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1   **Title:** Objective Logic of Consciousness

2   **Authors:** Venkata Rayudu Posina and Sisir Roy

3   **ORCID IDs:** 0000-0002-3040-9224 and 0000-0002-5328-2713

4   **Article Category:** Original Research Article

7   **Word Count:** 8777

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9   **Affiliations:** SR is Senior Homi Bhabha Visiting Professor, National Institute of Advanced

10   Studies, Bengaluru. VRP is an Independent scientist.

11   **Emails:** posinavrayudu@gmail.com; sisir.sisirroy@gmail.com

12   **Abstract**

13   We define consciousness as the category of all conscious experiences. This immediately raises

14   the question: What is the essence in which every conscious experience in the category of

15   conscious experiences partakes? We consider various abstract essences of conscious experiences

16   as theories of consciousness. They are: (i) conscious experience is an action of memory on

17   sensation, (ii) conscious experience is experiencing a particular as an exemplar of a general, (iii)

18   conscious experience is an interpretation of sensation, (iv) conscious experience is referring

19   sensation to an object as its cause, and (v) conscious experience is a model of stimulus.

20   Corresponding to each one of these theories we obtain a category of models of conscious

21   experiences: (i) category of actions, (ii) category of idempotents, (iii) category of two sequential

22   maps, (iv) category of brain-generalized figures, and (v) functor categories with intuition as base

23   and conceptual repertoire as exponent, respectively. For each theory of consciousness we also

24   calculate its truth value object and characterize the objective logic intrinsic to the corresponding

25   category of models of consciousness experiences.

26 **Introduction**

27 What is consciousness? Consciousness, according to Koch, "is everything you experience. It is

28 the tune stuck in your head, the sweetness of chocolate mousse, the throbbing pain of a

29 toothache, the fierce love for your child and the bitter knowledge that eventually all feelings will

30 end" (Koch, 2018, p. S9). This raises two foundational questions:

31 1. What is the nature of conscious experiences?

32 2. What is the nature of consciousness?

33 How are we to think of the totality of conscious experiences i.e., consciousness? How are we to

34 think of the constituents of consciousness i.e., conscious experiences? One obvious answer:

35 Conscious experiences are objects of the category of all conscious experiences and

36 consciousness is the category of conscious experiences. In other words, every conscious

37 experience has the essence of the category of conscious experiences, whatever the essence(s)

38 maybe. This characterization is in the spirit of asserting that a chair is an object of the category

39 of chairs.

40 Let us consider a visual experience: a face. A first-order approximation would represent the

41 experience as a point in a feature-space or as a set of features i.e., Face = {eyes, nose, mouth}

42 (Fodor, 1998). Sensory features are obviously structured, unlike the structureless elements of sets

43 (Lawvere and Rosebrugh, 2003, p. 1). Equally importantly, sensory features of a visual object

44 are related to one another in specific ways resulting in a cohesive object that is conscious

45 experience, which cannot be modelled as a set with its zero internal cohesion (Lawvere and

46   Schanuel, 2009, p. 146). Elementism, notwithstanding the Gestalt demonstrations (Albright et

47   al., 2000, p. S34), continues to be the default terminology as in analysing "perceptual experience

48   into a collection of simple sensory elements" (Albright, 2013a, p. 19). Along similar lines, mind

49   is defined as a set of brain functions (Bunge, 1981, p. 68; Kandel, 2013, p. 546). The claim that

50   'mind is a set' is repeatedly asserted in the textbook Principles of Neural Science (Kandel et al.,

51   2013, p. 5, 334, 384), which takes on added significance in light of its pedagogical value in

52   training neuroscientists. Of course, this terminology does not reflect any failure to recognize that,

53   in terms of the above example of face perception, the constituent eyes, nose, and mouth, unlike

54   the structureless elements of a set, are figures of various shapes; these figures constituting a face

55   are related to one another in specific ways (cf. Croner and Albright, 1999). Nevertheless, it does

56   highlight the absence and the significance of having a conceptual repertoire that fits the reality of

57   conscious experiences. Here we put forward mathematical category as a construct suited for the

58   study of consciousness (Lawvere, 1994; Lawvere and Schanuel, 2009, p. 21, 135-148). In line

59   with the commonplace understanding of the notion of category, a mathematical category consists

60   of objects all of which partake in the essence that is characteristic of the category; since every

61   object of the category partakes in the essence, the transformations of objects preserve the essence

62   (e.g. in the category of dogs, a transformation of an young dog into an old dog preserves the

63   "dogness"). We find that defining conscious experience as an object of the category of conscious

64   experiences, instead of as cohesion-less set of structure-less elements, provides the conceptual

65   repertoire—basic shapes, figures, and incidence relations—needed to reason about the essence of

66   conscious experiences and the essence-preserving transformations of conscious experiences.

67

68 **Theory of Conscious Experiences**

69 What is the essence of conscious experiences? Continuing with our example of face perception,

70 an experience of a face can be said to consist of figures of various shapes: two eye-shaped

71 figures, one nosed-shaped figure, and one mouth-shaped figure. Of these shapes, we can say that

72 eye, nose, and mouth are the basic shapes, and their incidence relations determine the mutual

73 relations between various basic-shaped figures constituting the face (Lawvere and Schanuel,

74 2009, pp. 82-83, 250-253, 369-371). When considering conscious experience in general, we may

75 treat sensory features (e.g. colour, shape), modalities (visual, tactile, etc.), and emotion, among

76 others, as basic shapes. For illustration, anger (in conscious experience) can be considered as an

77 emotion-shaped figure (in the experience) just as redness can be thought of as a colour-shaped

78 figure. The mutual relations between basic shapes, say, emotion and colour, determine the

79 mutual relations between figures of the corresponding shapes (anger and redness).

80 Basic shapes along with their incidence relations constitute the abstract essence or theory of the

81 category of conscious experiences (Lawvere, 2003, p. 215, 217; Lawvere, 2004a, pp. 10-12;

82 Lawvere and Rosebrugh, 2003, pp. 154-155, 235-236; Lawvere and Schanuel, 2009, pp. 149-

83 151, 369-371). First, every experience has the essence [of the category of conscious experiences]

84 given by the basic shapes and their incidence relations. Next, every experience can be

85 represented as a structure formed of basic-shaped figures and their mutual relations induced by

86 the incidences of basic shapes (see Fig. 4 in Posina, Ghista, and Roy, 2017). Since every

87 experience has the essence of experiences, transformations of experiences are required to

88 preserve the essence of experiences, and as such are natural transformations (Lawvere and

89 Schanuel, 2009, p. 378). Geometrically speaking, natural transformations 'do not tear' the

90 structure transformed (ibid, p. 210). Philosophically, a natural transformation is: Becoming

91 consistent with Being (e.g. biological growth; Posina, 2016).

92   What are we to make of the totality of all conscious experiences along with their essence-

93 preserving transformations? Objects along with essence-preserving morphisms of objects form a

94 category. With experiences as objects [with a given structural essence] and essence-preserving

95 transformations of experiences as structure-preserving morphisms of objects, consciousness—the

96 totality of conscious experiences—can be construed as a category of conscious experiences. Note

97 that any experience can remain the same (identity transformation). If I went from sad to happy

98 and from happy to detached, then I went from sad to detached (composition of transformations of

99 experiences). Along these lines, other axioms and laws, which are required to be satisfied in

100 order for us to talk about a category of experiences, can be verified (Lawvere and Schanuel,

101 2009, p. 21, 149-160). Within this categorical framework, the structure of consciousness is an

102 external reflection of the structural essence of conscious experiences (Lawvere, 1972, p. 10).

103 More immediately, a category embodies a mode of cohesion (Lawvere and Schanuel, 2009, p.

104 146), which is the most basic attribute of conscious experience. For example, parts of a body

105 (hands, legs, etc.) have a mode of cohesion, which is different from the mode of cohesion of

106 parts of a perceptual object (colour, shape). Note that 'part' is both itself and its relationship to

107 the whole (Lawvere, 1994, p. 53). Formally, a part of an object is not merely a subobject, but a

108 monomorphism specifying the inclusion of the subobject into the object (Lawvere and Schanuel,

109 2009, p. 335).

