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# European Options in a Nonlinear Incomplete Market Model with Default*

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**Abstract.** We study the superhedging prices and the associated superhedging strategies for European options in a nonlinear incomplete market model with default. The underlying market model consists of one risk-free asset and one risky asset, whose price may admit a jump at the default time. The portfolio processes follow nonlinear dynamics with a nonlinear driver $f$. By using a dynamic programming approach, we first provide a dual formulation of the seller's (superhedging) price for the European option as the supremum, over a suitable set of equivalent probability measures $Q \in \mathcal{Q}$, of the $f$-evaluation/expectation under $Q$ of the payoff. We also establish a characterization of the seller's (superhedging) price as the initial value of the minimal supersolution of a constrained backward stochastic differential equation with default. Moreover, we provide some properties of the terminal profit made by the seller, and some results related to replication and no-arbitrage issues. Our results rely on first establishing a nonlinear optional and a nonlinear predictable decomposition for processes which are $\mathcal{E}^f$-strong supermartingales under $Q$ for all $Q \in \mathcal{Q}$.

**Key words.** incomplete market, superhedging, nonlinear option pricing, constrained BSDE, control problem with $f$-expectation, nonlinear optional decomposition, pricing-hedging duality

**AMS subject classifications.** 93E20, 91B28, 60H10, 60H30

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## 1. Introduction.

We study the European option valuation problem in a *nonlinear incomplete* market model with default. This model allows us to take into account various market imperfections via the nonlinearity of the portfolio dynamics. These include, in particular, credit and funding costs, appearing in recent papers on nonlinear valuation in markets with default (see [1, 4, 5, 6]), as well as the impact of a large seller on the default probability (see [9]). The papers [10, 11] study the pricing of European options (as well as American and game options) in a nonlinear market model with default which is *complete*. In this case, the seller's (hedging) price of the European option with payoff $\eta$ and maturity $T$ is given by the nonlinear $f$-evaluation (expectation) of $\eta$, denoted by $\mathcal{E}^f(\eta)$, where $f$ is the nonlinear driver of the portfolio value process. The aim of the present paper is to study the superhedging evaluation of European options when the nonlinear default market is *incomplete*. The incompleteness question has been raised in [5, 6].

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We study a market model containing one risky asset whose price dynamics are driven by a one-dimensional Brownian motion and a compensated default martingale. Our market is incomplete, in the sense that not every contingent claim can be replicated by a portfolio. In this framework, we are interested in the problem of pricing and (super)hedging of European options, from the point of view of the seller and of the buyer. Since contingent claims are not necessarily replicable, we are led to defining the *seller's (superhedging) price* of the option. The seller's (superhedging) price at time 0, denoted by $\mathbf{v}_0$, is the minimal initial capital which allows her to build a (nonlinear) portfolio whose terminal value dominates the payoff $\eta$ of the option.

We provide a dual formulation of this price as the supremum, over a suitable set of equivalent probability measures $Q \in \mathcal{Q}$, of the $(f, Q)$-evaluation,[1] denoted by $\mathcal{E}^f_Q$, of the payoff $\eta$, that is,

$$\mathbf{v}_0 = \sup_{Q \in \mathcal{Q}} \mathcal{E}^f_{Q,0,T}(\eta). \tag{1.1}$$

More precisely, the set $\mathcal{Q}$ is the set of *$f$-martingale probability measures*, defined as the equivalent probability measures such that the wealth processes are *$(f, Q)$-martingales*;[2] it can be shown that the set $\mathcal{Q}$ is related to the set of the so-called martingale probability measures. In the case when $f$ is linear, our result reduces to the well-known dual representation via the set of martingale probability measures from the literature on linear incomplete markets (cf. [12, 13]).

We also show that the supremum in (1.1) is attained if and only if the option with payoff $\eta$ is replicable and, in this case, $\mathbf{v}_0 = \mathcal{E}^f_{Q,0,T}(\eta)$ for all $f$-martingale probability measures $Q$. We study arbitrage issues, and, when the option is not replicable, we provide some properties of the terminal profit realized by the seller, by investing the amount $\mathbf{v}_0$ in the market according to a superhedging strategy. By symmetry, we derive corresponding results for the *buyer's superhedging price*.

A crucial step in the proof of the above results is to establish a *nonlinear $\mathcal{E}^f$-optional decomposition* for processes which are $(f, Q)$-supermartingales for all $Q \in \mathcal{Q}$. This decomposition is the analogue in our framework of the well-known optional decomposition from the linear case (cf. [12, 13]). We also provide a *nonlinear $\mathcal{E}^f$-predictable decomposition,* which is used to characterize the seller's (superhedging) price as the initial value of the minimal *supersolution of a constrained backward stochastic differential equation (BSDE)* with default.

**Brief literature review on superhedging in incomplete markets.** The superhedging problem (and related optional decomposition) in a *linear incomplete* market model has been much studied (cf., e.g., [12, 13, 24]). Some recent works (cf. [3, 25]) study this problem in a linear incomplete market with model ambiguity of nondominated type, in the case when the trading is in discrete time in [3], and in the continuous case in [25]. In [25], the author works on the canonical Skorokhod space $D([0, T], \mathbb{R}^n)$, and his model includes, in particular, the case when the price of the underlying risky asset is a Lévy process. There are very few papers which consider the case of a *nonlinear incomplete* market model. The paper [2, section 4] considers

[1] In other terms, this is the $f$-evaluation/expectation under the probability measure $Q$.

[2] In other terms, these are $\mathcal{E}^f_Q$-martingales.

a nonlinear incomplete market model in a Brownian framework. In [27], the authors consider a nonlinear incomplete market model with uncertainty (of nondominated type) in which the underlying price process is a (Brownian) diffusion; they address the superhedging problem by using some fine techniques of analysis of [25] together with 2BSDE techniques (working on the canonical space of continuous functions $\mathcal{C}([0,T],\mathbb{R}^n)$ or the product space $\mathcal{C}([0,T],\mathbb{R}^n)^2)$. This framework does not include the case when the underlying price process is not necessarily continuous. To the best of our knowledge, our paper is the first to address the case of a nonlinear incomplete market model when the underlying price process is not necessarily continuous.

The paper is organized as follows: in section 2, we introduce some notation and definitions. In section 3, we present our market model; moreover, we introduce the (new) notion of $f$-martingale probability measure, and provide a characterization as well as useful properties of these $f$-martingale probability measures. In section 4, we present the main results of the paper: the optional and predictable $\mathcal{E}^f$-decompositions for processes which are $(f,Q)$-supermartingales for all $Q \in \mathcal{Q}$ (subsection 4.1), the pricing-hedging duality formula (subsection 4.2), and the characterization of the seller's superhedging price $\mathbf{v_0}$ as the initial value of the minimal supersolution of a constrained BSDE with default (subsection 4.3). In subsection 4.4, we provide results related to replication and profit realized by the seller, and we discuss no-arbitrage issues. In section 5, we study the dual nonlinear control problem associated with the superhedging price, from which some of the results from the previous section are derived. We also provide an extension of the pricing-hedging duality formula (1.1), which holds under weaker integrability conditions. The appendices contain results which are interesting in their own right, besides being useful in the proofs of some results of this paper. In Appendix A, we give some results of strong $\mathcal{E}$-supermartingale families and processes. Appendix B is devoted to the important nonlinear optional and nonlinear predictable decompositions. Appendix C provides some properties on BSDEs with a nonpositive jump at the default time, in particular a nonlinear dual representation. Appendix D gathers some useful lemmas.

## 2. Notation and definitions.

Let $(\Omega, \mathcal{G}, P)$ be a complete probability space equipped with a unidimensional standard Brownian motion $W$ and a jump process $N$ defined by $N_t = \mathbf{1}_{\vartheta \le t}$ for all $t \ge 0$, where $\vartheta$ is a random variable which models a default time. We assume that $P(\vartheta \ge t) > 0$ for all $t \ge 0$. We denote by $\mathbb{G} = \{\mathcal{G}_t, t \ge 0\}$ the *augmented filtration* associated with $W$ and $N$, and by $\mathcal{P}$ the predictable $\sigma$-algebra. We suppose that $W$ is a $\mathbb{G}$-Brownian motion. Let $(\Lambda_t)$ be the predictable compensator of the nondecreasing process $(N_t)$. Note that $(\Lambda_{t\wedge\vartheta})$ is then the predictable compensator of $(N_{t\wedge\vartheta}) = (N_t)$. By uniqueness of the predictable compensator, $\Lambda_{t\wedge\vartheta} = \Lambda_t$, $t \ge 0$ a.s. We assume that $\Lambda$ is absolutely continuous w.r.t. Lebesgue's measure, so that there exists a nonnegative process $\lambda$, called the intensity process, such that $\Lambda_t = \int_0^t \lambda_s ds$, $t \ge 0$. To simplify the presentation, we suppose that $\lambda$ is bounded. Since $\Lambda_{t\wedge\vartheta} = \Lambda_t$, $\lambda$ vanishes after $\vartheta$. Let $M$ be the compensated martingale defined by $M_t := N_t - \int_0^t \lambda_s ds$. Recall that in this setup, we have a martingale representation theorem with respect to $W$ and $M$ (see, e.g., [18]). Let $T > 0$ be the terminal time. We define the following sets:

- $\mathbb{H}^2$ is the set of $\mathbb{G}$-predictable processes $Z$ such that $\|Z\|^2 := \mathbb{E}[\int_0^T |Z_t|^2 dt] < \infty$.
- $\mathbb{H}^2_\lambda := L^2(\Omega \times [0,T], \mathcal{P}, \lambda_t \, dP \otimes dt)$, equipped with the norm $\|U\|^2_\lambda := \mathbb{E}[\int_0^T |U_t|^2 \lambda_t dt] < \infty$.

Note that, without loss of generality, we may assume that if $U \in \mathbb{H}^2_\lambda$, then $U$ vanishes after $\vartheta$.

• $S^2$ is the space of right-continuous with left limits (RCLL) adapted processes $\phi$ with $\|\phi^2\|_{S^2} := \mathbb{E}[\sup_{t\in[0,T]} |\phi_t|^2] < \infty$.

• $\mathcal{A}^2$ is the set of real-valued nondecreasing RCLL $\mathbb{G}$-predictable processes $A$ with $A_0 = 0$ and $\mathbb{E}(A_T^2) < \infty$.

• $\mathcal{C}$ is the set of real-valued nondecreasing RCLL $\mathbb{G}$-optional processes $h$ with $h_0 = 0$ and $\mathbb{E}(h_T^2) < \infty$.

• $\mathcal{T}$ is the set of stopping times $\tau$ such that $\tau \in [0,T]$ a.s. and for $S$ in $\mathcal{T}$, $\mathcal{T}_S$ is the set of stopping times $\tau$ such that $S \le \tau \le T$ a.s.

• $\mathbb{S}^2$ is the vector space of $\mathbb{R}$-valued optional (not necessarily càdlàg) processes $\phi$ such that $|||\phi|||^2_{\mathbb{S}^2} := \mathbb{E}[ess\sup_{\tau\in\mathcal{T}} |\phi_\tau|^2] < \infty$. Note that $S^2$ is the subspace of RCLL processes of $\mathbb{S}^2$.

Definition 2.1 (driver, *λ-admissible* driver). *A function $g$ is said to be a* driver *if $g : \Omega \times [0,T] \times \mathbb{R}^3 \to \mathbb{R}$; $(\omega,t,y,z,k) \mapsto g(\omega,t,y,z,k)$ is $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^3)$-measurable, and $g(.,0,0,0) \in \mathbb{H}^2$. A driver $g$ is said to be* $\lambda$-admissible *if, moreover, there exists a constant $C \ge 0$, called* $\lambda$-constant, *such that for $dP\otimes dt$-almost every $(\omega,t)$ for all $(y_1,z_1,k_1)$, $(y_2,z_2,k_2)$,*

$$|g(\omega,t,y_1,z_1,k_1) - g(\omega,t,y_2,z_2,k_2)| \le C(|y_1-y_2| + |z_1-z_2| + \sqrt{\lambda_t(\omega)}|k_1-k_2|). \tag{2.1}$$

By condition (2.1) and since $\lambda_t = 0$ on $]\vartheta,T]$, $g$ does not depend on $k$ on $]\vartheta,T]$.

Let $g$ be a *λ-admissible driver*. For all $\eta \in L^2(\mathcal{G}_T)$, there exists a unique solution $(X(T,\eta),\ Z(T,\eta),\ K(T,\eta))$ (denoted simply by $(X,Z,K)$) in $S^2\times\mathbb{H}^2\times\mathbb{H}^2_\lambda$ of the *BSDE with default* (cf. [9]):

$$-\,dX_t = g(t,X_t,Z_t,K_t)dt - Z_t dW_t - K_t dM_t; \quad X_T = \eta. \tag{2.2}$$

We denote by $\mathcal{E}^g$ the *g-conditional expectation* operator (called $g$-conditional evaluation in [26]), defined for each $T' \in [0,T]$ and for each $\eta \in L^2(\mathcal{G}_{T'})$ by $\mathcal{E}^g_{t,T'}(\eta) := X_t(T',\eta)$ a.s. for all $t\in[0,T']$.

We introduce the following assumption which ensures the strict monotonicity of the operator $\mathcal{E}^g$ (see [9, section 3.3]).

*Assumption* 2.2. There exists a bounded map $\gamma : \Omega\times[0,T]\times\mathbb{R}^4 \to \mathbb{R}$; $(\omega,t,y,z,k_1,k_2) \mapsto \gamma_t^{y,z,k_1,k_2}(\omega)$ which is $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^4)$-measurable and satisfies $dP\otimes dt$-a.s. for all $(y,z,k_1,k_2)\in\mathbb{R}^4$,

$$g(t,y,z,k_1) - g(t,y,z,k_2) \ge \gamma_t^{y,z,k_1,k_2}(k_1-k_2)\lambda_t, \quad \gamma_t^{y,z,k_1,k_2} > -1.$$

Definition 2.3. *Let $Y\in\mathbb{S}^2$. The process $(Y_t)$ is said to be a* strong $\mathcal{E}^g$-supermartingale *(resp.,* martingale*) if $\mathcal{E}^g_{\sigma,\tau}(Y_\tau) \le Y_\sigma$ (resp., $= Y_\sigma$) a.s. on $\sigma\le\tau$ for all $\sigma,\tau\in\mathcal{T}$.*

By the flow property of BSDEs, for each $\eta\in L^2(\mathcal{G}_T)$, the process $\mathcal{E}^g_{\cdot,T}(\eta)$ is an $\mathcal{E}^g$-martingale. The converse also holds since, if $(Y_t)$ is an $\mathcal{E}^g$-martingale, then $Y_\cdot = \mathcal{E}^g_{\cdot,T}(Y_T)$.

## 3. Market model and $f$-martingale probability measures.

**3.1. The market model $\mathcal{M}^f$.** We consider a financial market with one risk-free asset, whose price process $S^0 = (S^0_t)_{0\le t\le T}$ satisfies $dS^0_t = S^0_t r_t dt$, and one risky asset, whose price process $S = (S_t)_{0\le t\le T}$ may admit a discontinuity at time $\vartheta$, and evolves according to the equation

$$dS_t = S_{t^-}(\mu_t dt + \sigma_t dW_t + \beta_t dM_t). \tag{3.1}$$

The processes $\sigma$ $r, \mu, \beta$ are supposed to be predictable (that is, $\mathcal{P}$-measurable), satisfying $\sigma_t > 0$ $dP \otimes dt$ a.s., $\beta_\vartheta > -1$ a.s., and such that $\sigma$, $\sigma^{-1}$, $\beta$ are bounded. We consider an investor, endowed with an initial wealth $x$, who can invest her wealth in the two assets of the market. At each time $t$, she chooses the amount $\varphi_t$ of wealth invested in the risky asset. A process $\varphi = (\varphi_t)_{0 \le t \le T}$ is called a *portfolio strategy* if it belongs to $\mathbb{H}^2$. The value of the associated portfolio (also called *wealth*) at time $t$ is denoted by $V_t^{x,\varphi}$ (or simply by $V_t$).

In the classical linear case, the wealth process $(V_t^{x,\varphi})$ satisfies the linear forward SDE:

$$dV_t = (r_t V_t + \varphi_t(\mu_t - r_t))dt + \varphi_t \sigma_t dW_t + \varphi_t \beta_t dM_t, \quad V_0 = x. \tag{3.2}$$

Here we suppose that the *wealth* process $(V_t^{x,\varphi})$ satisfies the following *nonlinear* forward SDE:

$$dV_t = -f(t, V_t, \varphi_t \sigma_t)dt + \varphi_t \sigma_t dW_t + \varphi_t \beta_t dM_t, \quad V_0 = x, \tag{3.3}$$

where $f(t, y, z)$ is a driver, which does not depend on $k$. We suppose that $f$ is uniformly Lipschitz with respect to $(y, z)$ $dP \otimes dt$-a.s. (which implies that it is $\lambda$-*admissible*), and that $f(t, 0, 0) = 0$ (so that $\mathcal{E}^f_{\cdot,T}(0) = 0$). In the linear case, we have $f(t, y, z) = -r_t y - z\theta_t$, where $\theta_t := \sigma_t{}^{-1}(\mu_t - r_t)$.

Using the following change of variables which maps a process $\varphi \in \mathbb{H}^2$ to $Z \in \mathbb{H}^2$ defined by $Z_t = \varphi_t \sigma_t$, the wealth process $V_t^{x,\varphi} (= V_t^{x,\sigma^{-1}Z})$ is then the unique process $(V_t)$ satisfying

$$dV_t = -f(t, V_t, Z_t)dt + Z_t dW_t + Z_t \sigma_t^{-1} \beta_t dM_t, \quad V_0 = x. \tag{3.4}$$

*Example* 3.1. Consider a market with default under credit and funding constraints. Suppose that the borrowing rate $R_t$ is different from the lending rate $r_t$ (with $R_t \ge r_t$). Suppose moreover that there is a repo market with different repo rates for long and short positions in the risky asset, denoted by $l_t$ and $b_t$ (see, e.g., [4]). The wealth process associated with strategy $\varphi$ satisfies

$$dV_t = (r_t(V_t - \varphi_t)^+ - R_t(V_t - \varphi_t)^-)dt + (l_t \varphi_t^- - b_t \varphi_t^+)dt + \varphi_t(\mu_t dt + \sigma_t dW_t + \beta_t dM_t). \tag{3.5}$$

Other types of nonlinear wealth dynamics due to funding costs can be found in the papers [1, 6], which address the problem of pricing and hedging the Credit Valuation Adjustment (CVA) in a market with default(s).

