepsilon^{-2})\lambda(\varepsilon)\,\widehat{a}^-(\xi))}\widehat{a}^-(\xi)\,d\xi + r(\varepsilon) = 0,$$

where $r(\varepsilon) \to 0$, $\varepsilon \to 0$, is given by (6.3)

Let $\delta \in \bigl(0, \min\{\lambda_2, 1\}\bigr)$ be such that, cf. (4.1), (5.8),

$$(\lambda_2 + \delta)\delta^2 W(\delta^{-2}) < \frac{\varkappa^+ - m}{\varkappa^-}, \tag{6.7}$$

and let also $r(\varepsilon)$ be continuous on $(0, \delta)$. Note that the function in the left hand side of (6.7) is increasing in $\delta > 0$. We define then r on $(-\delta, 0]$ as in (5.9). For $\lambda \in (\lambda_2 - \delta, \lambda_2 + \delta)$, $\varepsilon \in (-\delta, \delta)$, $\varepsilon \ne 0$, we consider the function

$$\begin{aligned} F(\lambda, \varepsilon) :&= \frac{1}{\varkappa^- W(\varepsilon^{-2})}\int_{\mathbb{R}^2} \frac{\widehat{a}^+(\xi) - \lambda\varepsilon^2 W(\varepsilon^{-2})\,\widehat{a}^-(\xi)}{\varkappa^+ - \widehat{a}^+(\xi) + \lambda\varepsilon^2 W(\varepsilon^{-2})\,\widehat{a}^-(\xi)}\widehat{a}^-(\xi)\,d\xi - \lambda - r(\varepsilon) \\ &= \frac{\varkappa^+}{\varkappa^- W(\varepsilon^{-2})}\int_{\mathbb{R}^2} \frac{1}{\varkappa^+ - \widehat{a}^+(\xi) + \lambda\varepsilon^2 W(\varepsilon^{-2})\,\widehat{a}^-(\xi)}\widehat{a}^-(\xi)\,d\xi \\ &\quad - \frac{1}{\varkappa^- W(\varepsilon^{-2})}\int_{\mathbb{R}^2} \widehat{a}^-(\xi)\,d\xi - \lambda - r(\varepsilon). \end{aligned} \tag{6.8}$$

Clearly, F is continuous for $\lambda \in (\lambda_2 - \delta, \lambda_2 + \delta)$, $\varepsilon \in (-\delta, \delta)$, $\varepsilon \neq 0$.

By the same arguments as above,

$$\lim_{\varepsilon \to 0} F(\lambda, \varepsilon) = -\lambda + \lim_{\varepsilon \to 0} \frac{1}{W(\varepsilon^{-2})} \sigma(\varepsilon, \delta, \lambda), \tag{6.9}$$

where, for a small enough $\delta > 0$,

$$\sigma(\varepsilon, \delta, \lambda) := \frac{\varkappa^+}{\varkappa^-} \int_{\Delta_\delta} \frac{1}{\varkappa^+ - \widehat{a}^+(\xi) + \lambda \varepsilon^2 W(\varepsilon^{-2})\, \widehat{a}^-(\xi)} \widehat{a}^-(\xi)\, d\xi,$$

and Δ_δ is given by (5.15).

Recall that the inequality (4.10), applied for $B = D$, implies that $o(|D\xi|^2) = o(|\xi|^2)$ for $|\xi| \to 0$. Then, cf. (5.16), by (4.8), for any $\rho \in (0,1)$ there exists $\delta_\rho > 0$ small enough such that, for $|\xi| \leq \delta < \delta_\rho$,

$$\begin{gathered} 2\pi^2(1-\rho)|\xi|^2 \leq \varkappa^+ - \widehat{a}^+(D\xi) = 2\pi^2|\xi|^2 + o(|\xi|^2) \leq 2\pi^2(1+\rho)|\xi|^2, \\ (1-\rho)\varkappa^- \leq \widehat{a}^-(D\xi) = \varkappa^- + o(1) \leq \varkappa^-. \end{gathered} \tag{6.10}$$