110   As illustrations of theory of a category and its basic shapes, we present simple theories

111 (abstract essences) of conscious experiences (in the spirit of Lawvere, 1999). More explicitly, the

112   mathematical method, according to F. William Lawvere, "consists of taking the main structure

113   [of an object] by itself as a first approximation to a theory of the object, i.e. mentally operating as

114   though all further structure of the object simply did not exist" (Lawvere, 1972, pp. 9-10). An

115   example of an abstract theory of conscious experiences is 'particular as an exemplar of a general'

116   (cf. categorical perception; Albright, 2013b, pp. 628-630; Grossberg, 1976), whose models form

117   a category of idempotents (Lawvere and Schanuel, 2009, pp. 99-106), with exemplar as its basic

118   shape. The truth value object of the category of idempotents has three global truth values. With

119   'interpretation of sensation' (Albright and Stoner, 1995; Croner and Albright, 1999; Schlack and

120   Albright, 2007) as a theory of conscious experiences, we obtain a category of two sequential

121   processes as the category of conscious experiences. Here, the basic shapes are physical stimuli,

122   neural sensation of stimuli, and conscious interpretation of sensation. With conscious experience

123   as an object of the category of two sequential functions, we find that the objective logic intrinsic

124   to consciousness has four truth values (Posina, Ghista, and Roy, 2017, pp. 172-174). We also

125   consider 'action of memory on sensation' (Albright, 2012, Fig. 5, 8; Hopfield, 1982; Lawvere

126   and Schanuel, 2009, p. 218), 'referring sensation to an object as its cause' (Albright, 2015, p. 22;

127   Lawvere and Rosebrugh, 2003, pp. 125-126, 148-152), and 'model of stimulus' (Chalmers,

128   2006; Posina, Ghista, and Roy, 2017) as theories of conscious experiences.

129   Given a category of experiences, how do we abstract the theory (essence) of experiences?

130   Theorization begins with measurements of properties of the objects of the given category.

131   Oftentimes, we find that there is small subcategory of properties (and their determinations)

132   within the category of all properties that constitutes the abstract essence shared by all objects of

133   the given category. This abstract essence in which every object of a given category partakes is

134   the theory of the given category (Lawvere, 1994, pp. 44-47; Lawvere and Rosebrugh, 2003, pp.

135   154-155; Lawvere and Schanuel, 2009, pp. 149-150; see also Fig. 5 in Posina, Ghista, and Roy,

136   2017). In geometric terminology, we consider a subcategory of basic shapes and their incidence

137   relations, and examine if figures with objects in the subcategory as shapes are adequate to

138   completely characterize every object of the category and tell apart transformations between

139   objects (Lawvere, 1994, p. 49; Lawvere and Schanuel, 2009, pp. 369-371). In the following we

140   focus on the calculation of truth value objects corresponding to various theories of conscious

141   experiences and the subsequent characterization of the objective logic intrinsic to various

142   categories of models of conscious experiences (ibid, pp. 335-357).

143

Action of Memory on Sensation

145   Conscious experience of a given physical stimulus can be thought of as an action of memory on

146   the sensation elicited by the stimulus (for a vivid illustration of the action of memory on

147   sensation, see Fig. 5 and 8 in Albright, 2012). A formalization of conscious experience as an

148   action of memory on sensation is provided by Hopfield (1982). Here sensation S is a $1 \times n$

149   feature vector, with each one of the elements of the vector S representing the activity of each one

150   of the n feature-selective neurons. Memory M, or the $n \times n$ synaptic weight matrix, is a result of

151   associative learning, and can be expressed as a product of the sensation S with its transpose S^T,

152   i.e.

153 $M = S^T \times S$

154   Conscious experience C(S) corresponding to sensation S is:

155    $$C(S) = S \times M.$$

156   For a given memory, conscious experiences corresponding to various sensations have the

157   structure of idempotents (as discussed in detail in appendix A1). Categorical perception, wherein

158   particulars (stimuli) are perceived as exemplars of a general (category; Albright, 2013b, pp. 628-

159   630), also has the structure of idempotents. The abstract essence or theory of the category of

160   idempotents consists of one basic shape: exemplar, along with an idempotent endomap as the

161   structural map. Unlike the classical Boolean logic of sets, we find that the truth value object of

162   the category of idempotents consists of three truth values. Also, two dual forms of negation—

163   *not*, *non*—can be defined (Lawvere, 1986, 1991). We find that double negation can be greater or

164   less than identity depending on the exact nature of negation. Furthermore, the category of

165   idempotents admits logical contradiction (Lawvere, 2003, p. 214-215; or boundary operation

166   defined as the intersection of a part with its negation; Lawvere, 1994, p. 48; Lawvere and

167   Rosebrugh, 2003, p. 201).

168

169   **Interpretation of Sensation**

170   Conscious experience involves two sequential processes of sensation (of stimulus) followed by

171   interpretation of the sensation. A classic illustration of the two sequential processes involved in

172   conscious experience is R. C. James's image (Miller, 1999). When looking at the image one

173   initially sees black and white blobs of various sizes and shapes, which subsequently, in light of

174   the concept DALMATIAN, is perceptually interpreted as a dog. That conscious experience is

175   mediated by the two processes of sensation followed by interpretation is well-established in

176   various perceptual modalities (e.g. Albright and Stoner, 1995; Croner and Albright, 1999). Thus

177   the abstract theory of consciousness consists of three basic shapes i.e. objects (Physical Stimuli,

178   Neural Codes, and Conscious Experiences) and two incidence relations i.e. maps (*sensation* and

179   *interpretation*) organized as shown below:

180   Physical Stimuli –*sensation*→ Neural Codes –*interpretation*→ Conscious Experiences

181   The truth value object of the category of models of conscious experiences of the above theory of

182   consciousness consists of four global truth values (Fig. 6c in Posina, Ghista, and Roy, 2017

183   depicts the internal diagram of the truth value object; see Appendix A2 in Posina and Roy, 2018

184   for the calculation of the truth value object; see also Linton, 2005).

185

186   **Brain-generalized Figures**

187   Everyday experience of effectively interacting with objects of conscious experience indicates

188   that conscious experience of objects is recovery of the objects based on the sensation elicited by

189   the objects (i.e. constructing an object isomorphic to the cause of sensation; Albright, 2015, p.

190   22; Lawvere and Rosebrugh, 2003, pp. 125-126, 148-152). In reconstructing objects in conscious

191   experience, we encounter the possibility of not only the commonplace isomorphism between

192   objects and conscious experience of objects, but also illusory conscious experiences with no

193   correspondent underlying objects. We now present a mathematical framework rich enough to

194   capture both veridical perception and illusions (Lawvere, 2004b). Given a stimulus A, we define

195   sensation p as a brain V-valued property of the stimulus A, i.e.

$$p: A \rightarrow V$$

196

197   Next, we consider brain V-valued properties of sensations, i.e.

$$q: V^A \rightarrow V$$

198

199   which may be considered as a special case of interpretation of sensation, i.e. a neural measure of

200   sensation. Perceived element is defined as a brain V-generalized point of A satisfying naturality

201   conditions (discussed in detail in appendix A2). Within this mathematical framework, we find

202   that defining brain V as neuronal states of 'firing' and 'not firing', i.e. as a two-element property

203   type, can give rise to illusions. However, defining brain V in terms of all possible changes in

204   neuronal firing rate, i.e. as a three-element set V = {decreased firing rate, constant firing rate,

205   increased firing rate}, ensures illusion-free perception, or isomorphism between elements of a

206   stimulus set and perceived elements. As we have been emphasizing, physical stimuli, neural

207   sensations, and conscious experiences are much more structured than structureless sets; as such

208   we are working on refining the mathematical framework to accommodate conscious experience

209   defined as brain-generalized figure of stimuli (for basic shapes and their incidences of the

210   categories corresponding to physical stimuli and neural sensation).

