

# Gravity waves in parity-violating Copernican Universes

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In recent work minimal theories allowing the variation of the cosmological constant,  $\Lambda$ , by means of a balancing torsion, have been proposed. It was found that such theories contain *parity violating* homogeneous and isotropic solutions, due to a torsion structure called the Cartan spiral staircase. Their dynamics are controlled by Euler and Pontryagin *quasi*-topological terms in the action. Here we show that such theories predict a dramatically different picture for gravitational wave fluctuations in the parity violating branch. If the dynamics are ruled solely by the Euler-type term, then linear tensor mode perturbations are entirely undetermined, hinting at a new type of gauge invariance. The Pontryagin term not only permits for phenomenologically sounder background solutions (as found in previous literature), but for realistic propagation of gravitational wave modes. These have the general property that the right and left handed gravitational waves propagate with different speeds. More generally they imply modified dispersion relations for the graviton, with both parity violating and non-violating deformations, including an effective mass for both gravitational wave polarizations. We discuss the observational constraints and predictions of these theories.

## I. INTRODUCTION

The cosmological constant,  $\Lambda$ , and the Copernican principle are two cornerstones of modern cosmology. In this paper we explore the implications of the fact that their story may be more intricate than it is usually assumed. That the cosmological “constant” does not actually need to be constant in theories with torsion has been noted, for example, in [1, 2]. It is not new that torsion can change dramatically the perspective of many problems (for a selection of examples see [3–14]). It has also been noted [5, 15] that under the shadow of torsion, homogeneity and isotropy do not imply parity invariance. The Copernican principle therefore has a choice between incorporating parity invariance or not. Parity odd solutions in homogeneous and isotropic models employ a geometrical structure which has been known since the inception of General Relativity: Cartan’s spiral staircase [5, 16]. Thus, a varying Lambda may go hand in hand with parity violating Copernican models, creating an interesting synergy.

Within the theories considered in [1, 15] the inverse of Lambda becomes canonically conjugate to the Chern-Simons invariant<sup>1</sup> [20, 21]. The radical implications of this fact in quantum cosmology were examined in [21] (see [22, 23] for the background problem). In the context of classical solutions, the dynamics are then ruled by two topological invariants, of which the Chern-Simons functional is the density. Depending

on whether one considers the real or imaginary parts of the Chern-Simons term, these are the Pontryagin and the Euler (or Gauss-Bonnet) invariants. Since these terms appear in the action multiplied by  $\Lambda^{-1}$ , they are only topological invariants if  $\Lambda$  is a constant. The variability of Lambda disrupts their topological nature, and so they are *quasi*-topological terms (to use the terminology of [1]).

The theories considered in [1, 2, 15] have the virtue that they do not add new parameters to gravity with respect to Einstein’s theory with a cosmological constant. The coefficient of the Euler term is fully fixed by the Bianchi identities (from solutions without matter), so that the only true new parameter is the numerical coefficient of the Pontryagin term, should we consider it. However, for these theories  $\Lambda$  is longer a free parameter, as it is in Einstein’s theory. Hence, a theory with the Euler term alone would have *fewer* free parameters than General Relativity, as argued in [1]. As explained in [2] such a theory conflicts dramatically with basic Hot Big Bang cosmology (it refuses to accept a radiation epoch). The introduction of the Pontryagin term allows for a viable expansion history (as studied in [15]), leaving the working theory with the *same* number of free parameters as General Relativity.

It was found in [15] that the parity-even and parity-odd Copernican solutions belong to separate branches of the dynamics. Indeed, a Hamiltonian analysis revealed a different structure of constraints and consequently a different number of degrees of freedom. We are therefore talking about different phases of the same non-perturbative theory. The underlying gauge symmetry associated with the new constraint of the theory is a form of conformal invariance (generalized for theories with torsion). Lambda appears to be pure gauge with regards to this symmetry in the parity-even branch (in the absence of matter). The parity-odd branch breaks conformal invariance even in the absence of matter, giving a varying Lambda a physical meaning. Non-conformal matter does the

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<sup>1</sup> Models where  $\Lambda$  is directly conjugate to the Chern-Simons invariant (or similar quantities) have been considered within the context of unimodular gravity [17–19].

same in the parity-even branch, but then Lambda becomes a slave to matter (much in the spirit of [24]). It is interesting to note that the (odd parity) Pontryagin term is only relevant for the homogeneous and isotropic dynamics in the parity-odd branch of the solutions.

A preliminary investigation [15] revealed that phenomenology in these theories (which, we stress, often have *fewer* free parameters than General Relativity, and rarely can be made to have more) shows a preference for the parity-odd branch in the presence of Pontryagin dynamics. These considerations concerned only the background solution, which is already very rich in the parity-odd branch. The next obvious step is to investigate the propagation of gravitational waves in the same branch. Such is the purpose of the current investigation.

The plan of this paper is as follows. In Section II we start by reviewing previous results that will be needed in this paper, translating them into the notation we found most useful for establishing a perturbation calculation. In Section III we set up the tensor perturbation variables and work out the linearized equations in various forms (tetrad index and space-time index forms, and then decomposed in Fourier and helicity modes). The equations in general look ominous: we have to contend with first order equations in three variables – metric, and parity even and odd components of the connection – but in subsection III we condense them in a more aesthetically pleasing form, and lay out a strategy for their solution.

The rest of the paper is spent on working out solutions for various parameter settings. In Section IV we briefly discuss general properties of the perturbed equations. Next we discuss a number of limiting cases of interest. As a sanity check we find the General Relativity limit in Section V, with reassuring results. In Section VI we consider the case where the dynamics are ruled purely by an Euler pseudo-topological term. We unveil our first surprise: the tensor mode perturbation is left

entirely undetermined by the equations of motion. This could well signify that they have become a gauge degree of freedom in this case.

The introduction of the Pontryagin term changes the picture. Physical propagating tensor modes now do exist, but they are endowed with chiral modified dispersion relations. We concentrate on two limiting forms - in Section VII the propagation of gravitational waves in the late universe is discussed, whilst in Section VIII their propagation at earlier stages when the evolution is dominated by matter and radiation components is discussed. Finally in Section IX we summarize our results and discuss prospects for further development.

## II. REVIEW OF PREVIOUS RESULTS

Here we shall review some results, adapting the notation in previous literature to the notation that shall be more useful in this paper. Specifically, we shall use the following conventions for indices:

- $A, B, C, D$ :  $SO(1, 3)$  gauge indices.
- $I, J, K, L$ :  $SO(3)$  gauge indices.
- $\mu, \nu, \alpha, \beta$ : spacetime coordinate indices.
- $t$ : time coordinate index.
- $i, j, k, l$ : spatial coordinate indices.

### A. The full theory and its equations

The theories we analyze can be written as:

$$S^g[e, \omega, \Lambda] = - \int \frac{3}{2\Lambda} \left( \epsilon_{ABCD} + \frac{2}{\gamma} \eta_{AC} \eta_{BD} \right) \left( R^{AB} - \frac{\Lambda}{3} e^A e^B \right) \left( R^{CD} - \frac{\Lambda}{3} e^C e^D \right) - \frac{2}{\gamma} \int T^A T_A. \quad (1)$$

where  $R^{AB} \equiv d\omega^{AB} + \omega^A_C \omega^{CB}$ ,  $T^A \equiv de^A + \omega^A_B e^B$  and unless otherwise stated, multiplication of differential forms is via the wedge product<sup>2</sup>. The action can be rewritten as proportional to four terms  $S^g = S_{Pal} + S_{Eul} + S_{NY} + S_{Pont}$ ,

with

$$S_{Pal} = \int \epsilon_{ABCD} \left( e^A e^B R^{CD} - \frac{\Lambda}{6} e^A e^B e^C e^D \right), \quad (2)$$

$$S_{Eul} = -\frac{3}{2} \int \frac{1}{\Lambda} \epsilon_{ABCD} R^{AB} R^{CD}, \quad (3)$$

$$S_{NY} = \frac{2}{\gamma} \int e^A e^B R_{AB} - T^A T_A, \quad (4)$$

$$S_{Pont} = -\frac{3}{\gamma} \int \frac{1}{\Lambda} R^{AB} R_{AB}. \quad (5)$$

The first term is the Palatini action, though differs from that of the Einstein-Cartan theory in that we allow  $\Lambda$  vary as a dynamical field rather than fixing it to be a constant. The second term is the quasi-Euler term of [1]. The third term is the Nieh-Yan topological invariant (replacing the Holst term should there be torsion). The last term is the quasi-Pontryagin term studied

<sup>2</sup> If the parameter  $\gamma \rightarrow \infty$  and  $\Lambda$  is constrained to be a constant, the resulting theory is the Einstein-Cartan theory alongside an Euler boundary term; the particular coefficient of this boundary term has been found to be associated with interesting properties of Noether charges in gravity [25, 26]

in [15]. We stress that the connection proposed here between  $\gamma$  and the pre-factor of the quasi-Pontryagin term can be broken, and is not strictly needed. More generally, we could also look at theories with arbitrary numerical factors in front of the quasi-Euler and quasi-Pontryagin terms.