Proposition 3.2. *For each $(x, \varphi) \in \mathbb{R} \times \mathbb{H}^2$, the wealth process $(V_t^{x,\varphi})$ is an $\mathcal{E}^f$-martingale.*

*Proof.* Let $(x, \varphi) \in \mathbb{R} \times \mathbb{H}^2$ be given. The process $(V_t^{x,\varphi}, \varphi_t \sigma_t, \varphi_t \beta_t)$ is the solution of the BSDE with default jump associated with driver $f$ and terminal condition $V_T^{x,\varphi}$. The result then follows from the flow property of BSDEs. ∎

*Remark* 3.3. Let $(x, \varphi) \in \mathbb{R} \times \mathbb{H}^2$ be given. By the above proposition, if $V_T^{x,\varphi} \ge 0$ a.s. and $P(V_T^{x,\varphi} > 0) > 0$, then $x = \mathcal{E}^f_{0,T}(V_T^{x,\varphi}) > \mathcal{E}^f_{0,T}(0)$, where the last inequality follows from the strict monotonicity of the operator $\mathcal{E}^f$. Moreover, since $f(t, 0, 0) = 0$, we have $\mathcal{E}^f_{0,T}(0) = 0$. We thus get $x > 0$. Hence, our market model is arbitrage-free.

Let $\eta$ in $L^2(\mathcal{G}_T)$ be the payoff of the option with maturity $T$. It is called *replicable* (for the seller) if there exists $x \in \mathbb{R}$ and $\varphi \in \mathbb{H}^2$ such that $\eta = V_T^{x,\varphi}$ a.s. or, equivalently, such that the process $(V_t^{x,\varphi}, \varphi_t\sigma_t, \varphi_t\beta_t)$ is the solution of the BSDE with default associated with driver $f$, terminal time $T$, and terminal condition $\eta$. This is also equivalent to the existence of $(X, Z) \in S^2 \times \mathbb{H}^2$ such that

$$-dX_t = f(t, X_t, Z_t)dt - Z_t dW_t - Z_t\sigma_t^{-1}\beta_t dM_t, \quad X_T = \eta. \tag{3.6}$$

In this case, the replication strategy $(x, \varphi)$ is unique, and is given by $(x, \varphi) = (X_0, \sigma^{-1}Z)$.

The market, denoted by $\mathcal{M}^f$, is incomplete, since it is clear that all contingent claims are not necessarily replicable.

*Nonlinear pricing in a* complete *market with default.* We end this section by some insights into the nonlinear complete case. Suppose that there is an additional tradable risky asset with price process $S^1$ following the dynamics

$$dS_t^1 = S_t^1[\mu_t^1 dt + \sigma_t^1 dW_t]$$

where $\mu^1$, $\sigma^1$ are predictable processes with $\sigma^1 > 0$. The price process $S_t$ of the nondefaultable asset (with a jump at the default time) is now denoted by $S_t^2$, and its dynamics is now written

$$dS_t^2 = S_{t^-}^2[\mu_t^2 dt + \sigma_t^2 dW_t + \beta_t dM_t]$$

with $\beta_\vartheta \neq 0$ and $\beta_\vartheta > -1$ a.s. We suppose that all coefficients of the model are bounded as well as $(\sigma_t^1)^{-1}$, $\lambda_\vartheta^{-1}$, and $\beta_\vartheta^{-1}$. A portfolio strategy consists of a process $(\varphi_t^1, \varphi_t^2)$ in $\mathbb{H}^2 \times \mathbb{H}^2_\lambda$, where for each $i = 1, 2$, $\varphi_t^i$ represents the amount invested in the asset with price $S^i$. Given an initial capital $x$, the associated wealth process $V_t^{x,\varphi}$ (or simply $V_t$) satisfies

$$-dV_t = g(t, V_t\,,\, \varphi_t^1\sigma_t^1 + \varphi_t^2\sigma_t^2\,,\, \beta_t\varphi_t^2)dt - (\varphi_t^1\sigma_t^1 + \varphi_t^2\sigma_t^2)dW_t - \beta_t\varphi_t^2 dM_t, \tag{3.7}$$

where $g(t, y, z, k)$ is a driver, supposed to be $\lambda$-admissible.[3] The $\lambda$-*admissibility* of $g$ ensures that for each $\eta \in L^2(\mathcal{G}_T)$, there exists a unique solution $(X, Z, K) \in S^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ of the BSDE with default (2.2) associated with driver $g$ and terminal condition $\eta$ (cf. [9, Proposition 2]). Hence, for an option with payoff $\eta$, there exists a unique replicating portfolio for the seller, which is characterized by its initial value $X_0$, and its associated risky-asset strategy $\varphi = (\varphi^1, \varphi^2)$ given by: $\varphi_t^2 = K_t/\beta_t$ and $\varphi_t^1 = (Z_t - \varphi_t^2\sigma_t^2)/\sigma_t^1$. The market is thus complete. The (hedging) price for the seller of the option with payoff $\eta$ is thus equal to $X_0 = \mathcal{E}^g_{0,T}(\eta)$. When $g$ satisfies Assumption 2.2, the nonlinear pricing system $\mathcal{E}^g$ is strictly increasing, which implies that the market is *arbitrage-free* (by Remark 3.3 with $f$ replaced by $g$). By symmetry, the price for the buyer is given by $-\mathcal{E}^g_{0,T}(-\eta)$.

In the particular case when the market is *linear*, the driver $g$ is given by

$$g(t, y, z, k) = -r_t y - \theta_t^1 z - (\mu_t^2 - r_t - \sigma_t^2\theta_t^1)\beta_t^{-1}k,$$

where $\theta_t^1 := (\mu_t^1 - r_t)(\sigma_t^1)^{-1}$. In order for $g$ to be $\lambda$-admissible, the coefficients of the model must satisfy the condition

$$\mu_t^2 - r_t - \sigma_t^2\theta_t^1 = 0, \quad \vartheta < t \leq T \;\; dP \otimes dt - \text{a.s.},$$

[3]Since $\lambda_\vartheta^{-1}$ is here supposed to be bounded, the $\lambda$-*admissibility* assumption on the driver $g$ is equivalent to the assumption: $dP \otimes dt$-a.e., $g$ is uniformly Lipschitz with respect to $(y, z, k)$, and does not depend on $k$ on $]\vartheta, T]$. We note that when $g$ depends on $k$ on $]\vartheta, T]$, the existence and uniqueness result for the BSDE (2.2) does not generally hold (in other terms, the market is not necessarily complete).

so that $g$ does not depend on $k$ on $]\vartheta, T]$. This corresponds to the usual condition required to ensure that this linear market is complete (cf. [19, Proposition 3.7.3.1]). Moreover, in order for $g$ to satisfy Assumption 2.2, we require that $(\mu_t^2 - r_t - \sigma_t^2\theta_t^1)\beta_t^{-1}\lambda_t^{-1}\mathbf{1}_{\lambda_t \neq 0} < 1\ dP \otimes dt$-a.s., which is the usual condition for this linear market model to be arbitrage-free.

**3.2. The set $\mathcal{Q}$ of $f$-martingale probability measures.** We recall that in the linear (incomplete) case, that is, when $f(t, y, z) = -r_t y - \theta_t z$, a dual representation of the superhedging price can be achieved via a martingale approach based on the following notion of martingale probability measures: a probability measure $R$ equivalent to $P$ is called a martingale probability measure if the discounted risky-asset price $(e^{-\int_0^t r_s ds} S_t)$ is a martingale under $R$. This is equivalent to the following definition given, for example, in [29]: a probability measure $R$ is a martingale probability measure if the discounted (linear) wealth processes are $R$-martingales, that is, for all $x \in \mathbb{R}, \varphi \in \mathbb{H}^2$, the process $(e^{-\int_0^t r_s ds}\bar{V}_t^{x,\varphi})$ (where $\bar{V}^{x,\varphi}$ follows the linear dynamics (3.2)) is a martingale under $R$.

In our nonlinear framework, by analogy with the linear case, we are naturally led to introducing the notion of $\mathcal{E}^f$-martingale property under a given probability measure $Q$. To this aim, we first introduce the notion of $f$-evaluation under $Q$. Let $Q$ be a probability measure, equivalent to $P$. From the $\mathbb{G}$-martingale representation theorem (cf., e.g., [18]), its density process $(\zeta_t)$ satisfies

$$d\zeta_t = \zeta_{t^-}(\alpha_t dW_t + \nu_t dM_t); \zeta_0 = 1, \tag{3.8}$$

where $(\alpha_t)$ and $(\nu_t)$ are predictable processes with $\nu_{\vartheta\wedge T} > -1$ a.s. By Girsanov's theorem, the process $W_t^Q := W_t - \int_0^t \alpha_s ds$ is a Brownian motion under $Q$, and the process $M_t^Q := M_t - \int_0^t \nu_s\lambda_s ds$ is a martingale under $Q$. We define the spaces $S_Q^2$, $\mathbb{H}_Q^2$, and $\mathbb{H}_{Q,\lambda}^2$ similarly to $S^2, \mathbb{H}^2$, and $\mathbb{H}_\lambda^2$, but under probability $Q$ instead of $P$.

Definition 3.4. *We call* $f$-evaluation under $Q$, *or* $(f, Q)$-evaluation *in short, denoted by* $\mathcal{E}_Q^f$, *the operator defined for each* $T' \in [0, T]$ *and for each* $\eta \in L_Q^2(\mathcal{G}_{T'})$ *by* $\mathcal{E}_{Q,t,T'}^f(\eta) := X_t$ *for all* $t \in [0, T']$, *where* $(X, Z, K)$ *is the (unique) solution in* $S_Q^2 \times \mathbb{H}_Q^2 \times \mathbb{H}_{Q,\lambda}^2$ *of the* BSDE *under* $Q$ *associated with driver* $f$, *terminal time* $T'$, *terminal condition* $\eta$, *and driven by* $W^Q$ *and* $M^Q$, *that is,*[4]

$$-dX_t = f(t, X_t, Z_t)dt - Z_t dW_t^Q - K_t dM_t^Q, \quad X_{T'} = \eta.$$

We note that $\mathcal{E}_P^f = \mathcal{E}^f$.

Definition 3.5. *Let* $Y \in S_Q^2$. *The process* $(Y_t)$ *is said to be a (strong)* $\mathcal{E}_Q^f$*-martingale, or an* $(f, Q)$*-martingale, if* $\mathcal{E}_{Q,\sigma,\tau}^f(Y_\tau) = Y_\sigma$ *a.s. on* $\sigma \le \tau$ *for all* $\sigma, \tau \in \mathcal{T}$.

We now introduce the concept of $f$-martingale probability measure.

Definition 3.6. *A probability measure* $Q$ *equivalent to* $P$ *is called an* $f$*-martingale probability measure if for all* $x \in \mathbb{R}$ *and for all portfolio strategies* $\varphi \in \mathbb{H}_Q^2$, *the wealth process* $V^{x,\varphi}$ *is a strong* $\mathcal{E}_Q^f$*-martingale or, in other terms, an* $(f, Q)$*-martingale.*[5]

[4] Since we have a representation theorem for $(Q, \mathbb{G})$-martingales with respect to $W^Q$ and $M^Q$ (see, e.g., Proposition 6 in the appendix of [9]), this BSDE admits a unique solution $(X, Z, K)$ in $S_Q^2 \times \mathbb{H}_Q^2 \times \mathbb{H}_{Q,\lambda}^2$.

[5] Note that $P$ is an $f$-martingale probability measure (cf. Proposition 3.2), but is not the only one.

*Remark* 3.7 (linear incomplete case). Let $R^0$ be the martingale probability measure, with density $\zeta^0$ satisfying $d\zeta^0_t = -\zeta^0_t\,\theta_t dW_t$ with $\zeta^0_0 = 1$. Suppose $f(t, y, z) = -r_t y - \theta_t z$. Then, the $(f, P)$-martingale property of the (linear) wealth processes (cf. Proposition 3.2) is equivalent to the well-known $R^0$-martingale property of the discounted wealth processes. In other terms, the $f$-martingale probability property of $P$ corresponds to the (well-known) martingale probability property of $R^0$.

*Notation.* We denote by $\mathcal{Q}$ the set of $f$-martingale probability measures $Q$ such that the coefficients $(\alpha_t)$ and $(\nu_t)$ associated with its density (3.8) with respect to $P$ are bounded. We note that $P \in \mathcal{Q}$.

Let $\mathcal{V}$ be the set of bounded predictable processes $\nu$ such that $\nu_{\vartheta\wedge T} > -1$ a.s., which is equivalent to $\nu_t > -1$ for all $t \in [0, T]$ $\lambda_t dP \otimes dt$-a.e. (cf. [9, Remark 9]).

Proposition 3.8 (characterization of $\mathcal{Q}$). *Let $Q$ be a probability measure equivalent to $P$, such that the coefficients $\alpha$ and $\nu$ of its density* (3.8) *with respect to $P$ are bounded. The three following assertions are equivalent:*

(i) *$Q \in \mathcal{Q}$, that is, $Q$ is an $f$-martingale probability measure.*

(ii) *There exists $\nu \in \mathcal{V}$ such that $Q = Q^\nu$, where $Q^\nu$ is the probability measure which admits $\zeta^\nu_T$ as density with respect to $P$ on $\mathcal{G}_T$, where $\zeta^\nu$ satisfies*

$$d\zeta^\nu_t = \zeta^\nu_{t^-}(-\nu_t\lambda_t\beta_t\sigma_t^{-1}dW_t + \nu_t dM_t); \zeta^\nu_0 = 1. \tag{3.9}$$

(iii) *The stochastic integral $\int_0^\cdot(\sigma_s dW_s + \beta_s dM_s)$ is a $Q$-martingale.*

*Remark* 3.9. The mapping $\nu \mapsto Q^\nu$ is a one-to-one mapping that carries $\mathcal{V}$ onto $\mathcal{Q}$. So we have $\mathcal{Q} = \{Q^\nu,\ \nu \in \mathcal{V}\}$. For $\nu = 0$, we have $Q^\nu = Q^0 = P$. Note also that $\mathcal{Q}$ does not depend on $f$.

*Proof.* Let $Q$ be a probability measure equivalent to $P$, such that the coefficients $\alpha$ and $\nu$ of its density (3.8) with respect to $P$ are bounded. Let $x \in \mathbb{R}$ and $\varphi \in \mathbb{H}^2 \cap \mathbb{H}^2_Q$. The associated wealth process $V = V^{x,\varphi}$ satisfies (3.4). Since $dW_t = dW^Q_t + \alpha_t dt$ and $dM_t = dM^Q_t + \nu_t\lambda_t dt$, we have

$$\sigma_t dW_t + \beta_t dM_t = \sigma_t dW^Q_t + \beta_t dM^Q_t + \sigma_t(\alpha_t + \nu_t\lambda_t\beta_t\sigma_t^{-1})dt, \tag{3.10}$$

$$-dV^{x,\varphi}_t = f(t, V^{x,\varphi}_t, \varphi_t\sigma_t)dt - \varphi_t\sigma_t(\alpha_t + \nu_t\lambda_t\beta_t\sigma_t^{-1})dt - \varphi_t\sigma_t dW^Q_t - \varphi_t\beta_t dM^Q_t. \tag{3.11}$$

Suppose that $Q$ satisfies (ii), that is $\alpha_t = -\nu_t\lambda_t\beta_t\sigma_t^{-1}$ $dP \otimes dt$-a.e. By (3.10), the stochastic integral $\int_0^\cdot(\sigma_s dW_s + \beta_s dM_s)$ is then a $Q$-martingale, which corresponds to (iii). Moreover, by (3.11), for each $x \in \mathbb{R}$ and each $\varphi \in \mathbb{H}^2 \cap \mathbb{H}^2_Q$, the process $(V^{x,\varphi}_t, \varphi_t\sigma_t, \varphi_t\beta_t)$ is the solution of the BSDE under $Q$ associated with driver $f$ and terminal condition $V^{x,\varphi}_T$, which implies that the wealth process $V^{x,\varphi}$ is an $(f, Q)$-martingale. In other terms, $Q$ is an $f$-martingale probability measure, that is, (i) holds.

Suppose now that (iii) holds, that is, the process $\int_0^\cdot(\sigma_s dW_s + \beta_s dM_s)$ is a $Q$-martingale. By (3.10), we thus have $\sigma_t(\alpha_t + \nu_t\lambda_t\beta_t\sigma_t^{-1}) = 0$ $dP \otimes dt$-a.e. Since, by assumption, $\sigma_t > 0$, we get $\alpha_t = -\nu_t\lambda_t\beta_t\sigma_t^{-1}$ $dP \otimes dt$-a.e., which corresponds to property (ii).

Suppose (i), that is, $Q \in \mathcal{Q}$. Since $Q$ is an $f$-martingale probability measure, by definition of an $f$-martingale probability measure, for all $x \in \mathbb{R}$ and $\varphi \in \mathbb{H}^2 \cap \mathbb{H}^2_Q$, the wealth process $V^{x,\varphi}$ is an $(f,Q)$-martingale. By (3.11) and Lemma D.1 in Appendix D, the finite variational process $\int_0^\cdot \varphi_t \sigma_t(\alpha_t + \nu_t\lambda_t\beta_t\sigma_t^{-1})dt$ is thus equal to 0. Since this holds for all $\varphi \in \mathbb{H}^2 \cap \mathbb{H}^2_Q$ and since $\sigma_t > 0$, we derive that $\alpha_t = -\nu_t\lambda_t\beta_t\sigma_t^{-1}$ $dP \otimes dt$-a.e., that is, (ii) holds. The proof is thus complete. ∎

We now provide a connection between $f$-martingale probabilities and martingale probabilities. Let $R$ be a probability measure equivalent to $P$ such that the coefficients $\alpha$ and $\nu$ of its density with respect to $P$ (cf. (3.8)) are bounded. Proceeding as in the proof of Proposition 3.8, we derive that $R$ is a martingale probability measure if and only if there exists $\nu \in \mathcal{V}$ such that $R = R^\nu$, where $R^\nu$ is the probability measure with density process $\tilde{\zeta}^\nu$ (with respect to $P$) satisfying

$$d\tilde{\zeta}_t^\nu = \tilde{\zeta}_{t^-}^\nu((-\theta_t - \nu_t\lambda_t\beta_t\sigma_t^{-1})dW_t + \nu_t dM_t); \tilde{\zeta}_0^\nu = 1. \tag{3.12}$$

We denote by $\mathcal{P}$ the set of all such probability measures.

