Non-principal surface waves

in deformed incompressible materials

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Abstract

The Stroh formalism is applied to the analysis of infinitesimal surface wave propagation in a statically, finitely and homogeneously deformed isotropic half-space. The free surface is assumed to coincide with one of the principal planes of the primary strain, but a propagating surface wave is not restricted to a principal direction. A variant of Taziev's technique [*Sov. Phys. Acoust.* **35** (1989) 535] is used to obtain an explicit expression of the secular equation for the surface wave speed, which possesses no restrictions on the form of the strain energy function. Albeit powerful, this method does not produce a unique solution and additional checks are necessary. However, a class of materials is presented for which an exact secular equation for the surface wave speed can be formulated. This class includes the well-known Mooney-Rivlin model. The main results are illustrated with several numerical examples.

1 Introduction

The study of small-amplitude surface waves propagating in finitely and homogeneously deformed hyperelastic materials has quite a long history, dating back to the classical paper by Hayes and Rivlin [1] in 1961. The interest in using the theory of small motions superimposed on a large static deformation of a hyperelastic half-space is manifold, for once the problem is solved the results are applicable to various advanced topics. These, in particular, include the non-destructive evaluation of solids (see Guz' and Makhort [2] for a review), the incremental stability analysis of the loaded surface of a deformed material (see Guz' [3] for comprehen-

sive review and bibliography), and the acousto-elastic effect (second-order theory of linear elasticity) where the pre-deformation is also considered small (see Pao et al. [4] for a review of both experimental and theoretical results).

The results of Hayes and Rivlin, valid for a compressible hyperelastic material, have since been extended to the case of hyperelastic materials subject to incompressibility [5] or to a generic isotropic internal constraint [6]. However, these works are limited to the consideration of *principal* surface waves, i.e. surface waves that propagate and attenuate along principal directions of pre-strain. We note that Connor and Ogden [7] and Destrade and Ogden [8] considered two-partial (non-principal) surface waves polarized in a principal plane of pre-strain. Nevertheless, for *three-partial* non-principal surface waves very few explicit results exist and the scope of their applicability is limited. Specifically, Flavin [9] considered the problem for a Mooney-Rivlin material with one material parameter much smaller than the other; Willson [10, 11] studied materials subject to equi-biaxial pre-deformations; Gerard [12], Gerhart [13], and Iwashimizu and Kobori [14] worked with the linearized theory of second-order acousto-elasticity; Chadwick and Jarvis [15], Mase and Johnson [16], and Chadwick [17] used the Stroh-Barnett-Lothe integral formalism; Rogerson and Sandiford [18] used numerical methods for an implicit secular equation; etc.

This paper presents an explicit secular equation for the speed of a surface wave propagating in a principal plane, but not in a principal direction, of a tri-axially deformed, general hyperelastic incompressible material. This result is achieved by using methods first developed by Taziev [19, 20] for surface waves propagating in the symmetry plane of a crystal. Although similar, the analysis presented here

reveals some features particular to the context of nonlinear elasticity. In general, the surface wave consists of a linear combination of three partial modes, each proportional to $\exp\{ik(\boldsymbol{n}\cdot\boldsymbol{x}+q_\alpha\boldsymbol{m}\cdot\boldsymbol{x}-vt)\}$, $\alpha=1,2,3$, where k is the wave number, v the wave speed, $\boldsymbol{n}$ the direction of propagation, $\boldsymbol{m}$ the normal to the surface, and q_α an attenuation coefficient. For a general strain energy function, the q_α are the roots of a bi-cubic equation with positive imaginary parts. Their explicit analytical expressions are awkward and the method of Taziev proves useful (Section 4), because it does not require such expressions. However, for a whole class of hyperelastic incompressible materials, inclusive of the Mooney-Rivlin model, the coefficients of attenuation q_α are obtained analytically, for one is always equal to $\mathrm{i}=\sqrt{-1}$, and so the two others are roots of a bi-quadratic (Section 3). This property has been touched upon by Flavin but has been only recently examined in detail (Pichugin [21]). Here, it leads to the derivation of an *exact and explicit* secular equation for surface waves, which possesses no more than one root for the speed. In the general case, the method of Taziev leads to a *rationalized* secular equation (a polynomial of degree 12 in the squared wave speed), with spurious roots to be discarded.

Before these two main results are developed, we summarize in Section 2 the basic governing equations and present the equations of motion as a first-order linear differential system. The derivation of this system is a lengthy process and is not obvious at all. Thanks to some shorthand notations and to the Stroh formalism, the system can however be presented in quite a compact form. Its resolution, coupled to the appropriate boundary conditions for surface waves (vanishing of the wave away from the interface; no incremental traction on the interface), leads to an implicit

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secular equation, to be made explicit in the subsequent two Sections. Section 3 is devoted to the special class of incompressible materials which is associated with a factorized propagation condition. Numerical examples in this section involve a Mooney-Rivlin material characterized by the same material parameters, state of tri-axial pre-strain, and normal load as the one considered by Rogerson and Sandiford [18]. A connection is made with their numerical results for the surface wave speed versus the angle of propagation. Some new features are highlighted; in addition, the attenuation coefficients for the partial displacements are presented. Section 4 covers the derivation of the secular equation for the general strain energy density of an incompressible material, using a variant of Taziev's technique. To illustrate the method, we investigate numerically the case of a deformed Varga material where a surface wave propagates in any direction in the plane of shear, with a view to the non-destructive acoustic evaluation of a deformed rubber insulator.