By change of variables and (6.10), for each $\rho \in (0,1)$, there exists $\delta_\rho < \lambda_2$, such that, for all $\delta \in (0, \delta_\rho)$,

$$\frac{1}{1-\rho} \tau(\varepsilon, \delta, \lambda) \geq \sigma(\varepsilon, \delta, \lambda) \geq \frac{1-\rho}{1+\rho} \tau(\varepsilon, \delta, \lambda), \tag{6.11}$$

where, for $D = (B^+)^{-1}$,

$$\begin{aligned} \tau(\varepsilon, \delta, \lambda) :&= \varkappa^+ \det(D) \int_{|\xi| \leq \delta} \frac{1}{2\pi^2|\xi|^2 + \varkappa^- \lambda \varepsilon^2 W(\varepsilon^{-2})}\, d\xi \\ &= 2\varkappa^+ \pi \det(D) \int_0^\delta \frac{1}{2\pi^2 r^2 + \varkappa^- \lambda \varepsilon^2 W(\varepsilon^{-2})}\, r\, dr \\ &= \frac{\varkappa^+}{2\pi} \det(D) \log\biggl(1 + \frac{2\pi^2\delta^2}{\varkappa^- \lambda \varepsilon^2 W(\varepsilon^{-2})}\biggr). \end{aligned}$$

Note that $W(\varepsilon^{-2}) e^{W(\varepsilon^{-2})} = \varepsilon^{-2}$, i.e. $\varepsilon^2 W(\varepsilon^{-2}) = e^{-W(\varepsilon^{-2})}$. Set $R := W(\varepsilon^{-2}) \to +\infty$, $\varepsilon \to 0$. Then

$$\begin{aligned} &\frac{1}{W(\varepsilon^{-2})} \tau(\varepsilon, \delta, \lambda) = \frac{\varkappa^+ \det(D)}{2\pi R} \log\biggl(1 + \frac{2\pi^2\delta^2}{\varkappa^- \lambda} e^R\biggr) \\ &= \frac{\varkappa^+ \det(D)}{2\pi R} \biggl(\log \frac{2\pi^2\delta^2}{\varkappa^- \lambda} + R + \log\biggl(1 + \frac{\varkappa^- \lambda}{2\pi^2\delta^2} e^{-R}\biggr)\biggr) \to \frac{\varkappa^+ \det(D)}{2\pi}, \end{aligned}$$

as $R \to +\infty$, i.e. as $\varepsilon \to 0$. Combining this with (6.9) and (6.11), we conclude that, for each $\rho \in (0,1)$,

$$\lim_{\varepsilon \to 0} F(\lambda, \varepsilon) + \lambda \in \biggl(\frac{\varkappa^+ \det(D)}{2\pi} \frac{1-\rho}{1+\rho}, \frac{\varkappa^+ \det(D)}{2\pi} \frac{1}{1-\rho}\biggr).$$

By sending ρ to 0, we get

$$\lim_{\varepsilon \to 0} F(\lambda, \varepsilon) = -\lambda + \frac{\varkappa^+ \det(D)}{2\pi} = -\lambda + \frac{\varkappa^+}{2\pi \det(B^+)} = -\lambda + \lambda_2,$$

where λ_2 is given by (6.5), since $(B^+)^2 = A^+$ implies $\det(B^+) = \sqrt{\det(A^+)}$.

Therefore, if we set

$$F(\lambda, 0) := \lambda_2 - \lambda, \qquad \lambda \in (\lambda_2 - \delta, \lambda_2 + \delta),$$

then $F(\lambda, \varepsilon)$ becomes a continuous function on

$$E_\delta := (\lambda_2 - \delta, \lambda_2 + \delta) \times (-\delta, \delta)$$

with a small enough $\delta \in (0, \lambda_2)$. Moreover, $F(\lambda_2, 0) = 0$. Next, since

$$\frac{F(\lambda + h, 0) - F(\lambda, 0)}{h} = -1, \qquad \lambda, \lambda + h \in (\lambda_2 - \delta, \lambda_2 + \delta),$$

we have

$$\frac{\partial}{\partial \lambda} F(\lambda, 0) = -1 \neq 0, \qquad \lambda \in (\lambda_2 - \delta, \lambda_2 + \delta).$$