211

212

213   **Model of Physical Stimulus**

214   Of the various abstract essences (theories) of conscious experiences, the most basic

215   characterization of conscious experience is: conscious experience of a stimulus is a model of the

216   stimulus (Chalmers, 2006). This immediately suggests functorial semantics, which provides a

217   mathematical account of constructing models of particulars, as an abstract theory of conscious

218   experience (Lawvere, 1994, 2004a). Given a category of physical stimulus, a model or conscious

219   experience of the stimulus is calculated by abstracting the essence (mental concepts) of its brain-

220   valued properties i.e. sensation. Thus abstracted mental concepts are then interpreted into a

221   background category of intuition to obtain models of the physical stimuli or conscious

222   experiences. The objective logic of conscious experiences construed as functor categories, with

223   intuition as base and mental concepts as exponent, is not classical (Posina, Ghista, and Roy,

224   2017). Furthermore, subjectivity (understood as viewpoint; cf. Sen, 1993) is captured by the

225   framework of functorial semantics; more specifically, the mathematical construct of monad

226   determines how a category of particulars is [subjectively] generalized into the adjoint pair of

227   functors: mental concepts (theories) and conscious percepts (defined as functorial interpretation

228   of concepts into a background of intuition or models; Eilenberg and Moore, 1965; Lawvere,

229   1994, 2004a). As such, functorial semantics is the objective logic (as defined in Lawvere, 1994,

230   p. 43) of consciousness.

231

232

233

Conclusions

We defined conscious experience as an object of the category of conscious experiences, which aligns with the intuitions engendered by our everyday experiences with objects (cf. a table is an object of a category of tables). It is fascinating to note that the most advanced scientific understanding of object (as an object of a category of objects; Lawvere, 2015) is in accord with our ordinary experience. The category of conscious experiences provides the conceptual repertoire—basic shapes, figures, and incidences—needed to develop an adequately explicit theory of conscious experience. Given that the objective logic intrinsic to conscious experiences is not classical for a variety of abstract essences of consciousness that we considered, it would be interesting to compare the objective logic of consciousness with quantum logic that was found to better account for cognition compared to the classical logic (Roy, 2016).

252 **Appendices**

253 **A1. Category of Idempotents**

254 Let's consider, within the framework of Hopfield model (1982), a neural network consisting of

255 two neurons coding for two features. For a given stimulus, a neuron can respond with a decrease,

256 constant, or increase in its firing rate. So, we have nine 1×2 vectors as possible sensations S:

257 $S_1 = [\text{-}1 \ \text{-}1]$

258 $S_2 = [\text{-}1 \ 0]$

259 $S_3 = [\text{-}1 \ +1]$

260 $S_4 = [0 \ \text{-}1]$

261 $S_5 = [0 \ 0]$

262 $S_6 = [0 \ +1]$

263 $S_7 = [+1 \ \text{-}1]$

264 $S_8 = [+1 \ 0]$

265 $S_9 = [+1 \ +1]$

266 We define memory as a synaptic weight matrix M with entries given by associative learning:

267 $$m_{ij} = s_i \times s_j$$

268 where (subscripts) i, j index the two neurons. For example, memory M_1 of sensation S_1 is:

269 $$M_1 = S_1{}^T \times S_1$$

270 (T denotes transpose), which is a 2×2 weight matrix. Substituting the values of sensation S_1, we

271 find:

272 $M_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

273 Next, we define perception P as an action of memory M on sensation S:

274 $$P = S \times M$$

275 With memory $M = M_1$, we find the percepts resulting from the action of memory M_1 on the nine

276 sensations:

277 $P_1 = S_1 \times M_1 = 2[\text{-}1 \ \text{-}1]$

278 $P_2 = S_2 \times M_1 = [\text{-}1 \ \text{-}1]$

279 $P_3 = S_3 \times M_1 = [0 \ 0]$

280 $P_4 = S_4 \times M_1 = [\text{-}1 \ \text{-}1]$

281 $P_5 = S_5 \times M_1 = [0 \ 0]$

282 $P_6 = S_6 \times M_1 = [+1 \ +1]$

283 $P_7 = S_7 \times M_1 = [0 \ 0]$

284 $P_8 = S_8 \times M_1 = [+1 \; +1]$

285 $P_9 = S_9 \times M_1 = 2[+1 \; +1]$

286 Note that P_1, P_5, and P_9 are fixed-points (with, say, S_1 as initial state and $P_1 (= S_1)$ as final state in

287 the language of dynamical systems), while sensations S_1, S_2, and S_4 perceived as P_1; sensations

288 S_3, S_5, and S_7 perceived as P_5; sensations S_6, S_8, and S_9 perceived as P_9. The dynamics of action

289 of memory on sensation has the structure of idempotents as displayed below:

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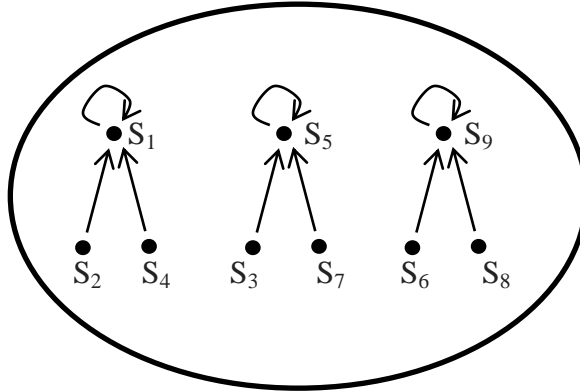
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294 The above neural network can be formalized with action P of memory M on sensation S defined

295 as a map:

296 $$P: S \times M \to S$$

297 For a neural network of N neurons, memories M are $N \times N$ matrices, which when thought of

298 endomaps $N \to N$, and with matrix multiplication as composition (of endomaps; MacLane, 1998,

299 p. 11), form a monoid (Lawvere and Rosebrugh, 2003, p. 167). The $N \times N$ matrix with all

300 diagonal elements as 1 serves as monoid identity (ibid, p. 77), while the $N \times N$ matrix with all

301   entries as 0 is a constant C, since $C \times M = C$ for all memories M. Since this is the only constant

302   of the monoid of memories, the category of actions (Lawvere and Schanuel, 2009, p. 360):

303    $$P: S \times M \to S$$

304   forms a topos of idempotents (ibid, p. 367), i.e. a category with truth value object. The truth

305   value object of the category of idempotents has three truth values:
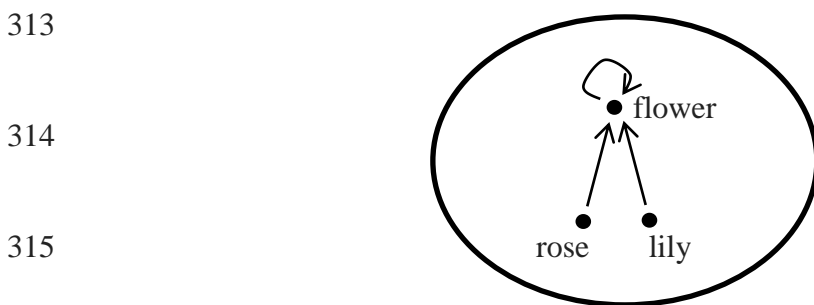
306    $$V = \{false, u, true\}$$

307   equipped with an idempotent endomap:

308    $$v: V \to V$$

309   defined as

310    $v(false) = false$, $v(u) = true$, and $v(true) = true$.

311   Categorical perception, wherein a particular (rose) is perceived as an exemplar of a general

312   (flower), also has the structure of idempotents (see also Lawvere and Schanuel, 2009, p. 106):

313

314

315

316   Given the splitting of idempotent endomaps into section-retract pairs, the category of

317   idempotents can also be characterized in terms of the opposite pair of section-retract maps,

318   wherein the basic shapes are particulars and generals, while sorting and exemplifying are the

319   incidence relations between the two basic shapes (Kathpalia, Posina, and Nagaraj, 2017).

320   We now calculate the truth value object of the category of idempotents. Truth value object of

321   the category of idempotents is an object of the category of idempotents (just as in the case of

322   sets, where a two-element set **2** = {false, true} is the truth value object of the category of sets).