By this observation together with Proposition 3.8, we derive the following result.

Proposition 3.10. *There exists a one-to-one mapping from $\mathcal{Q}$ onto $\mathcal{P}$. More precisely, the mapping $T_\theta$, which, for each $\nu \in \mathcal{V}$, maps the $f$-martingale probability $Q^\nu$ (with density $\zeta^\nu$ given by (3.9)) onto the martingale probability measure $R^\nu$ (with density $\tilde{\zeta}^\nu$) is a one-to-one correspondence between $\mathcal{Q}$ and $\mathcal{P}$. We have $T_\theta(P) = R^0$.*

## 4. Main results.

We define the following spaces: $\mathbf{L^2} := \mathbf{L^2}(\mathcal{G}_T) := \cap_{Q\in\mathcal{Q}} L^2_Q(\mathcal{G}_T)$, $\mathbf{H^2} := \cap_{Q\in\mathcal{Q}}\mathbb{H}^2_Q$, $\mathbf{H}^2_\lambda := \cap_{Q\in\mathcal{Q}}\mathbb{H}^2_{Q,\lambda}$, and $\mathbf{S^2} := \cap_{Q\in\mathcal{Q}} S^2_Q$. We restrict ourselves to portfolio strategies belonging to $\mathbf{H^2}$.

We consider an option with maturity $T$ and payoff $\eta \in \mathbf{L^2}(\mathcal{G}_T)$.

We introduce the *superhedging price for the seller* of this option defined as the minimal initial capital which allows her to build a superhedging strategy, that is,

$$\mathbf{v}_0 := \inf\{x \in \mathbb{R} :\ \exists \varphi \in \mathbf{H}^2 \ \text{ s.t. } V_T^{x,\varphi} \geq \eta \text{ a.s.}\} = \inf\{x \in \mathbb{R} :\ \exists \varphi \in \mathcal{A}_0(x)\}, \tag{4.1}$$

where for each $x \in \mathbb{R}$, $\mathcal{A}_0(x) = \{\varphi \in \mathbf{H^2} \ \text{ s.t. } V_T^{x,\varphi} \geq \eta \text{ a.s.}\}$.[6]

By convention, $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$. Note that $\mathbf{v_0} \in \overline{\mathbb{R}}$.

### 4.1. Optional and predictable $\mathcal{E}^f$-decompositions.

We first provide a nonlinear optional decomposition and a nonlinear predictable decomposition for RCLL optional processes $(Y_t)$ which are strong $(f,Q)$-supermartingale for all $f$-martingale probability measure $Q$. These decompositions are crucial to prove two characterizations of superhedging prices of options: one via a pricing-hedging duality formula (cf. Theorem 4.8) and a second one via a constrained BSDE with default (cf. Theorem 4.12).

Theorem 4.1 (optional $\mathcal{E}^f$-decomposition). *Let $(Y_t)$ be an RCLL optional process belonging to $\mathbf{S^2}$. Suppose that $(Y_t)$ is a strong $(f,Q)$-supermartingale for all $f$-martingale probability*

[6] We shall see in section 5 that $\mathbf{v_0} = v_0$, where $v_0$ is defined similarly to $\mathbf{v_0}$ from (4.1), with $\mathbf{H^2}$ replaced by $\mathbb{H}^2$.

*measure $Q$. Then, there exists a unique $Z \in \mathbf{H^2}$, and a unique $h \in \mathcal{C} \cap \mathbf{S^2}$ such that*

$$-dY_t = f(t, Y_t, Z_t)dt - Z_t\sigma_t^{-1}(\sigma_t dW_t + \beta_t dM_t) + dh_t. \tag{4.2}$$

*Moreover, the converse statement holds.*

*Proof.* The first assertion follows from Theorem B.2 (with $C = 0$ since $Y$ is RCLL) and Remark B.4, together with Remark 5.2. Let us show the second one. Suppose that there exists $Z \in \mathbf{H^2}$, and a nondecreasing optional RCLL process $h$ in $\mathbf{S^2}$ with $h_0 = 0$ such that (4.2) holds. Let $Q \in \mathcal{Q}$. Let $\sigma \in \mathcal{T}$ and let $\tau \in \mathcal{T}_\sigma$. We have to show that $\mathcal{E}^f_{Q,\sigma,\tau}(Y_\tau) \leq Y_\sigma$ a.s. By definition, the process $\mathcal{E}^f_{Q,\cdot,\tau}(Y_\tau)$ is the solution of the $Q$-BSDE associated with driver $f$, terminal time $\tau$, and terminal condition $Y_\tau$. Now, the process $(Y_{t\wedge\tau})$ is the solution of the $Q$-BSDE associated with generalized (optional) driver $f(t, y, z)dt + dh_t$, terminal time $\tau$, and terminal condition $Y_\tau$. By the comparison theorem for BSDEs with default jump and generalized drivers (cf. [9, Theorem 3]), we derive that $\mathcal{E}^f_{Q,\sigma,\tau}(Y_\tau) \leq Y_\sigma$ a.s. Hence, $Y$ is an $\mathcal{E}^f_Q$-strong supermartingale for all $Q \in \mathcal{Q}$. ■

Theorem 4.2 (predictable $\mathcal{E}^f$-decomposition). *Let $(Y_t)$ be an RCLL optional process belonging to $\mathbf{S^2}$. Suppose that $(Y_t)$ is a strong $(f, Q)$-supermartingale for all $f$-martingale probability measure $Q$. There exists a unique process $(Z, K, A) \in \mathbf{H^2} \times \mathbf{H^2_\lambda} \times \mathcal{A}^\mathbf{2}$ such that*

$$-dY_t = f(t, Y_t, Z_t)dt - Z_t dW_t - K_t dM_t + dA_t, \tag{4.3}$$

(4.4)
$$A_\cdot + \int_0^\cdot (K_s - \beta_s\sigma_s^{-1}Z_s)\lambda_s ds \in \mathcal{A}^2 \quad \text{and} \quad (K_t - \beta_t\sigma_t^{-1}Z_t)\lambda_t \leq 0,\ t \in [0, T],\ dP \otimes dt - \text{a.e.}$$

*Moreover, the converse statement holds.*

This result follows from Theorem B.1.[7]

**4.2. Pricing-hedging duality formula.** By the $(f, Q)$-martingale property of the wealth processes under any $f$-martingale probability measure $Q$, it is straightforward to get the following result.

Lemma 4.3. *We have*

$$\mathbf{v_0} \geq \sup_{Q\in\mathcal{Q}} \mathcal{E}^f_{Q,0,T}(\eta). \tag{4.5}$$

*Proof.* Let $\mathcal{H}$ be the set of initial capitals which allow the seller to be "superhedged," that is, $\mathcal{H} := \{x \in \mathbb{R} : \exists\varphi \in \mathcal{A}_0(x)\}$. We have $\mathbf{v}_0 = \inf \mathcal{H}$. If $\mathcal{H} = \emptyset$, then $\mathbf{v_0} = +\infty$ and, hence, inequality (4.5) holds. Suppose now that $\mathcal{H} \neq \emptyset$. Let $x \in \mathcal{H}$. There exists $\varphi \in \mathbf{H^2}$ such that $V_T^{x,\varphi} \geq \eta$ a.s. Let $Q \in \mathcal{Q}$. Since $Q$ is an $f$-martingale probability measure, the wealth process $V^{x,\varphi}$ is an $(f, Q)$-martingale. By taking the $(f, Q)$-evaluation/expectation in the inequality $V_T^{x,\varphi} \geq \eta$, we thus obtain $x = \mathcal{E}^f_{Q,0,T}(V_T^{x,\varphi}) \geq \mathcal{E}^\nu_{0,T}(\eta)$. As $Q \in \mathcal{Q}$ is arbitrary, we get $x \geq \sup_{Q\in\mathcal{Q}} \mathcal{E}^f_{Q,0,T}(\eta)$. This holds for all $x \in \mathcal{H}$. By taking the infimum over $x \in \mathcal{H}$, we derive the desired inequality. ■

[7]Note that by [9, Remark 9], the condition $(K_t - \beta_t\sigma_t^{-1}Z_t)\lambda_t \leq 0,\ t \in [0, T],\ dP \otimes dt -$ a.e. is equivalent to $K_\vartheta - \beta_\vartheta\sigma_\vartheta^{-1}Z_\vartheta \leq 0$, $P$-a.s.

To prove the converse inequality in (4.5), we introduce the *associated (dynamic) dual problem*.

For each $S \in \mathcal{T}$, let $X(S)$ be the *value of the dual problem* at time $S$ defined by

$$X(S) := ess \sup_{Q \in \mathcal{Q}} \mathcal{E}^f_{Q,S,T}(\eta). \tag{4.6}$$

We suppose that

$$\mathbb{E}_Q[ess \sup_{S \in \mathcal{T}_0} X(S)^2] < +\infty \ \ \forall Q \in \mathcal{Q}. \tag{4.7}$$

*Remark* 4.4. This integrability condition holds, e.g., for a call option, that is, when $\eta = (S_T - k)^+$.

The theorem below can be seen as a dynamic programming principle for the dual value problem.

Theorem 4.5 (the dual value process. Minimality characterization). *There exists a unique (right-continuous) process $(X_t) \in \mathbf{S^2}$, called the* dual value process*, which aggregates the value family $(X(S))$, that is, for each $S \in \mathcal{T}_0$, $X(S) = X_S$ a.s. The process $(X_t)$ is a strong $(f, Q)$-supermartingale for all $f$-martingale probability measures $Q$, and satisfies $X_T = \eta$ a.s. Moreover, the process $(X_t)$ is* the smallest *process in $\mathbf{S^2}$ satisfying these properties.*

*Proof.* The proof is given in section 5 (cf. Theorem 5.7, together with Remark 5.2). ■

Corollary 4.6 (optional $\mathcal{E}^f$-decomposition of the dual value process $(X_t)$). *There exists a unique $Z \in \mathbf{H^2}$, and a unique $h \in \mathcal{C} \cap \mathbf{S^2}$ such that*

$$X_t = X_0 - \int_0^t f(s, X_s, Z_s)ds + \int_0^t \sigma_s^{-1} Z_s(\sigma_s dW_s + \beta_s dM_s) - h_t, \ 0 \leq t \leq T \quad \text{a.s.} \tag{4.8}$$

*Moreover, the portfolio strategy $\varphi^* := \sigma^{-1}Z$ belongs to $\mathcal{A}_0(X_0)$, that is, $V_T^{X_0,\varphi^*} \geq \eta$ a.s*

*Proof.* The first assertion follows from Theorem 4.5 together with Theorem 4.1. The proof of the second assertion relies on a forward argument. By (3.3)–(3.4), the process $(V_t^{X_0,\varphi^*})$ satisfies the forward equation:

$$V_t^{X_0,\varphi^*} = X_0 - \int_0^t f(s, V_s^{X_0,\varphi^*}, Z_s)ds + \int_0^t \sigma_s^{-1} Z_s(\sigma_s dW_s + \beta_s dM_s), \ 0 \leq t \leq T \quad \text{a.s.} \tag{4.9}$$

Moreover, the value process $(X_t)$ satisfies the forward SDE (4.8). Since $(h_t)$ is nondecreasing, by the comparison result for forward differential equations, we get $V_T^{X_0,\varphi^*} \geq X_T = \eta$ a.s. ■

Corollary 4.7 (superhedging equivalences). *The integrability condition (4.7) is equivalent to the existence of $x_0 \in \mathbb{R}$ and $\psi \in \mathbf{H}^2$ such that $\eta \leq V_T^{x_0,\psi}$ a.s. (which is also equivalent to $\mathbf{v_0} < +\infty$).*

*Proof.* Suppose that (4.7) holds. By Corollary 4.6, we have $V_T^{X_0,\varphi^*} \geq \eta$ a.s. Let us show the converse. Suppose that $\eta \leq V_T^{x_0,\psi}$ a.s., where $x_0 \in \mathbb{R}$ and $\psi \in \mathbf{H^2}$. We first show that for all $S \in \mathcal{T}$, we have $X(S) \leq V_S^{x_0,\psi}$ a.s. Let $S \in \mathcal{T}$. Since $\eta \leq V_T^{x_0,\psi}$ a.s. for each $Q \in \mathcal{Q}$, we have $\mathcal{E}^f_{Q,S,T}(\eta) \leq \mathcal{E}^f_{Q,S,T}(V_T^{x_0,\psi}) = V_S^{x_0,\psi}$ a.s., where the last equality holds since $Q$ is an $f$-martingale probability measure. Taking the essential supremum over $Q \in \mathcal{Q}$ in this inequality, we get $X(S) = ess\sup_{Q\in\mathcal{Q}} \mathcal{E}^f_{Q,S,T}(\eta) \leq V_S^{x_0,\psi}$ a.s.

We now show (4.7). Let $Q \in \mathcal{Q}$. We have $\mathcal{E}^f_{Q,S,T}(\eta) \leq X(S) \leq V_S^{x_0,\psi}$ a.s. Since $\mathcal{E}^f_{Q,\cdot,T}(\eta) \in S^2_Q$ and $V^{x_0,\psi} \in S^2_Q$, we thus get $\mathbb{E}_Q[ess\sup_{S\in\mathcal{T}_0} X(S)^2] < +\infty$. ∎

Using the above results, we now prove the pricing-hedging duality formula.

Theorem 4.8 (pricing-hedging duality and superhedging strategy). *The superhedging price $\mathbf{v_0}$ for the seller of the option with payoff $\eta$ and maturity $T$ satisfies the equality $\mathbf{v_0} = X_0$, that is,*

$$\mathbf{v_0} = \sup_{Q\in\mathcal{Q}} \mathcal{E}^f_{Q,0,T}(\eta). \tag{4.10}$$

*Moreover, the portfolio strategy $\varphi^* := \sigma^{-1}Z$, where $Z$ is the process from the optional $\mathcal{E}^f$-decomposition (4.8), is a superhedging strategy for the seller associated with $\mathbf{v_0}$, that is, $V_T^{\mathbf{v_0},\varphi^*} \geq \eta$ a.s.*

*Proof.* The notation is the same as in the proof of Lemma 4.3. Let us show the inequality $\mathbf{v_0} \leq \sup_{Q\in\mathcal{Q}} \mathcal{E}^f_{Q,0,T}(\eta)$, that is, $X_0 \geq \mathbf{v_0}$. By Corollary 4.6, $\varphi^* \in \mathcal{A}_0(X_0)$, which implies $X_0 \in \mathcal{H}$ . Since $\mathbf{v_0} = \inf \mathcal{H}$, it follows that $X_0 \geq \mathbf{v_0}$. Now, by Lemma 4.3, we have $X_0 \leq \mathbf{v_0}$. Hence, $X_0 = \mathbf{v_0}$. Moreover, since $V_T^{X_0,\varphi^*} \geq \eta$ a.s., we get $V_T^{\mathbf{v_0},\varphi^*} \geq \eta$ a.s. ∎

*Remark* 4.9. By similar arguments, it can be proven that, for each $S \in \mathcal{T}$,

$$X_S = ess\sup_{Q\in\mathcal{Q}} \mathcal{E}^f_{Q,S,T}(\eta) = \operatorname{ess\,inf}\{X \in \mathbf{L^2}(\mathcal{G}_S),\ \exists\varphi \in \mathbf{H^2} \text{ s.t. } V_T^{S,X,\varphi} \geq \eta \text{ a.s.}\} \text{ a.s.}$$

Let us now consider the *buyer's* superhedging price $\tilde{\mathbf{v}}_\mathbf{0}$ (cf., e.g., [23]), which is defined as the maximal initial price which allows her to build a superhedging strategy, i.e.,

$$\tilde{\mathbf{v}}_0 := \sup\{x \in \mathbb{R} :\ \exists\varphi \in \mathbf{H}^2 \text{ s.t. } V_T^{-x,\varphi} + \eta \geq 0 \text{ a.s.}\}. \tag{4.11}$$

We note that $\tilde{\mathbf{v}}_\mathbf{0}$ is equal to the opposite of the superhedging price for the seller of the option with payoff $-\eta$. We thus derive, using Theorem 4.8, that, if $\tilde{\mathbf{v}}_\mathbf{0} > -\infty$, then the following dual representation result for $\tilde{\mathbf{v}}_\mathbf{0}$ holds:

$$\tilde{\mathbf{v}}_\mathbf{0} = \inf_{Q\in\mathcal{Q}}(-\mathcal{E}^f_{Q,0,T}(-\eta)).$$

**4.3. Constrained BSDE characterization.** We see in this subsection that the seller's superhedging price $\mathbf{v_0}$ is characterized as the value at time 0 of the minimal supersolution of a default BSDE with driver $f$ and suitably defined *constraints*. We suppose that the integrability assumption (4.7) holds.

Definition 4.10. *Let $\eta \in \mathbf{L^2}(\mathcal{G}_\mathbf{T})$. A process $X' \in \mathbf{S}^2$ is said to be a* supersolution *of the* constrained BSDE *with driver $f$ and terminal condition $\eta$ if there exists a process $(Z', K')$ in*

$\mathbf{H^2} \times \mathbf{H^2_\lambda}$ *and a predictable nondecreasing process* $A'$ *in* $\mathbf{S^2}$ *with* $A'_0 = 0$ *such that* $(X', Z', K', A')$ *satisfies*

$$-dX'_t = f(t, X'_t, Z'_t)dt + dA'_t - Z'_t dW_t - K'_t dM_t; \quad X'_T = \eta \text{ a.s.,} \tag{4.12}$$

$$A'_\cdot + \int_0^\cdot (K'_s - \beta_s \sigma_s^{-1} Z'_s)\lambda_s ds \text{ is nondecreasing,} \quad \text{and} \quad K'_\vartheta - \beta_\vartheta \sigma_\vartheta^{-1} Z'_\vartheta \le 0 \quad \text{a.s.} \tag{4.13}$$

*Remark* 4.11. When $\beta = 0$, the second condition of (4.13) reduces to $K'_\vartheta \le 0$, which means that the jump of $X'$ at the default time $\vartheta$ is nonpositive. In [21] the particular case of BSDEs with nonpositive jumps is studied using a different approach (via penalization). See Appendix C.