2   Preliminaries

2.1   Equations of motion

Consider a half-space, composed of a homogeneous pre-stressed hyperelastic in-compressible material with mass density ρ, characterized by strain energy function W. Let (O, x_1, x_2, x_3) be a fixed rectangular Cartesian coordinate system such that the body occupies the region $x_2 \geqslant 0$ and that the principal stretches coincide with the $\mathcal{O}x_i$ directions, with corresponding stretch ratios $\lambda_1, \lambda_2, \lambda_3$ ($\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$ and $\lambda_1 \lambda_2 \lambda_3 = 1$). The half-space is maintained in the static state of deformation

by the application of the constant tractions σ_1, σ_2, σ_3 at infinity, given by

$$\sigma_i = \lambda_i W_i - p, \tag{2.1}$$

(no summation assumed here) where $W_i := \partial W / \partial \lambda_i$ and p is a constant scalar, introduced by the constraint of incompressibility.

Then consider the superposition of a small-amplitude motion $\boldsymbol{u}(x_1, x_2, x_3, t)$ upon the primary large deformation. The corresponding incremental nominal stress $\boldsymbol{s}$ has the following components [22],

$$s_{ij} = B_{ijkl} u_{l,k} + p u_{i,j} - p^* \delta_{ij}, \tag{2.2}$$

where p^* is a Lagrange multiplier, corresponding to an increment in p, and the non-zero components of the fourth-order elasticity tensor $\boldsymbol{B}$ are (no summation on repeated i, j indexes assumed here)

$$
\begin{aligned}
B_{iijj} &= \lambda_i \lambda_j W_{ij}, \\
B_{ijij} &= (\lambda_i W_i - \lambda_j W_j) \lambda_i^2 / (\lambda_i^2 - \lambda_j^2), \\
B_{ijji} &= B_{jiij} = B_{ijij} - \lambda_i W_i.
\end{aligned}
\tag{2.3}
$$

The linearized incremental equations of motion and incompressibility condition read

$$s_{ij,i} = \rho u_{j,tt}, \quad u_{j,j} = 0. \tag{2.4}$$

We specialize our analysis to the consideration of a surface (Rayleigh) wave, propagating in a principal plane but not in a principal direction, see Figure 1. Specifically, it is assumed that the wave is traveling with phase velocity v and wave number k over the surface $x_2 = 0$ in a direction which makes an angle θ with
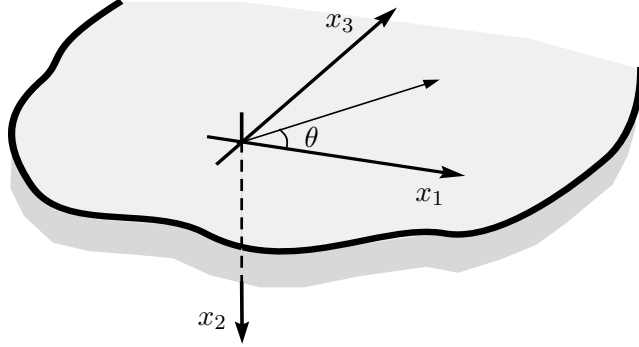
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Figure 1: A surface wave propagating in a principal plane but not in a principal direction.

$\mathcal{O}x_1$; it decays exponentially away from the boundary $x_2 = 0$, and produces no incremental traction at the boundary.

Hence we model this motion by

$$\{\boldsymbol{u}, p^*, \boldsymbol{s}\} = \{\boldsymbol{U}(kx_2), \mathrm{i}kP(kx_2), \mathrm{i}k\boldsymbol{S}(kx_2)\}e^{\mathrm{i}k(c_\theta x_1 + s_\theta x_3 - vt)}, \qquad (2.5)$$

where $c_\theta := \cos\theta$, $s_\theta := \sin\theta$, and $\boldsymbol{U}$, P, $\boldsymbol{S}$ are functions of kx_2 alone. By substituting these forms of the mechanical displacements and the tractions into the incremental constitutive equation (2.2) and using (2.1), the incremental equations of motion and the incompressibility constraint (2.4) can be cast as a homogeneous linear system of six first-order differential equations,

$$\boldsymbol{\xi}' = \mathrm{i}\boldsymbol{N}\boldsymbol{\xi}, \quad \text{where} \quad \boldsymbol{\xi}(kx_2) := [U_1, U_2, U_3, S_{21}, S_{22}, S_{23}]^{\mathrm{T}}, \qquad (2.6)$$

within which the prime denotes differentiation with respect to the variable kx_2.