323   An object of the category of idempotents (modelled in the category of sets) is a set A equipped

324   with an idempotent endomap

325 $a: A \rightarrow A$

326   satisfying

327 $a \circ a = a.$

328   where '$\circ$' denotes composition of maps. A map f from an idempotent a to an idempotent b

329   (where $b: B \rightarrow B$ satisfying $b \circ b = b$) is a function

330 $f: A \rightarrow B$

331   satisfying

332 $f \circ a = b \circ f.$

333   The truth value object of a category is defined as an object representing every part

334   (monomorphism) of any object of the category. The truth value object can be calculated in terms

335   of the inverse images of parts of basic shapes along incidence relations (structural maps). Basic

336   shapes along with incidence relations constitute the abstract essence or the theory of a given

337   category. What are the basic shapes and their incidences constituting the theory of category of

338   idempotents? The theory of idempotents consists of a generic idempotent E, along with an [non-

339   identity] idempotent endomap *e* on E (as shown below):

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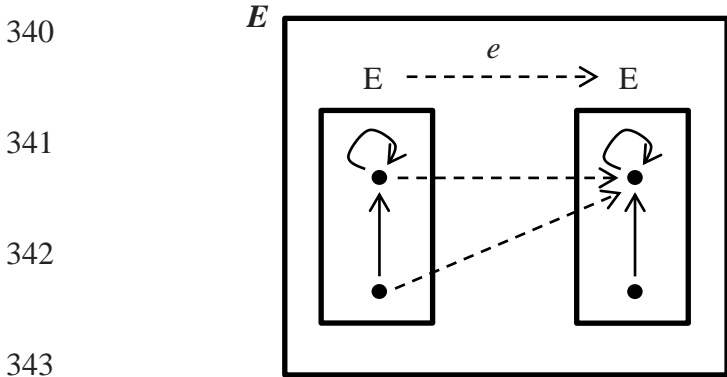
344   The basic shape E along with the incidence relation *e* together constitute the theory subcategory

345   ***E*** of the category of idempotents. The basic shape E has three parts *false*, *u*, and *true* (shown

346   below), which, by the definition of truth value object, correspond to three E-shaped figures (truth

347   values) in the truth value object of the category of idempotents.

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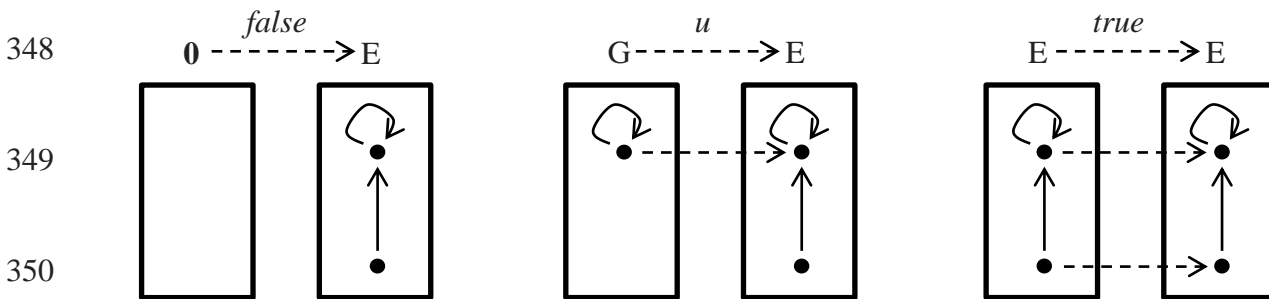
351   The idempotent endomap *v*: V → V, where V = {*false*, *u*, *true*}, is given by the inverse images of

352   the three parts along the incidence relation *e*: E → E. The inverse image of the part *false* along *e*

353   is *false*; the inverse image of the part *u* along *e* is *true*; and the inverse image of the part *true*

354   along *e* is also *true*. Putting it all together we obtain the truth value object *v* (depicted below) of

355   the category of idempotents.

$$v: V \rightarrow V$$

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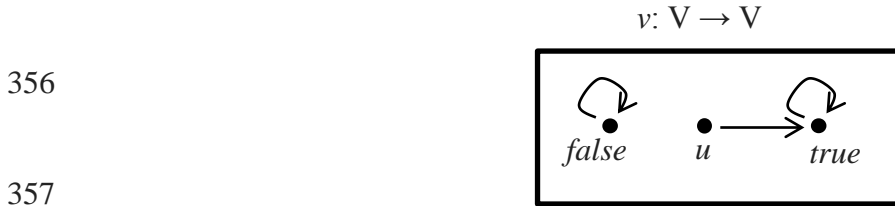
358   Based on the truth value object *v*, logical operations negation, *AND*, and *OR* can be defined.

359   Given a part P of an object A (of the category of idempotents), the familiar negation *not*(P) is

360   defined as the largest part of A whose intersection with P is empty. Dually, another negation

361   *non*(P) is defined as the smallest part of A whose union with P is the whole object A. In the

362   category of sets, both negation operations—*not*, *non*—are identical. However, in the category of

363   idempotents, these two negation operations can give different results (as shown below). The

364   negation operation *non* allows logical contradiction: P *AND non*(P), which is boundary in

365   geometric terminology (Lawvere, 1986, 1991; Lawvere and Rosebrugh, 2003, p. 201).
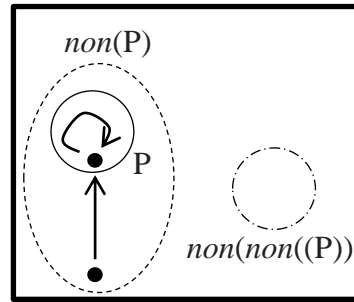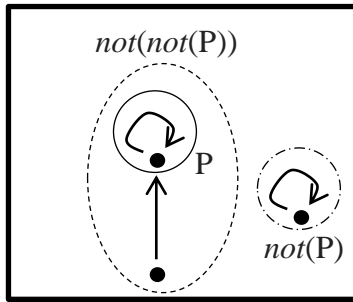
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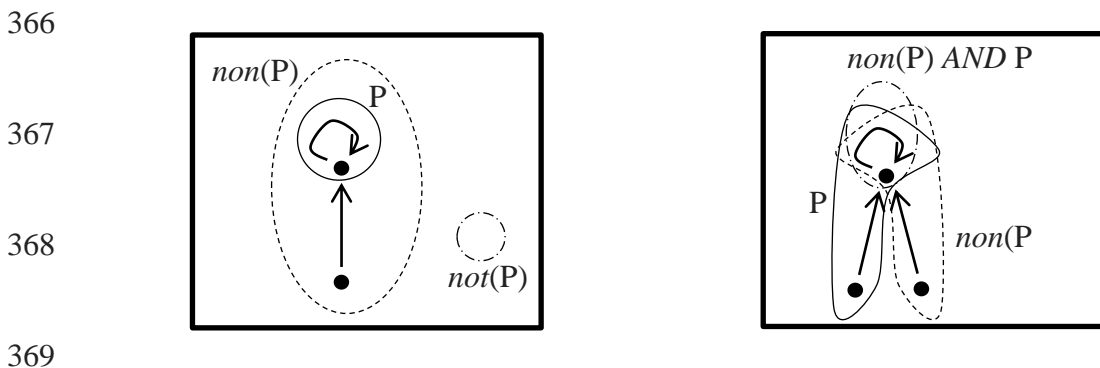
370   Furthermore, double negation can be bigger or smaller than identity depending on the nature of

371   negation as shown below:

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376

377 **A2. Illusion-free Perception**

378 Consider a set A and a set V of values of properties of A. A function

379 $$p: A \rightarrow V$$

380 is a V-valued property of the set A. The set of all V-valued properties of A (or more broadly, the

381 set of all functions from the domain set A to the codomain set V) is the map set V^A. The set V^A

382 of properties, in turn, has properties, which are functionals

383 $$q: V^A \rightarrow V$$

384 where the functional q is a V-valued property of the set V^A of all V-valued properties of A. Our

385 objective is to reconstruct the set A from the set $V^{(V^A)}$ of properties of its V-valued properties

386 (Lawvere and Rosebrugh, 2003, pp. 125-126, 148-152). Towards this end, we define generalized

387 point. First recollect that a point of a set A is a function

388 $$a: \mathbf{1} \rightarrow A$$

389   where $\mathbf{1} = \{*\}$ is the terminal set of the category of sets, i.e. a single-element set; hence points of

390   a set correspond to its elements. Since elements of set completely characterize a set, we define

391   generalized point so as to establish a 1-1 correspondence between points and generalized points

392   (perceived elements).