By Theorem 4.5 together with the predictable $\mathcal{E}^f$-decomposition (Theorem 4.2), we derive that the value process admits a predictable $\mathcal{E}^f$-decomposition of the form from Theorem 4.2, and that it is the minimal one which admits such a decomposition. Using also Theorem 4.8, we obtain the following result, written in terms of the above constrained BSDE.

Theorem 4.12 (characterization of $X$ and $\mathbf{v_0}$ in terms of a constrained BSDE with default). *The dual value process* $(X_t)$ *is* a supersolution of the constrained BSDE *associated with driver* $f$ *and terminal condition* $\eta$*, that is, there exists a unique process* $(Z, K) \in \mathbf{H^2} \times \mathbf{H^2_\lambda}$ *and a unique predictable nondecreasing process* $A$ *in* $\mathbf{S^2}$ *with* $A_0 = 0$ *such that* $(X, Z, K, A)$ *satisfies* (4.12)–(4.13)*. Moreover, the process* $(X_t)$ *is* the minimal *supersolution of the above constrained BSDE.*

*In particular, the seller's superhedging price* $\mathbf{v_0}$ $(= X_0)$ *is characterized as the value at time* $0$ *of the minimal supersolution of the constrained BSDE.*

### 4.4. Results on replication and the profit realized by the seller.

Let $\eta \in \mathbf{L}^2(\mathcal{G}_T)$ be a given payoff. We first provide replication criteria for the associated option.

Proposition 4.13 (replication criteria). *The following four assertions are equivalent:*

(i) *The option with payoff* $\eta$ *is replicable, that is, there exists* $(x, \varphi) \in \mathbb{R} \times \mathbf{H^2}$ *with* $\eta = V_T^{x,\varphi}$ *a.s.*

(ii) *The nondecreasing optional process* $h$ *from the optional* $\mathcal{E}^f$*-decomposition* (4.8) *of the dual value process* $(X_t)$ *is equal to* $0$*.*

(iii) *The supremum in* (4.10) *is attained.*

(iv) *For all* $f$*-martingale probability measure* $Q$*, we have* $\mathcal{E}^f_{Q,0,T}(\eta) = \mathcal{E}^f_{P,0,T}(\eta)$*.*

*In this case, the replication strategy is given by* $(x, \varphi) = (X_0, \sigma^{-1} Z)$*, where* $Z$ *is the process from the optional* $\mathcal{E}^f$*-decomposition* (4.8) *of the dual value process* $(X_t)$*.*[8]

*Proof.* Suppose (ii). Then, by (4.8), we get $X_\cdot = V_\cdot^{X_0, \sigma^{-1}Z}$. In particular, $X_T = V_T^{X_0, \sigma^{-1}Z}$. Since $X_T = \eta$, we get $\eta = V_T^{X_0, \sigma^{-1}Z}$. Hence, (i) holds and $(X_0, \sigma^{-1}Z)$ is the replication strategy.

Suppose now (i), that is, there exists $x \in \mathbb{R}$ and $\varphi \in \mathbf{H^2}$ such that $\eta = V_T^{x,\varphi}$ a.s. Let $Q \in \mathcal{Q}$. Since $Q \in \mathcal{Q}$, the wealth process $V_\cdot^{x,\varphi}$ is an $(f, Q)$-martingale. Hence, by taking the

[8] $(X, Z, \sigma^{-1}Z\beta)$ is the solution of the BSDE with default associated with $(f, \eta)$ or, equivalently, $(X, Z)$ satisfies (3.6).

$(f,Q)$-expectation in the equality $\eta = V_T^{x,\varphi}$, we get $\mathcal{E}^f_{Q,0,T}(\eta) = x$. Since this equality holds for all $Q \in \mathcal{Q}$, we get (iv).

If (iv) holds, then the supremum in (4.10) is attained at any $Q \in \mathcal{Q}$, that is, (iii) holds.

Suppose (iii). Hence, there exists $\hat{Q} \in \mathcal{Q}$ such that $X_0 = \mathcal{E}^f_{\hat{Q},0,T}(\eta)$. Now, the process $\mathcal{E}^f_{\hat{Q},\cdot,T}(\eta)$ is the solution of the $\hat{Q}$-BSDE associated with driver $f$, terminal time $T$, and terminal condition $\eta$. Moreover, the dual value process $(X_t)$ admits the optional $\mathcal{E}^f$-decomposition (4.8) from Corollary 4.6. Since $X_T = \eta$ and $\sigma_t dW_t + \beta_t dM_t = \sigma_t dW_t^{\hat{Q}} + \beta_t dM_t^{\hat{Q}}$, we get that $(X_t, Z_t, \sigma_t^{-1} Z_t \beta_t)$ is the solution of the $\hat{Q}$-BSDE associated with generalized driver $f(t,y,z)dt + dh_t$, terminal time $T$, and terminal condition $\eta$. By the strict comparison theorem (cf. [9, Theorem 3(ii)]), since $h$ is a nondecreasing optional process and $X_0 = \mathcal{E}^f_{\hat{Q},0,T}(\eta)$, we derive that $h = 0$, that is, (ii) holds. ■

We now introduce the notion of *arbitrage opportunity for the seller (resp., for the buyer)*.

Definition 4.14. *Let $x \in \mathbb{R}$. A strategy $\varphi$ in $\mathbf{H}^2$ is said to be an arbitrage opportunity for the seller*[9] *(resp., for the buyer)*[10] *of the option with initial price $x$ if*

$$V_T^{x,\varphi} - \eta \geq 0 \text{ a.s. and } P(V_T^{x,\varphi} - \eta > 0) > 0$$
$$(\text{resp., } V_T^{-x,\varphi} + \eta \geq 0 \text{ a.s. and } P(V_T^{-x,\varphi} + \eta > 0) > 0).$$

Proposition 4.15. *Let $x \in \mathbb{R}$. If $x > \mathbf{v_0}$ (resp., $x < \tilde{\mathbf{v}}_\mathbf{0}$), then there exists an arbitrage opportunity for the seller (resp., for the buyer) of the option with price $x$.*

*If $x < \mathbf{v_0}$ (resp., $x > \tilde{\mathbf{v}}_\mathbf{0}$), then there exists no arbitrage opportunity for the seller (resp., for the buyer) of the option with price $x$.*

*Proof.* The proof relies on Theorem 4.8. Suppose that $x$ is the price of the option.

Suppose that $x < \mathbf{v_0}$. By the definition of $\mathbf{v_0}$, there exists no arbitrage opportunity for the seller.

Suppose now that $x > \mathbf{v_0}$. By Theorem 4.8, the portfolio strategy $\varphi^* := \sigma^{-1} Z$, where $Z$ is the process from the optional $\mathcal{E}^f$-decomposition (4.8), satisfies $V_T^{\mathbf{v_0},\varphi^*} \geq \eta$ a.s. By the "strict comparison" property for forward differential equations, get $V_T^{x,\varphi^*} > V_T^{\mathbf{v_0},\varphi^*} \geq \eta$ a.s. Hence, $\varphi^*$ is an arbitrage opportunity for the seller.

By symmetry, we get the result for the buyer, who can be seen as the seller of the option with payoff $-\eta$ and with superhedging price equal to $-\tilde{\mathbf{v}}_\mathbf{0}$. ■

Definition 4.16. *A real number $x$ is called an* arbitrage-free price for the option *if there exists no arbitrage opportunity, neither for the seller nor for the buyer.*[11]

[9] This means that the seller sells the option at the price $x$ and, by using the strategy $\varphi$, she makes the profit $V_T^{x,\varphi} - \eta \geq 0$ at time $T$ with $P(V_T^{x,\varphi} - \eta > 0) > 0$. Note that this strategy is an arbitrage opportunity in the extended market in the classical sense (given in the literature on linear market models).

[10] This means that the buyer buys the option at the price $x$ which, borrowed at time 0, allows her to recover her debt at time $T$ (by using the strategy $\varphi$) and even to make the profit $V_T^{-x,\varphi} + \eta \geq 0$ a.s. with $P(V_T^{-x,\varphi} + \eta > 0) > 0$). Note that this strategy is an arbitrage opportunity in the extended market in the classical sense (given in the literature on linear market models).

[11] Note that an *arbitrage opportunity* $\varphi$ in the sense of the literature on markets with constraints (see, e.g., [20, 23, 10]) is also an arbitrage opportunity in our sense, but the converse does not necessarily hold. Hence, the set of *arbitrage-free prices* in those works is larger than ours.

Using Theorem 4.8 and Theorem 4.13(iv), we show the following result.

Proposition 4.17 (arbitrage-free interval). *If $\mathbf{v}_0 < \tilde{\mathbf{v}}_0$, there does not exist any arbitrage-free price for the option. Suppose that $\tilde{\mathbf{v}}_0 \leq \mathbf{v}_0$. The set of arbitrage-free prices for the option is an interval, which is called the* arbitrage-free *interval for the option. It is of the form $(\tilde{\mathbf{v}}_0, \mathbf{v}_0)$.*

*The interval is closed on the right (resp., on the left) if and only if $\eta$ (resp., $-\eta$) is replicable.*

*Proof.* The two first assertions follow from Proposition 4.15. It remains to show the third one.

Suppose that the option is replicable. Suppose that there exists an arbitrage opportunity $\varphi$ for the seller of the option with price $\mathbf{v}_0$, that is, such that $V_T^{\mathbf{v}_0,\varphi} \geq \eta$ a.s. and $P(V_T^{\mathbf{v}_0,\varphi} > \eta) > 0$. By the $(f,P)$-martingale property of the wealth process $V_T^{\mathbf{v}_0,\varphi}$, we get $\mathbf{v}_0 = \mathcal{E}^f_{P,0,T}(V_T^{\mathbf{v}_0,\varphi}) > \mathcal{E}^f_{P,0,T}(\eta)$, where the last inequality follows from the strict monotonicity of the $(f,P)$-evaluation. On the other hand, since $\eta$ is replicable, by Theorem 4.13(iv), we have $\mathcal{E}^f_{P,0,T}(\eta) = \mathbf{v}_0$, which leads to a contradiction. Hence, there does not exist any arbitrage opportunity for the seller of the option with price $\mathbf{v}_0$, which implies that the arbitrage-free interval $(\tilde{\mathbf{v}}_0, \mathbf{v}_0)$ is closed on the right.

Suppose now that the option is not replicable. By Theorem 4.8, we have $\varphi^* := \sigma^{-1}Z \in \mathcal{A}_0(\mathbf{v_0})$. We thus have $V_T^{\mathbf{v_0},\varphi^*} \geq \eta$ a.s. Now, since the option is not replicable, we have $P(V_T^{\mathbf{v_0},\varphi^*} > \eta) > 0$. The strategy $\varphi^*$ is thus an arbitrage opportunity for the seller of the option with price $\mathbf{v}_0$, which implies that the arbitrage-free interval $(\tilde{\mathbf{v}}_0, \mathbf{v}_0)$ is opened on the right.[12]

By symmetry, we get the result for the lower bound $\tilde{\mathbf{v}}_0$ of the arbitrage-free interval. ■

For each $\eta \in \mathbf{L^2}$, we denote by $\mathbf{v_0}(\eta)$ (resp., $\tilde{\mathbf{v}}_\mathbf{0}(\eta)$) the superhedging price for the seller (resp., the buyer) of the option with payoff $\eta$. By Proposition 4.17, we derive the following result.

Proposition 4.18. *Suppose that the payoff $\eta$ satisfies the inequality $\tilde{\mathbf{v}}_\mathbf{0}(\eta) \leq \mathbf{v_0}(\eta)$. Then, for each $Q \in \mathcal{Q}$, the quantities $\mathcal{E}^f_{Q,0,T}(\eta)$ and $-\mathcal{E}^f_{Q,0,T}(-\eta)$ are arbitrage-free prices of the option.*

*Remark* 4.19. It may happen that $\eta$ satisfies the strict inequality $\mathbf{v_0}(\eta) < \tilde{\mathbf{v}}_\mathbf{0}(\eta)$ and, hence, that there does not exist an arbitrage-free price for the option with payoff $\eta$. A simple example is given by $f(t,y,z) = -|y|$ and $\eta = 1$. In this case, we have $\mathbf{v_0}(\eta) = e^{-T}$ and $\tilde{\mathbf{v}}_\mathbf{0}(\eta) = e^T$.

Suppose now that $f(t,y,z) \geq -f(t,-y,-z)$ for all $t,y,z$ (which is satisfied, for example, when $f$ is convex with respect to $(y,z)$). Setting $\tilde{f}(t,y,z) := -f(t,-y,-z)$, by the comparison theorem together with Theorem 4.8, we get that for all $\eta \in \mathbf{L^2}$, for all $Q \in \mathcal{Q}$, $\mathbf{v_0}(\eta) \geq \mathcal{E}^f_{Q,0,T}(\eta) \geq \mathcal{E}^{\tilde{f}}_{Q,0,T}(\eta) = -\mathcal{E}^f_{Q,0,T}(-\eta) \geq \tilde{\mathbf{v}}_\mathbf{0}(\eta)$. Hence, for all $\eta \in \mathbf{L^2}$, we have $\mathbf{v_0}(\eta) \geq \tilde{\mathbf{v}}_\mathbf{0}(\eta)$.

[12]From a financial point of view, this property makes sense. Note that, using the definitions of arbitrage opportunities from [20], this property does not hold, and $\mathbf{v}_0$ is always an arbitrage-free price, even in the case when the option is not replicable (cf. [20]).

We recall that, by Theorem 4.8, the strategy $\varphi^* := \sigma^{-1}Z$, where $Z$ is the process from the optional $\mathcal{E}^f$-decomposition (4.8), belongs to $\mathcal{A}_0(\mathbf{v_0})$. When the price of the option is equal to $\mathbf{v_0}$, by investing this amount in the market according to the strategy $\varphi^*$, the seller makes the profit $V_T^{\mathbf{v_0},\varphi^*} - \eta \geq 0$ at time $T$. In the following, we provide a "minimality" property for the seller's profit, as well as a characterization of the superhedging price $\mathbf{v_0}$.

*The linear case.* We start by gathering some observations in the simpler case when the model market is linear with $r = 0$, that is, when $f(t, y, z) = -\theta_t z$. We then have, for each $\nu$ in $\mathcal{V}$, $\mathcal{E}^f_{Q^\nu,0,T}(\eta) = \mathbb{E}_{R^\nu}(\eta)$ (see (3.9) and (3.12) for the definitions of $Q^\nu$ and $R^\nu$). The pricing-hedging duality formula (from Theorem 4.8) reduces here to the classical formula: $\mathbf{v_0} = X_0 = \sup_{R\in\mathcal{P}} \mathbb{E}_R(\eta)$. Furthermore, the dual value process $(X_t)$ satisfies, for each $S \in \mathcal{T}_0$, $X_S = ess \sup_{R\in\mathcal{P}} \mathbb{E}_R(\eta \,|\mathcal{G}_S)$ a.s. , and it admits the linear optional decomposition $X_t = V_t^{\mathbf{v_0},\sigma^{-1}Z} - h_t$. Since $X_T = \eta$, by investing the amount $\mathbf{v_0}$ according to the strategy $\varphi^* := \sigma^{-1}Z$, the seller makes the profit $V_T^{\mathbf{v_0},\varphi^*} - \eta = h_T \geq 0$.

Suppose now that $\sup_{R\in\mathcal{P}} \mathbb{E}_R[\sup_{0\leq t\leq T} X_t^2] < +\infty$. For each $R \in \mathcal{P}$, using the $R$-martingale property of the wealth process, we have $\mathbb{E}_R(V_T^{\mathbf{v_0},\varphi^*} - \eta) = \mathbf{v_0} - \mathbb{E}_R(\eta)$. By taking the infimum over $R \in \mathcal{P}$ in this equality, using the pricing-hedging dual formula, we get that the seller's profit $V_T^{\mathbf{v_0},\varphi^*} - \eta$ satisfies the minimality condition: $\inf_{R\in\mathcal{P}} \mathbb{E}_R(V_T^{\mathbf{v_0},\varphi^*} - \eta) = 0$. Let now $x \in \mathbb{R}$ be such that there exists a strategy $\varphi$ with $\sup_{R\in\mathcal{P}} \|\varphi\|^2_{\mathbb{H}^2_R} < +\infty$ and $V_T^{x,\varphi} \geq \eta$ a.s. For all $R \in \mathcal{P}$, we have $\mathbb{E}_R(V_T^{x,\varphi} - \eta) = x - \mathbb{E}_R(\eta)$. Taking the infimum over $R \in \mathcal{P}$ in this equality, since $\mathbf{v_0} = \sup_{R\in\mathcal{P}} \mathbb{E}_R(\eta)$, we derive that $x = \mathbf{v_0}$ if and only if $\inf_{R\in\mathcal{P}} \mathbb{E}_R(V_T^{x,\varphi} - \eta) = 0$.

*The nonlinear case.* Consider now the case of a nonlinear $f$. We provide analogous results on the seller's profit in terms of the set $\mathcal{Q}$. We assume in the remainder of the section that $\eta$ satisfies the integrability condition $\sup_{Q\in\mathcal{Q}} \mathbb{E}_Q[\sup_{0\leq t\leq T} X_t^2] < +\infty$.

As seen above, $\varphi^* = \sigma^{-1}Z \in \mathcal{A}_0(\mathbf{v_0})$. Note that in general the profit $V_T^{\mathbf{v_0},\varphi^*} - \eta$ is not equal to $h_T$. When the option is not replicable, we have $V_T^{\mathbf{v_0},\varphi^*} - \eta \not\equiv 0$. However, when the seller invests the amount $\mathbf{v_0}$ according to the strategy $\varphi^*$, her gain satisfies the following minimality condition:

$$\inf_{Q\in\mathcal{Q}} \mathbb{E}_Q(V_T^{\mathbf{v_0},\varphi^*} - \eta) = 0. \tag{4.14}$$

This property is a consequence of the following more general result.

