Knot Graphs

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Abstract

We consider the equivalence classes of graphs induced by the unsigned versions of the Reidemeister moves on knot diagrams. Any graph which is reducible by some finite sequence of these moves, to a graph with no edges is called a knot graph. We show that the class of knot graphs strictly contains the set of delta-wye graphs. We prove that the dimension of the intersection of the cycle and cocycle spaces is an effective numerical invariant of these classes.

1 Introduction

The problems discussed in this paper stem from the paper of Schwärzler and Welsh [6] where an attempt is made to decide how well the Jones polynomial, or equivalently the bracket polynomial, of a knot diagram detects whether the diagram represents the unknot. As far as we know this problem is still open and is one of the foremost unsolved questions in the theory of link polynomials. Most of [6] is concerned with *signed graphs*, that is ordinary undirected graphs, in which each edge is given a plus or minus sign. Here we are primarily interested in the unsigned case and unless otherwise specified, graphs will be unsigned. The notation is fairly standard. We use O_k to denote the graph with k vertices and no edges. A graph with no edges is called an *empty graph*. $G \setminus A$ (G/A) is formed by deleting (contracting) all the edges in A; $G|A$ is $G \setminus (E - A)$. The class of all graphs is denoted by $\mathcal{G}$. A graph is *simple* if it contains no loops or parallel edges; it is *cosimple* if it contains no isthmuses or vertices of degree 2. The number of connected components of a graph is denoted by $k(G)$. The *rank* of a set A of edges is given by $r(A) = |V(G)| - k(G|A)$.

In [6] a class of graphs is studied which are called *Reidemeister graphs*. These are the $\mathcal{R}$-graphs defined below. The graphic versions of Reidemeister moves are clearly related to the relatively well studied delta-wye moves (see Robertson, Seymour and Thomas [5]) which define the class $\mathcal{D}$, the *delta-wye* graphs. Much of this paper is concerned with transformations related to Reidemeister and delta-wye moves on unsigned graphs. We are especially interested in the class of knot graphs, denoted by $\mathcal{K}$, which consists of those graphs which are reducible to an empty graph by the unsigned version of the Reidemeister graph moves. We show that delta-wye graphs form a proper subset of the class of knot graphs. We show in Section 4, a relationship between the span of the bracket polynomial of a knot and $\mathcal{K}$-equivalence and it is tempting to believe that $\mathcal{K}$-reducibility might be related to linkless or flat embeddings in $\mathbb{R}^3$ as described in [5]. However any connection cannot be too straightforward. A prime reason for this is that each of these geometric properties is closed under minors, whereas the class of knot graphs certainly is not.

In the original version of this paper we stated as a conjecture the belief that the "smallest" graphs which are not knot graphs are the seven graphs of the Petersen family. This was posed as a problem by the second author at the DIMACS meeting in July 1998 and settled in the affirmative there by a multipronged attack, consisting of D. Archdeacon, N. Robertson, P. D. Seymour, R. Thomas and D. L. Vertigan [2], who showed that K_6 is not a knot graph. The argument is not straightforward.

2 Reidemeister moves

In [6] the allowable moves on signed graphs consist of the following moves and their inverses.

A Delete any loop, contract any isthmus (or coloop).

B Delete any pair of edges e, f with opposite signs and such that $\{e, f\}$ is a circuit; contract any pair of edges e, f with opposite signs and such that $\{e, f\}$ is a cocircuit.

C Replace a triangle which has edges of both signs as shown in Figure 1.

Two signed graphs G, H are called *Reidemeister equivalent*, $G \overset{R}{\sim} H$, if it is possible to transform G to an isomorphic copy of H by some finite sequence of moves (A),(B),(C) and their inverses.

For two *unsigned* graphs G and H we say that G is *Reidemeister reducible, ($\mathcal{R}$-reducible)* to H, denoted by $G \overset{\mathcal{R}}{\to} H$ if there exist signings ω_1,

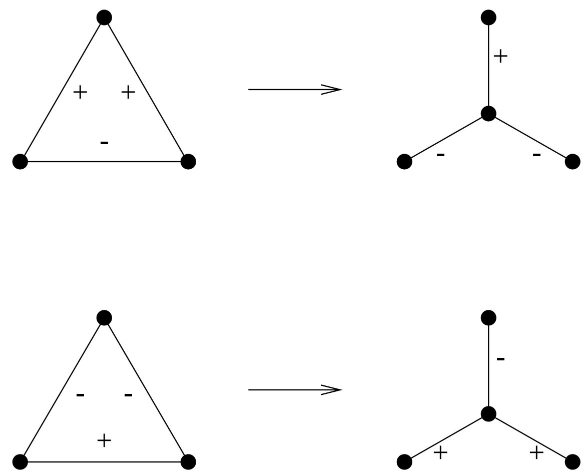

Figure 1: Moves of type C.