393   A V-generalized point of the set A is a functional

394   $$q\colon V^A \to V$$

395   such that for each V-valued property of A

396   $$p\colon A \to V$$

397   and for every endomap of the property type V

398   $$e\colon V \to V$$

399   the following equation is satisfied:

400   $$q\,(e \circ p) = e\,(q\,(p))$$

401   where '$\circ$' denotes composition of functions and parentheses denote evaluation of functions.

402   Let's first consider the left-hand side

403   $$q\,(e \circ p)$$

404   The V-valued property of A

405   $$p\colon A \to V$$

406   is an element of the set V^A of all V-valued properties of A. Furthermore, since the codomain V

407   of p is same as the domain V of

408 $$e: V \to V$$

409   we can compose them to obtain a composite

410 $$e \circ p = A \to V \to V = A \to V$$

411   which is also an element of the set V^A of all V-valued properties of A; and hence when the

412   functional

413 $$q: V^A \to V$$

414   is evaluated at '$e \circ p$' gives an element 'v' of the set V of values

415 $$q\,(e \circ p) = v$$

416   Let us now consider the right-hand side

417 $$e\,(q\,(p))$$

418   Once again

419 $$p: A \to V$$

420   is a V-valued property of A, i.e. an element of the set V^A of all V-valued properties of A. Hence

421   evaluating the functional

422 $$q: V^A \to V$$

423   at p gives a value

424 $$q\,(p)$$

425 which is an element of the set V of values. Hence the endomap

426 $$e\colon V \to V$$

427 when evaluated at the element '$q\,(p)$' of domain set V gives an element of V, i.e.

428 $$e\,(q\,(p)) = v'$$

429 Summing up, the functional

430 $$q\colon V^A \to V$$

431 is a V-generalized point of A (perceived element of A) if for every

432 $$p\colon A \to V$$

433 and for every

434 $$e\colon V \to V$$

435 we find that

436 $$q\,(e \circ p) = e\,(q\,(p))$$

437 or in terms of our above example

438 $$v = v'.$$

439   Returning to our main objective, i.e., establishing an isomorphism between points and

440   perceived elements (generalized points) involves finding a property type V such that for every

441   set A, the V-generalized points of A, i.e. the functionals

442
$$q\colon V^A \to V$$

443   satisfying

444
$$q\,(e \circ p) = e\,(q\,(p))$$

445   for every

446
$$p\colon A \to V$$

447   and for every

448
$$e\colon V \to V$$

449   are in 1-1 correspondence with the points

450
$$a\colon \mathbf{1} \to A$$

451   of the set A.

452   We give an example of perceived elements (generalized points) corresponding to points of a set.

453   Let A = {a1, a2} be the object of our investigation and V = {0, 1} be the property type. There are

454   two points in A

455
$$a1\colon \mathbf{1} \to A$$

456   and

457 $$a2\colon \mathbf{1} \to A$$

458 Now let's calculate the number of generalized points. First, there are four functions from A to V,

459 i.e., four V-valued properties of A, $p\colon A \to V$ ($V^A = 2^2 = 4$)

460 $$p1\colon A \to V;\ p1(a1) = 0,\ p1(a2) = 0$$

461 $$p2\colon A \to V;\ p2(a1) = 1,\ p2(a2) = 0$$

462 $$p3\colon A \to V;\ p3(a1) = 0,\ p3(a2) = 1$$

463 $$p4\colon A \to V;\ p4(a1) = 1,\ p4(a2) = 1$$

464 Thus,

465 $$V^A = \{p1, p2, p3, p4\}$$

466 Next, there are 16 functionals, $q\colon V^A \to V$ ($V^{(V^A)} = 2^{(2^2)} = 16$)

467 $$q1\colon V^A \to V;\ q1(p1) = 0,\ q1(p2) = 0,\ q1(p3) = 0,\ q1(p4) = 0$$

468 $$q2\colon V^A \to V;\ q2(p1) = 1,\ q2(p2) = 0,\ q2(p3) = 0,\ q2(p4) = 0$$

469 $$q3\colon V^A \to V;\ q3(p1) = 0,\ q3(p2) = 1,\ q3(p3) = 0,\ q3(p4) = 0$$

470 $$q4\colon V^A \to V;\ q4(p1) = 1,\ q4(p2) = 1,\ q4(p3) = 0,\ q4(p4) = 0$$

471 $$q5\colon V^A \to V;\ q5(p1) = 0,\ q5(p2) = 0,\ q5(p3) = 1,\ q5(p4) = 0$$

472 $$q6\colon V^A \to V;\ q6(p1) = 1,\ q6(p2) = 0,\ q6(p3) = 1,\ q6(p4) = 0$$

473 $$q7\colon V^A \to V;\ q7(p1) = 0,\ q7(p2) = 1,\ q7(p3) = 1,\ q7(p4) = 0$$

474             $q8: V^A \rightarrow V; q8(p1) = 1, q8(p2) = 1, q8(p3) = 1, q8(p4) = 0$

475             $q9: V^A \rightarrow V; q9(p1) = 0, q9(p2) = 0, q9(p3) = 0, q9(p4) = 1$

476             $q10: V^A \rightarrow V; q10(p1) = 1, q10(p2) = 0, q10(p3) = 0, q10(p4) = 1$

477             $q11: V^A \rightarrow V; q11(p1) = 0, q11(p2) = 1, q11(p3) = 0, q11(p4) = 1$

478             $q12: V^A \rightarrow V; q12(p1) = 1, q12(p2) = 1, q12(p3) = 0, q12(p4) = 1$

479             $q13: V^A \rightarrow V; q13(p1) = 0, q13(p2) = 0, q13(p3) = 1, q13(p4) = 1$

480             $q14: V^A \rightarrow V; q14(p1) = 1, q14(p2) = 0, q14(p3) = 1, q14(p4) = 1$

481             $q15: V^A \rightarrow V; q15(p1) = 0, q15(p2) = 1, q15(p3) = 1, q15(p4) = 1$

482             $q16: V^A \rightarrow V; q16(p1) = 1, q16(p2) = 1, q16(p3) = 1, q16(p4) = 1$

483   Of these 16 functionals, there are only two V-generalized points of A corresponding to the two

484   points of A = {a1, a2}. The two V-generalized points of A are

485             $q11: V^A \rightarrow V; q11(p1) = 0, q11(p2) = 1, q11(p3) = 0, q11(p4) = 1$

486   and

487             $q13: V^A \rightarrow V; q13(p1) = 0, q13(p2) = 0, q13(p3) = 1, q13(p4) = 1$

488   i.e. they both satisfy

489                                 $q (e \circ p) = e (q (p))$

490   Let's consider the functional

491     $q11: V^A \to V; q11(p1) = 0, q11(p2) = 1, q11(p3) = 0, q11(p4) = 1$

492   In order for the functional $q11$ to be a V-generalized point of A, it has to satisfy

493     $$q11 \ (e \circ p) = e \ (q11 \ (p))$$

494   for all four p: A $\to$ V and all four endomaps e: V $\to$ V. They are

495     $$e1: V \to V; e1(0) = 0, e1(1) = 0$$

496     $$e2: V \to V; e2(0) = 1, e2(1) = 0$$

497     $$e3: V \to V; e3(0) = 0, e3(1) = 1$$

498     $$e4: V \to V; e4(0) = 1, e4(1) = 1$$

499   So, the set of all endomaps of property type V is $V^V = \{e1, e2, e3, e4\}$.

500   Thus, there are 16 cases we have to evaluate to show that the functional

501     $q11: V^A \to V; q11(p1) = 0, q11(p2) = 1, q11(p3) = 0, q11(p4) = 1$

502   is a V-generalized point of A.