As usual, matter can be added to the gravitational action, to yield a total action:

$$S = \frac{1}{2\kappa} S^g(e, \omega, \Lambda) + S_M(\Phi, e, \omega, \Lambda), \quad (6)$$

where  $\Phi$  denote matter fields. The full gravitational equations of motion of this theory are then:

$$\epsilon_{ABCD} \left( e^B R^{CD} - \frac{1}{3} \Lambda e^B e^C e^D \right) = -2\kappa \tau_A \quad (7)$$

$$T^{[A} e^{B]} + \frac{3}{2\Lambda^2} d\Lambda R^{AB} - \frac{3}{4\gamma\Lambda^2} \epsilon^{ABCD} d\Lambda R_{CD} = \kappa \mathcal{S}^{AB} \quad (8)$$

$$\epsilon_{ABCD} \left( R^{AB} R^{CD} - \frac{1}{9} \Lambda^2 e^A e^B e^C e^D \right) + \frac{2}{\gamma} R^{AB} R_{AB} = -2\kappa \mathcal{J} \quad (9)$$

where  $\kappa \equiv 8\pi G$  and we have defined energy momentum 3-form  $\tau_A = \frac{1}{2} \frac{\delta S_M}{\delta e^A}$ , the spin-current 3-form  $\mathcal{S}^{AB} \equiv -(1/2) \epsilon^{ABCD} \delta S_M / \delta \omega^{CD}$  and the  $\Lambda$ -source 4-form  $\mathcal{J} \equiv (2/3) \delta S / \delta \Lambda$ . They are obtained by varying (1) together with the action for matter with respect to  $e$ ,  $\omega$  and  $\Lambda$ , respectively. A key property of these models is that Einstein's equation (7) takes the same form in the Einstein-Cartan formulation of gravity (where  $\Lambda = cst.$ ). Any dynamics for  $\Lambda$  will arise from the gravitational field  $\omega^{AB}$  rather than via the addition of explicit kinetic terms for  $\Lambda$  in the Lagrangian.

In this paper we will confine ourselves to situations where the quantities  $\mathcal{S}^{AB}$  and  $\mathcal{J}$  both are negligible. For standard 'minimal' coupling between fermions and the spin connection, the quantity  $\mathcal{S}^{AB}$  is sourced by the axial spinor current; much of our focus will be on the behaviour of certain cosmological perturbations in 'recent' post-recombination cosmological history where this quantity is expected to be negligible [3]. The assumption that  $\mathcal{J}$  is negligible must be regarded as a simplifying assumption and more detailed analysis is needed to determine its expected coupling to matter. For the particular cosmological consequences of the theory examined in this paper, it will suffice to that the matter content is describable in terms of perfect fluids. By way of example, a perfect fluid with density  $\rho$ , pressure  $p$  and four-velocity  $U^\mu = e_A^\mu U^A$  will have stress-energy 3-form:

$$\tau^A = -\frac{1}{6}(\rho + p)U^A \epsilon_{BCDE} U^B e^C e^D e^E - \frac{1}{6} p \epsilon^A{}_{BCD} e^B e^C e^D \quad (10)$$

## B. The background solution

We now look at the behaviour of the theory in situations where spacetime has Friedmann-Robertson-Walker (FRW) symmetry. This symmetry is widely considered to well approximate the geometry of the universe on large scales and there exist strong constraints on the evolution of the universe

within this framework. We will henceforth refer to possible solutions with this symmetry as 'background' solutions as later we will consider the behaviour of small perturbations around them. It is important then to demonstrate that the combined action (6) yields solutions that are consistent with these constraints.

We shall denote all background quantities by a bar over the respective variable. For simplicity we assume that the background spatial curvature is zero, so that we can use Cartesian coordinates with

$$\bar{e}^0 = N(t) dt \quad (11)$$

$$\bar{e}^I = a(t) \delta_i^I dx^i \quad (12)$$

where  $N(t)$  is the lapse function ( $N = 1$  for proper time) and  $a(t)$  is the expansion factor. Note that  $\bar{e}_i^I = a \delta_i^I$  and  $\bar{e}_I^i = a^{-1} \delta_I^i$ . Then, the spin connection will be given by:

$$\bar{\omega}^{0I} = g(t) a(t) \delta_i^I dx^i \quad (13)$$

$$\bar{\omega}^{IJ} = -P(t) a(t) \epsilon^{IJ}{}_{K} \delta_k^K dx^k \quad (14)$$

where  $g$  and  $P$  are its parity even and odd components, respectively. A connection of the form (14) was considered by Cartan as an extension to Riemannian geometry, with parallel transport according to this connection yielding a rotation of vectors with a 'handedness' dictated by the sign of  $P$ . This effect has been termed Cartan's spiral staircase and we will see that all parity violating effects in this gravitational model appear only when  $P \neq 0$ . The torsion associated with (13) and (14) is given by:

$$\bar{T}^0 = 0 \quad (15)$$

$$\bar{T}^I = T \bar{e}^I \bar{e}^0 + P \epsilon^{IJK} \bar{e}_J \bar{e}_K \quad (16)$$

with the parity even component  $T$  related to  $g$  by:

$$T = \left( g - \frac{1}{N} \frac{\dot{a}}{a} \right). \quad (17)$$

The field strength is:

$$\bar{R}^{0I} = \frac{1}{N} \left( \dot{g} + \frac{\dot{a}}{a} g \right) \bar{e}^0 \bar{e}^I + g P \epsilon^{IJK} \bar{e}_J \bar{e}_K \quad (18)$$

$$\bar{R}^{IJ} = \frac{1}{N} \left( \dot{P} + P \frac{\dot{a}}{a} \right) \epsilon^{IJ} \bar{e}^K \bar{e}^0 + (g^2 - P^2) \bar{e}^I \bar{e}^J \quad (19)$$

$$g^2 - P^2 = \frac{\Lambda + \kappa\rho}{3} \quad (20)$$

$$\frac{(ag)^\cdot}{a} = \frac{\Lambda}{3} - \frac{\kappa}{6}(\rho + 3p) \quad (21)$$

$$T = \frac{\dot{\Lambda}}{2\Lambda^2} \left( \Lambda + \kappa\rho - \frac{6}{\gamma} gP \right) \quad (22)$$

$$P = \frac{3\dot{\Lambda}}{\Lambda^2} \left( gP + \frac{\Lambda + \kappa\rho}{6\gamma} \right) \quad (23)$$

$$(\Lambda + \kappa\rho) \left( \Lambda - \frac{\kappa}{2}(\rho + 3p) \right) - \Lambda^2 = 18gP \frac{(aP)^\cdot}{a} + \frac{9}{\gamma} \left( \frac{\Lambda + \kappa\rho}{3} \frac{(aP)^\cdot}{a} + \frac{2}{3} \left( \Lambda - \kappa \frac{\rho + 3p}{2} \right) gP \right) \quad (24)$$

As shown in Appendix A, this system can be cast in the form of a first-order system of evolution equations for  $\{a, g, \Lambda, P\}$  plus a constraint (the Hamiltonian constraint/Friedmann's equation). Reference to these background equations will be made at several points in this paper, to simplify the perturbation equations.