ω_2 such that $\omega_1(G) \overset{R}{\sim} \omega_2(H)$. We denote by $\mathcal{R}_k$ the class of unsigned graphs which are $\mathcal{R}$-reducible to O_k and let $\mathcal{R} = \bigcup_{k\geq 1} \mathcal{R}_k$. Although it is easy to see that $\mathcal{R}$-equivalence is an equivalence relation on signed graphs, we do not know whether $\mathcal{R}$-reducibility is an equivalence relation on unsigned graphs.

As pointed out in [6], the moves (A) – (C) are not the exact equivalents of the Reidemeister moves on knots. We now define a weaker notion of equivalence which is closer to the spirit of the Reidemeister moves. Replace the set of moves (B) by (D).

D Delete any pair of parallel edges e, f with opposite signs; contract any pair of edges e, f with opposite signs and e, f incident with a common vertex of degree 2.

Call two signed graphs G and H, *$\mathcal{Q}$-equivalent*, $G \overset{Q}{\sim} H$ if G can be reduced to an isomorphic copy of H by some finite sequence of the moves (A),(C),(D) and their inverses.

For unsigned graphs, G is *$\mathcal{Q}$-reducible* to H, $G \overset{Q}{\to} H$ if there are signings ω_1, ω_2 such that $\omega_1(G)$ and $\omega_2(G)$ are Q-equivalent. Again we do not know whether $\mathcal{Q}$-reducibility is an equivalence relation.

Since (D) is a special case of (B) it is clear that the following is true.

Observation 1. *If G is $\mathcal{Q}$-reducible to H then G is $\mathcal{R}$-reducible to H.*

Similarly, $\mathcal{Q}_k$, defined as the class of graphs $\mathcal{Q}$-reducible to O_k, is contained in $\mathcal{R}_k$.

For our next result we need a few basic facts from knot theory. A *link* with k components is a subset of $\mathbb{R}^3$ which is homeomorphic to the disjoint union of k circles. Two links K, L are *ambient isotopic* if there exists a homotopy $h_t : \mathbb{R}^3 \to \mathbb{R}^3\,(0 \leq t \leq 1)$ such that h_0 is the identity map, each h_t is a homeomorphism and $h_1(K) = L$. A knot is usually described by a projection onto a plane. A projection of K is a *regular projection* if it contains only finitely many multiple points and all multiple points are double points and these are crossing points. A regular projection and the specification of which is the under/over crossing for each crossing point determines the knot.

Given any link diagram it is easy to prove that we can colour the faces black and white so that no two faces with a common edge receive the same colour. Conventionally the outer face is coloured black and this colouring is known as a *Tait colouring*. From this we can get a canonical signed graph $S(D)$, its vertices are the black faces of the Tait colouring and two vertices are joined by a signed edge if they share a crossing. The sign of the edge is determined by the rule shown in Figure 2. Given any signed plane graph we can construct a link diagram in a canonical way. Given a graph G, the *medial graph* $m(G)$ has vertices at the midpoint of each edge of G and two vertices are adjacent in $m(G)$ if they are on consecutive edges of a face in G. Therefore $m(G)$ is 4-regular and the vertices are the crossings of the link diagram D. The sign of the edge of G determines whether the corresponding crossing is under/over in D. This construction determines a 1-1 correspondence between link diagrams and plane signed graphs. The fundamental theorem of Reidemeister, [4], can be stated in terms of plane signed graphs.

Theorem 2. *Two plane signed graphs G_1, G_2, represent links which are ambient isotopic if and only if G_1 can be transformed to a graph isomorphic to G_2 by a finite sequence of moves (A), (C), (D) and their inverses.*

A link is said to be *descending* if for each component we can find a point on the string and trace along the string and always take the under crossing the first time we arrive at a particular crossing. It is easy to see that each component of this link is ambient isotopic to the unknot and that the link diagram of the unknot is just O_1. See [8] for more information on knot theory.