Proposition 4.20 (characterization of the superhedging price $\mathbf{v_0}$). *Let $x \in \mathbb{R}$ be such that there exists $\varphi \in \mathcal{A}_0(x)$ with $\sup_{Q\in\mathcal{Q}} \|\varphi\|^2_{\mathbb{H}^2_Q} < +\infty$. Then, $x = \mathbf{v_0}$ if and only if $\inf_{Q\in\mathcal{Q}} \mathbb{E}_Q(V_T^{x,\varphi} - \eta) = 0$. In other terms, an initial capital $x$ which allows the seller to build a superhedging strategy $\varphi$, is equal to $\mathbf{v_0}$, if and only if, the terminal profit $V_T^{x,\varphi} - \eta$ realized by the seller satisfies the minimality condition $\inf_{Q\in\mathcal{Q}} \mathbb{E}_Q(V_T^{x,\varphi} - \eta) = 0$.*[13]

The proof of this result relies on the following lemma, whose proof is given in Appendix D.

[13]Note that, by Lemma D.4, $\varphi^* := \sigma^{-1}Z$ satisfies $\sup_{Q\in\mathcal{Q}} \|\varphi^*\|^2_Q < +\infty$. Since $\varphi^* \in \mathcal{A}_0(\mathbf{v_0})$, the minimality property (4.14) holds. Moreover, we may have $\{\varphi^*\} \subsetneq \mathcal{A}_0(\mathbf{v_0})$. By Proposition 4.20, the property (4.14) holds for any superhedging strategy $\varphi \in \mathcal{A}_0(\mathbf{v_0})$ such that $\sup_{Q\in\mathcal{Q}} \|\varphi\|^2_Q < +\infty$.

Lemma 4.21 (estimates). *Let $(x,\varphi) \in \mathbb{R} \times \mathbb{H}^2$ with $\sup_{Q\in\mathcal{Q}} \|\varphi\|_Q^2 < +\infty$. Suppose $V_T^{x,\varphi} \geq \eta$ a.s. There exist positive constants $m$ and $m'$ such that, for all $Q \in \mathcal{Q}$,*

$$m'(\mathbb{E}_Q(V_T^{x,\varphi} - \eta))^2 \leq x - \mathcal{E}^f_{Q,0,T}(\eta) \leq m(\mathbb{E}_Q(V_T^{x,\varphi} - \eta))^{1/2}. \tag{4.15}$$

*Proof of Proposition* 4.20. By taking the infimum over $Q \in \mathcal{Q}$ in the estimates (4.15), using the dual formula $\mathbf{v_0} = \sup_{Q\in\mathcal{Q}} \mathcal{E}^f_{Q,0,T}(\eta)$ (cf. Theorem 4.8), we derive the desired result. ∎

When the option is not replicable, by Proposition 4.13, there does not exist any $f$-martingale probability measure $Q$ such that $\mathbf{v_0} = \mathcal{E}^f_{Q,0,T}(\eta)$. However, we have the following property.

Theorem 4.22. *There exists a sequence $(Q_n)_{n\in\mathbb{N}}$ of $f$-martingale probability measures such that*

(i) *the seller's price $\mathbf{v_0}$ is equal to the limit of the $f$-evaluation under $Q^n$ as $n$ goes to $+\infty$, that is,*
$$\mathbf{v_0} = \lim_{n\to+\infty} \mathcal{E}^f_{Q_n,0,T}(\eta);$$

(ii) $(Q_n)_{n\in\mathbb{N}}$ converges weakly *to a non-negative measure $Q^* \ll P$ (in the sense that the sequence of the densities with respect to $P$ on $\mathcal{G}_T$ converges $P$-a.s. to the density of $Q^*$), under which the option is "replicable," and $\eta = V_T^{\mathbf{v_0},\sigma^{-1}Z}$ $Q^*$-a.e., where $Z$ is the process from the optional $\mathcal{E}^f$-decomposition* (4.8).

*Proof.* The proof relies on the estimates from Lemma 4.21. We denote by $G$ the gain made by the seller, that is, $G := V_T^{\mathbf{v_0},\sigma^{-1}Z} - \eta$. As seen above, $G$ satisfies the minimality condition (4.14). Denoting by $D$ the set of the densities of the $f$-martingale probability measures, that is, $D := \{\zeta_T^\nu,\ \nu \in \mathcal{V}\}$, we have $\inf_{\zeta\in D} \mathbb{E}[\zeta G] = 0$. There exists a sequence $(\bar{\zeta}^n)_{n\in\mathbb{N}}$ with $\bar{\zeta}^n \in D$ for all $n$, such that the sequence $(\mathbb{E}[\bar{\zeta}^n G])_{n\in\mathbb{N}}$ is nonincreasing and satisfies $\lim_{n\to\infty} \mathbb{E}[\bar{\zeta}^n G] = 0$. Now, for all $n \in \mathbb{N}$, $\mathbb{E}[\bar{\zeta}^n] = 1$. Hence, by Komlós' theorem [22], there exists a sequence $(\zeta^n)_{n\in\mathbb{N}}$ of convex combinations of $\bar{\zeta}^n$ such that, for all $n \in \mathbb{N}$, $\zeta^n \in Conv\{\bar{\zeta}^n, \bar{\zeta}^{n+1}, \dots\}$ which converges $P$- a.s. to a random variable denoted by $\zeta$. Since $\mathcal{V}$ is convex, we can show that $D = \{\zeta^\nu,\ \nu \in \mathcal{V}\}$ is convex, by using the form of $\zeta^\nu$ (cf. (3.9)). We thus have $\zeta^n \in D$ for all $n \in \mathbb{N}$. We also have $\mathbb{E}[\zeta^n] = 1$ for all $n \in \mathbb{N}$. Hence, by Fatou's lemma, $\mathbb{E}[\zeta] \leq 1$. Since the sequence $(\mathbb{E}[\bar{\zeta}^n G])_{n\in\mathbb{N}}$ is nonincreasing, we get $0 \leq \mathbb{E}[\zeta^n G] \leq \mathbb{E}[\bar{\zeta}^n G]$ for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \mathbb{E}[\bar{\zeta}^n G] = 0$, we derive that $\lim_{n\to\infty} \mathbb{E}[\zeta^n G] = 0$. For each $n \in \mathbb{N}$, let $Q^n$ be the $f$-martingale probability measure with density $\zeta^n$. We have $\lim_{n\to\infty} \mathbb{E}_{Q^n}[G] = 0$. By using the estimates (4.15) , we deduce $\lim_{n\to+\infty} \mathcal{E}^f_{Q_n,0,T}(\eta) = X_0$. Moreover, by Fatou's lemma, we get $0 \leq \mathbb{E}[\zeta G] \leq \lim_{n\to\infty} \mathbb{E}[\zeta^n G] = 0$, from which we get $\mathbb{E}[\zeta G] = 0$. Denoting by $Q^*$ the measure with density $\zeta$, we get $\int G dQ^* = 0$. The result follows. ∎

*Remark* 4.23. In the linear case, this result becomes, there exists a sequence $(R_n)_{n\in\mathbb{N}}$ of martingale probability measures such that $\mathbf{v_0} = \lim_{n\to+\infty} \mathbb{E}_{R_n}(\eta)$ and $(R_n)_{n\in\mathbb{N}}$ converges weakly to a measure $R^* \ll P$, satisfying $\eta = V_T^{\mathbf{v_0},\sigma^{-1}Z}$, $R^*$-a.e.

**5. Study of the dual value problem.** The study in this section is done under the primitive probability $P$, which is more tractable. Moreover, working under $P$ allows us to assume weaker integrability conditions (compared to those of the previous section). Recall that $\mathcal{V}$ is the set of bounded predictable processes $\nu$ such that $\nu_{\vartheta\wedge T} > -1$ a.s. We introduce a family of drivers $(f^\nu, \nu \in \mathcal{V})$.

Definition 5.1 (driver $f^\nu$ and $\mathcal{E}^\nu$-expectation). *For $\nu \in \mathcal{V}$, we define*

$$f^\nu(\cdot, t, y, z, k) := f(\cdot, t, y, z) + \nu_t \lambda_t\big(k - \beta_t \sigma_t^{-1} z\big). \tag{5.1}$$

*The mapping $f^\nu$ is a $\lambda$-admissible driver. The associated nonlinear family of operators, denoted by $\mathcal{E}^{f^\nu}$ or, simply, $\mathcal{E}^\nu$, is defined as follows: for each $S \le T$ and each $\eta \in L^2(\mathcal{G}_S)$,*

$$\mathcal{E}^\nu_{\cdot,S}(\eta) := X^\nu_\cdot,$$

*where $(X^\nu, Z^\nu, K^\nu) = (X, Z, K)$ is the unique solution in $S^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ of the BSDE*

$$-\,dX_t = \big(f(t, X_t, Z_t) + \nu_t \lambda_t (K_t - \beta_t \sigma_t^{-1} Z_t)\big)\, dt - Z_t dW_t - K_t dM_t,\ \ X_S = \eta. \tag{5.2}$$

*Remark* 5.2. By Proposition 3.8, we derive that, for each $\nu \in \mathcal{V}$, for each $\eta \in L^2(\mathcal{G}_T) \cap L^2_{Q^\nu}(\mathcal{G}_T)$, the $(f^\nu, P)$-evaluation of $\eta$ is equal to its $(f, Q^\nu)$-evaluation, that is,

$$\mathcal{E}^\nu_{\cdot,T}(\eta) = \mathcal{E}^f_{Q^\nu,\cdot,T}(\eta).$$

Hence, a process $X$ in $S^2 \cap S^2_{Q^\nu}$ is an $\mathcal{E}^\nu$-supermartingale (resp., $\mathcal{E}^\nu$-martingale) if and only if it is an $\mathcal{E}^f_{Q^\nu}$-supermartingale (resp., $\mathcal{E}^f_{Q^\nu}$-martingale).

We consider an option with maturity $T$ and payoff $\eta \in L^2(\mathcal{G}_T)$. The superhedging price $v_0$ of the option is defined similarly to $\mathbf{v_0}$ from (4.1), with $\mathbf{H^2}$ replaced by $\mathbb{H}^2$, that is,

$$v_0 := \inf\{x \in \mathbb{R} :\ \exists \varphi \in \mathbb{H}^2 \ \text{ s.t. } V_T^{x,\varphi} \ge \eta \text{ a.s.}\}. \tag{5.3}$$

For each $S \in \mathcal{T}$, we define *the dual value at time $S$* as the $\mathcal{F}_S$-measurable random variable,

$$X(S) := ess \sup_{\nu \in \mathcal{V}_S} \mathcal{E}^\nu_{S,T}(\eta), \tag{5.4}$$

where $\mathcal{V}_S$ is the set of bounded predictable processes $\nu$ defined on $[S, T]$, such that $\nu_t > -1$, $S \le t \le T$, $\lambda_t dP \otimes dt$-a.e., and $\mathcal{E}^\nu$ is defined in Definition 5.1. The family of random variables $(X(S),\ S \in \mathcal{T})$ is called the *dual value family.*[14] We suppose that the value family satisfies

$$\mathbb{E}[ess \sup_{S \in \mathcal{T}} X(S)^2] < +\infty. \tag{5.5}$$

We shall see below that this integrability condition is equivalent to the existence of $x_0 \in \mathbb{R}$ and $\psi \in \mathbb{H}^2$ such that $\eta \le V_T^{x_0,\psi}$ a.s. (which means that $v_0 < +\infty$).

Let us recall the definition of an admissible family of random variables indexed by stopping times in $\mathcal{T}$ (or $\mathcal{T}$-system in the vocabulary of Dellacherie and Lenglart [7]).

[14] If $\eta \in \mathbf{L^2}(\mathcal{G}_T)$, then, by Remark 5.2, $ess \sup_{\nu \in \mathcal{V}_S} \mathcal{E}^\nu_{S,T}(\eta) = ess \sup_{Q \in \mathcal{Q}} \mathcal{E}^f_{Q,S,T}(\eta)$ a.s. This *dual value* thus coincides with the value defined by (4.6). Hence, the notation is consistent.

Definition 5.3. *We say that a family* $Y = (Y(S),\ S \in \mathcal{T})$ *is* admissible *if it satisfies*

1. *for all* $S \in \mathcal{T}$, $Y(S)$ *is a real-valued* $\mathcal{G}_S$*-measurable random variable;*
2. *for all* $S, S' \in \mathcal{T}$, $Y(S) = Y(S')$ *a.s. on* $\{S = S'\}$.

*Moreover, we say that* $Y$ *is* uniformly square-integrable *if* $\mathbb{E}[ess\sup_{S\in\mathcal{T}} Y(S)^2] < \infty$.

Lemma 5.4. *The value family* $(X(S),\ S \in \mathcal{T})$ *is an admissible family.*

*Moreover, for each* $S \in \mathcal{T}$, *there exists a sequence of controls* $(\nu^n)_{n\in\mathbb{N}}$ *with* $\nu^n \in \mathcal{V}_S$ *for all* $n$, *such that the sequence* $(\mathcal{E}^{\nu^n}_{S,T}(\eta))_{n\in\mathbb{N}}$ *is nondecreasing and satisfies*

$$X(S) = \lim_{n\to\infty} \mathcal{E}^{\nu^n}_{S,T}(\eta) \text{ a.s.} \tag{5.6}$$

*Proof.* For each $S \in \mathcal{T}$, by definition (5.4), $X(S)$ is $\mathcal{G}_S$-measurable as the essential supremum of $\mathcal{G}_S$-measurable random variables. Let $S, S' \in \mathcal{T}$ such that $S = S'$ a.s. We have $\mathcal{E}^{\nu}_{S,T}(\eta) = \mathcal{E}^{\nu}_{S',T}(\eta)$ a.s. for all $\nu \in \mathcal{V}$. Hence, $ess\sup_{\nu\in\mathcal{V}} \mathcal{E}^{\nu}_{S,T}(\eta) = ess\sup_{\nu\in\mathcal{V}} \mathcal{E}^{\nu}_{S',T}(\eta)$ a.s. From this, together with (5.4), we get $X(S) = X(S')$ a.s. The value family is thus admissible.

Let us show the second assertion. By a classical result on essential suprema, it is sufficient to prove that, for each $S \in \mathcal{T}$, the set $\{\mathcal{E}^{\nu}_{S,T}(\eta), \nu \in \mathcal{V}_S\}$ is stable under pairwise maximization. Indeed, let $\nu, \nu' \in \mathcal{V}_S$. Set $A := \{\, \mathcal{E}^{\nu'}_{S,T}(\eta) \le \mathcal{E}^{\nu}_{S,T}(\eta)\,\}$. We have $A \in \mathcal{F}_S$. Set $\tilde{\nu} := \nu\mathbf{1}_A + \nu'\mathbf{1}_{A^c}$. Then $\tilde{\nu} \in \mathcal{V}_S$. We have $\mathcal{E}^{\tilde{\nu}}_{S,T}(\eta)\mathbf{1}_A = \mathcal{E}^{f^{\tilde{\nu}}\mathbf{1}_A}_{S,T}(\eta\mathbf{1}_A) = \mathcal{E}^{f^{\nu}\mathbf{1}_A}_{S,T}(\eta\mathbf{1}_A) = \mathcal{E}^{\nu}_{S,T}(\eta)\mathbf{1}_A$ a.s. and, similarly, on $A^c$. It follows that $\mathcal{E}^{\nu}_{S,T}(\eta) = \mathcal{E}^{\nu}_{S,T}(\eta)\mathbf{1}_A + \mathcal{E}^{\nu'}_{S,T}(\eta)\mathbf{1}_{A^c} = \mathcal{E}^{\nu}_{S,T}(\eta)\vee \mathcal{E}^{\nu'}_{S,T}(\eta)$ a.s. The proof is complete. ∎

Let $g$ be a $\lambda$-admissible driver satisfying Assumption 2.2. We give the definition of an $\mathcal{E}^g$*-supermartingale* (resp., $\mathcal{E}^g$*-submartingale*, $\mathcal{E}^g$*-martingale*) family.

Definition 5.5. *A uniformly square integrable admissible family* $(Y(S),\ S \in \mathcal{T})$ *is said to be an* $\mathcal{E}^g$*-supermartingale (resp.,* $\mathcal{E}^g$*-submartingale,* $\mathcal{E}^g$*-martingale) family if for all* $S, S' \in \mathcal{T}$ *such that* $S \ge S'$ *a.s.,* $\mathcal{E}^g_{S',S}(Y(S)) \le$ *(resp.,* $\ge, =$*)* $Y(S')$ *a.s.*

Lemma 5.6. *The value family* $(X(S))$ *is the smallest admissible family such that for all* $\nu \in \mathcal{V}$, *it is an* $\mathcal{E}^{\nu}$*-supermartingale family (that is an* $\mathcal{E}^{f^{\nu}}$*-supermartingale family) satisfying* $X(T) = \eta$ *a.s.*

*Proof.* We first note that, by the definition of $X(T)$, we have $X(T) = \eta$ a.s.

Fix $S \in \mathcal{T}_{S'}$ a.s. There exists an optimizing sequence of controls $(\nu^n)_{n\in\mathbb{N}}$ with $(\nu^n)$ in $\mathcal{V}_S$ such that equality (5.6) holds. Let $\nu \in \mathcal{V}$. By the continuity of $\mathcal{E}^{\nu}$, we have $\mathcal{E}^{\nu}_{S',S}(X(S)) = \lim_{n\to\infty} \mathcal{E}^{\nu}_{S',S}(\mathcal{E}^{\nu^n}_{S,T}(\eta))$ a.s. We define for each $n$ the control $\tilde{\nu}^n_t := \nu_t\mathbf{1}_{[S',S]}(t) + \nu^n_t\mathbf{1}_{[S,T]}(t)$, which belongs to $\mathcal{V}_{S'}$. Notice that $f^{\tilde{\nu}^n} = f^{\nu}\mathbf{1}_{[S',S]} + f^{\nu^n}\mathbf{1}_{[S,T]}$, which implies that $\mathcal{E}^{\nu}_{S',S}(\mathcal{E}^{\nu^n}_{S,T}(\eta)) = \mathcal{E}^{\tilde{\nu}^n}_{S',S}(\mathcal{E}^{\tilde{\nu}^n}_{S,T}(\eta)) = \mathcal{E}^{\tilde{\nu}^n}_{S',T}(\eta)$ a.s. Hence, we obtain $\mathcal{E}^{\nu}_{S',S}(X(S)) = \lim_{n\to\infty} \mathcal{E}^{\tilde{\nu}^n}_{S',T}(\eta) \le X(S')$ a.s., where the last inequality follows from the definition of $X(S')$. We now show the minimality property. Let $(X'(S), S \in \mathcal{T})$ be an admissible family such that for each $\nu \in \mathcal{V}$, it is an $\mathcal{E}^{\nu}$-supermartingale family satisfying $X'(T) = \eta$ a.s. By the properties of $X'$, for all $S \in \mathcal{T}$, and all $\nu \in \mathcal{V}$, we have $X'(S) \ge \mathcal{E}^{\nu}_{S,T}(X'(T)) = \mathcal{E}^{\nu}_{S,T}(\eta)$ a.s. Taking the essential supremum over $\nu \in \mathcal{V}_S$, we deduce $X'(S) \ge X(S)$ a.s. ∎

Using the above lemma together with Proposition A.6, we get the following result.