Next, for $(\lambda, \varepsilon) \in E_\delta$, $\varepsilon \neq 0$, we have, by (6.8),

$$\begin{aligned}
\frac{\partial}{\partial \lambda} F(\lambda, \varepsilon) &= -1 - \frac{1}{\varkappa^- W(\varepsilon^{-2})} \int_{\mathbb{R}^2} \frac{\varepsilon^2 W(\varepsilon^{-2})\, \widehat{a}^-(\xi)}{\bigl(\varkappa^+ - \widehat{a}^+(\xi) + \lambda \varepsilon^2 W(\varepsilon^{-2})\, \widehat{a}^-(\xi)\bigr)^2} \widehat{a}^-(\xi)\, d\xi \\
&= -1 - \frac{\varepsilon^2}{\varkappa^-} \int_{\mathbb{R}^2} \frac{\bigl(\widehat{a}^-(\xi)\bigr)^2}{\bigl(\varkappa^+ - \widehat{a}^+(\xi) + \lambda \varepsilon^2 W(\varepsilon^{-2})\, \widehat{a}^-(\xi)\bigr)^2}\, d\xi.
\end{aligned}$$

By the same arguments as above,

$$\lim_{\varepsilon \to 0} \frac{\partial}{\partial \lambda} F(\lambda, \varepsilon) = -1 - \frac{1}{\varkappa^-} \lim_{\varepsilon \to 0} \varepsilon^2 h(\varepsilon, \delta),$$

where

$$h(\varepsilon, \delta) = \int_{\Delta_\delta} \frac{\bigl(\widehat{a}^-(\xi)\bigr)^2}{\bigl(|B^+\xi|^2 + o(|\xi|^2) + \lambda \varepsilon^2 W(\varepsilon^{-2})(\varkappa^- + o(1))\bigr)^2}\, d\xi,$$

where Δ_δ is given by (5.15). By change and variables and (5.16), we get that, for some $C_1, C_2, C_3 > 0$ and for small enough $\delta > 0$,

$$\begin{aligned}
0 < h(\varepsilon, \delta) &\leq \int_{\{|\xi| \leq \delta\}} \frac{C_1}{\bigl(|\xi|^2 + \lambda \varepsilon^2 W(\varepsilon^{-2}) C_2\bigr)^2}\, d\xi \\
&\leq C_3 \int_0^\delta \frac{r}{\bigl(r^2 + \lambda \varepsilon^2 W(\varepsilon^{-2}) C_2\bigr)^2}\, dr \\
&= \frac{C_3}{2C_2} \frac{\delta^2}{\lambda \varepsilon^2 W(\varepsilon^{-2}) \bigl(\delta^2 + \lambda \varepsilon^2 W(\varepsilon^{-2}) C_2\bigr)},
\end{aligned}$$

and hence $\varepsilon^2 h(\varepsilon, \delta) \to 0$, $\varepsilon \to 0$, that yields

$$\lim_{\varepsilon \to 0} \frac{\partial}{\partial \lambda} F(\lambda, \varepsilon) = -1,$$

and therefore, $\frac{\partial}{\partial \lambda} F$ is continuous on E_δ.

Again, the implicit function theorem states that there exists a unique function $\lambda = \lambda(\varepsilon)$, $\varepsilon \in (-\delta, \delta)$ (with, possibly, smaller δ), such that $\lambda(0) = \lambda_2$ and

$$F(\lambda(\varepsilon), \varepsilon) = 0, \qquad \varepsilon \in (-\delta, \delta);$$

moreover, $\lambda(\varepsilon)$ is *continuous* in $\varepsilon \in (-\delta, \delta)$. Therefore, $\lambda(\varepsilon) = \lambda_2 + o(1)$, $\varepsilon \to 0$; hence, from (6.6) and (4.1), we get (6.4). □

Corollary 6.3. *If function a^+ in Theorem 6.1 is radially symmetric, i.e. $a^+(x) = b^+(|x|)$ for some $b^+ : \mathbb{R} \to \mathbb{R}$, then $a_{12} = a_{21} = 0$ and*

$$a_{11} = a_{22} = \int_{\mathbb{R}^2} x_1^2 a^+(x)\,dx = \frac{1}{2}\int_{\mathbb{R}^2} |x|^2 a^+(x)\,dx,$$

so that

$$\lambda_2 = \frac{\varkappa^+}{\pi \int_{\mathbb{R}^2} |x|^2 a^+(x)\,dx}.$$

Remark 6.4. It can be checked that $F(\lambda, \varepsilon)$ defined by (6.8) is not continuously differentiable in ε at $\varepsilon = 0$ (even if we assume that r is); hence, in general, one can not expect that $\lambda(\varepsilon)$ is continuously differentiable at $\varepsilon = 0$. Hence, the question about the next term of the assymptotic in (6.4) remains open.