Proposition 3. *$\mathcal{Q}$ contains all planar graphs.*

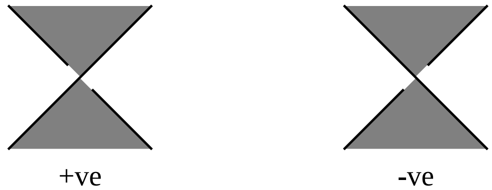

Figure 2: Positive and negative crossings.

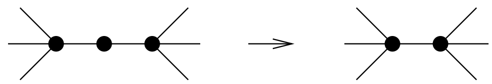

Figure 3: Move 2.

Proof. Given a planar graph G give it an arbitrary signing ω. Let $L(G)$ be the associated link. Now successively change the under/over crossings of $L(G)$ so that it becomes the descending link L' with say k components. Change the signs of the edges of G to get a graph G' such that $L' = L(G')$ and then G' is the signed version of G which is Q-equivalent to O_k. $\square$

However $\mathcal{Q}$ is a larger class than this, for example $K_5 \in \mathcal{Q}$. In fact, as pointed out in [6] there are 260 signings of K_5 which show this. In the next section we relate $\mathcal{Q}$ and $\mathcal{R}$ with classes that do not rely on signings.

3 ΔY-Equivalence and knot graphs.

We start by recalling the notion of ΔY-equivalence from [5]. Let G be a graph with a vertex v of degree 3 which has distinct neighbours. Let H be obtained from G by deleting v and its incident edges and adding an edge between each pair of neighbours of v. We say that H is obtained from G by a $Y\Delta$-*exchange* and that G is obtained from H by a ΔY-*exchange*. Two graphs are ΔY-*equivalent* if one can be obtained from a graph isomorphic to the other by a finite sequence of the following operations and their inverses.

1. Deleting a vertex of degree 1.

2. Suppressing a vertex of degree 2 (that is contracting an edge incident with a vertex of degree 2) as in Figure 3.

3. Deleting a parallel edge or a loop.

4. $Y\Delta$-exchange.

We define the class of graphs $\mathcal{D}_k$ by $G \in \mathcal{D}_k$ if G is ΔY-equivalent to O_k. We let $\mathcal{D} = \bigcup_{k=1}^{\infty} \mathcal{D}_k$ and if $G \in \mathcal{D}$ we say G is a *delta-wye graph*. Applying any of moves $1-4$ does not affect the number of connected components and so a delta-wye graph is in $\mathcal{D}_k$ if and only if it has k components.

We now define the unsigned graphs G and H to be $\mathcal{K}$-*equivalent* if one can be obtained from an isomorphic copy of the other by a finite sequence of the following moves and their inverses.

I. Contract an isthmus.

II. Delete a loop.

III. Delete a pair of parallel edges.

IV. Contract a pair of edges which are not parallel if both are incident with the same vertex of degree 2.

V. $Y\Delta$-exchange.

We write $G \overset{\mathcal{K}}{\sim} H$ if G and H are $\mathcal{K}$-equivalent. Clearly this is an equivalence relation. As we said earlier, the motivation for this is that this set of moves can be regarded as the unsigned graphical version of the Reidemeister moves on knots. For each positive integer k, we define the class $\mathcal{K}_k$ by, $G \in \mathcal{K}_k$ if and only if $G \overset{\mathcal{K}}{\sim} O_k$. Then the class of *knot graphs* is given by $\mathcal{K} = \bigcup_{k=1}^{\infty} \mathcal{K}_k$.

A sequence of moves reducing G to an empty graph is called a *reduction sequence*. To make it clear that the moves in a reduction sequence are for instance the $\mathcal{K}$ moves we will sometimes refer to a reduction sequence as a $\mathcal{K}$-reduction sequence.

Observation 4. *If G,H belong to $\mathcal{K}_j$,$\mathcal{K}_k$ respectively and have disjoint vertex sets, then $G \cup H$ belongs to $\mathcal{K}_{j+k}$.*

Our first results link $\mathcal{K}$ with existing classes.