Here the 6×6 real matrix $\boldsymbol{N}$ follows the usual block decomposition [23] of

linear anisotropic elasticity,

$$\boldsymbol{N} = \begin{bmatrix} \boldsymbol{N}_1 & \boldsymbol{N}_2 \\ \boldsymbol{N}_3 + X\boldsymbol{1} & \boldsymbol{N}_1^{\mathrm{T}} \end{bmatrix}, \quad X := \rho v^2, \tag{2.7}$$

where $\boldsymbol{N}_1$, $\boldsymbol{N}_2 \equiv \boldsymbol{N}_2^{\mathrm{T}}$, and $\boldsymbol{N}_3 \equiv \boldsymbol{N}_3^{\mathrm{T}}$ are 3×3 matrices. However, the components of the $\boldsymbol{N}_i$ are specific to the theory of small motions superposed on large static deformations [17]. To present them in a compact form, we introduce the short-hand notations

$$\gamma_{ij} := (\lambda_i W_i - \lambda_j W_j)\lambda_i^2/(\lambda_i^2 - \lambda_j^2) \equiv \gamma_{ji} + \lambda_i W_i - \lambda_j W_j,$$

$$2\beta_{ij} := \lambda_i^2 W_{ii} - 2\lambda_i \lambda_j W_{ij} + \lambda_j^2 W_{jj} + 2(\lambda_i W_j - \lambda_j W_i)\lambda_i \lambda_j/(\lambda_i^2 - \lambda_j^2) \equiv 2\beta_{ji}. \tag{2.8}$$

Then $-\boldsymbol{N}_1$ and $\boldsymbol{N}_2$ are given by

$$\begin{bmatrix} 0 & c_\theta(\gamma_{21} - \sigma_2)/\gamma_{21} & 0 \\ c_\theta & 0 & s_\theta \\ 0 & s_\theta(\gamma_{23} - \sigma_2)/\gamma_{23} & 0 \end{bmatrix}, \quad \begin{bmatrix} 1/\gamma_{21} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/\gamma_{23} \end{bmatrix}, \tag{2.9}$$

respectively, and

$$-\boldsymbol{N}_3 = \begin{bmatrix} \eta & 0 & -\kappa \\ 0 & \nu & 0 \\ -\kappa & 0 & \mu \end{bmatrix}, \tag{2.10}$$

where

$$\eta := 2c_\theta^2(\beta_{12} + \gamma_{21} - \sigma_2) + s_\theta^2 \gamma_{31},$$

$$\nu := c_\theta^2[\gamma_{12} - (\gamma_{21} - \sigma_2)^2/\gamma_{21}] + s_\theta^2[\gamma_{32} - (\gamma_{23} - \sigma_2)^2/\gamma_{23}],$$

$$\mu := c_\theta^2 \gamma_{13} + 2s_\theta^2(\beta_{23} + \gamma_{23} - \sigma_2),$$

$$\kappa := c_\theta s_\theta(\beta_{13} - \beta_{12} - \beta_{23} - \gamma_{21} - \gamma_{23} + 2\sigma_2), \tag{2.11}$$

(see Destrade [24] for the case $\sigma_2 = 0$, and Destrade and Ogden [8] for the case of a wave polarized in the symmetry plane of a stretched and sheared material).

2.2 Propagation condition

Now the requirement of exponential decay away from the surface $x_2 = 0$ is expressed by choosing the following form for the wave,

$$\boldsymbol{\xi}(kx_2) = \boldsymbol{\xi^o}e^{\mathrm{i}kqx_2}, \quad \Im(q) > 0, \tag{2.12}$$

where $\boldsymbol{\xi^o}$ is a constant vector and q is an attenuation coefficient. Then the equations of motion (2.6) become the eigenvalue problem: $\boldsymbol{N}\boldsymbol{\xi^o} = q\boldsymbol{\xi^o}$. The associated characteristic equation $\det(\boldsymbol{N} - q\boldsymbol{1}) = 0$, is the *propagation condition*. This equation is a cubic in q^2, as demonstrated by Rogerson and Sandiford [18]:

$$\gamma_{21}\gamma_{23}q^6 - [(\gamma_{21}+\gamma_{23})X - c_1]\,q^4 + (X^2 - c_2X + c_3)\,q^2 + (X-c_4)(X-c_5) = 0, \tag{2.13}$$

with

$$c_1 := (\gamma_{21}\gamma_{13} + 2\beta_{12}\gamma_{23})c_\theta^2 + (\gamma_{23}\gamma_{31} + 2\beta_{23}\gamma_{21})s_\theta^2,$$

$$c_2 := (\gamma_{23} + \gamma_{13} + 2\beta_{12})c_\theta^2 + (\gamma_{21} + \gamma_{31} + 2\beta_{23})s_\theta^2,$$