### C. Background evolution

We now discuss solutions to equations (20)-(24) with an emphasis on solutions that appear likely to be most consistent with the observed expansion history of the universe. Care must be taken here as many probes of background quantities are additionally sensitive to details of cosmological perturbations. For example, the position of the first peak of temperature anisotropies in the cosmic microwave background (CMB) is sensitive to both the distance to last scattering (a background quantity) and the sound horizon at last scattering (a quantity which additionally depends on the form of equations describing cosmological perturbations) [27].

The system of equations (20)-(24) is rather complicated and must be solved numerically. However, relevant approximate solutions do exist, which we will now discuss.

#### 1. Early times

There is strong evidence that the universe has undergone an early period ('the radiation era') where relativistic species (such as photons and relativistic neutrinos) dominate the evolution of the universe for a time before the universe cools down enough such that the gravitational effect of near-pressureless/dustlike matter (baryons and dark matter) dominates ('the matter era'), before eventually a new source of en-

ergy - typically termed dark energy - begins to dominate and cause the expansion of the universe to accelerate [28]. We will look to see whether the theory (6) permits this kind of cosmological history, whilst ascribing the recent cosmological acceleration to - now dynamical -  $\Lambda$ . An important part of this is that the gravitational effect of new degrees of freedom quantities such as  $\Lambda$  and the torsion  $P$  do not contradict the above picture.

It can be shown that when  $|\gamma| \ll 1$ , to first order in  $\gamma$  there exists a solution for the field  $P$  in the limit  $\Lambda \rightarrow 0$

$$P = P_{(\rho)} = \frac{\gamma}{3} \sqrt{\frac{\kappa\rho}{3}} \quad (25)$$

We see then that when this solution holds, the torsion field  $P$  is proportional to  $\gamma$  and so a smaller value of  $\gamma$  suppresses torsion in the cosmological background. Neglecting the contribution of  $\Lambda$  is expected to be a good approximation in the earlier universe where the 'dark energy' is a sub-dominant contributor to the universe's expansion. When (25) holds it may be shown that the Friedmann equation can be recovered in approximation:

$$3 \left( \frac{\dot{a}}{a} \right)^2 = \left( 1 - \frac{\gamma^2}{9} \right) \kappa\rho + \mathcal{O}(\gamma^3) \quad (26)$$

where we have adopted the  $N = 1$  spacetime gauge. Hence the solution (25) acts to rescale the bare Newton's constant  $G$  during times when the effect of  $\Lambda$  is negligible. The degree to which this effect is observable depends on how the value of Newton's constant  $G_N$  measured in tabletop experiments is related to  $G$ . If  $G \neq G_N$ , then the rate of expansion  $\dot{a}/a$  due to a given  $\rho$  will be different from as is the case in General Relativity and so in principle  $\gamma$  could be constrained by probes of the expansion rate during big bang nucleosynthesis [29].

However, importantly, the solution (25) is not stable. By way of illustration, we may consider the evolution of small, homogeneous perturbations  $P = P_{(\rho)}(1 + \delta_P(t))$ . It can be shown that deep in the radiation era where  $\kappa\rho/3 \sim H_0^2\Omega_r/a^4$  - where  $H_0$  is the Hubble constant today - that

$$P_{(\rho)} = \frac{\gamma}{3a^2}H_0\sqrt{\Omega_r} \quad (27)$$

$$\delta_P = Ca^3 \quad (28)$$

Therefore  $\delta_P$  grows as  $a$  increases. By way of example, if  $\delta_P \ll 1$  at  $a \sim 10^{-15}$  then for it to remain smaller than unity at  $a \sim 10^{-5}$  we must have  $\delta_P(a = 10^{-15}) < 10^{-30}$ . This indicates that significant fine-tuning of initial data is required for the spiral staircase field  $P$  to find itself following the solution (25).

If  $P$  deviates considerably from the tracking solution, the tendency is for  $P$  to evolve to *dominate* the evolution of the universe. In this case it may be shown that  $P = P_0/a$  where  $P_0$  is a constant and  $a \sim (1+\gamma)(t-t_0)$  - here the evolution of the universe due to  $P$  resembles a General-Relativistic *empty* universe with negative spatial curvature. It is hard to see how such a universe could be consistent with experiment. This is the case even if  $P$  is initially negligible. Therefore, the phenomenological viability of the model rests on  $P$  being able to initially find itself sufficiently close to the form (25) to avoid dominating the evolution of the universe.

## 2. Late times

During late time cosmological evolution for realistic cosmologies we expect that universe to begin accelerating and we look to ascribe this to  $\Lambda$  and  $P$  beginning to dominate cosmic evolution. Again assuming  $|\gamma| \ll 1$  and now assuming  $P^2 \ll \Lambda$  and taking the limit  $\rho \rightarrow 0$  we have the following evolution equations for  $\Lambda$  and  $P$ :

$$\frac{dP}{d \ln a} = -3P, \quad \frac{d\Lambda}{d \ln a} = 2\sqrt{3}\gamma P\sqrt{\Lambda} \quad (29)$$

which possess solutions

$$P = \frac{P_i}{a^3} \quad (30)$$

$$\Lambda = \Lambda_0 - \frac{2\gamma}{\sqrt{3}a^3}P_i\sqrt{\Lambda_0} \quad (31)$$

So, asymptotically for large  $a$ ,  $\Lambda \rightarrow \Lambda_0$  and  $P \rightarrow 0$ , leading to a confluence with the current standard cosmological picture of the late-time universe's evolution being dominated by a cosmological constant of magnitude  $\Lambda_0$ . The contribution of  $P$  to the Hamiltonian constraint goes as  $\sim P^2$  so we see that in this regime  $P$  evolves like a shear component, its energy density diluting as  $a^{-6}$ . For realistic cosmologies a typical value for  $P_i$  will be given by its value when  $\Lambda$  begins to dominate the evolution of the universe at a scale factor

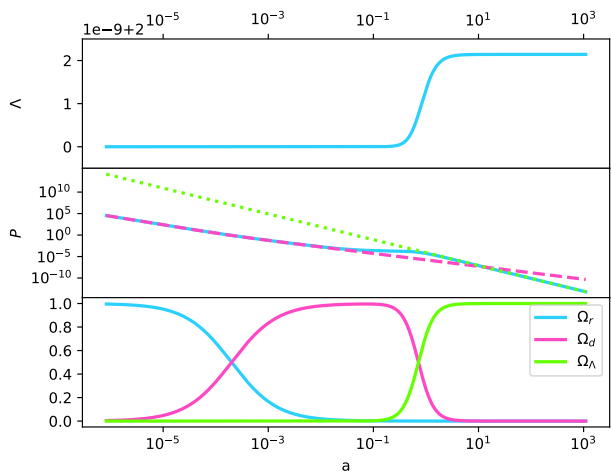


FIG. 1. The numerical evolution of various quantities for the parameter choice  $\gamma = 10^{-5}$ . In the upper plot the evolution of  $\Lambda$  is shown; it can be seen that the field changes by roughly one part in  $10^9$  over cosmic history. In the middle plot the solid line shows the exact evolution of the torsion field  $P$  whilst dashed line shows the solution (25) and the dotted line shows the solution (30). The lower plot shows evolution of  $\Omega$  quantities (here defined as fractional contributions to  $g^2$  in the Hamiltonian constraint) and  $P$  (solid lines) as a function of  $\ln a$  for a realistic universe. Subscripts  $d$  and  $r$  denote dust and radiation-like components of the universe. The scale factor is fixed to be  $a = 1$  at the present moment, and units where the present day Hubble parameter  $H_0$  is set to unity are used.

$a \sim a_i$  following a period of matter domination during which  $P \sim (\gamma/3)H_0a_i^{-3/2}$  (from the solution (25)). We expect then  $P_i \sim (\gamma/3)H_0a_i^{3/2}$ . This has important implications: if phenomenologically viable cosmologies involve  $P$  staying on the tracking solution (25) for an appreciable amount of time, this means that fixing  $\gamma$  fixes the size of  $P$  during matter domination, and the size of  $P_i$  as the cosmological constant begins to dominate.

We now discuss the evolution of  $\Lambda$ . It can be seen from (25) that the tracking solution can exist if  $\text{sign}(P) = \text{sign}(\gamma)$ . Recall that the  $\Lambda$  equation of motion is  $\dot{\Lambda} = 2\gamma\Lambda^2P/(6\gamma gP + \Lambda + \kappa\rho)$  and therefore in the earlier universe if  $\kappa\rho$  is initially greater than  $\Lambda$  it will tend to suppress time variation of  $\Lambda$ . Furthermore, we will have  $\dot{\Lambda} > 0$  throughout, meaning that  $\Lambda$  must be of smaller magnitude in the past than today. A typical evolution of  $\Lambda$  and  $P$  are shown in Figure 1.