7 Asymptotics of the critical mortality: $d = 1$

Let $d = 1$ and (4.5) hold. We firstly proceed again heuristically. Similarly to the arguments at the beginning of Section 6, if (4.6) holds then we may expect, for $\varepsilon \to 0$,

$$q^*(\varepsilon) \approx \varepsilon l(\varepsilon), \tag{7.1}$$

where

$$l(\varepsilon) \sim p^*(\varepsilon) \to \infty, \qquad \varepsilon l(\varepsilon) \to 0.$$

Then, by (4.3), the ansatz (7.1) implies

$$\begin{aligned} l(\varepsilon) &\sim \frac{1}{\varkappa^-}\int_{\mathbb{R}} \frac{\widehat{a}^+(\xi) - \varepsilon l(\varepsilon)\,\widehat{a}^-(\xi)}{\varkappa^+ - \widehat{a}^+(\xi) + \varepsilon l(\varepsilon)\,\widehat{a}^-(\xi)}\widehat{a}^-(\xi)\,d\xi \\ &= \frac{1}{\varkappa^+\varkappa^-}\int_{\mathbb{R}} \widehat{a}^-(\xi)\,d\xi - \frac{1}{\varkappa^-}\int_{\mathbb{R}} \frac{1}{\varkappa^+ - \bigl(\widehat{a}^+(\xi) - \varepsilon l(\varepsilon)\widehat{a}^-(\xi)\bigr)}\widehat{a}^-(\xi)\,d\xi. \end{aligned}$$

By the same arguments as above, the singularity of the latter expression as $\varepsilon \to 0$ is fully determined by the integral

$$\begin{aligned} &\frac{1}{\varkappa^-}\int_{-\delta}^{\delta} \frac{1}{\varkappa^+ - \bigl(\widehat{a}^+(\xi) - \varepsilon l(\varepsilon)\widehat{a}^-(\xi)\bigr)}\widehat{a}^-(\xi)\,d\xi \\ &\sim c_1 \cdot \int_0^{\delta} \frac{1}{r^2 + c_2\varepsilon l(\varepsilon)}\,dr = \frac{c_3}{\sqrt{\varepsilon l(\varepsilon)}} \arctan\frac{\delta}{\sqrt{c_2\varepsilon l(\varepsilon)}}, \end{aligned}$$

for small enough $\delta > 0$ and some $c_1, c_2, c_3 > 0$; here we used (4.8). As a result, heuristically,

$$l(\varepsilon)\sqrt{\varepsilon l(\varepsilon)} \approx \text{const}\,,$$

and hence $l(\varepsilon) \approx \text{const}\,\varepsilon^{-\frac{1}{3}}$, $\varepsilon \to 0$. Again, it gives us a hint to define $\lambda(\varepsilon)$ in the proof of the following theorem.

Theorem 7.1. *Let $d = 1$ and (**A1**)–(**A4**) hold. Let (4.5) hold with $o(\varepsilon)$ such that the function*

$$r(\varepsilon) := \varepsilon^{-\frac{2}{3}} o(\varepsilon) \tag{7.2}$$

is continuous for small $\varepsilon > 0$*. Then*

$$q^*(\varepsilon) = \lambda_1 \varepsilon^{\frac{2}{3}} + o\bigl(\varepsilon^{\frac{2}{3}}\bigr), \tag{7.3}$$

where

$$\lambda_1 := \left(\frac{(\varkappa^+)^2}{2\varkappa^- \displaystyle\int_{\mathbb{R}} x^2 a^+(x)\,dx} \right)^{\frac{1}{3}}.$$

Remark 7.2. Note that, in contrast to the cases $d \geq 3$ and $d = 2$, cf. Remarks 5.2, 6.2, λ_1 depends effectively on (the ratio of) $\varkappa^+$ and $\varkappa^-$.