$$c_3 := (\gamma_{12}\gamma_{23} + 2\beta_{12}\gamma_{13})c_\theta^4 + (\gamma_{21}\gamma_{32} + 2\beta_{23}\gamma_{31})s_\theta^4$$

$$+ [\gamma_{12}\gamma_{21} + \gamma_{13}\gamma_{31} + \gamma_{23}\gamma_{32} - (\beta_{13} - \beta_{12} - \beta_{23})^2 + 4\beta_{12}\beta_{23}]c_\theta^2 s_\theta^2,$$

$$c_4 := \gamma_{12}c_\theta^2 + \gamma_{32}s_\theta^2,$$

$$c_5 := \gamma_{13}c_\theta^4 + 2\beta_{13}c_\theta^2 s_\theta^2 + \gamma_{31}s_\theta^4. \tag{2.14}$$

Note that the roots q_1^2, q_2^2, q_3^2 of the bicubic are such that

$$q_1^2 + q_2^2 + q_3^2 = [(\gamma_{21} + \gamma_{23})X - c_1]/(\gamma_{21}\gamma_{23}),$$

$$q_1^2 q_2^2 + q_2^2 q_3^2 + q_3^2 q_1^2 = (X^2 - c_2 X + c_3)/(\gamma_{21}\gamma_{23}),$$

$$q_1^2 q_2^2 q_3^2 = -(X - c_4)(X - c_5)/(\gamma_{21}\gamma_{23}). \tag{2.15}$$

2.3 Implicit secular equation for surface waves

For the three roots q_1, q_2, q_3 of the propagation condition (2.13) with positive imaginary parts, the corresponding eigenvalue problems yield three linearly independent eigenvectors $\boldsymbol{\xi}^1$, $\boldsymbol{\xi}^2$, $\boldsymbol{\xi}^3$, respectively. Then the general solution to the equations of motion (2.6) may be written as

$$\boldsymbol{\xi}(kx_2) = \gamma_1 e^{iq_1 kx_2}\boldsymbol{\xi}^1 + \gamma_2 e^{iq_2 kx_2}\boldsymbol{\xi}^2 + \gamma_3 e^{iq_3 kx_2}\boldsymbol{\xi}^3, \tag{2.16}$$

for some constants γ_1, γ_2, γ_3. Explicitly, $\boldsymbol{\xi}^i$ are given by columns of the matrix adjoint to $\boldsymbol{N} - q_i \boldsymbol{1}$. Taking, for example, the second such column gives $\boldsymbol{\xi}^i$ in the form:

$$\boldsymbol{\xi}^i = \begin{bmatrix} a_4 q_i^4 + a_2 q_i^2 + a_0 \\ -q_i^5 + b_3 q_i^3 + b_1 q_i \\ d_4 q_i^4 + d_2 q_i^2 + d_0 \\ h_3 q_i^3 + h_1 q_i \\ (\nu - X)(q_i^4 + m q_i^2 + n) \\ g_3 q_i^3 + g_1 q_i \end{bmatrix}, \tag{2.17}$$

where expressions for the constants a_i, b_i, d_i, h_i, and g_i are too lengthy to be reproduced here and

$$
\begin{aligned}
m = & \left(\frac{1}{\gamma_{21}} + \frac{(\gamma_{21} - \sigma_2)^2}{\gamma_{21}^2(\nu - X)} c_\theta^2 \right) [\eta - X] + \left(\frac{1}{\gamma_{23}} + \frac{(\gamma_{23} - \sigma_2)^2}{\gamma_{23}^2(\nu - X)} s_\theta^2 \right) [\mu - X] \\
& - 2\kappa \frac{(\gamma_{21} - \sigma_2)(\gamma_{23} - \sigma_2)}{\gamma_{21}\gamma_{23}(\nu - X)} c_\theta s_\theta, \\
n = & \left\{ 1 + \left[\frac{(\gamma_{21} - \sigma_2)^2}{\gamma_{21}} c_\theta^2 + \frac{(\gamma_{23} - \sigma_2)^2}{\gamma_{23}} s_\theta^2 \right] (\nu - X)^{-1} \right\} \\
& \times [(\mu - X)(\eta - X) - \kappa^2]/(\gamma_{21}\gamma_{23}).
\end{aligned} \tag{2.18}
$$

The boundary condition of zero incremental tractions at the plane surface $x_2 = 0$ means that

$$
\boldsymbol{\xi}(0) = [U_1(0), U_2(0), U_3(0), 0, 0, 0]^{\mathrm{T}}. \tag{2.19}
$$

By comparing this expression with (2.16) evaluated for $x_2 = 0$, we conclude that γ_1, γ_2, γ_3 are solutions to a homogeneous linear system of three equations. The corresponding secular equation is given by

$$
(\nu - X) \begin{vmatrix} h_3 q_1^3 + h_1 q_1 & h_3 q_2^3 + h_1 q_2 & h_3 q_3^3 + h_1 q_3 \\ q_1^4 + m q_1^2 + n & q_2^4 + m q_2^2 + n & q_3^4 + m q_3^2 + n \\ g_3 q_1^3 + g_1 q_1 & g_3 q_2^3 + g_1 q_2 & g_3 q_3^3 + g_1 q_3 \end{vmatrix} = 0. \tag{2.20}
$$