In summary then, numerical exploration suggests that unless  $P$  finds itself on the tracking solution (25) for much of cosmic history, it will tend to dominate the evolution of the universe and therefore very likely in conflict with cosmological data. This requires fine tuning of the initial value of  $P$  so that it begins close to the tracking solution. During the tracking stage, the effect of  $P$  is to rescale Newton's constant  $G$ . We will find later that deviations of gravitational wave speed from unity tend to be of order  $\gamma^2$ . This justifies our assumption that  $|\gamma| \ll 1$  and - in conjunction with recent constraints on the speed of gravity - restricts the fractional rescal-

ing of  $G$  to be  $\mathcal{O}(10^{-15})$ , which is well within bounds that will be placed by BBN constraints for the foreseeable future [29]. Additionally we see that a smaller value of  $\gamma$  tends to lower the total time variation of  $\Lambda$  over cosmic history, making it more difficult to distinguish from a genuine cosmological constant.

### III. THE PERTURBED EQUATIONS OF MOTION FOR TENSOR MODES

We now look at the evolution of small perturbations to the cosmological background. We perturb the tetrad and connection as:

$$\delta e^0 = 0 \quad (32)$$

$$\delta e^I = \frac{1}{2} \mathcal{H}^{IJ} \bar{e}_J \quad (33)$$

$$\delta \omega^{0I} = \frac{1}{2} \mathcal{E}^{IJ} \bar{e}_J \quad (34)$$

$$\delta \omega^{IJ} = \frac{1}{2} \epsilon^{IJK} \mathcal{B}^{KL} \bar{e}_L \quad (35)$$

where  $\mathcal{H}^{[IJ]} = \mathcal{E}^{[IJ]} = \mathcal{B}^{[IJ]} = 0$  and  $\mathcal{H}^I{}_I = \mathcal{E}^I{}_I = \mathcal{B}^I{}_I = 0$ . In addition we apply the restriction of looking at tensor (transverse traceless) modes, so that we impose:

$$\bar{D}_I \mathcal{H}^{IJ} = \bar{D}_I \mathcal{E}^{IJ} = \bar{D}_I \mathcal{B}^{IJ} = 0. \quad (36)$$

where  $\bar{D}_I \equiv \bar{e}_I^i \bar{D}_i$  and  $\bar{D}_i$  is the covariant derivative according to  $\bar{\omega}^{IJ}{}_{i}$ . Note the field  $P$  does not contribute to the expressions (36) and so the equations are equivalent to  $\bar{e}_i^j \partial_i \mathcal{H}^{IJ} = \bar{e}_i^j \partial_i \mathcal{E}^{IJ} = \bar{e}_i^j \partial_i \mathcal{B}^{IJ} = 0$ .

Given a quantity  $Y_{IJ}$  that represents a small perturbation, it can be converted into a tensor  $Y_{ij}$  in the spatial coordinate basis via  $Y_{ij} \equiv \bar{e}_i^I \bar{e}_j^J Y_{IJ}$ . Given our assumption of vanishing spatial curvature, a ‘co-moving’ tensor  $\tilde{Y}_{ij} = Y_{ij}/a^2$  can further be constructed.

The linearly-perturbed form of equations (7)-(9) can be written as linear partial differential equations in  $(t, x^i)$  and they are written in this form in Appendix C. For simplicity we decompose these perturbations into plane-wave Fourier components labelled by wave number  $k$ . As a further simplification we decompose each co-moving tensor mode Fourier mode into helicity eigenstates:

$$\tilde{\mathcal{H}}_{ij} = \sum_{\pm} \mathcal{H}^{\pm}(k, t) e^{ik_i x^i} \tilde{\mathcal{P}}_{ij}^{\pm} \quad (37)$$

$$\tilde{\mathcal{E}}_{ij} = \sum_{\pm} \mathcal{E}^{\pm}(k, t) e^{ik_i x^i} \tilde{\mathcal{P}}_{ij}^{\pm} \quad (38)$$

$$\tilde{\mathcal{B}}_{ij} = \sum_{\pm} \mathcal{B}^{\pm}(k, t) e^{ik_i x^i} \tilde{\mathcal{P}}_{ij}^{\pm}. \quad (39)$$

Here  $\tilde{\mathcal{P}}_{ij}^{\pm}$  are co-moving polarization tensors for  $+$  and  $-$  helicity components. For a plane-wave perturbation with wavenumber  $k^i$  we have the important identities,  $ik^m \bar{\epsilon}_{im}{}^l \tilde{\mathcal{P}}_{lj}^{\pm} = \pm k \tilde{\mathcal{P}}_{ij}^{\pm}$  and  $\tilde{\mathcal{P}}_{ij}^{\lambda} \tilde{\mathcal{P}}^{\lambda' ij} = 2\delta^{\lambda\lambda'}$  where  $\lambda = +, -$  and  $\bar{\epsilon}_{ijk}$  is the co-moving three dimensional Levi-Civita

symbol. Indices of co-moving tensors are taken to be raised and lowered with the Kronecker delta symbol.

After some algebra, it can be shown that the spin connection equations of motion yield the following equations:

$$\begin{pmatrix} \mathcal{B}^{\pm} \\ \mathcal{E}^{\pm} \end{pmatrix} = \frac{1}{A^2 + B^2} \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \left( \dot{\mathcal{H}}^{\pm} + \left( 2\frac{\dot{a}}{a} - g \right) \mathcal{H}^{\pm} \right) - k_P^{\pm} \mathcal{H}^{\pm} \quad (40)$$

where here and subsequently we choose the spacetime gauge  $N = 1$  (proper time) and where

$$A = 1 - \frac{3\dot{\Lambda}}{\Lambda^2} \left( g - k_P^{\pm} \gamma^{-1} \right) \quad (41)$$

$$B = \frac{3\dot{\Lambda}}{\Lambda^2} \left( k_P^{\pm} + g \gamma^{-1} \right) \quad (42)$$

We have introduced the polarization-dependent torsion-adjusted proper wavenumber  $k_P^{\pm}$  according to:

$$k_P^{\pm} \equiv \pm \frac{k}{a} - P \quad (43)$$

For reference, in the usual Einstein-Cartan theory we have  $A = 1, B = 0$ ; in that case,  $\mathcal{B}^{\pm}$  is related to spatial derivatives of  $\mathcal{H}^{\pm}$  and  $\mathcal{E}^{\pm}$  is related to time variations of  $\mathcal{H}^{\pm}$ . All modifications to the relation between  $\{\mathcal{E}^{\pm}, \mathcal{B}^{\pm}\}$  and  $\mathcal{H}^{\pm}$  stem from non-constancy of  $\Lambda$ . Hence the connection equations imply that in general the parity even and odd components of the connection ( $\mathcal{E}$  and  $\mathcal{B}$ ) can be obtained from their Einstein-Cartan expressions via a rotation, with an angle  $\theta$  satisfying:

$$\tan \theta = \frac{B}{A} = \frac{\frac{3\dot{\Lambda}}{\Lambda^2} (g + \gamma k_P^{\pm}) \gamma^{-1}}{1 - \frac{3\dot{\Lambda}}{\Lambda^2} (g - k_P^{\pm} \gamma^{-1})} \quad (44)$$

followed by a dilatation by  $1/\sqrt{A^2 + B^2}$ .

Then we may look to find the ‘‘second order’’ evolution equation for  $\mathcal{H}^{\pm}$  by inserting the solution (40) for  $\mathcal{B}^{\pm}$  and  $\mathcal{E}^{\pm}$  into the Einstein equation:

$$0 = \dot{\mathcal{E}}^{\pm} + \left( \frac{\dot{a}}{a} + g \right) \mathcal{E}^{\pm} - k_P^{\pm} \mathcal{B}^{\pm} - \left( \frac{2}{3} \Lambda + \frac{\kappa}{6} (\rho - 3p) \right) \mathcal{H}^{\pm} \quad (45)$$

We now look at solutions to the system (40) and (45).