Proposition 5. *$\mathcal{K}$ contains $\mathcal{Q}$.*

This is easy and follows from

Proposition 6. *For each positive integer k, $\mathcal{Q}_k \subseteq \mathcal{K}_k$.*

Proof. Suppose $G \in Q_k$. Then there exists some signing ω, such that $\omega(G) \overset{\mathcal{Q}}{\sim} O_k$. Now follow the $\mathcal{Q}$-reduction sequence $\omega(G) \overset{\mathcal{Q}}{\sim} \cdots \overset{\mathcal{Q}}{\sim} O_k$ replacing each signed move from the set $\{(A), (C), (D)\}$ together with their inverses with its unsigned equivalent from moves I – V and their inverses giving a $\mathcal{K}$-reduction from G to O_k. □

Every planar graph is contained in $\mathcal{Q}$ by Proposition 3 and so we have the following corollary.

Corollary 7. *The class of knot graphs contains all planar graphs.*

Moreover this containment is strict as we have the following result.

Proposition 8. *The class of knot graphs contains all delta-wye graphs.*

The proof of this result needs the following lemmata which may be known but we have been unable to find in the literature, although the proof ideas are certainly contained in [7]. A similar result can be found in [1].

Lemma 9. *Suppose G is in $\mathcal{D}$, and there is a $\mathcal{D}$-reduction sequence for G in which the number of edges never increases. Now let H be a minor of G, then H is in $\mathcal{D}$ and there exists a $\mathcal{D}$-reduction sequence for H in which the number of edges never increases.*

Proof. The proof is by induction on the number of edges in H plus the number of moves in the $\mathcal{D}$-reduction sequence for G. The result is clearly true if H has no edges. Otherwise we may assume that H is simple and cosimple, if not we could apply one of moves $1-4$ to remove an edge from H and we could then use induction. Let G' be the first graph obtained in the $\mathcal{D}$-reduction sequence of G. If G' is obtained from G by one of moves 1 – 4 then H is a minor of G' because H is simple and cosimple and H is a minor of G. Suppose the first move in the $\mathcal{D}$-reduction sequence of G is a $Y\Delta$-exchange. If the three edges involved are contained in H then because H is simple and cosimple they join a vertex of degree three to three distinct vertices so apply the same $Y\Delta$-exchange to H giving H'. Now H' is a minor of G'. If any of the three edges involved is not present in H then H is a minor of G'. The case when the first move in the $\mathcal{D}$-reduction sequence of G is a ΔY-exchange is similar. Either one of the three edges involved is not present in H in which case H is a minor of G' or the three edges form a triangle and so applying a ΔY-exchange to H gives a graph H' which is a minor of G'. In each case we have obtained a graph with the same number of edges as H and which is a minor of G', and so by induction H can be reduced to the empty graph without increasing the number of edges at any stage. $\square$

Lemma 10. *If $G \in \mathcal{D}$ then there is a $\mathcal{D}$-reduction sequence for G in which the number of edges never increases.*

Proof. If the lemma is not true for G then any $\mathcal{D}$-reduction sequence for G must contain a move where the number of edges increases. Consider a $\mathcal{D}$-reduction sequence for G where the number of moves which increase the number of edges is minimised. Suppose this $\mathcal{D}$-reduction sequence is of the form $G \to \ldots \to H \to H' \to \ldots \to O_{k(G)}$ and the transition $H \to H'$ is the last move which increases the edge number. Hence H is a minor of H' and there is a decreasing $\mathcal{D}$-reduction sequence for H' so applying the preceding lemma implies that there is a $\mathcal{D}$-reduction sequence for H in which the number of edges never increases. This contradicts the choice of $\mathcal{D}$-reduction sequence for G. $\square$

The following lemma is well known folklore and is immediate from the two preceding lemmata.

Lemma 11. *The class $\mathcal{D}$ is closed under minors.*

Hence from Robertson-Seymour theory we know that there is a finite collection of forbidden minors. This set is known to include K_6 and the Petersen family, that is those graphs which can be obtained from K_6 by a finite sequence of $Y\Delta$ and ΔY moves but seems to be much larger.