Theorem 5.7. *There exists an RCLL adapted process $(X_t) \in S^2$, called the dual value process which aggregates the value family $(X(S))$. The process $(X_t)$ is a strong $\mathcal{E}^\nu$-supermartingale for all $\nu \in \mathcal{V}$ and $X_T = \eta$ a.s. Moreover, $(X_t)$ is the smallest process in $S^2$ satisfying these properties.*

*Proof.* Since $0 \in \mathcal{V}$, Lemma 5.6 implies that the value family $(X(S))$ is a strong $\mathcal{E}^0$-supermartingale family. By Lemma A.1, there exists a right upper-semicontinuous (r.u.s.c.) optional process $(X_t)$ such that $\mathbb{E}[ess\sup_{S\in\mathcal{T}} X_S^2] < \infty$ which aggregates the family $(X(S),\ S \in \mathcal{T})$ with $X_T = \eta$ a.s. Moreover, by Lemma 5.6, $(X_t)$ is the minimal optional process which is a strong $\mathcal{E}^\nu$-supermartingale for all $\nu \in \mathcal{V}$, with terminal value greater than or equal to $\eta$. Using this minimality property, we show in the appendix (cf. Proposition A.6), that $(X_t)$ is a right-continuous process belonging to $S^2$. ■

From this result together with Theorem B.2 applied to the right-continuous process $(X_t)$, we derive that the value process $(X_t)$ admits the optional $\mathcal{E}^f$-decomposition from Theorem B.2. More precisely, we have the analogous Corollary 4.6 under the weaker integrability assumption (5.5).

Corollary 5.8 (optional $\mathcal{E}^f$-decomposition of the dual value process). *There exists a unique $Z \in \mathbb{H}^2$ and a unique nondecreasing optional RCLL process $h$ with $h_0 = 0$ and $\mathbb{E}[h_T^2] < \infty$ such that (4.8) holds. Moreover, the portfolio strategy $\varphi^* := \sigma^{-1}Z$ satisfies $V_T^{X_0,\varphi^*} \geq \eta$ a.s.*

Corollary 5.9. *The integrability condition (5.5), that is $\mathbb{E}[ess\sup_{S\in\mathcal{T}} X(S)^2] < +\infty$, is equivalent to the existence of $x_0 \in \mathbb{R}$ and $\psi \in \mathbb{H}^2$ such that $\eta \leq V_T^{x_0,\psi}$ a.s.*

*Proof.* Suppose that $\mathbb{E}[ess\sup_{S\in\mathcal{T}} X(S)^2] < +\infty$. By Corollary 5.8, we have $V_T^{X_0,\varphi^*} \geq \eta$. Suppose now that there exists $x_0 \in \mathbb{R}$ and $\psi \in \mathbb{H}^2$ such that $\eta \leq V_T^{x_0,\psi}$ a.s. By similar arguments as in the proof of Corollary 4.7, we get $X(S) \leq V_S^{x_0,\psi}$ a.s. On the other hand, as $0 \in \mathcal{V}$, we have $X(S) \geq \mathcal{E}^0_{S,T}(\eta)$ a.s. Since $\mathcal{E}^0_{\cdot,T}(\eta) \in S^2$ and $V^{x_0,\psi} \in S^2$, we thus have $\mathbb{E}[ess\sup_{S\in\mathcal{T}} X(S)^2] < +\infty$. ■

Using the optional $\mathcal{E}^f$-decomposition of the dual value process from Corollary 5.8, we derive the following dual representation for the seller's superhedging price $v_0$, which extends the pricing-hedging duality result (Theorem 4.8) to the case when the payoff $\eta$ satisfies only integrability conditions under the primitive probability measure $P$.

Theorem 5.10 (pricing-hedging duality via the set $\mathcal{V}$). *Suppose that condition (5.5) holds. The* seller's superhedging price *$v_0$ of the option satisfies*

$$v_0 = \sup_{\nu\in\mathcal{V}} \mathcal{E}^\nu_{0,T}(\eta). \tag{5.7}$$

*Moreover, the portfolio strategy $\varphi^* := \sigma^{-1}Z$, where the process $Z$ is the one from the optional $\mathcal{E}^f$-decomposition of the dual value process $(X_t)$ (cf. Corollary 5.8), is a superhedging strategy for the seller, that is, $V_T^{v_0,\varphi^*} \geq \eta$ a.s.*

*Proof.* Using the $\mathcal{E}^\nu$-martingale property of the wealth processes and proceeding as in the proof of Lemma 4.5, we get $v_0 \geq \sup_{\nu\in\mathcal{V}} \mathcal{E}^\nu_{0,T}(\eta)$. The converse inequality follows from Corollary 5.8 and the same arguments as those from the proof of Theorem 4.8 on the pricing-hedging duality for $\mathbf{v_0}$. ■

Under the stronger integrability assumptions on $\eta$ from section 4, we have $v_0 = \mathbf{v_0}$. More precisely, we have the following.

Corollary 5.11. *If* $\mathbb{E}_Q(\eta^2) < \infty$ *for all* $Q \in \mathcal{Q}$*, and* $\mathbb{E}_Q[ess\sup_{S\in\mathcal{T}_0} X(S)^2] < \infty$*, then* $v_0 = \mathbf{v_0}$*.*

*Proof.* By Remark 5.2 and the pricing-hedging duality theorem for $v_0$, we have $v_0 = \sup_{Q\in\mathcal{Q}} \mathcal{E}^f_{Q,0,T}(\eta)$. By the pricing-hedging duality theorem for $\mathbf{v_0}$ (cf. Theorem 4.8), we derive that $v_0 = \mathbf{v_0}$. ∎

The *infinitesimal characterization* of the dual value process $(X_t)$ given in section 4 under Assumption (4.7) (cf. Theorem 4.12) still holds under the weaker assumptions from this section, if we replace the spaces $\mathbf{L^2}$ by $L^2$, $\mathbf{H^2}$ by $H^2$, $\mathbf{H^2_\lambda}$ by $H^2_\lambda$, and $\mathbf{S^2}$ by $S^2$.

## 6. Concluding remarks.

We have studied the superhedging price for the seller of a European option in a nonlinear incomplete market model with default. We introduce the concept of $f$-martingale probability measures, for which we give a characterization via their densities with respect to $P$. We establish a pricing-hedging duality formula for this price via the set $\mathcal{Q}$ of $f$-martingale probability measures. The proof relies on the nonlinear $\mathcal{E}^f$-optional decomposition for the processes which are $(f,Q)$-supermartingales for all $Q \in \mathcal{Q}$ (see Theorem B.2 in the RCLL case). We also provide a characterization of the seller's (superhedging) price as the initial value of the minimal supersolution of a constrained BSDE with default. The proof of this characterization relies on the nonlinear $\mathcal{E}^f$-predictable decomposition (see Theorem B.1). Moreover, we provide a replication criterium and some properties of the terminal gain realized by the seller. By a form of symmetry, we derive corresponding results for the buyer. We study the case of the American option in [17], both from the point of view of the seller and the buyer,[15] when the payoff process is completely irregular, which raises a lot of technical issues. The study relies on many results from the present paper, in particular the nonlinear predictable and optional decompositions.

## Appendix A. Some results on $\mathcal{E}^g$-supermartingale families and processes.

Let $g$ be a $\lambda$-admissible driver (cf. Definition 2.1) which may depend on $k$ (contrary to the driver $f$ considered in this paper), and which satisfies Assumption 2.2 (with only $\gamma_t^{y,z,k_1,k_2} \geq -1$) so that $\mathcal{E}^g$ is monotone. We provide some results on $\mathcal{E}^g$-supermartingale families and $\mathcal{E}^g$-supermartingale processes, which are useful in the paper.

Let $\mathbb{C}^2$ be the set of real-valued purely discontinuous nondecreasing RCLL $\mathbb{G}$-optional processes $C$ with $C_{0^-} = 0$ and $\mathbb{E}(C_T^2) < \infty$.

Lemma A.1. *Let* $(X(S),\ S \in \mathcal{T})$ *be an* $\mathcal{E}^g$*-supermartingale family. Then, there exists an r.u.s.c. optional process* $(X_t)$ *such that* $\mathbb{E}[ess\sup_{S\in\mathcal{T}} X_S^2] < \infty$ *which aggregates the family* $(X(S),\ S \in \mathcal{T})$*, that is, such that* $X(S) = X_S$ *a.s. for all* $S \in \mathcal{T}$*. Moreover, the process* $(X_t)$ *is a strong* $\mathcal{E}^g$*-supermartingale, that is, for all* $S, S' \in \mathcal{T}$ *such that* $S \geq S'$ *a.s.,* $\mathcal{E}^g_{S',S}(X_S) \leq X_{S'}$ *a.s.*

*Proof.* By [15, Lemma 8.3], the $\mathcal{E}^g$-supermartingale family $(X(S),\ S \in \mathcal{T})$ is r.u.s.c. (along stopping times). It follows from [7, Theorem 4] that there exists an r.u.s.c. optional process

[15]Contrary to the European case, there is no symmetry between them.

$(X_t)$ which aggregates the family $(X(S),\ S \in \mathcal{T})$. The process $(X_t)$ is clearly a strong $\mathcal{E}^g$-supermartingale. ∎

*Remark* A.2. As a consequence, we recover that a strong $\mathcal{E}^g$-supermartingale is necessarily r.u.s.c.

We now recall the $\mathcal{E}^g$-Mertens decomposition of an $\mathcal{E}^g$- supermartingale proved in [14], and provide some estimates of the norm of the associated Mertens process, which are analogous to the nonlinear case of the inequalities of the potential theory [8].

Theorem A.3 ($\mathcal{E}^g$-Mertens decomposition of $\mathcal{E}^g$-supermartingales). *Let $(Y_t)$ be an optional process in $\mathbb{S}^2$. Then $(Y_t)$ is an $\mathcal{E}^g$-submartingale if and only if there exists a nondecreasing right continuous and predictable processes $A$ in $\mathcal{A}^2$, a nondecreasing adapted right continuous and purely discontinuous processes $C$ in $\mathbb{C}^2$, and $(Z,K) \in \mathbb{H}^2 \times \mathbb{H}^2_\nu$ such that*

$$-dY_s = g(s, Y_s, Z_s, K_s)ds - Z_s dW_s - K_t dM_t + dA_s + dC_{s^-}. \tag{A.1}$$

*Moreover, this decomposition is unique. We also have the following estimate:*

$$\mathbb{E}(A_T^2) + \mathbb{E}(C_T^2) + \|Z\|^2 + \|\ K\|_\lambda^2 \le c\left(\|X\|_{S^2}^2 + \|g(\cdot,0,0,0)\|^2\right), \tag{A.2}$$

*where $c$ is a universal positive constant depending only on $T$ and the $\lambda$-*constant *$C$ of $g$ (see (2.1)).*

*Remark* A.4. As a consequence, we recover that a strong $\mathcal{E}^g$-supermartingale admits left and right limits.

*Proof.* The proof of (A.1) is given in [14]. Let us show (A.2). By classical computations (see, e.g., the proof of a priori estimates for BSDEs with default in [9]), applying Itô's formula to $X_t^2$, and using the $\lambda$-Lipschitz property of $g$, we obtain

(A.3)
$$X_0^2 + \|Z\|^2 + \|\ K\|_\lambda^2$$
$$\le \mathbb{E}\left[X_T^2 + \int_0^T 2C|X_s|(|X_s| + |Z_s| + \sqrt{\lambda_s}|K_s|)ds + \int_0^T 2|X_s|(dA_s + dC_s) + \int_0^T 2|X_s g_s^0|ds\right],$$

where $C$ is the $\lambda$-constant of $g$, and where $g_s^0 := g(s,0,0,0)$. Let now $\varepsilon > 0$. Noting that for all $x,y \in \mathbb{R}$, $2Cxy \le \varepsilon^{-1}C^2x^2 + \varepsilon y^2$, and using the inequality $\|X\|^2 \le T\|X\|_{S^2}^2$, we get

$$\mathbb{E}\left[\int_0^T 2C|X_s|(|Z_s| + \sqrt{\lambda_s}|K_s|)ds\right] \le 2\varepsilon^{-1}C^2T\|X\|_{S^2}^2 + \varepsilon(\|Z\|^2 + \|\ K\|_\lambda^2). \tag{A.4}$$

Now, by (A.1), since $g$ is $\lambda$-admissible and thus satisfies (2.1), using classical martingale inequalities, we derive that there exists a constant $\bar{C}$ which does not depend on $\varepsilon$ such that

$$\mathbb{E}(A_T^2) \le \bar{C}(\|X\|_{S^2}^2 + \|g^0\|^2 + \|Z\|^2 + \|\ K\|_\lambda^2). \tag{A.5}$$

Hence, since $\mathbb{E}[\int_0^T 2|X_s|dA_s] \le \varepsilon^{-1}\|X\|_{S^2}^2 + \varepsilon\mathbb{E}(A_T^2)$, we get

$$\mathbb{E}\left[\int_0^T 2|X_s|dA_s\right] \le \varepsilon^{-1}\|X\|_{S^2}^2 + \varepsilon\bar{C}(\|X\|_{S^2}^2 + \|g^0\|^2 + \|Z\|^2 + \|\ K\|_\lambda^2), \tag{A.6}$$

and a similar estimate for $\mathbb{E}[\int_0^T 2|X_s|dC_s]$. Using (A.3) and (A.4), we derive that for all $\varepsilon \in ]0,1]$,

$$\|Z\|^2 + \|\ K\|_\lambda^2 \leq \tilde{C}\varepsilon^{-1}(\|X\|_{S^2}^2 + \|g^0\|^2) + \varepsilon(1+2\bar{C})(\|Z\|^2 + \|\ K\|_\lambda^2),$$

where $\tilde{C}$ is a constant depending only on $T$ and $C$. Choosing $\varepsilon := \frac{1}{2(1+2\bar{C})}$, we deduce $\|Z\|^2 + \|\ K\|_\lambda^2 \leq \underline{C}(\|X\|_{S^2}^2 + \|g^0\|^2)$, where $\underline{C}$ is a constant depending only on $T$ and $C$. Using (A.5), we derive the desired estimate (A.2). ∎

Lemma A.5. *If $(X_t)_{t\in[0,T]}$ is a strong $\mathcal{E}^g$-supermartingale, then the process of right-limits $(X_{t+})_{t\in[0,T]}$ (where, by convention, $X_{T+} := X_T$) is a strong $\mathcal{E}^g$-supermartingale.*

*Proof.* Since $(X_t)$ is a strong $\mathcal{E}^g$-supermartingale, $(X_t)$ has right-limits (cf. Remark A.4). Let us show that the process $(X_{t+})$ is a strong $\mathcal{E}^g$-supermartingale. Let $S, \theta$ be two stopping times in $\mathcal{T}$ with $S \leq \theta$ a.s. There exist two nondecreasing sequences of stopping times $(S_n)$ and $(\theta_n)$ such that for each $n$, $S_n > S$ a.s. on $\{S < T\}$, and $\theta_n > \theta$ a.s. on $\{\theta < T\}$. Replacing if necessary $S_n$ by $S_n \wedge \theta_n$, we can suppose that for each $n$, $S_n \leq \theta_n$ a.s. Let $\nu \in \mathcal{V}$. Since $(X_t)$ is a strong $\mathcal{E}^g$-supermartingale, for each $n$, $\mathcal{E}^g_{S_n,\theta_n}(X_{\theta_n}) \leq X_{S_n}$ a.s. By the monotonicity property of $\mathcal{E}^g$, we derive that, for each $n$, $\mathcal{E}^g_{S,S_n}(\mathcal{E}^g_{S_n,\theta_n}(X_{\theta_n})) \leq \mathcal{E}^g_{S,S_n}(X_{S_n})$ a.s., which, by the consistency property of $\mathcal{E}^g$ implies $\mathcal{E}^g_{S,\theta_n}(X_{\theta_n}) \leq \mathcal{E}^g_{S,S_n}(X_{S_n})$ a.s. By letting $n$ tend to $\infty$ in this inequality and applying the continuity property (with respect to terminal time and terminal condition) of BSDEs with default (cf. [9]), we get $\mathcal{E}^g_{S,\theta}(X_{\theta+}) \leq \mathcal{E}^g_{S,S}(X_{S+}) = X_{S+}$ a.s. Hence,$(X_{t+})$ is a strong $\mathcal{E}^g$-supermartingale. ∎

Let $\mathcal{N}$ be a nonempty set. Let $(g^\nu,\ \nu \in \mathcal{N})$ be a family of $\lambda$-admissible drivers satisfying Assumption 2.2 (with only $\gamma_t^{\nu,y,z,k_1,k_2} \geq -1$) so that $\mathcal{E}^{g^\nu}$ is monotone. The following result is used to prove the right-continuity of the value process (see the proof of Proposition 5.7).