### IV. GENERAL FEATURES

Generally, if the solution (40) is inserted into (45) then the resulting coefficient of  $\dot{\mathcal{H}}^{\pm}$  is proportional to:

$$\left( 1 + \frac{6P}{\Lambda + \kappa\rho} k_P^{\pm} \right). \quad (46)$$

We see that the coefficient is not positive-definite and hits zero when  $k = k_*^\pm$ :

$$k_*^\pm = \mp \frac{a}{6P} \left( \Lambda + \kappa\rho - 6P^2 \right) \quad (47)$$

signalling a divergence in the frequency. For example in the very late universe we may expect  $\Lambda \sim \Lambda_0 = cst.$  to dominate the evolution of the universe hence then:

$$k_*^\pm \sim \mp \frac{a}{6P} \Lambda_0 \quad (48)$$

where we've assumed that  $P^2/\Lambda_0 \ll 1$ .

Following the arguments proposed in Subsections IIC 1 and IIC 2, we have that  $k_*^\pm \sim \mp 2a^4\Lambda_0/\gamma$  for realistic cosmologies. Reaching  $k_*^\pm$  will correspond to  $\omega_\pm^2(k, t)$  and  $f_\pm(k, t)$  diverging and therefore likely signals a breakdown in the applicability of linear perturbation theory. As for the case of  $P(t)$ , we see that a key parameter for the size of  $k_*^\pm$  is  $\gamma$ .

When  $k \neq k_*^\pm$  and with the important exception of the limit  $|\gamma| \rightarrow \infty$  (see Section VI), it is possible to write the Einstein equation (45) in the following form :

$$\ddot{\mathcal{H}}^\pm = -\omega_\pm^2(k, t)\mathcal{H}^\pm - f_\pm(k, t)\dot{\mathcal{H}}^\pm \quad (49)$$

For arbitrarily values of  $\gamma$ , the form of  $\omega_\pm^2(k, t)$  and  $f_\pm(k, t)$  will be extremely complicated and so we will concentrate in detail on how (49) looks in relevant, limiting cases.

## V. THE EINSTEIN-CARTAN LIMIT

We start by finding the Einstein-Cartan limit of these theories, noting that when  $\gamma$  is finite and  $\rho = p = 0$  there are solutions where  $P = 0$ ,  $g = \dot{a}/a$  and  $\Lambda$  is constant [15]. Taking these background solutions we should obtain the Einstein-Cartan limit for our theory, which is equivalent to General Relativity in this situation. Inserting these conditions into the formalism just developed, we find  $\dot{\Lambda} = 0$ , and so  $A = 1$  and  $B = 0$ , as already announced in the previous Section. The connection equations are therefore:

$$\mathcal{B}^\pm = -k_P^\pm \mathcal{H}^\pm \quad (50)$$

$$\mathcal{E}^\pm = \left( \frac{d}{dt} + \frac{\dot{a}}{a} \right) \mathcal{H}^\pm. \quad (51)$$

Note that since  $P = 0$  we have  $k_P^\pm = \pm k/a$ , and so for gravity waves in Einstein-Cartan theory, the parity-odd connection perturbation,  $\mathcal{B}$ , is a spatial gradient of the metric, whereas the parity-even component,  $\mathcal{E}$ , is a time derivative of the metric (cf. Eqs (35) and (34)). Inserting these expressions into the Einstein equation (45), as prescribed, we find:

$$\ddot{\mathcal{H}}^\pm + 3\frac{\dot{a}}{a}\dot{\mathcal{H}}^\pm + \left( \dot{g} + 2g^2 - \frac{2}{3}\Lambda \right) \mathcal{H}^\pm + (k_P^\pm)^2 \mathcal{H}^\pm = 0 \quad (52)$$

where the dot denotes derivative with respect to the background proper time. In the Einstein-Cartan theory we have

$T = 0$  in the absence of background sources of torsion, so  $g = \dot{a}/a$ , and the background equations of motion read (see (20) and (21)):

$$g^2 = \frac{\Lambda}{3} \quad (53)$$

$$\dot{g} + \frac{\dot{a}}{a}g = \dot{g} + g^2 = \frac{\Lambda}{3}. \quad (54)$$

Therefore we find:

$$\begin{aligned} \ddot{\mathcal{H}}^\pm + 3\frac{\dot{a}}{a}\dot{\mathcal{H}}^\pm &= -(k_P^\pm)^2 \mathcal{H}^\pm \\ &= -\left( \frac{k}{a} \right)^2 \mathcal{H}^\pm \end{aligned} \quad (55)$$

Thus, our formalism for gravity waves reduces to the textbook equations for gravity waves in General Relativity in this limit.

## VI. EULER THEORY ( $\gamma \rightarrow \infty$ ) IN A PARITY-ODD BACKGROUND ( $P \neq 0$ )

Our first surprise arises when we consider a theory with the Euler pseudo-topological term only, by letting  $\gamma \rightarrow \infty$ , but with a background with  $P \neq 0$ . Then, as the background Equation (23) shows (with  $P \neq 0$  and  $\gamma \rightarrow \infty$ ), we must have  $3\dot{\Lambda}g = \Lambda^2$ . Therefore, the definitions of  $A$  and  $B$  (Eqns. (41) and (42)) lead to:

$$A = 0, \quad B = \frac{k_P^\pm}{g} \quad (56)$$

These are orthogonal to the Einstein-Cartan values, in the sense that for the latter the matrix (40) is diagonal, whereas here the matrix is purely off-diagonal. Indeed the rotation part of the transformation is now  $\theta = \pi/2$ . This is reflected in the way the connection is related to the metric. the Einstein-Cartan case (cf. Eqns. (50) and (51)) we have:

$$\mathcal{E}^\pm = g\mathcal{H}^\pm \quad (57)$$

$$k_P^\pm \mathcal{B}^\pm = g \left( \dot{\mathcal{H}}^\pm + \left( \frac{2\dot{a}}{a} - g \right) \mathcal{H}^\pm \right) \quad (58)$$

Inserting into the Einstein equation (45) we find that not only does this imply an absence of second order time derivatives for  $\mathcal{H}^\pm$ , but the first time derivatives cancel out. In addition the algebraic equation obtained is

$$\left( \frac{1}{2}\dot{g} + g^2 - \frac{g\dot{a}}{2a} - \frac{\Lambda}{3} + \frac{1}{12}(3p - \rho) \right) \mathcal{H}^\pm = 0 \quad (59)$$

The term in brackets in (59) vanishes due to the equations of motion, therefore the tensor mode perturbation is  $\mathcal{H}^\pm$  completely undetermined by the perturbed equations of motion<sup>3</sup>.

<sup>3</sup> This would appear to contradict the result found in [30] which says that tensor modes propagate luminally as in General Relativity in a model with tensor mode perturbation equations that should be mappable to the ones considered here.

One may wonder to what extent this is a result of the particular choice for our action. For example, if the coefficient  $-\frac{3}{2}$  in the term (3) is replaced by  $-\frac{3}{2\xi}$  then it can be shown that Einstein's equation instead becomes:

$$\frac{1}{\xi} \left(1 - \xi\right) \left(4\Lambda + \rho - 3p\right) \mathcal{H}^\pm = 0 \quad (60)$$

Thus in the case when  $\Lambda \neq 0$ ,  $\rho \neq 3p$  and  $\xi \neq 1$ , the perturbation  $\mathcal{H}^\pm$  is not undetermined but fixed to vanish. It appears that the presence of the Euler term in the absence of the Pontryagin term is sufficient to nullify the dynamics of the perturbation  $\mathcal{H}^\pm$  with the special case  $\xi = 1$ , which leaves them undetermined by the perturbed equations of motion. Note that the case of simultaneous vanishing of the Euler and Pontrya-

gin term ( $\xi \rightarrow \infty$ ) does not correspond to General Relativity. In fact such limit yields a rather exotic background solution  $a = 0$  due to  $\Lambda$  being a dynamical field.