503   Case 1: $(p1, e1)$

504     $$q11 \ (e1 \circ p1) = e1 \ (q11 \ (p1))$$

505   LHS: $q11 \ (e1 \circ p1) = q11 \ (p1) = 0$

506   RHS: $e1 \ (q11 \ (p1)) = e1 \ (0) = 0$

507   Case 2: $(p2, e1)$

508 $\qquad q11\,(e1 \circ p2) = e1\,(q11\,(p2))$

509 LHS: $q11\,(e1 \circ p2) = q11\,(p1) = 0$

510 RHS: $e1\,(q11\,(p2)) = e1\,(1) = 0$

511 Case 3: $(p3, e1)$

512 $\qquad q11\,(e1 \circ p3) = e1\,(q11\,(p3))$

513 LHS: $q11\,(e1 \circ p3) = q11\,(p1) = 0$

514 RHS: $e1\,(q11\,(p3)) = e1\,(0) = 0$

515 Case 4: $(p4, e1)$

516 $\qquad q11\,(e1 \circ p4) = e1\,(q11\,(p4))$

517 LHS: $q11\,(e1 \circ p4) = q11\,(p1) = 0$

518 RHS: $e1\,(q11\,(p4)) = e1\,(1) = 0$

519 $\qquad q11\colon V^A \to V;\ q11(p1) = 0,\ q11(p2) = 1,\ q11(p3) = 0,\ q11(p4) = 1$

520 Case 5: $(p1, e2)$

521 $\qquad q11\,(e2 \circ p1) = e2\,(q11\,(p1))$

522 LHS: $q11\,(e2 \circ p1) = q11\,(p4) = 1$

523 RHS: $e2\,(q11\,(p1)) = e2\,(0) = 1$

524 Case 6: $(p2, e2)$

525
$$q11\ (e2 \circ p2) = e2\ (q11\ (p2))$$

526 LHS: $q11\ (e2 \circ p2) = q11\ (p3) = 0$

527 RHS: $e2\ (q11\ (p2)) = e2\ (1) = 0$

528 Case 7: $(p3, e2)$

529
$$q11\ (e2 \circ p3) = e2\ (q11\ (p3))$$

530 LHS: $q11\ (e2 \circ p3) = q11\ (p2) = 1$

531 RHS: $e2\ (q11\ (p3)) = e2\ (0) = 1$

532 Case 8: $(p4, e2)$

533
$$q11\ (e2 \circ p4) = e2\ (q11\ (p4))$$

534 LHS: $q11\ (e2 \circ p4) = q11\ (p1) = 0$

535 RHS: $e2\ (q11\ (p4)) = e2\ (1) = 0$

536
$$q11\colon V^A \to V;\ q11(p1) = 0,\ q11(p2) = 1,\ q11(p3) = 0,\ q11(p4) = 1$$

537 Case 9: $(p1, e3)$

538
$$q11\ (e3 \circ p1) = e3\ (q11\ (p1))$$

539 LHS: $q11\ (e3 \circ p1) = q11\ (p1) = 0$

540 RHS: $e3\ (q11\ (p1)) = e3\ (0) = 0$

541 Case 10: $(p2, e3)$

542 $$q11\ (e3 \circ p2) = e3\ (q11\ (p2))$$

543 LHS: $q11\ (e3 \circ p2) = q11\ (p2) = 1$

544 RHS: $e3\ (q11\ (p2)) = e3\ (1) = 1$

545 Case 11: $(p3, e3)$

546 $$q11\ (e3 \circ p3) = e3\ (q11\ (p3))$$

547 LHS: $q11\ (e3 \circ p3) = q11\ (p3) = 0$

548 RHS: $e3\ (q11\ (p3)) = e3\ (0) = 0$

549 Case 12: $(p4, e3)$

550 $$q11\ (e3 \circ p4) = e3\ (q11\ (p4))$$

551 LHS: $q11\ (e3 \circ p4) = q11\ (p4) = 1$

552 RHS: $e3\ (q11\ (p4)) = e3\ (1) = 1$

553 $$q11: V^A \rightarrow V;\ q11(p1) = 0,\ q11(p2) = 1,\ q11(p3) = 0,\ q11(p4) = 1$$

554 Case 13: $(p1, e4)$

555 $$q11\ (e4 \circ p1) = e4\ (q11\ (p1))$$

556 LHS: $q11\ (e4 \circ p1) = q11\ (p4) = 1$

557 RHS: $e4\ (q11\ (p1)) = e4\ (0) = 1$

558 Case 14: $(p2, e4)$

559 $$q11 \, (e4 \circ p2) = e4 \, (q11 \, (p2))$$

560 LHS: $q11 \, (e4 \circ p2) = q11 \, (p4) = 1$

561 RHS: $e4 \, (q11 \, (p2)) = e4 \, (1) = 1$

562 Case 15: $(p3, e4)$

563 $$q11 \, (e4 \circ p3) = e4 \, (q11 \, (p3))$$

564 LHS: $q11 \, (e4 \circ p3) = q11 \, (p4) = 1$

565 RHS: $e4 \, (q11 \, (p3)) = e4 \, (0) = 1$

566 Case 16: $(p4, e4)$

567 $$q11 \, (e4 \circ p4) = e4 \, (q11 \, (p4))$$

568 LHS: $q11 \, (e4 \circ p4) = q11 \, (p4) = 1$

569 RHS: $e4 \, (q11 \, (p4)) = e4 \, (1) = 1$

570 Thus, the functional

571 $$q11: V^A \rightarrow V; \, q11(p1) = 0, \, q11(p2) = 1, \, q11(p3) = 0, \, q11(p4) = 1$$

572 is a V-generalized point of A. Along similar lines, we can show that the functional

573 $$q13: V^A \rightarrow V; \, q13(p1) = 0, \, q13(p2) = 0, \, q13(p3) = 1, \, q13(p4) = 1$$

574 is another V-generalized point of A. These two perceived elements (generalized points)

575 correspond to the two points

576 $$a1: \mathbf{1} \to A$$

577 $$a2: \mathbf{1} \to A$$

578 of the set A = {a1, a2}.

579   Next, we give an example of a functional (one of the remaining 14 functionals of the total 16

580 functionals) which is not a generalized point. Consider the functional

581 $$q12: V^A \to V; \; q12(p1) = 1, \; q12(p2) = 1, \; q12(p3) = 0, \; q12(p4) = 1$$

582 In order to be a V-generalized point of A, the functional

583 $$q12: V^A \to V$$

584 must satisfy

585 $$q12 \, (e \circ p) = e \, (q12 \, (p))$$

586 for all p in $V^A = \{p1, p2, p3, p4\}$ and for all e in $V^V = \{e1, e2, e3, e4\}$, i.e., for all 16 cases we

587 evaluated earlier. Let's consider the case of $p = p1$ and $e = e1$. We have to check for the equality

588 $$q12 \, (e1 \circ p1) = e1 \, (q12 \, (p1))$$

589 LHS: $q12 \, (e1 \circ p1) = q12 \, (p1) = 1$

590 RHS: $e1 \, (q12 \, (p1)) = e1 \, (1) = 0$

591 Since LHS is not equal to RHS, the functional

592 $$q12: V^A \to V; \; q12(p1) = 1, \; q12(p2) = 1, \; q12(p3) = 0, \; q12(p4) = 1$$

593   is not a V-generalized point of A.

594   Now we spell out how each point of a set gives rise to a generalized point (perceived element).

595   Consider a set A and a type V. Since generalized point is a functional

596
$$q: V^A \rightarrow V$$

597   we first construct a functional called evaluation functional for each element 'a' of the set A.

598   Recollect that the elements of domain set V^A of the functional q are V-valued properties of A,

599   i.e.

600
$$p: A \rightarrow V$$

601   and the functional q assigns to each p in V^A an element 'v' of the codomain set V of the

602   functional q. An evaluation functional corresponding to an element 'a' of the set A is defined as

603
$$q_a: V^A \rightarrow V$$

604   with

605
$$q_a(p) = p(a)$$

606   Now we show that this evaluation functional is a generalized point, i.e. satisfies

607
$$q_a(e \circ p) = e\,(q_a(p))$$

608   where

609
$$e: V \rightarrow V$$

610   is an endomap on the type V of property.