As noted by Taziev [19] in the context of linear anisotropic elasticity, this determinant factorizes greatly. By omitting the factors $\nu - X$, $q_i - q_j$ and $h_1 g_3 - h_3 g_1$, we are left with

$$
n\omega_{\mathrm{I}} = \omega_{\mathrm{III}}(m - \omega_{\mathrm{II}}), \tag{2.21}
$$

where

$$
\omega_{\mathrm{I}} := -(q_1 + q_2 + q_3), \quad \omega_{\mathrm{II}} := q_1 q_2 + q_2 q_3 + q_3 q_1, \quad \omega_{\mathrm{III}} := -q_1 q_2 q_3. \tag{2.22}
$$

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Eq. (2.21) is the secular equation for surface waves in deformed incompressible materials; it remains implicit as long as the explicit expressions for the ω_α are not known.

3   Factorization of the propagation condition

Before moving on to the derivation of an explicit secular equation, a special case must be treated separately. A simple analysis shows that, for a certain class of incompressible hyperelastic materials maintained in a state of large homogeneous deformation (strain-induced anisotropy), the propagation condition (2.13) factorizes into the product of a term linear in q^2 and a term quadratic in q^2, thus leading to simple explicit expressions for the q_α and eventually, for the secular equation. This class of materials includes the well-known Mooney-Rivlin model, often used to describe finite deformations of rubber. The described factorization does not in general occur for linear elastic materials such as crystals (intrinsic anisotropy).

3.1   Conditions on the strain energy function and explicit secular equation for surface waves

Following Pichugin [21], we seek a solution to the propagation condition (2.13) of the form $q^2 = C$, where C is a constant independent of v. By substituting $q^2 = C$ into (2.15), we obtain three equations for the two quantities S and P, the

respective sum and product of the remaining q_α^2, namely,

$$S + C = [(\gamma_{21} + \gamma_{23})X - c_1]/(\gamma_{21}\gamma_{23}),$$

$$CS + P = (X^2 - c_2 X + c_3)/(\gamma_{21}\gamma_{23}), \qquad (3.1)$$

$$CP = -(X - c_4)(X - c_5)/(\gamma_{21}\gamma_{23}).$$

After solving $(3.1)_1$ for S and then $(3.1)_3$ for P, the substitutions of S and P into $(3.1)_2$ allow for the identification of like-powers of X and c_θ^2 on both sides of the resulting equation. Since the identity must hold for all X and θ we obtain that $C = -1$, provided the following relationships are satisfied

$$\gamma_{ij} + \gamma_{ji} = 2\beta_{ij}. \qquad (3.2)$$

Then the associated propagation condition factorizes into

$$(q^2 + 1)(q^4 - Sq^2 + P) = 0, \qquad (3.3)$$

where

$$S = \left(\frac{1}{\gamma_{21}} + \frac{1}{\gamma_{23}}\right) X - \left(\frac{\gamma_{12}}{\gamma_{21}} + \frac{\gamma_{13}}{\gamma_{23}}\right) c_\theta^2 - \left(\frac{\gamma_{31}}{\gamma_{21}} + \frac{\gamma_{32}}{\gamma_{23}}\right) s_\theta^2,$$

$$P = (X - \gamma_{12}c_\theta^2 - \gamma_{32}s_\theta^2)(X - \gamma_{13}c_\theta^2 - \gamma_{31}s_\theta^2)/(\gamma_{21}\gamma_{23}). \qquad (3.4)$$

To sum up, if the conditions (3.2) on the strain energy function are satisfied, then the bi-cubic (2.13) factorizes into the product of a term linear in q^2 and a term quadratic in q^2. It follows that the roots q_α with positive imaginary parts and hence the corresponding ω_I, ω_{II}, and ω_{III} defined in (2.22), can be found explicitly: $q_1 = i$, $q_2 q_3 = -\sqrt{P}$, $q_2 + q_3 = i\sqrt{2\sqrt{P} - S}$, so that

$$\omega_I = -i\left(1 + \sqrt{2\sqrt{P} - S}\right), \quad -\omega_{II} = \sqrt{P} + \sqrt{2\sqrt{P} - S}, \quad \omega_{III} = i\sqrt{P}. \qquad (3.5)$$

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The end result is that equation (2.21) is now an *explicit secular equation for surface waves in deformed incompressible materials satisfying* (3.2), namely

$$n\left(1 + \sqrt{2\sqrt{P} - S}\right) + \sqrt{P}\left(m + \sqrt{P} + \sqrt{2\sqrt{P} - S}\right) = 0. \qquad (3.6)$$

Note that the conditions (3.2) impose restrictions upon the strain energy function of the incompressible material. Explicitly, they read ($i \neq j$)

$$(\lambda_i^2 + 3\lambda_j^2)\lambda_i W_i - (3\lambda_i^2 + \lambda_j^2)\lambda_j W_j - (\lambda_i^2 - \lambda_j^2)(\lambda_i^2 W_{ii} - 2\lambda_i\lambda_j W_{ij} + \lambda_j^2 W_{jj}) = 0. \quad (3.7)$$

For example, the Mooney-Rivlin strain energy function,

$$W = \mathcal{D}_1(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)/2 + \mathcal{D}_2(\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 - 3)/2, \qquad (3.8)$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are constants, satisfies this condition. This case is treated in the following subsection.