Proof of Proposition 3.5: We use induction on the number of edges in G. If G has no edges then clearly G is a knot graph so suppose G has k edges and $G \in \mathcal{D}$. Find any $\mathcal{D}$-reduction sequence for G in which the number of edges never increases. This sequence will begin with a possibly empty sequence of $Y\Delta$ and ΔY-exchanges to form H and then make a move that reduces the number of edges. Suppose the next move is to delete a loop e from H. Now $H \setminus e \in \mathcal{D}$ and has $k-1$ edges and so by induction $H \setminus e$ is a knot graph but this means that H is a knot graph. The same argument works if the next move is to contract a pendant edge. Now suppose the next move is to delete e where e and f are a pair of parallel edges. We have that $H \setminus \{e, f\} \in \mathcal{D}$ because $\mathcal{D}$ is closed under minors and $H \setminus \{e, f\}$ is a minor of H. Now $H \setminus \{e, f\}$ has $k-2$ edges and so by induction it is a knot graph which means that H and hence G are knot graphs. The case where the next move is to suppress a vertex of degree two is similar.

$\square$

However $\mathcal{K}$ is much larger than $\mathcal{D}$ for the following reasons. The *2-thickening* of G, denoted by $G^{(2)}$, is formed by replacing every edge of G by two parallel edges and the *2-stretch* of G is formed by replacing every edge of G by two edges in series. An obvious fact is

Proposition 12. *For any graph, both its 2-thickening and 2-stretch are knot graphs.*

Example 13. For any $k \geq 6$, $K_k^{(2)}$ is a knot graph and is not a delta-wye graph.

The question of whether every graph is a knot graph was resolved by Archdeacon, Robertson, Seymour, Thomas and Vertigan [2]. See the remarks in Section 1.

Theorem 14. *K_6 is not a knot graph.*

Corollary 15. *The smallest graphs which are not knot graphs are the seven graphs of the Petersen family.*

Proof. The Petersen family is the class of graphs that can be obtained from K_6 using a sequence of $Y\Delta$ and ΔY moves so if any of them is a knot graph then all are. Since K_6 is not a knot graph none of them are. Straightforward checking shows that every graph with fewer than fifteen edges is a knot graph. $\square$

Combining Proposition 12 and Theorem 14 give the following Corollary.

Corollary 16. *The class of knot graphs is not closed under either deletion or contraction.*

4 The bracket and Tutte polynomials

We start by recalling the definition of the *Tutte polynomial*. This is the two-variable polynomial defined by

$$T(G; x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)}$$

where E is the edge set of G. One evaluation of T which is particularly relevant here is

$$|T(G; -1, -1)| = 2^{\dim(\mathcal{C} \cap \mathcal{C}^*)}$$

where $\mathcal{C}$ and $\mathcal{C}^*$ are respectively the cycle and cocycle spaces of G, (see [6]). The point $(-1,-1)$ is one of the very few points in the (x, y) plane at which there exists a polynomial time algorithm to evaluate T, see [3].

In [6] the definition of the bracket polynomial of a knot is extended to any matroid and hence to any graph. As far as this paper is concerned for a signed graph G the *bracket polynomial* is a Laurent polynomial in the variable A given by

$$\begin{aligned} < G; A > &= A^{|E^-|-|E^+|-2r(E)}(-A^2 - A^{-2})^{k(G)-1} \\ &\cdot \sum_{X \subseteq E} A^{4(r(X)-|X^-|)}(-A^4 - 1)^{r(E)+|X|-2r(X)} \end{aligned}$$

where X^+ (X^-) denotes the set of positive (negative) edges of X.

When G is planar and monosigned, along the hyperbola $xy = 1$, T evaluates the bracket polynomial of the alternating link $L(G)$ determined by G, (see [6]).

First we give a necessary condition for G to belong to $\mathcal{K}_k$ in terms of the Tutte polynomial. For any G define

$$\mu(G) = \dim(\mathcal{C} \cap \mathcal{C}^*) + k(G).$$

Theorem 17. *The integer μ is an invariant of $\mathcal{K}$-equivalence.*

Corollary 18. *For any positive integer k a necessary condition for a graph G to belong to $\mathcal{K}_k$ is that $k = \log_2(|T(G; -1, -1)|) + k(G)$.*

Proof. This is immediate from Theorem 17 because $T(O_k; -1, -1) = 1$. $\square$

Corollary 19. *Given a knot graph G, we have a polynomial time algorithm to decide which class $\mathcal{K}_k$ contains G.*