## VII. THE LEADING ORDER SOLUTION FOR THE GENERAL CASE IN THE LATE UNIVERSE

We now consider a general finite value of  $\gamma$  and look at the perturbed equations in a regime where the evolution of the universe is dominated by  $\Lambda$ . We define a dimensionless parameter  $\epsilon_P \equiv P/\sqrt{\Lambda}$  which is expected to be of magnitude much smaller than unity in the late universe. Furthermore we assume that  $|\gamma| \ll 1$ . Inserting the solutions for  $\mathcal{E}^\pm$  and  $\mathcal{B}^\pm$  from (40) into the Einstein equations and keeping only terms up to second order in  $\{\epsilon_P, \gamma\}$  we find:

$$\begin{aligned} \dot{\mathcal{H}}^\pm &= -\omega_\pm^2(k, t) \mathcal{H}^\pm - f_\pm(k, t) \dot{\mathcal{H}}^\pm \quad (61) \\ \omega_\pm^2(k, t) &\equiv \left[ 4\Lambda\epsilon_P^2 \mp 4\sqrt{\Lambda}\epsilon_P \left(\frac{k}{a}\right) + \left(1 + \frac{(\kappa\Lambda\rho - 3\kappa\Lambda p - \kappa^2\rho^2 - 3\kappa^2 p\rho)}{(\kappa\rho + \Lambda)^2} \gamma^2 \right. \right. \\ &\quad \left. \left. + \frac{8 \left( \sqrt{3}\Lambda^{5/2} + \sqrt{3}\kappa\Lambda^{3/2}\rho - \sqrt{3}\kappa^2\sqrt{\Lambda}\rho^2 - 3\sqrt{3}\kappa^2\sqrt{\Lambda}p\rho - 3\sqrt{3}\kappa\Lambda^{3/2}p \right)}{(\kappa\rho + \Lambda)^{5/2}} \gamma\epsilon_P + \frac{42(2\Lambda^2 - \kappa\Lambda\rho - 3\kappa\Lambda p)}{(\kappa\rho + \Lambda)^2} \epsilon_P^2 \right) \left(\frac{k}{a}\right)^2 \right] \\ &\quad + \mathcal{O}(\epsilon_P, \gamma)^3 \quad (62) \\ f_\pm(k, t) &\equiv \sqrt{3(\Lambda + \kappa\rho)} \pm \left[ \frac{4\sqrt{3}\sqrt{\Lambda}\sqrt{\kappa\rho + \Lambda}(2\Lambda - \kappa\rho)\epsilon_P + \kappa\rho(\kappa\rho - \Lambda)\gamma + 3\kappa p \left( (\kappa\rho + \Lambda)\gamma - 4\sqrt{3}\sqrt{\Lambda}\sqrt{\kappa\rho + \Lambda}\epsilon_P \right)}{(\kappa\rho + \Lambda)^2} \right] \left(\frac{k}{a}\right) \\ &\quad + \mathcal{O}(\epsilon_P, \gamma)^2 \quad (63) \end{aligned}$$

where it is assumed that  $|k| \ll k_*$ . Roughly speaking, positivity of both  $\omega_\pm^2(k, t)$  and  $f_\pm(k, t)$  imply that  $\mathcal{H}^\pm$  evolves in a stable manner.

We immediately see from (61) that novel features are generally present in the propagation of  $\mathcal{H}^\pm$ . In the limit  $k \rightarrow 0$  we see that

$$\lim_{k \rightarrow 0} \omega_\pm^2(k, t) = 4\Lambda\epsilon_P^2 \equiv \mathcal{M}_G^2(P) \quad (64)$$

i.e. the non-Riemannian background curvature provided by the spiral staircase field  $P$  gives what may be interpreted as contributing to a non-zero effective mass to the graviton.

At non-zero  $k$  we see that there exist terms linear in  $k$  in  $\omega_\pm^2(k, t)$  alongside the term proportional to  $k^2$  familiar from General Relativity present in equation (55). We may consider the wavenumber  $k_{(\omega)12}$  at which the term linear in  $k$  is of comparable size to the term quadratic in  $k^2$ . It can be calculated to be, to leading order in small quantities:

$$k_{(\omega)12}^\pm \equiv \pm 4a\epsilon_P\sqrt{\Lambda} = \pm 4aP \quad (65)$$

For realistic background cosmologies we expect in this regime

that  $P \approx (\gamma/3)H_0 a_i^{3/2} a^{-3}$  so  $k_{(\omega)12}^\pm \approx \pm 4(\gamma/3)a_i^{3/2} a^{-2} H_0$ , where  $a_i$  will be the scale factor where  $\Lambda$  begins to dominate the evolution of the universe. As  $|\gamma| \ll 1$  then  $k_{(\omega)12}^\pm$  is expected to be on scales far larger than the characteristic cosmological horizon scale  $k_H \equiv H_0$  today. We note that the mass term and chirality-dependent leading term linear in  $k$  are due to the tensor perturbation  $\mathcal{H}^{ij}$  coupling to the parity-violating torsionful generalization of the Laplacian operator.

Following [31] (see also [32–34]) the speed of monochromatic tensor modes today  $c_T^\pm$  (taking  $a = 1$ ) is given by  $c_T^\pm = \frac{\omega_\pm}{k}$ . In general our expression for  $c_T$  will be rather complicated but it is instructive to detail the order of magnitude of terms appearing in its expressions. Given how we expect  $P(t)$  to scale with  $\gamma$  from the results of Section II C 2 we find that:

$$\begin{aligned} c_T^\pm &\sim 1 \pm \mathcal{O}\left(\frac{H_0}{k}\gamma\right) + \mathcal{O}\left(\gamma^2\right) + \mathcal{O}\left(\frac{H_0^2}{k^2}\gamma^2\right) \\ &\quad \pm \mathcal{O}\left(\frac{H_0^3}{k^3}\gamma^3\right) \pm \mathcal{O}\left(\frac{k}{k_*}\gamma^2\right) + \mathcal{O}\left(\frac{k^2}{k_*^2}\gamma^2\right) + \dots \quad (66) \end{aligned}$$

The above shows the leading contribution to each  $k$  depen-



dence; these get further corrections by higher powers of  $\gamma$  as appropriate. Note that the terms involving  $k_*$ , which are important in terms of telling us when breakdown happens for higher  $k$ , appear at leading order cubic in  $\{\epsilon_P, \gamma\}$  and that the next-to-leading contribution to  $c_T^\pm$  involving  $k_*$  is a factor  $(k/k_*)$  smaller than the leading one, implying that the leading term is the dominant one as long as  $|k/k_*| \ll 1$ . Constraints from the LIGO experiment roughly constrain the deviation of  $c_T^\pm$  from unity by approximately  $10^{-15}$ . This will generally imply that each of the terms in (66) that cause deviations from unity should be no bigger than  $10^{-15}$ . If we take a typical wavelength of gravitational waves probed by LIGO to be  $\lambda_{LIGO} \sim 1000 \text{ km}$  then  $k_{LIGO} = 2\pi/\lambda_{LIGO} \sim 6 \times 10^{-3} \text{ km}^{-1}$ . Taking a value  $H_0 \sim (2/3)10^{-23} \text{ km}^{-1}$  (corresponding to  $H_0 \sim 70 \text{ km/s/Mpc}$ ) we have  $H_0/k_{LIGO} \sim 10^{-21}$ . As we expect  $\gamma^2 < 1$ , the leading  $H_0/k$  chirality-dependent modification of the speed of gravity is not constrained by existing data. The constraint on the speed of gravitational wave speed then places the following restriction on  $\gamma$ :

$$\gamma^2 \lesssim \mathcal{O}(10^{-15}) \quad (67)$$

Given this constraint and the small value of  $H_0/k_{LIGO}$ , the

remaining immediate constraint from  $c_T^\pm$  is that

$$|k_*^\pm| \gg \mathcal{O}(k_{LIGO}) \quad (68)$$

which is necessary for the consistency of our use of the linearly perturbed equations of motion; as the breakdown of the applicability of these equations is approached, significant deviations of  $c_T^\pm$  from unity are expected. We can translate this into a constraint on  $\gamma$  by assuming as above that  $P \sim (\gamma/3)H_0 a_i^{3/2} a^{-3}$  and so using equation 47 we have  $k_*^\pm \sim \mp H_0/\gamma$  and so