Proposition A.6. *Let $\eta$ be a given random variable belonging to $L^2(\mathcal{G}_T)$. Let $(X_t)_{t\in[0,T]}$ be an optional process such that $(X_t)$ is a strong $\mathcal{E}^{g^\nu}$-supermartingale for all $\nu \in \mathcal{N}$ and such that $X_T \geq \eta$ a.s. Assume moreover that $(X_t)$ is* minimal, *that is, $(X_t)$ is the smallest optional process satisfying these properties. Then, the process $(X_t)$ is right-continuous.*

*Proof.* Since $(X_t)$ is a strong $\mathcal{E}^{g^\nu}$-supermartingale, it is r.u.s.c. and has right limits (cf. Remarks A.2 and A.4). We thus have $X_{t+} \leq X_t$ for all $t \in [0,T]$ a.s. Since $(X_t)$ is a strong $\mathcal{E}^{g^\nu}$-supermartingale for all $\nu \in \mathcal{N}$, it follows by Lemma A.5 that $(X_{t+})$ is a strong $\mathcal{E}^{g^\nu}$-supermartingale for all $\nu \in \mathcal{N}$. We also note that $X_{T+} = X_T \geq \eta$ a.s. Using the *minimality* property of $(X_t)$, we thus get $X_t \leq X_{t+}$ for all $t \in [0,T]$ a.s. We conclude that $X_{t+} = X_t$ for all $t \in [0,T]$ a.s., which ends the proof. ∎

*Remark* A.7. This property still holds when the (terminal) constraint $X_T \geq \eta$ a.s. is replaced by the constraint $X_t \geq \xi_t$ for all $t \in [0,T]$ a.s., where $(\xi_t)$ is a given *right lower-semicontinuous* (r.l.s.c.) process belonging to $\mathbb{S}^2$. The proof is analogous to the proof of Proposition A.6. The inequality $X_{T+} \geq \eta$ is to be replaced by $X_{t+} \geq \xi_t$ for all $t \in [0,T]$ a.s., which holds due to the right lower-semicontinuity of $(\xi_t)$.

**Appendix B. Nonlinear predictable and optional $\mathcal{E}^f$-decompositions.** We provide predictable and optional $\mathcal{E}^f$-decompositions for processes which are strong $\mathcal{E}^\nu$-supermartingales for all $\nu \in \mathcal{V}$. We use the notation introduced in section 3.1. We also recall that $\mathbb{C}^2$ is the set of purely discontinuous nondecreasing RCLL processes $C$ with $C_{0^-} = 0$ and $\mathbb{E}(C_T^2) < \infty$.

Theorem B.1 (predictable $\mathcal{E}^f$-decomposition). *Let $(X_t) \in \mathbb{S}^2$ be a strong $\mathcal{E}^\nu$-supermartingale for all $\nu \in \mathcal{V}$. There exists a unique process $(Z, K, A, C) \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2 \times \mathbb{C}^2$ such that*

$$-dX_t = f(t, X_t, Z_t)dt - Z_t dW_t - K_t dM_t + dA_t + dC_{t-}, \tag{B.1}$$

(B.2)

$$A_\cdot + \int_0^\cdot (K_s - \beta_s \sigma_s^{-1} Z_s)\lambda_s ds \in \mathcal{A}^2 \quad \text{and} \quad (K_t - \beta_t \sigma_t^{-1} Z_t)\lambda_t \le 0,\ t \in [0,T],\ dP \otimes dt - \text{a.e.}$$

*Conversely, if $(X_t) \in \mathbb{S}^2$ satisfies (B.1) and (B.2) (with $(Z, K, A, C) \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2 \times \mathbb{C}^2$), then $(X_t)$ is a strong $\mathcal{E}^\nu$-supermartingale for all $\nu \in \mathcal{V}$.*

*Proof.* As $(X_t)$ is a strong $\mathcal{E}^0$-supermartingale, by the $\mathcal{E}^0$-Mertens decomposition (see Theorem A.3), there exists a unique process $(Z, K, A, C)$ in $\mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2 \times \mathbb{C}^2$ such that (B.1) holds. Let $\nu \in \mathcal{V}$. Since $(X_t)$ is a strong $\mathcal{E}^\nu$- supermartingale, by the $\mathcal{E}^\nu$-Mertens decomposition, (see Theorem A.3), there exists a unique process $(Z^\nu, K^\nu, A^\nu, C^\nu)$ in $\mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2 \times \mathbb{C}^2$ such that

$$-dX_t = \big(f(t, X_t, Z_t^\nu) + (K_t^\nu - \beta_t \sigma_t^{-1} Z_t^\nu)\nu_t \lambda_t\big)\, dt - Z_t^\nu dW_t - K_t^\nu dM_t + dA_t^\nu + dC_{t-}^\nu. \tag{B.3}$$

Since $X$ satisfies (B.1) and (B.3), by the uniqueness of the canonical decomposition of a special optional semimartingale (cf. Lemma D.2), and the uniqueness of the representation of the martingale part as the sum of two stochastic integrals (with respect to $W$ and $M$), we get $Z_t = Z_t^\nu$ $dP \otimes dt$-a.e. and $K_t = K_t^\nu$ $dP \otimes dt$-a.e., $C_{t-} = C_{t-}^\nu$, for all $t$ a.s. Hence, equality (B.3) can be written

$$-dX_t = \big(f(t, X_t, Z_t) + (K_t - \beta_t \sigma_t^{-1} Z_t)\nu_t \lambda_t\big)\, dt - Z_t dW_t - K_t dM_t + dA_t^\nu + dC_{t-}. \tag{B.4}$$

From (B.1) and (B.4), we derive that

$$dA_t^\nu = dA_t - (K_t - \beta_t \sigma_t^{-1} Z_t)\nu_t \lambda_t dt. \tag{B.5}$$

Let us show that this implies that $(K_t - \beta_t \sigma_t^{-1} Z_t)\lambda_t \le 0$ $dP \otimes dt$-a.e. Suppose by contradiction that there exists a predictable set $A \subset [0,T] \times \Omega$ such that $(dP \otimes dt)(A) > 0$ and $(K_t - \beta_t \sigma_t^{-1} Z_t)\lambda_t > 0$, $t \in [0,T]$, $dP \otimes dt$ − a.e. on $A$. For each $n \in \mathbb{N}$, set $\nu_t^n := n\mathbf{1}_A$. Note that $(\nu_t^n)$ is a bounded predictable process with $\nu_t^n > -1$. Hence, $\nu^n \in \mathcal{V}$. Using equality (B.5), we derive that for $n$ sufficiently large, we have $\mathbb{E}[A_T^{\nu^n}] = \mathbb{E}[A_T - n\int_0^T (K_t - \beta_t \sigma_t^{-1} Z_t)\lambda_t \mathbf{1}_A dt] < 0$. We thus get a contradiction with the nondecreasing property of $A^{\nu^n}$. Hence, $(K_t - \beta_t \sigma_t^{-1} Z_t)\lambda_t \le 0$ $dP \otimes dt$-a.s.

Let us show that condition (B.5) implies that the process $A_\cdot + \int_0^\cdot (K_s - \beta_s \sigma_s^{-1} Z_s)\lambda_s ds$ is nondecreasing. Suppose by contradiction that there exists $B \in \mathcal{G}_T$ with $P(B) > 0$, as well as $\varepsilon > 0$ and $(t,s) \in [0,T]^2$ with $t < s$, such that $\int_t^s (dA_r + (K_r - \beta_r \sigma_r^{-1} Z_r)\lambda_r dr) \le -\varepsilon$ a.s. on $B$. For each $n \in \mathbb{N}^*$, set $\nu^n := -1 + \frac{1}{n}$. Note that $\nu^n \in \mathcal{V}$. From (B.5), we derive that $\int_t^s (dA_r + (K_r - \beta_r \sigma_r^{-1} Z_r)(-1 + \frac{1}{n})\lambda_r dr) \ge 0$ a.s. We thus get that for all $n \in \mathbb{N}^*$,

$$-\varepsilon \ge \int_t^s (dA_r + (K_r - \beta_r \sigma_r^{-1} Z_r)\lambda_r dr) \ge \frac{1}{n}\int_t^s (K_r - \beta_r \sigma_r^{-1} Z_r)\lambda_r dr \text{ a.s. on B.}$$

By letting $n$ tend to $+\infty$, we obtain a contradiction. Hence, the process

$$A_\cdot + \int_0^\cdot (K_s - \beta_s \sigma_s^{-1} Z_s)\lambda_s ds$$

is nondecreasing. The uniqueness of the decomposition follows by Lemma D.2. ∎

Theorem B.2 (optional $\mathcal{E}^f$-decomposition). *Let $(X_t)$ be an optional process belonging to $\mathbb{S}^2$. Suppose that it is an $\mathcal{E}^\nu$-strong supermartingale for each $\nu \in \mathcal{V}$. Then, there exists a unique $Z \in \mathbb{H}^2$, a unique $C \in \mathbb{C}^2$, and a unique nondecreasing optional RCLL process $h$, with $h_0 = 0$ and $\mathbb{E}[h_T^2] < \infty$ such that*

$$-dX_t = f(t, X_t, Z_t)dt - Z_t(dW_t + \sigma_t^{-1}\beta_t dM_t) + dC_{t^-} + dh_t. \tag{B.6}$$

*Proof.* By Theorem B.1, there exists $(Z, K, A, C) \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2 \times \mathbb{C}^2$ satisfying (B.1)–(B.2). Set

$$h_t := A_t - \int_0^t (K_s - \beta_s \sigma_s^{-1} Z_s) dM_s. \tag{B.7}$$

Since $dM_t = dN_t - \lambda_t dt$, we have

$$h_t = A_t + \int_0^t (K_s - \beta_s \sigma_s^{-1} Z_s)\lambda_s ds - \int_0^t (K_s - \beta_s \sigma_s^{-1} Z_s) dN_s. \tag{B.8}$$

Now, by property (B.2), the process $A_\cdot + \int_0^\cdot (K_s - \beta_s \sigma_s^{-1} Z_s)\lambda_s ds$ is nondecreasing.

Moroever, the process $\int_0^\cdot (K_s - \beta_s \sigma_s^{-1} Z_s) dN_s$ is a purely discontinuous process which admits a unique jump, given by $K_\vartheta - \beta_\vartheta \sigma_\vartheta^{-1} Z_\vartheta$ (at time $\vartheta$). By (B.2), we have $K_\vartheta - \beta_\vartheta \sigma_\vartheta^{-1} Z_\vartheta \le 0$ a.s. We thus derive that the process $\int_0^\cdot (K_s - \beta_s \sigma_s^{-1} Z_s) dN_s$ is nonincreasing. Hence, by the equality (B.8), we derive that the process $(h_t)$ is nondecreasing. Using (B.1), we thus get (B.6).

It remains to prove the uniqueness of the processes $Z$, $C$, and $h$ in (B.6). We first show that if $X$ is decomposable as in (B.6), then the process $X'$ defined by $X'_t = X_t - \Delta X_\vartheta \mathbb{I}_{t \ge \vartheta}$ is a *special* optional semimartingale (cf. Lemma D.2). By (B.6), we have

$$\Delta X_\vartheta = Z_\vartheta \sigma_\vartheta^{-1} \beta_\vartheta - \Delta h_\vartheta. \tag{B.9}$$

Subtracting $\Delta X_\vartheta \mathbb{I}_{t \ge \vartheta}$ on both sides of (B.6), we get

$$X_t - \Delta X_\vartheta \mathbb{I}_{t \ge \vartheta} = X_0 - \int_0^t f(s, X_s, Z_s) ds + \int_0^t Z_s(dW_s + \sigma_s^{-1}\beta_s dM_s)) - C_{t^-} - h_t - \Delta X_\vartheta \mathbb{I}_{t \ge \vartheta}. \tag{B.10}$$

Using this and the expression (B.9) for $\Delta X_\vartheta$, we get

$$\begin{aligned} X_t - \Delta X_\vartheta \mathbb{I}_{t \ge \vartheta} &= X_0 - \int_0^t f(s, X_s, Z_s) ds \\ &\quad + \int_0^t Z_s(dW_s + \sigma_s^{-1}\beta_s dM_s) - C_{t^-} - h_t - Z_\vartheta \sigma_\vartheta^{-1} \beta_\vartheta \mathbb{I}_{t \ge \vartheta} + \Delta h_\vartheta \mathbb{I}_{t \ge \vartheta}. \end{aligned} \tag{B.11}$$

We set $B_t := h_t - \Delta h_\vartheta \mathbb{I}_{t\geq\vartheta}$. By Lemma D.3, the process $B$ is a (predictable) process in $\mathcal{A}^2$. Recall that we have also set $X'_t = X_t - \Delta X_\vartheta \mathbb{I}_{t\geq\vartheta}$. With this notation, (B.11) becomes

$$X'_t = X'_0 - \int_0^t f(s,X_s,Z_s)ds + \int_0^t Z_s(dW_s + \sigma_s^{-1}\beta_s dM_s) - C_{t^-} - B_t - Z_\vartheta\sigma_\vartheta^{-1}\beta_\vartheta \mathbb{I}_{t\geq\vartheta}. \tag{B.12}$$

Since $dM_t = dN_t - \lambda_t dt$, we get

$$X_t = X_0 - \int_0^t f(s,X_s,Z_s)ds + \int_0^t Z_s dW_s - C_{t^-} - B_t - \int_0^t Z_s\sigma_s^{-1}\beta_s\lambda_s ds. \tag{B.13}$$

We conclude that $X'$ is a special optional semimartingale.

Let now $\tilde{Z}$, $\tilde{C}$, and $\tilde{h}$ be such that $\tilde{Z} \in \mathbb{H}^2$, $\tilde{C} \in \mathbb{C}^2$, and $\tilde{h}$ is a nondecreasing optional RCLL process with $\tilde{h}_0 = 0$ and $\mathbb{E}[\tilde{h}_T^2] < \infty$, and such that the decomposition (B.6) holds with $\tilde{Z}$, $\tilde{C}$, and $\tilde{h}$ (in place of $Z$, $C$, $h$). We show that $\tilde{Z} = Z$ in $\mathbb{H}^2$, $\tilde{C}_t = C_t$ for all $t$ a.s., and $\tilde{h}_t = h_t$ for all $t$ a.s. By the same reasoning as above, we have that (B.13) holds also with $Z$, $C$, and $B$ replaced by $\tilde{Z}$, $\tilde{C}$, and $\tilde{B}$, where $\tilde{B}$ is defined by $\tilde{B}_t := \tilde{h}_t - \Delta\tilde{h}_\vartheta \mathbb{I}_{t\geq\vartheta}$. Note that, due to (B.9), $\Delta\tilde{h}_\vartheta = \Delta h_\vartheta$. Showing the equality $\tilde{h}_t = h_t$ for all $t$ a.s. is thus equivalent to showing that $\tilde{B}_t = B_t$ for all $t$ a.s.

Now, as $X'$ is a special optional semimartingale admitting the decomposition (B.13) with $Z$, $C$, and $B$, on one hand, and with $\tilde{Z}$, $\tilde{C}$, and $\tilde{B}$, on the other hand, we have, by the uniqueness of the special optional semimartingale decomposition (cf. Lemma D.1), that $C = \tilde{C}$, $f(t,X_t,Z_t)dt + dB_t + Z_t\sigma_t^{-1}\beta_t\lambda_t dt = f(t,X_t,\tilde{Z}_t)dt + d\tilde{B}_t + \tilde{Z}_t\sigma_t^{-1}\beta_t\lambda_t dt$, and $Z_t dW_t = \tilde{Z}_t dW_t$. From the last equality, using the uniqueness of the martingale representation, we get $Z = \tilde{Z}$ in $\mathbb{H}^2$. This, together with the second equality, gives the equality of $B$ and $\tilde{B}$. The proof is thus complete. ∎

*Remark* B.3. The process $Z$ in Theorem B.2 is the same as in Theorem B.1, and the optional process $h$ in Theorem B.2 can be written in terms of the processes $(Z, K, A)$ from Theorem B.1 as follows: $h_t = A_t - \int_0^t (K_s - \beta_s\sigma_s^{-1}Z_s)dM_s$.

*Remark* B.4 (integrability property). Suppose that $X \in \mathbf{S^2}$. Then, the processes from the predictable $\mathcal{E}^f$-decomposition (B.1) and from the optional $\mathcal{E}^f$-decomposition (B.6) satisfy the following integrability properties: $Z \in \mathbf{H^2}$, $K \in \mathbf{H^2_\lambda}$, and $C, A, h \in \mathbf{S^2}$. Indeed, (B.4) can be written

$$-\, dX_t = f(t,X_t,Z_t)dt - Z_t dW_t^\nu - K_t dM_t^\nu + dA_t^\nu + dC_{t^-}, \tag{B.14}$$

where $dW_t^\nu = dW_t + \nu_t\lambda_t\beta_t\sigma_t^{-1}dt$ and $dM_t^\nu = dM_t + \nu_t\lambda_t dt$. Then, for each $\nu \in \mathcal{V}$, the processes $Z$, $K$, $A^\nu$, and $C$ correspond to the $\mathcal{E}^f_{Q^\nu}$-Mertens decomposition coefficients of the $Q^\nu$-square integrable strong $\mathcal{E}^f_{Q^\nu}$-supermartingale $X$. Since $X \in \mathbf{S^2}$, we derive that $Z \in \mathbf{H^2}$, $K \in \mathbf{H^2_\lambda}$, and $C \in \mathbf{S^2}$. Moreover, for all $\nu \in \mathcal{V}$, $A^\nu \in S^2_{Q^\nu}$. By (B.5), we get $A \in \mathbf{S^2}$. This with (B.7) implies $h \in \mathbf{S^2}$.