Proof of Theorem: Let $t(G) = T(G; -1, -1)$. We will show that $\mu(G)$ is preserved under moves I – V and their inverses. First recall the well-known deletion-contraction formulae for T. If e is a loop in G then

$$T(G; x, y) = y\, T(G \setminus e; x, y). \tag{20}$$

If e is a coloop in G then

$$T(G; x, y) = x\, T(G/e; x, y). \tag{21}$$

Otherwise

$$T(G; x, y) = T(G \setminus e; x, y) + T(G/e; x, y). \tag{22}$$

Suppose we are applying one of moves I – V to G and the move acts on the component H. Repeated use of Equations 20 – 22 gives the following.

Move I If e is an isthmus of H then $t(H/e) = -t(H)$.

Move II If e is a loop then $t(H \setminus e) = -t(H)$.

Move III If e and f are parallel edges, there are two cases depending on whether deleting e and f disconnects H. Suppose first that $H \setminus \{e, f\}$ is connected.

$$\begin{aligned} t(H) &= t(H \setminus e) + t(H/e) \\ &= t(H \setminus \{e, f\}) + t(H \setminus e/f) - t(H/e \setminus f) \\ &= t(H \setminus \{e, f\}). \end{aligned}$$

Now suppose that $H \setminus \{e, f\}$ is disconnected and that H_1 and H_2 are the connected components of $H \setminus \{e, f\}$.

$$\begin{aligned} t(H) &= t(H \setminus e) + t(H/e) \\ &= -t(H \setminus e/f) - t(H/e \setminus f) \\ &= -2t(H_1)t(H_2). \end{aligned}$$

Move IV If e and f are incident on a vertex of degree two and not in parallel.

$$\begin{aligned} t(H) &= t(H \setminus e) + t(H/e) \\ &= -t(H \setminus e/f) + t(H/e \setminus f) + t(H/\{e, f\}) \\ &= t(H/\{e, f\}). \end{aligned}$$

Move V Suppose v is a vertex of degree 3, with distinct neighbours x, y and z. Let e, f and g be the edges $\{v, x\}$, $\{v, y\}$ and $\{v, z\}$ respectively. Let H' denote the graph formed from H by applying the $Y\Delta$-exchange. Let e', f' and g' be the edges $\{y, z\}$, $\{x, z\}$ and $\{x, y\}$ respectively, that is those edges that are added in the $Y\Delta$-exchange. There are 3 cases to consider depending on how many of e, f and g are isthmuses. In each of the cases the expressions are not asymmetric because various terms have been cancelled, for instance in the first case $H \setminus \{e, g\}/f = H \setminus \{e, f\}/g$. First suppose none of e, f and g are isthmuses. Then

$$\begin{aligned} t(H) &= t(H \setminus \{e, f\}) + t(H \setminus e/f) + t(H \setminus f/e) + t(H/\{e, f\}) \\ &= t(H \setminus \{e, g\}/f) + t(H \setminus e/\{f, g\}) + t(H \setminus f/\{e, g\}) \\ &\quad + t(H \setminus g/\{e, f\}) + t(H/\{e, f, g\}) \end{aligned}$$

and

$$\begin{aligned} t(H') &= t(H' \setminus \{e', f'\}) + t(H' \setminus e'/f') + t(H' \setminus f'/e') + t(H/\{e', f'\}) \\ &= t(H' \setminus \{e', f', g'\}) + t(H' \setminus \{e', f'\}/g') + t(H' \setminus \{e', g'\}/f') \\ &\quad + t(H' \setminus \{f', g'\}/e') + t(H' \setminus f'/\{e', g'\}). \end{aligned}$$

So $t(H) = t(H')$. The second case is when e, say, is an isthmus but f and g are not. Then

$$\begin{aligned} t(H) &= -t(H/e) \\ &= t(H \setminus f/\{e, g\}) - t(H \setminus g/\{e, f\}) - t(H/\{e, f, g\}) \\ &= -t(H/\{e, f, g\}), \end{aligned}$$

and

$$\begin{aligned} t(H') &= t(H' \setminus f') + t(H'/f') \\ &= -t(H' \setminus f'/g') + t(H'/\{f', g'\}) + t(H' \setminus g'/f') \\ &= -t(H' \setminus e'/\{f', g'\}). \end{aligned}$$

So again $t(H) = t(H')$. It is not possible for exactly two of $\{e, f, g\}$ to be isthmuses so suppose they all are. In this case,

$$t(H) = -t(H/\{e, f, g\})$$

and

$$\begin{aligned} t(H') &= t(H' \setminus e') + t(H'/e') \\ &= t(H' \setminus e'/\{f', g'\}) - t(H' \setminus f'/\{e', g'\}) - t(H' \setminus g'/\{e', f'\}). \end{aligned}$$

This shows that $\mu = \log_2(|t(G)|) + k(G)$ is preserved under moves I – V and their inverses.