$$|\gamma| \ll \mathcal{O}(10^{-21}) \quad (69)$$

## VIII. PERFECT FLUID DOMINATION

In this limit, we consider the evolution of perturbations on a background where the evolution is dominated by a combination of perfect fluids. It was shown in I C that there exist solutions where  $\Lambda \sim 0$  and  $P^2 = \gamma^2 \kappa \rho / 27$  with  $\gamma \ll 1$  and that these seem to be the solutions that yield a realistic cosmology. Assuming that these solutions hold then to quartic order in the small parameter  $\gamma$  we have that:

$$\ddot{\mathcal{H}}^\pm = -\omega_\pm^2(k, t)\mathcal{H}^\pm - f_\pm(k, t)\dot{\mathcal{H}}^\pm \quad (70)$$

$$\begin{aligned} \omega_\pm^2(k, t) \equiv & \left[ \frac{4}{27} \left( 1 - \frac{1}{9}(1+3w)\gamma^2 \right) \gamma^2 \kappa \rho \mp \frac{4}{3} \sqrt{\frac{\kappa \rho}{3}} \gamma \left( \frac{k}{a} \right) + \left( 1 + \frac{1}{9}(1+3w)\gamma^2 + \frac{12}{243}(1+3w)\gamma^4 \right) \left( \frac{k}{a} \right)^2 \right. \\ & \left. \mp \frac{2}{9}(1+3w) \sqrt{\frac{3}{\kappa \rho}} \gamma^3 \left( \frac{k}{a} \right)^3 + \frac{20}{27} \frac{(1+3w)}{\kappa \rho} \gamma^4 \left( \frac{k}{a} \right)^4 \right] + \mathcal{O}(\gamma^5) \end{aligned} \quad (71)$$

$$\begin{aligned} f_\pm(k, t) \equiv & \left[ \sqrt{3\kappa\rho} \left( 1 - \frac{1}{18}\gamma^2 + \frac{217}{1944}\gamma^4 + \frac{13}{162}w\gamma^4 \right) \mp \frac{1}{9}(1+3w)(1+4\gamma^2)\gamma \left( \frac{k}{a} \right) \right. \\ & \left. + (1+3w) \sqrt{\frac{3}{\kappa\rho}} \frac{2}{9} \left( 1 + \frac{16}{9}\gamma^2 \right) \gamma^2 \left( \frac{k}{a} \right)^2 \mp \frac{4}{9} \frac{(1+3w)}{\kappa\rho} \gamma^3 \left( \frac{k}{a} \right)^3 + \frac{8}{27}(1+3w) \frac{\sqrt{3}}{(\kappa\rho)^{3/2}} \gamma^4 \left( \frac{k}{a} \right)^4 \right] + \mathcal{O}(\gamma^5) \end{aligned} \quad (72)$$

where  $w \equiv p/\rho$ . Note that as in the case of the late time solution discussed in Section VII the leading term in  $\omega_\pm^2$  to leading order in  $\gamma^2$  as  $k \rightarrow 0$  corresponds to  $\mathcal{M}_G^2 = 4P^2$ , and this term grows more quickly than  $(k/a)^2$  as the scale factor  $a$  decreases. By numerical inspection,  $\omega_\pm^2$  to arbitrary order in  $\gamma$  has a minimum during fluid domination (if  $a = 1$  today)

$$a \sim \frac{2 \left( \gamma^2 H_0^2 \Omega_d + \sqrt{\gamma^4 H_0^4 \Omega_d^2 + 9\gamma^2 H_0^2 k^2 \Omega_r} \right)}{9k^2} \quad (73)$$

where  $k > 0$  for the + polarization, and  $k < 0$  for the - polarization where we've defined  $\Omega_d$  and  $\Omega_r$  for dust and radiation respectively via  $\kappa \rho_d / 3 \equiv H_0^2 \Omega_d / a^3$  and  $\kappa \rho_r / 3 \equiv H_0^2 \Omega_r / a^4$ .

This behaviour can be seen for  $\omega_+^2$  in Figure 2 and appears to mark a brief transition between  $\omega_+^2$  being dominated by the mass term at earlier times and the more familiar  $(k/a)^2$  term at later times. Indeed, one can see that generally the effective mass term in (71) - proportional to  $\rho$  - grows more quickly than  $(k/a)^2$  as  $a$  decreases so will tend to dominate at early times.

## IX. OUTLOOK

In this paper we revisited models of the Universe where the cosmological constant is allowed to vary as a result of a balancing torsion. Such theories potentially have fewer free pa-

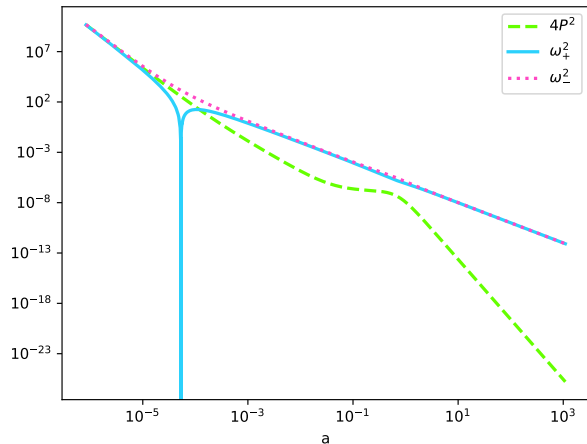


FIG. 2. Plot of the exact forms of  $\omega_+^2$  and  $\omega_-^2$  (correct up to any order in  $\gamma$ ) and  $4P^2$  for the background cosmology depicted in Figure 1 and with  $k = 10^{-3}H_0$ ,  $\gamma = 10^{-5}$ . It can be seen that both  $\omega_+^2$  and  $\omega_-^2$  asymptote at early times to the  $4P^2$ , whereas at later times each evolves as  $(k/a)^2$ .

rameters than General Relativity, but we need to consider parity violating backgrounds so that they display acceptable late time phenomenology even at the level of background cosmological evolution [15]. Going beyond the homogeneous and isotropic approximation, the most obvious question concerns the propagating modes of the theory, specifically gravitational waves. We found that indeed dramatic results and severe constraints arise in this respect.

We developed the required perturbation theory within the first order formulation, taking into account that the connection has parity-odd and -even components, with both potentially receiving zeroth order terms. We also proposed a strategy for solving the more involved equations one has to contend with in this setting. We recovered the usual result for gravity waves if we assume the Einstein-Cartan theory (or solutions to our theory that reduce to it). For a theory with a pure Euler term we found a remarkable result that the linearly perturbed equations of motion leave the tensor perturbations either entirely undetermined (or fixed to vanish, if the term has a factor different from the one imposed by self-duality). This may well be hinting at the fact that gravity waves become ‘pure gauge’ in this case (in analogy with what happens for a varying Lambda in the parity even branch of the background solutions).

In the more general case, with a Pontryagin-type quasi-topological term, the situation is more promising. There are exotic effects, but these need not contradict observations in particular if we restrict ourselves to viable background solutions that may be currently indistinguishable from the standard  $\Lambda$ CDM cosmological mode. At the level of perturbations, results will necessarily differ from General Relativity for some wavenumbers  $k$ , with the speed of tensor modes in the late universe receiving large modifications as  $|k| \sim \gamma H_0$  (the scale of the graviton mass) and as  $|k| \sim H_0/\gamma$  (the scale

$k_* \sim H_0/\gamma$  where linear perturbation theory is expected to break down in these models). As one approaches each of these values, the speed of gravity is predicted to diverge substantially from unity in a chirality-dependent manner. All these effects occur only when the parity-violating torsion field  $P(t) \neq 0$  and - via equation A3 - equivalently when the time derivative of  $\Lambda(t)$  is non-zero.

Thus, our results are potentially very useful as a new model relating observations on the accelerating Universe (possibly implying of a non-constant deceleration parameter) and other gravitational observations. But even more originally, our conclusions may be of great value for in phenomenological quantum gravity. Modified dispersion relations are a major feature of phenomenological approaches to quantum gravity (see, for example, [35–38]). Our work has added a layer to this approach by introducing *chiral* modified dispersion relations. It has been speculated that the concept of parity requires severe revision at the Planck scale [39]. Furthermore, our results supplement existing findings regarding how parity violation in theories of gravity involving to extensions to Riemannian geometry can affect the propagation of gravitational waves (for example see [30, 40] for cases where the gravitational connection field has torsion and non-metricity respectively).