Proposition B.5. *Let $(X_t) \in \mathbb{S}^2$. The process $(X_t)$ admits an optional $\mathcal{E}^f$-decomposition (that is of the form (B.6)) if and only if it admits a predictable $\mathcal{E}^f$-decomposition (of the form (B.1)–(B.2)).*

*Proof.* By the same arguments as those used in the proof of Theorem B.2, we derive that if $(X_t)$ admits a predictable $\mathcal{E}^f$-decomposition then it admits an optional $\mathcal{E}^f$-decomposition. Let us show the converse. Suppose there exists $(Z, C) \in \mathbb{H}^2 \times \mathbb{C}^2$ and a nondecreasing optional RCLL process $h$, with $h_0 = 0$ and $\mathbb{E}[h_T^2] < \infty$ such that (B.6) holds. By Lemma D.3, $h$ admits the decomposition: $h_t = B_t + \int_0^t \psi_s dN_s$, where $B$ is a predictable process in $\mathcal{A}^2$ and $\psi \in \mathbb{H}^2_\lambda$ with $\psi_t \lambda_t \geq 0$ $dP \otimes dt$-a.s. Let $(A_t)$ and $(K_t)$ be the processes defined by

$$A_t := B_t + \int_0^t \psi_s \lambda_s ds; \quad K_t := \beta_t \sigma_t^{-1} Z_t - \psi_t, \quad t \in [0, T]. \tag{B.15}$$

We have $A \in \mathcal{A}^2$ and $K \in \mathbb{H}^2_\lambda$. Since $\psi_t \lambda_t \geq 0$ $dP \otimes dt$-a.s., $(K_t - \beta_t \sigma_t^{-1} Z_t)\lambda_t \leq 0$ $dP \otimes dt$-a.s. By (B.15), we have $B_t = A_t + \int_0^t (K_s - \beta_s \sigma_s^{-1} Z_s)\lambda_s ds$. Since $B \in \mathcal{A}^2$, we derive that (B.2) holds. Moreover, since $N_t = M_t + \int_0^t \lambda_s ds$, by (B.15), we get $h_t = B_t + \int_0^t \psi_s dN_s = A_t + \int_0^t \psi_s dM_s$ a.s. By (B.6) and the second equality in (B.15), the process $(Z, K, A, C)$ satisfies (B.1). ■

Corollary B.6. *By this proposition together with the second assertion of Theorem B.1, we derive that a process $(X_t) \in \mathbb{S}^2$ admits an optional $\mathcal{E}^f$-decomposition (of the form (B.6)) if and only if $(X_t)$ is a strong $\mathcal{E}^\nu$-supermartingale for all $\nu \in \mathcal{V}$.*

**Appendix C. BSDEs with a nonpositive jump at the default time $\vartheta$.** Let $g$ be a $\lambda$-admissible driver, which may depend on $k$ (contrary to $f$). We suppose that $g$ satisfies Assumption 2.2 (with only $\gamma_t^{y,z,k_1,k_2} \geq -1$) so that $\mathcal{E}^g$ is monotone. Let $\mathcal{V}'$ be the set of bounded predictable processes $\nu$ such that $\nu_t \geq 0$ $dP \otimes dt$-a.e. Let $\eta \in L^2(\mathcal{G}_T)$. For each $\nu \in \mathcal{V}'$, we define

$$g^\nu(\cdot, t, y, z, k) := g(\cdot, t, y, z, k) + \nu_t \lambda_t k.$$

Note that $g^\nu$ is a $\lambda$-admissible driver. For each $S \in \mathcal{T}$, the value $X(S)$ at time $S$ is defined by

$$X(S) := ess \sup_{\nu \in \mathcal{V}'} \mathcal{E}^{g^\nu}_{S,T}(\eta). \tag{C.1}$$

Note that $X_T = \eta$ a.s. We suppose that the value family is uniformly square integrable, that is, $\mathbb{E}[ess \sup_{S \in \mathcal{T}} X(S)^2] < +\infty$. By similar arguments as in the proof of Theorem 5.7, using Proposition A.6 (which ensures the right-continuity of the value process), there exists an RCLL process $(X_t) \in S^2$ which aggregates the value family $(X(S))$, which is a strong $\mathcal{E}^{g^\nu}$-supermartingale for all $\nu \in \mathcal{V}'$ and $X_T \geq \eta$ a.s. Moreover, the process $(X_t)$ is the smallest process in $S^2$ satisfying these properties.

Using the same approach as that of the proof of Theorem B.1, it can be shown the following *nonlinear predictable representation* for processes which are $\mathcal{E}^{g^\nu}$-supermartingales for all $\nu \in \mathcal{V}'$.

Proposition C.1 (predictable $\mathcal{E}^g$-decomposition). *Let $(X_t) \in \mathbb{S}^2$. If $(X_t)$ is a strong $\mathcal{E}^{g^\nu}$-supermartingale for all $\nu \in \mathcal{V}'$, then there exists a unique process $(Z, K, A, C) \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda \times$*

*$\mathcal{A}^2 \times \mathbb{C}^2$ such that*

$$-dX_t = g(t, X_t, Z_t, K_t)dt - Z_t dW_t - K_t dM_t + dA_t + dC_{t-}$$
$$\text{and} \qquad K_t\lambda_t \leq 0,\ t \in [0,T],\ dP \otimes dt - \text{a.e.} \tag{C.2}$$

*Moreover, the converse statement holds.*

*Remark* C.2. The constraint (C.1) is equivalent to $K_\vartheta \leq 0$ a.s. It corresponds to the second constraint from (B.2) (in the case when $\beta = 0$). There is here only one constraint (C.1). This comes from the fact that here $\mathcal{V}'$ is the set of bounded predictable processes $\nu$ with $\nu_t \geq 0\ dP \otimes dt$-a.e., while in the (previous) case of $\mathcal{V}$, we had $\nu_t > -1\ dP \otimes dt$-a.e.

By similar arguments as those used in the proof of Theorem 4.12, it can be shown that the value process $(X_t)$ is a supersolution of the *constrained reflected BSDE* from Definition 4.10 with $f$ replaced by $g$ and the constraints (B.2) replaced by the constraint (C.1). We thus get the following.

Proposition C.3. *Let $(X_t) \in S^2$ be the RCLL process which aggregates the value family $(X(S))$ defined by (C.1). There exists a unique process $(Z, K, A) \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2$ such that*

$$-dX_t = g(t, X_t, Z_t, K_t)dt + dA_t - Z_t dW_t - K_t dM_t;\ X_T = \eta, \quad K_\vartheta \leq 0, \quad \text{a.e.}$$

*In other words, the value process $(X_t)$ is a supersolution of the above* constrained BSDE. *Moreover, it is the minimal one, that is, if $(X'_t)$ is another supersolution, then $X'_t \geq X_t$ for all $t \in [0,T]$ a.s.*

This result gives the existence of a minimal supersolution of a BSDE with nonpositive jumps and nonlinear driver $g$, which corresponds to a result shown in [21] by a penalization approach. In addition, it provides a dual representation (with nonlinear expectation) of this minimal supersolution.

## Appendix D. Some useful lemmas and proof of Lemma 4.21.

Lemma D.1. *Let $g$ be a $\lambda$-admissible driver. Let $(A_t)$ be an RCLL predictable process with square integrable total variation and $A_0 = 0$. Suppose $X$ is the first component of the solution of both the BSDE with generalized driver $g(\cdot, y, z, k)dt + dA_t$ and the BSDE with driver $g$ (with same terminal time $T$ and same terminal condition $\eta \in L^2(\mathcal{G}_T)$). We then have $A_t = 0$ for all $t \in [0,T]$ a.s.*

*Proof.* By assumption, there exists a unique process $(Z, K)$ in $\mathbb{H}^2 \times \mathbb{H}^2_\lambda$ such that $(X, Z, K)$ satisfies (2.2). Also, there exists a unique $(Z', K')$ in $\mathbb{H}^2 \times \mathbb{H}^2_\lambda$ such that $(X, Z', K')$ satisfies $-dX_t = g(t, X_t, Z'_t, K'_t)dt + dA_t - Z'_t dW_t - K'_t dM_t$. By the uniqueness of the decomposition of a special semimartingale together with the uniqueness in the martingale representation, we derive that $Z = Z'$ in $\mathbb{H}^2$ and $K = K'$ in $\mathbb{H}^2_\lambda$, and $dA_t = 0$. ■

Lemma D.2 (uniqueness of the canonical decomposition of a special optional semimartingale). *Let $X$ be an optional semimartingale with decomposition*[16]

$$X_t = X_0 + m_t - a_t - b_t, \text{ for all } t \in [0,T] \text{ a.s.} \tag{D.1}$$

*with $(m_t)$ a (right-continuous) local martingale, $(a_t)$ a* predictable *right-continuous process of finite variation, such that $a_0 = 0$, $(b_t)$ a* predictable *left-continuous process of finite variation, purely discontinuous and such that $b_{0-} = 0$. Then, the decomposition (D.1) is unique and will be called* the canonical decomposition of a special optional semimartingale.

*Proof.* Let $X_t = X_0 + m'_t - a'_t - b'_t$ for all $t \in [0,T]$ a.s., be (another) decomposition with $(m'_t)$, $(a'_t)$, and $(b'_t)$ as in the lemma. From this decomposition, it follows that $X_{t+} - X_t = -(b'_{t+} - b'_t)$ for all $t$ a.s. From (D.1), it follows that $X_{t+} - X_t = -(b_{t+} - b_t)$ for all $t$ a.s. Hence, $b'_{t+} - b'_t = b_{t+} - b_t$ for all $t$ a.s. As $b$ and $b'$ are purely discontinuous with the same initial value, we get $b'_t = b_t$, for all $t$ a.s. and the uniqueness of $b$ is proven. By (D.1), $(X_t + b_t)_t$ is a special right-continuous semimartingale. Hence, by [28, Theorem 30, Chapter III] the processes $(m_t)$ and $(a_t)$ are unique. ∎

Lemma D.3. *Let $h$ be a nondecreasing optional RCLL process $h$ with $h_0 = 0$ and $\mathbb{E}[h_T^2] < \infty$. Then, $h$ has at most one totally inaccessible jump and this jump is at $\vartheta$. All the other jumps of $h$ are predictable. Moreover, $h$ can be uniquely decomposed as follows: $h_t = B_t + \Delta h_\vartheta \mathbb{I}_{t\geq\vartheta} = B_t + \int_0^t \psi_s dN_s$, where $B$ is a (predictable) process in $\mathcal{A}^2$, and $\psi \in \mathbb{H}^2_\lambda$ with $\psi_\theta \geq 0$ a.s. on $\{\theta \leq T\}$.*

*Proof.* As $h$ is a square-integrable nondecreasing optional RCLL process, $h$ is a square-integrable RCLL submartingale. So, by the classical Doob–Meyer decomposition, $h$ can be uniquely decomposed as $h_t = a_t + m_t$ with $(a_t)$ a (predictable) process in $\mathcal{A}^2$ and $(m_t)$ a square-integrable martingale such that $m_0 = 0$. Now, by the martingale representation of $\mathbb{G}$-martingales and as $dM_s = dN_s - \lambda_s ds$, we get $m_t = \int_0^t \varphi_s dW_s - \int_0^t \psi_s \lambda_s ds + \int_0^t \psi_s dN_s$, where $\varphi \in \mathbb{H}^2$ and $\psi \in \mathbb{H}^2_\lambda$. Hence, $h_t = a_t + m_t = B_t + \int_0^t \psi_s dN_s = B_t + \psi_\vartheta \mathbb{I}_{t\geq\vartheta}$, where we have set $B_t := a_t + \int_0^t \varphi_s dW_s - \int_0^t \psi_s \lambda_s ds$. The process $(B_t)$ is clearly predictable (as the sum of three predictable processes). The equality $h_t = B_t + \psi_\vartheta \mathbb{I}_{t\geq\vartheta}$, together with the predictability of $B$ and the nondecreasingness of $h$, implies that $\Delta h_\vartheta = \psi_\vartheta \geq 0$ a.s. on $\{\theta \leq T\}$ and that $B$ is nondecreasing. The proof is thus complete. ∎

In the remainder of the appendix, we assume that the payoff $\eta$ satisfies the stronger integrability condition $\sup_{\nu\in\mathcal{V}} \mathbb{E}_{Q^\nu}[\sup_{0\leq t\leq T} X_t^2] < +\infty$, where $(X_t)$ is the associated dual value process.

For each $\nu \in \mathcal{V}$, denote by $W^\nu$ the $Q^\nu$-Brownian motion and by $M^\nu$ the $Q^\nu$-default martingale, where $Q^\nu$ is the probability measure with density $\zeta^\nu$ (with respect to $P$) given by (3.9). We have $dW_t^\nu = dW_t + \nu_t \lambda_t \beta_t \sigma_t^{-1} dt$ and $dM_t^\nu = dM_t + \nu_t \lambda_t dt$.

Lemma D.4. *Suppose that $\sup_{\nu\in\mathcal{V}} \mathbb{E}_{Q^\nu}[\sup_{0\leq t\leq T} X_t^2] < +\infty$, The process $Z$ from the optional $\mathcal{E}^f$-decomposition (4.8) of the value process $X$ satisfies $\sup_{\nu\in\mathcal{V}} \|Z\|^2_{Q^\nu} < +\infty$.*

[16] An optional semimartingale with a decomposition of this form (with $(a_t)$ and $(b_t)$ *predictable* processes) can be seen as a generalization of the notion of special semimartingale from the right-continuous to the general case.

*Proof.* Let $\nu \in \mathcal{V}$. The predictable $\mathcal{E}^f$-decomposition of $X$ can be written under $Q^\nu$ as follows:

$$-\,dX_t = f(t, X_t, Z_t)dt - Z_t dW_t^\nu - K_t dM_t^\nu + dA_t^\nu, \tag{D.2}$$

where $A^\nu$ is the RCLL nondecreasing predictable process given by (B.5). Applying the estimate (A.2) to the $\mathcal{E}^f$-supermartingale $X$ under $Q^\nu$, we deduce $\|Z\|^2_{Q^\nu} \leq c\, \mathbb{E}_{Q^\nu}[\sup_{0\leq t\leq T} X_t^2]$, where the constant $c$ does not depend on $\nu$. Taking the supremum over $\nu \in \mathcal{V}$, using the assumption $\sup_{\nu\in\mathcal{V}} \mathbb{E}_{Q^\nu}[\sup_{0\leq t\leq T} X_t^2] < +\infty$, we get the desired result. ∎

Lemma D.5. *Let $(x, \varphi) \in \mathbb{R}\times\mathbb{H}^2$ with $\sup_{\nu\in\mathcal{V}} \|\varphi\|^2_{Q^\nu} < +\infty$. We have $\sup_{\nu\in\mathcal{V}} \mathbb{E}_{Q^\nu}[(V_T^{x,\varphi})^2] < \infty$.*

*Proof.* Recall that the wealth process $V^{x,\varphi}$ is the solution of the forward SDE (3.3). When $f$ does not depend on $y$, using martingale inequalities, we derive $\sup_{\nu\in\mathcal{V}} \mathbb{E}_{Q^\nu}[(V_T^{x,\varphi})^2] < \infty$. When $f$ is linear with respect to $y$, that is, of the form $f(t, y, z) = -r_t y + g(t, z)$, by a discounting procedure, we get the desired integrability condition. In the general case, setting $f_y(s) := \mathbf{1}_{V_s\neq 0}(f(s, V_s, \sigma_s\varphi_s) - f(s, 0, \sigma_s\varphi_s))/V_s$, we get $dV_s = (f_y(s)V_s + f(s, 0, \sigma_s\varphi_s))ds + \varphi_s(\sigma_s dW_s + \beta_s dM_s)$ reduces to the case when $f$ is linear with respect to $y$ (since $f_y(s)$ is bounded). ∎

*Proof of Lemma* 4.21. We set $G := V_T^{x,\varphi} - \eta$. By Lemma D.5, we have $\sup_{\nu\in\mathcal{V}} \mathbb{E}_{Q^\nu}[G^2] < \infty$. Let $\nu \in \mathcal{V}$. Recall that the wealth process $V^{x,\varphi}$ is an $\mathcal{E}^f_{Q^\nu}$-martingale, which implies that $x = \mathcal{E}^f_{Q^\nu,0,T}(V_T^{x,\varphi})$. Now, the process $V^{x,\varphi} = \mathcal{E}^f_{Q^\nu,\cdot,T}(V_T^{x,\varphi})$ (resp., $\mathcal{E}^f_{Q^\nu,\cdot,T}(\eta)$) is the first coordinate of the solution of the BSDE with driver $f$, terminal time $T$, and terminal condition $V_T^{x,\varphi}$ (resp., $\eta$). By using a classical linearization method (see, e.g., the proof of [9, Theorem 3]), we obtain

$$x - \mathcal{E}^f_{Q^\nu,0,T}(\eta) = \mathcal{E}^f_{Q^\nu,0,T}(V_T^{x,\varphi}) - \mathcal{E}^f_{Q^\nu,0,T}(\eta) = \mathbb{E}_{Q^\nu}[\Gamma_T^\nu(V_T^{x,\varphi} - \eta)] = \mathbb{E}_{Q^\nu}[\Gamma_T^\nu G], \tag{D.3}$$

$\Gamma^\nu_\cdot$ being the solution of the SDE $d\Gamma_s^\nu = \Gamma_{s^-}^\nu\,[f_y(s)ds + f_z(s)dW_s^\nu]$; $\Gamma_0^\nu = 1$, where $f_y(s)$ (resp., $f_z(s)$) is the usual bounded increment rate of $f$ with respect to $y$ (resp., $z$). By the Cauchy–Schwarz inequality, we deduce

$$(\mathbb{E}_{Q^\nu}[G])^2 = (\mathbb{E}_{Q^\nu}[(\Gamma_T^\nu)^{1/2}G^{1/2}(\Gamma_T^\nu)^{-1/2}G^{1/2}])^2 \leq \mathbb{E}_{Q^\nu}[\Gamma_T^\nu\, G]\mathbb{E}_{Q^\nu}[(\Gamma_T^\nu)^{-1}\, G] \tag{D.4}$$

$$\text{and} \quad \mathbb{E}_{Q^\nu}[(\Gamma_T^\nu)^{-1}\, G] \leq (\mathbb{E}_{Q^\nu}[(\Gamma_T^\nu)^{-2}])^{1/2}\, (\mathbb{E}_{Q^\nu}[G^2])^{1/2}. \tag{D.5}$$

By classical computations, we get $\sup_{\nu\in\mathcal{V}} \mathbb{E}_{Q^\nu}[(\Gamma_T^\nu)^{-2}] < +\infty$. By (D.5), we derive that there exists a constant $c > 0$ such that $\sup_{\nu\in\mathcal{V}} \mathbb{E}_{Q^\nu}[(\Gamma_T^\nu)^{-1}\, G] \leq c$. Hence, setting $m' = 1/c$, using (D.3) and (D.4), we deduce $m'(\mathbb{E}_{Q^\nu}(G))^2 \leq x - \mathcal{E}^f_{Q^\nu,0,T}(\eta)$, which corresponds to the first inequality in (4.15).

It remains to show the second one. By (D.3), using the Cauchy–Schwarz inequality, we get

$$x - \mathcal{E}^f_{Q^\nu,0,T}(\eta) \leq (\mathbb{E}_{Q^\nu}[(\Gamma_T^\nu)^2 G])^{1/2}(\mathbb{E}_{Q^\nu}[G])^{1/2} \leq m(\mathbb{E}_{Q^\nu}[G])^{1/2},$$

where $m := \sup_{\nu\in\mathcal{V}_S}(\mathbb{E}_{Q^\nu}[(\Gamma_T^\nu)^2 G])^{1/2}$. Note that $m < \infty$. Now, we have $\sup_{\nu\in\mathcal{V}_S} \mathbb{E}_{Q^\nu}[(\Gamma_T^\nu)^4] < \infty$. By the Cauchy–Schwarz inequality, we derive that $m < \infty$, which completes the proof. ∎

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