611   LHS: $q_a(e \circ p) = (e \circ p)\,(\text{a}) = e(p(\text{a}))$

612   RHS: $e\,(q_a(p)) = e(p(\text{a}))$

613   Thus, the evaluation functional corresponding to each point of a set is a generalized point

614   (Lawvere and Rosebrugh, 2003, p. 150). We now give an example of this general result.

615   Consider a set A = {a1, a2, a3} and a property type V = {0, 1}. The set V^A of all V-valued

616   properties of A consists of 8 functions, i.e.

617    $$V^A = \{p1, p2, p3, p4, p5, p6, p7, p8\}$$

618   where

619    $$p1: A \rightarrow V;\ p1(\text{a}1) = 0,\ p1(\text{a}2) = 0,\ p1(\text{a}3) = 0$$

620    $$p2: A \rightarrow V;\ p2(\text{a}1) = 1,\ p2(\text{a}2) = 0,\ p2(\text{a}3) = 0$$

621    $$p3: A \rightarrow V;\ p3(\text{a}1) = 0,\ p3(\text{a}2) = 1,\ p3(\text{a}3) = 0$$

622    $$p4: A \rightarrow V;\ p4(\text{a}1) = 1,\ p4(\text{a}2) = 1,\ p4(\text{a}3) = 0$$

623    $$p5: A \rightarrow V;\ p5(\text{a}1) = 0,\ p5(\text{a}2) = 0,\ p5(\text{a}3) = 1$$

624    $$p6: A \rightarrow V;\ p6(\text{a}1) = 1,\ p6(\text{a}2) = 0,\ p6(\text{a}3) = 1$$

625    $$p7: A \rightarrow V;\ p7(\text{a}1) = 0,\ p7(\text{a}2) = 1,\ p7(\text{a}3) = 1$$

626    $$p8: A \rightarrow V;\ p8(\text{a}1) = 1,\ p8(\text{a}2) = 1,\ p8(\text{a}3) = 1$$

627   Let us now consider a point

628
$$a1: \mathbf{1} \to A$$

629 and the corresponding evaluation functional

630
$$q_{a1}: V^A \to V$$

631 defined as

632
$$q_{a1}(p) = p(\text{a}1)$$

633 for all p in V^A, i.e.

634
$$q_{a1}(p1) = p1(\text{a}1) = 0$$

635
$$q_{a1}(p2) = p2(\text{a}1) = 1$$

636
$$q_{a1}(p3) = p3(\text{a}1) = 0$$

637
$$q_{a1}(p4) = p4(\text{a}1) = 1$$

638
$$q_{a1}(p5) = p5(\text{a}1) = 0$$

639
$$q_{a1}(p6) = p6(\text{a}1) = 1$$

640
$$q_{a1}(p7) = p7(\text{a}1) = 0$$

641
$$q_{a1}(p8) = p8(\text{a}1) = 1$$

642 Now we have to show that this evaluation functional

643
$$q_{a1}: V^A \to V$$

644   satisfies

645
$$q_{a1}(e \circ p) = e(q_{a1}(p))$$

646   for all elements (V-valued properties of A) p in $V^A = \{p1, p2, p3, p4, p5, p6, p7, p8\}$, and for all

647   elements (endomaps of property type V) e in $V^V = \{e1, e2, e3, e4\}$ defined as

648
$$e1: V \rightarrow V; \; e1(0) = 0, \; e1(1) = 0$$

649
$$e2: V \rightarrow V; \; e2(0) = 1, \; e2(1) = 0$$

650
$$e3: V \rightarrow V; \; e3(0) = 0, \; e3(1) = 1$$

651
$$e4: V \rightarrow V; \; e4(0) = 1, \; e4(1) = 1$$

652   Thus we have to test for the equality

653
$$q_{a1}(e \circ p) = e(q_{a1}(p))$$

654   in 32 cases. They are:

655   Case 1: $(p1, e1)$

656
$$q_{a1}(e1 \circ p1) = e1(q_{a1}(p1))$$

657   LHS: $q_{a1}(e1 \circ p1) = q_{a1}(p1) = 0$

658   RHS: $e1(q_{a1}(p1)) = e1(0) = 0$

659   Case 2: $(p2, e1)$

660
$$q_{a1}(e1 \circ p2) = e1(q_{a1}(p2))$$

661   LHS: q_{a1} $(e1 \circ p2) = q_{a1}$ $(p1) = 0$

662   RHS: $e1$ $(q_{a1}$ $(p2)) = e1$ $(1) = 0$

663   Case 3: $(p3, e1)$

664$$q_{a1} (e1 \circ p3) = e1 (q_{a1} (p3))$$

665   LHS: q_{a1} $(e1 \circ p3) = q_{a1}$ $(p1) = 0$

666   RHS: $e1$ $(q_{a1}$ $(p3)) = e1$ $(0) = 0$

667   Case 4: $(p4, e1)$

668$$q_{a1} (e1 \circ p4) = e1 (q_{a1} (p4))$$

669   LHS: q_{a1} $(e1 \circ p4) = q_{a1}$ $(p1) = 0$

670   RHS: $e1$ $(q_{a1}$ $(p4)) = e1$ $(1) = 0$

671   Case 5: $(p5, e1)$

672$$q_{a1} (e1 \circ p5) = e1 (q_{a1} (p5))$$

673   LHS: q_{a1} $(e1 \circ p5) = q_{a1}$ $(p1) = 0$

674   RHS: $e1$ $(q_{a1}$ $(p5)) = e1$ $(0) = 0$

675   Case 6: $(p6, e1)$

676$$q_{a1} (e1 \circ p6) = e1 (q_{a1} (p6))$$

677   LHS: q_{a1} $(e1 \circ p6) = q_{a1}$ $(p1) = 0$

678   RHS: $e1\ (q_{a1}\ (p6)) = e1\ (1) = 0$

679   Case 7: $(p7, e1)$

680
$$q_{a1}\ (e1 \circ p7) = e1\ (q_{a1}\ (p7))$$

681   LHS: $q_{a1}\ (e1 \circ p7) = q_{a1}\ (p1) = 0$

682   RHS: $e1\ (q_{a1}\ (p7)) = e1\ (0) = 0$

683   Case 8: $(p8, e1)$

684
$$q_{a1}\ (e1 \circ p8) = e1\ (q_{a1}\ (p8))$$

685   LHS: $q_{a1}\ (e1 \circ p8) = q_{a1}\ (p1) = 0$

686   RHS: $e1\ (q_{a1}\ (p8)) = e1\ (1) = 0$

687   Case 9: $(p1, e2)$

688
$$q_{a1}\ (e2 \circ p1) = e2\ (q_{a1}\ (p1))$$

689   LHS: $q_{a1}\ (e2 \circ p1) = q_{a1}\ (p8) = 1$

690   RHS: $e2\ (q_{a1}\ (p1)) = e2\ (0) = 1$

691   Case 10: $(p2, e2)$

692
$$q_{a1}\ (e2 \circ p2) = e2\ (q_{a1}\ (p2))$$

693   LHS: $q_{a1}\ (e2 \circ p2) = q_{a1}\ (p7) = 0$

694   RHS: $e2\ (q_{a1}\ (p2)) = e2\ (1) = 0$

695   Case 11: ($p3$, $e2$)

696 $q_{a1} (e2 \circ p3) = e2 (q_{a1} (p3))$

697   LHS: $q_{a1} (e2 \circ p3) = q_{a1} (p6) = 1$

698   RHS: $e2 (q_{a1} (p3)) = e2 (0) = 1$

699   Case 12: ($p4$, $e2$)

700 $q_{a1} (e2 \circ p4) = e2 (q_{a1} (p4))$

701   LHS: $q_{a1} (e2 \circ p4) = q_{a1} (p5) = 0$

702   RHS: $e2 (q_{a1} (p4)) = e2 (1) = 0$

703   Case 13: ($p5$, $e2$)

704 $q_{a1} (e2 \circ p5) = e2 (q_{a1} (p5))$

705   LHS: $q_{a1} (e2 \circ p5) = q_{a1} (p4) = 1$

706   RHS: $e2 (q_{a1} (p5)) = e2 (0) = 1$

707   Case 14: ($p6$, $e2$)

708 $q_{a1} (e2 \circ p6) = e2 (q_{a1} (p6))$

709   LHS: $q_{a1} (e2 \circ p6) = q_{a1} (p3) = 0$

710   RHS: $e2 (q_{a1} (p6)) = e2 (1) = 0$

711   Case 15: ($p7$, $e2$)

712 $$q_{a1} (e2 \circ p7) = e2 (q_{a1} (p7))$$

713 LHS: $q_{a1} (e2 \circ p7) = q_{a1} (p2) = 1$

714 RHS: $e2 (q_{a1} (p7)) = e2 (0) = 1$

715 Case 16: $(p8, e2)$

716 $$q_{a1} (e2 \circ p8) = e2 (q_{a1} (p8))$$

717 LHS: $q_{a1} (e2 \circ p8) = q_{a1} (p1) = 0$

718 RHS: $e2 (q_{a1} (p8)) = e2 (1) = 0$

719 Case 17: $(p1, e3)$

720 $$q_{a1} (e3 \circ p1) = e3 (q_{a1} (p1))$$

721 LHS: $q_{a1} (e3 \circ p1) = q_{a1} (p1) = 0$

722 RHS: $e3 (q_{a1} (p1)) = e3 (0) = 0$

723 Case 18: $(p2, e3)$

724 $$q_{a1} (e3 \circ p2) = e3 (q_{a1} (p2))$$

725 LHS: $q_{a1} (e3 \circ p2) = q_{a1} (p2) = 1$

726 RHS: $e3 (q_{a1} (p2)) = e3 (1) = 1$

727 Case 19: $(p3, e3)$

728 $$q_{a1} (e3 \circ p3) = e3 (q_{a1} (p3))$$

729   LHS: $q_{a1}\,(e3 \circ p3) = q_{a1}\,(p3) = 0$

730   RHS: $e3\,(q_{a1}\,(p3)) = e3\,(0) = 0$

731   Case 20: $(p4,\, e3)$

732                                  $q_{a1}\,(e3 \circ p4) = e3\,(q_{a1}\,(p4))$

733   LHS: $q_{a1}\,(e3 \circ p4) = q_{a1}\,(p4) = 1$

734   RHS: $e3\,(q_{a1}\,(p4)) = e3\,(1) = 1$

735   Case 21: $(p5,\, e3)$

736                                  $q_{a1}\,(e3 \circ p5) = e3\,(q_{a1}\,(p5))$

737   LHS: $q_{a1}\,(e3 \circ p5) = q_{a1}\,(p5) = 0$

738   RHS: $e3\,(q_{a1}\,(p5)) = e3\,(0) = 0$

739   Case 22: $(p6,\, e3)$

740                                  $q_{a1}\,(e3 \circ p6) = e3\,(q_{a1}\,(p6))$

741   LHS: $q_{a1}\,(e3 \circ p6) = q_{a1}\,(p6) = 1$

742   RHS: $e3\,(q_{a1}\,(p6)) = e3\,(1) = 1$

743   Case 23: $(p7,\, e3)$

744                                  $q_{a1}\,(e3 \circ p7) = e3\,(q_{a1}\,(p7))$

745   LHS: $q_{a1}\,(e3 \circ p7) = q_{a1}\,(p7) = 0$

746   RHS: $e3\ (q_{a1}\ (p7)) = e3\ (0) = 0$

747   Case 24: $(p8, e3)$

748    $$q_{a1}\ (e3 \circ p8) = e3\ (q_{a1}\ (p8))$$

749   LHS: $q_{a1}\ (e3 \circ p8) = q_{a1}\ (p8) = 1$

750   RHS: $e3\ (q_{a1}\ (p8)) = e3\ (1) = 1$

751   Case 25: $(p1, e4)$

752    $$q_{a1}\ (e4 \circ p1) = e4\ (q_{a1}\ (p1))$$

753   LHS: $q_{a1}\ (e4 \circ p1) = q_{a1}\ (p8) = 1$

754   RHS: $e4\ (q_{a1}\ (p1)) = e4\ (0) = 1$

755   Case 26: $(p2, e4)$

756    $$q_{a1}\ (e4 \circ p2) = e4\ (q_{a1}\ (p2))$$

757   LHS: $q_{a1}\ (e4 \circ p2) = q_{a1}\ (p8) = 1$

758   RHS: $e4\ (q_{a1}\ (p2)) = e4\ (1) = 1$

759   Case 27: $(p3, e4)$

760    $$q_{a1}\ (e4 \circ p3) = e4\ (q_{a1}\ (p3))$$

761   LHS: $q_{a1}\ (e4 \circ p3) = q_{a1}\ (p8) = 1$

762   RHS: $e4\ (q_{a1}\ (p3)) = e4\ (0) = 1$

763   Case 28: ($p4$, $e4$)

764
$$q_{a1} (e4 \circ p4) = e4 (q_{a1} (p4))$$

765   LHS: $q_{a1} (e4 \circ p4) = q_{a1} (p8) = 1$

766   RHS: $e4 (q_{a1} (p4)) = e4 (1) = 1$

767   Case 29: ($p5$, $e4$)

768
$$q_{a1} (e4 \circ p5) = e4 (q_{a1} (p5))$$

769   LHS: $q_{a1} (e4 \circ p5) = q_{a1} (p8) = 1$

770   RHS: $e4 (q_{a1} (p5)) = e4 (0) = 1$

771   Case 30: ($p6$, $e4$)

772
$$q_{a1} (e4 \circ p6) = e4 (q_{a1} (p6))$$

773   LHS: $q_{a1} (e4 \circ p6) = q_{a1} (p8) = 1$

774   RHS: $e4 (q_{a1} (p6)) = e4 (1) = 1$

775   Case 31: ($p7$, $e4$)

776
$$q_{a1} (e4 \circ p7) = e4 (q_{a1} (p7))$$

777   LHS: $q_{a1} (e4 \circ p7) = q_{a1} (p8) = 1$

778   RHS: $e4 (q_{a1} (p7)) = e4 (0) = 1$

779   Case 32: ($p8$, $e4$)

780
$$q_{a1} (e4 \circ p8) = e4 (q_{a1} (p8))$$

781 LHS: $q_{a1} (e4 \circ p8) = q_{a1} (p8) = 1$

782 RHS: $e4 (q_{a1} (p8)) = e4 (1) = 1$

783 Thus the evaluation functional

784 q_{a1}: $V^A \to V$; $q_{a1}(p1)=0$, $q_{a1}(p2)=1$, $q_{a1}(p3)=0$, $q_{a1}(p4)=1$, $q_{a1}(p5)=0$, $q_{a1}(p6)=1$, $q_{a1}(p7)=0$, $q_{a1}(p8)=1$

785 satisfying

786
$$q_{a1} (e \circ p) = e (q_{a1} (p))$$

787 (for all p: A $\to$ V and for all e: V $\to$ V) is a V-generalized point of A corresponding to the point

788
$$a1: \mathbf{1} \to A$$

789 Along the same lines, we can show that each of remaining two points of the set A = {a1, a2, a3}

790 give rise to corresponding V-generalized points of A.

791   With property type V = **2** (two-element set), although there is a V-generalized point of A

792 corresponding to each point of A, there can also be generalized points that do not correspond to

793 any points of A, which we may call illusions (or ghost points). In order to obtain a 1-1

794 correspondence between points of any set A and V-generalized points of A, i.e. isomorphism

795 between objects and perceived objects, we need a 3-element set as the property type V. One of

796 our objectives is to calculate objects analogous to the 3-element set (in the category of sets) in

797 categories that are reflective of reality such as the category of categories. We plan to approach

798 this goal by calculating the basic types of knowing (objects analogous to 3-element set in the

799   category of sets) in more structured categories such as dynamical systems, functions, graphs, and

800   actions.

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