□

While proving that K_6 is not a knot graph is difficult it is much easier using Theorem 17 to prove the following result.

Proposition 23. *$K_6 \notin \mathcal{R}$.*

Proof. A graph $G \in \mathcal{R}_k$ if and only if there exists a signing ω such that $\omega(G) \overset{R}{\sim} O_k$. It is not difficult to show that if $G \overset{R}{\to} H$ then $\mu(G) = \mu(H)$ because the only extra move that must be checked in addition to the $\mathcal{K}$

moves is when we apply Move B to contract a pair of edges which form a cocircuit and checking this is essentially the same as showing that μ is preserved under Move II.

The *span* of a signed graph, G, is the difference between the largest and smallest powers with non-zero coefficients in the bracket polynomial of G. Since by [6] span is preserved under $\mathcal{R}$-moves we have that $\omega(G) \overset{R}{\sim} O_k$ implies $\operatorname{span}\omega(G) = \operatorname{span} O_k$. The bracket polynomial of O_k is

$$< O_k; A >= (-A^2 - A^{-2})^{k-1}$$

so that $\operatorname{span} O_k = 4(k-1)$. In [6], it was reported that a computer search of all possible signings of K_6 showed that no signing had span less than 24. However

$$T(K_6; -1, -1) = 16$$

and so if $K_6 \overset{R}{\sim} O_k$, then $k = 5$ but $\operatorname{span} O_5 = 16 < 24$ □

At the moment we know of no graphs which would show that any of the sets $\mathcal{K} \setminus \mathcal{Q}$, $\mathcal{K} \setminus \mathcal{R}$, $\mathcal{R} \setminus \mathcal{K}$ or $\mathcal{R} \setminus \mathcal{Q}$ is non-empty.

5 Conclusion: Which graphs are knot graphs ?

A natural hypothesis is that knot graphs are those graphs which are obtainable from some delta-wye graph by a finite sequence of moves I – IV and their inverses. However this is not true as the following example shows.

Example 24. Let G be the graph in Figure 4. If G is obtainable from some $H \in \mathcal{D}$ by a finite sequence of moves I – IV and their inverses then because G is simple and cosimple G must be a minor of H. It is easy to see that G contains K_6 as a minor and so H must also contain K_6 as a minor which contradicts the assumption that $H \in \mathcal{D}$ because the class $\mathcal{D}$ is closed under minors and K_6 is not contained in $\mathcal{D}$. However it is not difficult to show that G is a knot graph.

One of the main purposes of this paper has been to sort out the relationships between the classes of graphs defined by closely related families of moves. Two other open problems which are particularly frustrating us are the following.

1. If G is $\mathcal{K}$-equivalent to the empty graph does there exist a $\mathcal{K}$-reduction sequence which achieves this and never increases the number of edges? If this is true then $\mathcal{K}$ can contain no simple, cosimple, triangle free graph with minimum degree 4.

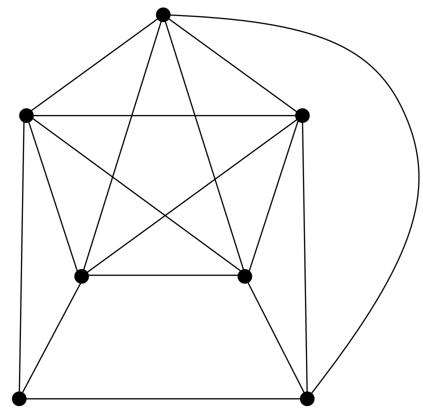

Figure 4: The graph G.

2. Is membership in $\mathcal{K}$ decidable? The answer is clearly yes if (1) is true, and it could even be decidable in polynomial time.

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