Proof. We set $\lambda(\varepsilon) := q^*(\varepsilon)\varepsilon^{-\frac{2}{3}}$ for $\varepsilon \in (0,1)$, and then rewrite (4.5) as follows

$$\lambda(\varepsilon) - \frac{\varepsilon^{\frac{1}{3}}}{\varkappa^-} \int_{\mathbb{R}} \frac{\widehat{a}^+(\xi) - \varepsilon^{\frac{2}{3}} \lambda(\varepsilon)\, \widehat{a}^-(\xi)}{\varkappa^+ - \widehat{a}^+(\xi) + \varepsilon^{\frac{2}{3}}\lambda(\varepsilon)\, \widehat{a}^-(\xi))} \widehat{a}^-(\xi)\, d\xi + r(\varepsilon) = 0,$$

where $r(\varepsilon) \to 0$, $\varepsilon \to 0$, is given by (7.2)

Let $\delta \in \bigl(0, \min\{\lambda_1, 1\}\bigr)$ be such that, cf. (4.1), (5.8), (6.7),

$$(\lambda_1 + \delta)\delta^{\frac{2}{3}} < \frac{\varkappa^+ - m}{\varkappa^-},$$

and let also $r(\varepsilon)$ be continuous on $(0, \delta)$. We define then r on $(-\delta, 0]$ as in (5.9). For $\lambda \in (\lambda_1 - \delta, \lambda_1 + \delta)$, $\varepsilon \in (-\delta, \delta)$, $\varepsilon \neq 0$, we consider the function

$$\begin{aligned} F(\lambda, \varepsilon) :&= \frac{\varepsilon^{\frac{1}{3}}}{\varkappa^-} \int_{\mathbb{R}^2} \frac{\widehat{a}^+(\xi) - \lambda \varepsilon^{\frac{2}{3}}\, \widehat{a}^-(\xi)}{\varkappa^+ - \widehat{a}^+(\xi) + \lambda \varepsilon^{\frac{2}{3}}\, \widehat{a}^-(\xi)} \widehat{a}^-(\xi)\, d\xi - \lambda - r(\varepsilon) \\ &= \frac{\varkappa^+ \varepsilon^{\frac{1}{3}}}{\varkappa^-} \int_{\mathbb{R}^2} \frac{1}{\varkappa^+ - \widehat{a}^+(\xi) + \lambda \varepsilon^{\frac{2}{3}}\, \widehat{a}^-(\xi)} \widehat{a}^-(\xi)\, d\xi \\ &\quad - \frac{\varepsilon^{\frac{1}{3}}}{\varkappa^-} \int_{\mathbb{R}^2} \widehat{a}^-(\xi)\, d\xi - \lambda - r(\varepsilon). \end{aligned} \tag{7.4}$$

Clearly, F is continuous for $\lambda \in (\lambda_1 - \delta, \lambda_1 + \delta)$, $\varepsilon \in (-\delta, \delta)$, $\varepsilon \neq 0$.

Let

$$B := \int_{\mathbb{R}} |x|^2 a^+(x)\, dx = 2 \int_0^\infty x^2 a^+(x)\, dx.$$

By (4.8) and the same arguments as in the proof of Theorem 6.1,

$$\begin{aligned} \lim_{\varepsilon \to 0} F(\lambda, \varepsilon) &= -\lambda + \frac{\varkappa^+}{\varkappa^-} \lim_{\varepsilon \to 0} \varepsilon^{\frac{1}{3}} \int_{-\delta}^{\delta} \frac{1}{\varkappa^+ - \widehat{a}^+(\xi) + \lambda \varepsilon^{\frac{2}{3}}\, \widehat{a}^-(\xi)} \widehat{a}^-(\xi)\, d\xi \\ &= -\lambda + 2\varkappa^+ \lim_{\varepsilon \to 0} \varepsilon^{\frac{1}{3}} \int_0^{\delta} \frac{1}{2\pi^2 B r^2 + \varkappa^- \lambda \varepsilon^{\frac{2}{3}}}\, dr \\ &= -\lambda + \frac{\sqrt{2}\varkappa^+}{\pi\sqrt{\lambda \varkappa^- B}} \lim_{\varepsilon \to 0} \arctan \frac{\sqrt{2B}\pi\delta}{\sqrt{\varkappa^- \lambda}\varepsilon^{\frac{1}{3}}} \\ &= -\lambda + \frac{\varkappa^+}{\sqrt{2\lambda \varkappa^- B}}. \end{aligned}$$