Note also that Pichugin [21] finds conditions under which the propagation condition (2.13) admits roots of the form $q^2 = CX + D$, where C, D are constants. It turns out this possibility arises when the half-space is subject to an *equi-biaxial* pre-deformation, whatever the strain energy function may be. Another simplification occurs when the bi-quadratic in (3.3) admits a double root (then $S^2 = 4P$); such is the case for the neo-Hookean form of strain energy function ($\mathcal{D}_2 = 0$ in (3.8)). The secular equations for surface waves in bi-axially deformed generic incompressible materials and in tri-axially deformed neo-Hookean materials were established by Willson [10] and Flavin [9], respectively, and are not investigated further here.

3.2 Example: Mooney-Rivlin materials

Now the case of Mooney-Rivlin materials is dealt with in such a way that a connection is made with the numerical results of Rogerson and Sandiford [18]. For the
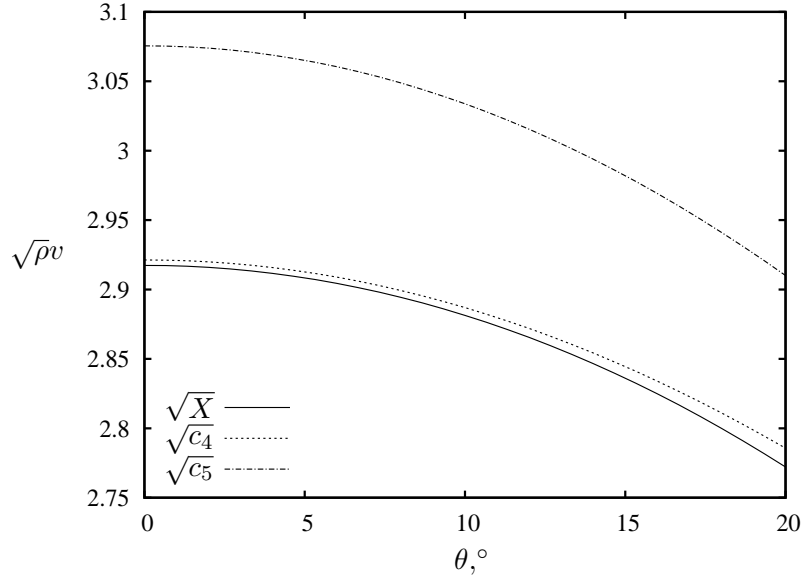
Figure 2: Scaled speeds of bulk and surface waves near the direction of greatest stretch in a Mooney-Rivlin material.

Mooney-Rivlin strain energy function (3.8) the quantities γ_{ij}, β_{ij}, defined in (2.8), yield the following forms,

$$\gamma_{ij} = (\mathcal{D}_1 + \mathcal{D}_2\lambda_k^2)\lambda_i^2, \quad 2\beta_{ij} = (\mathcal{D}_1 + \mathcal{D}_2\lambda_k^2)(\lambda_i^2 + \lambda_j^2) = \gamma_{ij} + \gamma_{ji}, \qquad (3.9)$$

where $k \neq i, j$.

Using these expressions, together with the material parameters [18] $\mathcal{D}_1 = 2$, $\mathcal{D}_2 = 0.8$, the stretch ratios given by $\lambda_1^2 = 3.695$, $\lambda_2^2 = 0.7$, $\lambda_3^2 = 0.387$, and the normal load $\sigma_2 = 0.8$, the secular equation (3.6) is solved numerically to give the variation of the surface wave speed with θ, and the graph of Rogerson and Sandiford [18] is reproduced with little effort. We take this opportunity to comment on their statement that "the surface wave degenerates into a shear wave as θ approaches 0 and $\pi/2$".