A number of open questions remain. Firstly, we have only considered tensor perturbations in this theory. It is expected that in the vector mode sector, there will be no new degrees of freedom present - as in the tensor mode case, a relic of the polynomial nature of the new  $\Lambda RR$  terms in the Lagrangian is that modifications will always only be enabled by a non-zero time derivative of  $\Lambda(t)$  which concomitantly implies that time derivatives of the spin-connection perturbation will not appear. In the scalar sector, a new scalar degree of freedom  $\delta\Lambda(x^i, t)$  is expected to propagate and it will be important to see its effect on the cosmic microwave background CMB) and the growth of large scale structure.

There are also several avenues to study further observational signatures of this parity violation. The gravitational wave waveform will show deviations from General Relativity in both the amplitude and phase, due to amplitude and velocity birefringence effects, respectively, which both arise as a result of parity violation [41, 42]. Some of these effects could potentially be constrained in second generation gravitational wave detectors, and it would be interesting to derive the modifications to the waveform due to these effects. In addition to observable signatures in propagation, there is potential to detect parity violation at the gravitational wave source through study of various types of astrophysical binary systems as proposed by [43, 44]. It has also been suggested that parity violation in the gravitational sector could leave distinct signatures in the CMB to be detected with future experiments (see e.g., [45–51]), which could also be worth further exploration in the context of our theory.

It remains to understand our results in the light of a Hamiltonian analysis of the theory to second order, or even non-perturbatively, to all orders. A clear direction of further development would be to present the Hamiltonian structure of the full theory, with an examination of its number of degrees of freedom. It is curious that the mini-superspace approximation

reveals two branches with different symmetries and degrees of freedom. We have now discovered that the fluctuations about them have rather exotic properties. In particular, the underdetermination of the perturbed tensor equations of motion for a theory with a pure Euler term hints that a novel type of gauge symmetry may exist.

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### Appendix A: Alternative form of the background equations of motion

It is possible to write down the field equations (7)-(9) as a system of first-order ordinary differential equations for variables  $\{P, g, \Lambda, a\}$  along with a constraint equation:

$$\dot{P} = \frac{-6P\kappa\rho(6\gamma P g^2 + 3g(\Lambda + 2P^2) - \Lambda\gamma P) + \kappa^2\rho(6P(3gp + \gamma P\rho) + \gamma\Lambda\kappa(\rho - 3p)) - \gamma\kappa^3\rho^2(3p + \rho)}{6\kappa\rho(6\gamma P g + \Lambda\kappa + \rho)} \quad (\text{A1})$$

$$\dot{g} = -\frac{6(\gamma^2 + 1)g^2 P^2}{6\gamma P g + \Lambda + \kappa\rho} - g^2 + g\gamma P + \frac{\Lambda}{3} - \frac{\kappa}{6}(\rho + 3P) \quad (\text{A2})$$

$$\dot{\Lambda} = \frac{2\gamma P \Lambda^2}{6\gamma P g + \Lambda + \kappa\rho} \quad (\text{A3})$$

$$\dot{a} = a \left( \frac{6(\gamma^2 + 1)gP^2}{6\gamma P g + \Lambda + \kappa\rho} + g - \gamma P \right) \quad (\text{A4})$$

$$\frac{\Lambda}{3} = g^2 - P^2 - \kappa \frac{\rho}{3} \quad (\text{A5})$$

It can be checked via differentiation of the constraint equation that  $\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p)$  as in the case of a perfect fluid in General Relativity. Using the constraint equation we can rewrite the evolution equation as  $\dot{\Lambda} = (2/3)\gamma P \Lambda^2 / (g^2 - P^2 + 2g\gamma P)$ . From this perspective, the dynamics for  $\Lambda$  can be seen as arising from the spin connection components  $g$  and  $P$ .

### Appendix B: Linearly perturbed field strength and torsion

Central objects in the field equations (7)-(9) are the curvature and torsion two-forms  $R^{AB}$  and  $T^A$ . Their linearly-perturbed forms around the cosmological background are found to be:

$$\delta R^{0I} = -\epsilon^I{}_{JK} g \mathcal{B}^J{}_L \bar{e}^L \bar{e}^K + \left( \frac{1}{N} \frac{\partial}{\partial t} \mathcal{E}^{IJ} + \frac{1}{N} \frac{\dot{a}}{a} \mathcal{E}^{IJ} \right) \bar{e}^0 \bar{e}_J + \bar{D}^K \mathcal{E}^{IJ} \bar{e}_K \bar{e}_J + P \mathcal{E}^I{}_J \epsilon^{JKL} \bar{e}_K \bar{e}_L \quad (\text{B1})$$

$$\delta R^{IJ} = 2g \mathcal{E}_L{}^{[I} \bar{e}^{|L|} \bar{e}^{J]} + \epsilon^{IJ}{}_K \left( \left( \frac{1}{N} \frac{\partial}{\partial t} \mathcal{B}^{KL} + \frac{1}{N} \frac{\dot{a}}{a} \mathcal{B}^{KL} \right) \bar{e}^0 \bar{e}_L + (\bar{D}_L \mathcal{B}^{KM}) \bar{e}^L \bar{e}_M + P \mathcal{B}^K{}_P \epsilon^{PMN} \bar{e}_M \bar{e}_N \right) \quad (\text{B2})$$

$$\delta T^I = \left( \frac{1}{N} \frac{\partial}{\partial t} \mathcal{H}^{IM} + \frac{1}{N} \frac{\dot{a}}{a} \mathcal{H}^{IM} - \mathcal{E}^{IM} \right) \bar{e}^0 \bar{e}_M + (\bar{D}^L \mathcal{H}^{IJ} + P \mathcal{H}^I{}_K \epsilon^{KLJ} + \epsilon^{IJ}{}_K \mathcal{B}^{KL}) \bar{e}_L \bar{e}_J \quad (\text{B3})$$

where  $\bar{D}_I \equiv e_I^i \bar{D}_i$  where  $\bar{D}_i$  is the covariant derivative according to  $\bar{\omega}^{IJ}$ .

### Appendix C: Perturbed equations of motion

For completeness we provide the full form of the perturbed equations of motion for the fields  $\{H^{IJ}, \mathcal{B}^{IJ}, \mathcal{E}^{IJ}\}$ :

$$-\left(\bar{D}^K H^{IM} \epsilon_{KM}^J + 2PH^{IJ}\right) - \left(\frac{\Lambda + 12P^2}{\Lambda + 6g\gamma P}\right) \mathcal{B}^{IJ} = -\frac{3}{\Lambda^2} \dot{\Lambda} \left(\bar{D}^K \mathcal{E}^{IM} \epsilon_{KM}^J + 2P\mathcal{E}^{IJ}\right) - \frac{3}{\Lambda^2 \gamma} \dot{\Lambda} \left(g\mathcal{E}^{IJ} - \bar{D}^K \mathcal{B}^{IM} \epsilon_{KM}^J\right) \quad (C1)$$

$$-\left(\frac{\partial}{\partial t} H^{IJ} + \left(2\frac{\dot{a}}{a} - g\right) H^{IJ}\right) + \left(\frac{\Lambda + 12P^2}{\Lambda + 6g\gamma P}\right) \mathcal{E}^{IJ} = -\frac{3}{\Lambda^2} \dot{\Lambda} \left(\bar{D}^K \mathcal{B}^{IM} \epsilon_{KM}^J + 2P\mathcal{B}^{IJ}\right) - \frac{3}{\Lambda^2 \gamma} \dot{\Lambda} \left(g\mathcal{B}^{IJ} + \bar{D}^K \mathcal{E}^{IM} \epsilon_{KM}^J\right) \quad (C2)$$

$$\frac{\partial}{\partial t} \mathcal{E}^{IJ} + \left(\frac{\dot{a}}{a} + g\right) \mathcal{E}^{IJ} = \frac{2}{3} \Lambda H^{IJ} + \bar{D}_K \mathcal{B}^{(I} \epsilon^{J)KL} + 2P\mathcal{B}^{IJ} + \left(\frac{2}{3} \Lambda + \frac{\kappa}{6} (\rho - 3p)\right) H^{IJ} \quad (C3)$$

where  $\epsilon_{IJK}$  is the three-dimensional Levi-Civita symbol and

we have chosen the  $N = 1$  spacetime gauge.

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