Therefore, if we set

$$F(\lambda,0):=-\lambda+\frac{\varkappa^+}{\sqrt{2\lambda\varkappa^-B}},\qquad \lambda\in(\lambda_1-\delta,\lambda_1+\delta),$$

then $F(\lambda,\varepsilon)$ becomes a continuous function on

$$E_\delta:=(\lambda_1-\delta,\lambda_1+\delta)\times(-\delta,\delta)$$

with a small enough $\delta\in(0,\lambda_1)$. Moreover, it is straightforward to check that

$$F(\lambda_1,0)=0.$$

For $\lambda\in(\lambda_1-\delta,\lambda_1+\delta)$, we have

$$\frac{\partial}{\partial\lambda}F(\lambda,0)=\lim_{h\to0}\frac{F(\lambda+h,0)-F(\lambda,0)}{h}=-1-\frac{\varkappa^+}{2\sqrt{2\varkappa^-B}}\lambda^{-\frac{3}{2}}<0,$$

and also

$$\frac{\partial}{\partial\lambda}F(\lambda_1,0)=-1-\frac{\varkappa^+}{2\sqrt{2\varkappa^-B}}\Big((\varkappa^+)^{\frac{2}{3}}(2\varkappa^-B)^{-\frac{1}{3}}\Big)^{-\frac{3}{2}}=-\frac{3}{2}\neq0.$$

Next, for $(\lambda,\varepsilon)\in E_\delta$, $\varepsilon\neq0$, we have, by (7.4),

$$\frac{\partial}{\partial\lambda}F(\lambda,\varepsilon)=-1-\frac{\varepsilon}{\varkappa^-}\int_{\mathbb{R}}\frac{\big(\widehat{a}^-(\xi)\big)^2}{\big(\varkappa^+-\widehat{a}^+(\xi)+\lambda\varepsilon^{\frac{2}{3}}\,\widehat{a}^-(\xi)\big)^2}\,d\xi.$$

By the same arguments as above,

$$\begin{aligned}\lim_{\varepsilon\to0}\frac{\partial}{\partial\lambda}F(\lambda,\varepsilon)&=-1-\frac{\varkappa^+}{\varkappa^-}\lim_{\varepsilon\to0}\varepsilon\int_{-\delta}^{\delta}\frac{\big(\widehat{a}^-(\xi)\big)^2}{\big(\varkappa^+-\widehat{a}^+(\xi)+\lambda\varepsilon^{\frac{2}{3}}\,\widehat{a}^-(\xi)\big)^2}\,d\xi\\&=-1-\varkappa^+\varkappa^-\lim_{\varepsilon\to0}\varepsilon\int_{-\delta}^{\delta}\frac{1}{\big(2\pi^2B|\xi|^2+\lambda\varkappa^-\varepsilon^{\frac{2}{3}}\big)^2}\,d\xi\end{aligned}$$

and by straightforward integration, one gets

$$\begin{aligned}&=-1-\varkappa^+\lim_{\varepsilon\to0}\frac{\delta\varepsilon^{\frac{1}{3}}}{2\pi^2B\delta^2\varkappa^-\lambda+\varepsilon^{\frac{2}{3}}(\varkappa^-)^2\lambda^2}\\&\quad-\varkappa^+\lim_{\varepsilon\to0}\frac{1}{\pi\sqrt{2B\varkappa^-}\lambda^{\frac{3}{2}}}\arctan\bigg(\sqrt{\frac{2B}{\varkappa^-\lambda}}\pi\delta\varepsilon^{-\frac{1}{3}}\bigg)\bigg)\\&=-1-\frac{\varkappa^+}{2\sqrt{2B\varkappa^-}}\lambda^{-\frac{3}{2}}=\frac{\partial}{\partial\lambda}F(\lambda,0).\end{aligned}$$

Therefore, $\frac{\partial}{\partial\lambda}F$ is continuous on E_δ.

Again, the implicit function theorem states that there exists a unique continuous function $\lambda=\lambda(\varepsilon)$ such that $\lambda(0)=\lambda_1$ and $F(\lambda(\varepsilon),\varepsilon)=0$, $\varepsilon\in(-\delta,\delta)$ $\varepsilon\in(-\delta,\delta)$ (with, possibly, smaller δ). Therefore, $\lambda(\varepsilon)=\lambda_1+o(1)$, $\varepsilon\to0$, that yields (7.3). □

Remark 7.3. Similarly to the case $d=2$, see Remark 6.4, function $F(\lambda,\varepsilon)$ defined by (7.4) is not continuously differentiable in ε at $\varepsilon=0$, hence the next term of the assymptotic in (7.3) remains an open problem.

Acknowledgements

I thank Otso Ovaskainen and Panu Somervuo for discussions that motivated the analyses presented in this paper.

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