Firstly, we find that at, and close to, the direction $\mathcal{O}x_1$ (also the direction of

greatest stretch), the surface wave speed $v = \sqrt{X/\rho}$ is *distinct* from the bulk shear wave $\sqrt{c_4/\rho}$, i.e. $c_5 > c_4$ in the neighborhood of $\theta = 0$. For instance, at $\theta = 0$ the *principal* surface wave propagation speed $\sqrt{X_0/\rho}$, where $\sqrt{X_0} = 2.917$, is found from Dowaikh and Ogden's [5] formula $X_0 = \gamma_{12} - \gamma_{21}\zeta^2$, where

$$\zeta^3 + \zeta^2 + \frac{2(\beta_{12} + \gamma_{21} - \sigma_2) - \gamma_{12}}{\gamma_{21}}\zeta - \frac{(\gamma_{21} - \sigma_2)^2}{\gamma_{21}} = 0\,, \qquad (3.10)$$

(or equivalently, is found from (3.6)), while the bulk shear waves propagate at speeds $\sqrt{c_5/\rho}$ and $\sqrt{c_4/\rho}$ where $\sqrt{c_5} = \sqrt{\gamma_{13}} = 3.076$ and $\sqrt{c_5} = \sqrt{\gamma_{12}} = 2.921$. Figure 2 shows the variations of these speeds in the $(0° - 20°)$ range. The top (dashed) curve is the graph of $\sqrt{c_5}$, the middle (dotted) curve is the graph of $\sqrt{c_4}$, and the bottom (solid) curve is the graph of $\sqrt{X}$.

Secondly, we find that, as the direction of propagation approaches the $\mathcal{O}x_3$ direction (direction of least stretch, $\theta = 90°$ here), the surface wave speed $v = \sqrt{X/\rho}$ is indeed tending to the bulk shear wave speed $\sqrt{c_5/\rho}$, so that the corresponding graphs are indistinguishable one from another in the approximative range $(82° - 90°)$. However, at $\theta = 90°$ exactly, there exists a two-partial principal surface wave whose speed is *intermediate* between the bulk shear wave speeds $\sqrt{c_5/\rho}$ and $\sqrt{c_4/\rho}$. Numerically, when $\theta = 90°$, $\sqrt{c_5} = 0.995$, $\sqrt{c_4} = 1.384$, and this two-partial principal surface wave propagates with speed $\sqrt{X_{90}/\rho}$ say, where $\sqrt{X_{90}} = 1.327$, the value found from Dowaikh and Ogden's [5] formula $X_{90} = \gamma_{32} - \gamma_{23}\zeta^2$, in which

$$\zeta^3 + \zeta^2 + \frac{2(\beta_{23} + \gamma_{23} - \sigma_2) - \gamma_{32}}{\gamma_{23}}\zeta - \frac{(\gamma_{23} - \sigma_2)^2}{\gamma_{23}} = 0\,. \qquad (3.11)$$

This peculiar situation is also encountered in cubic crystals with strong anisotropy, such as nickel [25]. In short, the subsonic two-partial surface wave must be slower than any in-plane bulk wave (such as the one propagating with speed $\sqrt{c_5/\rho}$), but

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is indifferent to the anti-plane wave propagating with speed $\sqrt{c_4/\rho}$. This principal two-partial surface wave is singular because it exists only in the direction $\theta = 90°$, although "pseudo-surface waves" may be found in its neighborhood.

Once $X = \rho v^2$ is known, the attenuation coefficients q_i are computed from (3.3) and the depth profiles follow naturally from (2.16). Figure 3 displays the imaginary part of the q_i, indicative of the penetration depth, as a function of θ. The horizontal straight top (dashed) line is for $q_1 = \mathrm{i}$. The two other curves (dotted and solid) are for the imaginary parts of q_2 and q_3. At $\theta = 0$, there are only two partial modes, one corresponding to $q_1 = \mathrm{i}$, the other corresponding to $q_3 = 0.119\mathrm{i}$. At $\theta \gtrsim 0$, a third partial mode appears, corresponding to a q_2 of the form $q_2 = \mathrm{i}\beta_2$, $\beta_2 \lesssim 0.52$. In the approximate ranges $(0° - 23°)$ and $(68° - 90°)$, the attenuation factors are of the form $q_1 = \mathrm{i}$, $q_2 = \mathrm{i}\beta_2$, $q_3 = \mathrm{i}\beta_3$ with $\beta_2 > 0$, $\beta_3 > 0$. In the approximate range $(23° - 68°)$, they are of the form $q_1 = \mathrm{i}$, $q_2 = \alpha + \mathrm{i}\beta$, $q_3 = -\alpha + \mathrm{i}\beta$ with $\beta > 0$. As θ approaches $90°$, and X approaches c_5, the imaginary part of the attenuation factor q_3 tends to zero, indicating a deeply penetrating quasi-bulk surface wave [26]. Finally at $\theta = 90°$, a singular two-partial principal surface wave exists, with one mode corresponding to $q_1 = \mathrm{i}$ and the other corresponding to $q_2 = 0.212\mathrm{i}$ (this latter value corresponds to a discontinuity in the representation of q_2 as a function of θ and cannot be represented on the graph).
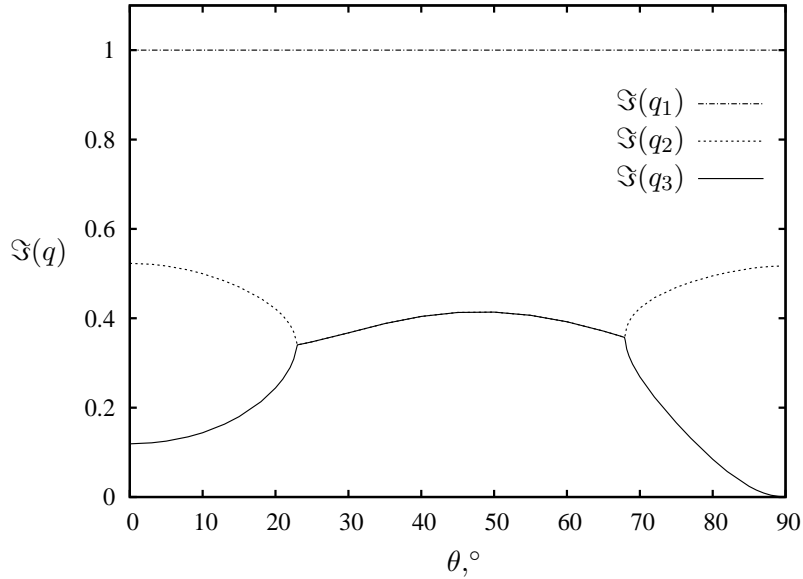
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Figure 3: Imaginary part of the attenuation coefficients for a surface wave in a deformed Mooney-Rivlin material.

4 General case

4.1 Explicit secular equation for surface waves

In the general case, where no factorization of the propagation condition occurs, a different treatment is required. The "method of the polarization vector", introduced by Currie [27], refined by Taziev [20], and recently revisited by Ting [28], proves to be a most effective mean for deriving the secular equation as a polynomial in $X = \rho v^2$. It relies on the equations

$$\overline{\boldsymbol{U}}(0) \cdot \boldsymbol{K}^{(n)} \boldsymbol{U}(0) = 0, \tag{4.1}$$

where $\boldsymbol{K}^{(n)}$ is the symmetric 3×3 lower left submatrix of the $\boldsymbol{N}^n$ matrix and n is an integer. Hence $\boldsymbol{K}^{(1)} = \boldsymbol{N}_3 + X\boldsymbol{1}$, $\boldsymbol{K}^{(2)} = \boldsymbol{K}^{(1)}\boldsymbol{N}_1 + \boldsymbol{N}_1^{\mathrm{T}}\boldsymbol{K}^{(1)}$, etc.

Also, for a wave propagating in the symmetry plane of a monoclinic crystal, Ting [28] showed that $\boldsymbol{U}(0)$ is of the form

$$\boldsymbol{U}(0) = U_1(0)[1, i\alpha_2, \beta_1]^{\mathrm{T}}, \qquad (4.2)$$

where α_2, β_1 are real numbers. We checked that $\boldsymbol{U}(0)$ is also of this form in the present case of a surface wave propagating in a principal plane of a deformed material.

Computing $\boldsymbol{N}^{-1}$ and $\boldsymbol{N}^3$, we find that $K_{12}^{(n)} = K_{23}^{(n)} = 0$, for $n = -1, 1, 3$. It follows that the equations (4.1) written for $n = -1, 1, 3$ reduce to the non-homogeneous system

$$\begin{bmatrix} K_{13}^{(-1)} & K_{33}^{(-1)} & K_{22}^{(-1)} \\ K_{13}^{(1)} & K_{33}^{(1)} & K_{22}^{(1)} \\ K_{13}^{(3)} & K_{33}^{(3)} & K_{22}^{(3)} \end{bmatrix} \begin{bmatrix} 2\beta_1 \\ \beta_1^2 \\ \alpha_2^2 \end{bmatrix} = \begin{bmatrix} -K_{11}^{(-1)} \\ -K_{11}^{(1)} \\ -K_{11}^{(3)} \end{bmatrix}. \qquad (4.3)$$

By Cramer's rule, we find $2\beta_1 = \Delta_1/\Delta$, $\beta_1^2 = \Delta_2/\Delta$ where

$$\Delta = \begin{vmatrix} K_{13}^{(-1)} & K_{33}^{(-1)} & K_{22}^{(-1)} \\ K_{13}^{(1)} & K_{33}^{(1)} & K_{22}^{(1)} \\ K_{13}^{(3)} & K_{33}^{(3)} & K_{22}^{(3)} \end{vmatrix},$$

$$\Delta_1 = \begin{vmatrix} -K_{11}^{(-1)} & K_{33}^{(-1)} & K_{22}^{(-1)} \\ -K_{11}^{(1)} & K_{33}^{(1)} & K_{22}^{(1)} \\ -K_{11}^{(3)} & K_{33}^{(3)} & K_{22}^{(3)} \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} K_{13}^{(-1)} & -K_{11}^{(-1)} & K_{22}^{(-1)} \\ K_{13}^{(1)} & -K_{11}^{(1)} & K_{22}^{(1)} \\ K_{13}^{(3)} & -K_{11}^{(3)} & K_{22}^{(3)} \end{vmatrix}, \qquad (4.4)$$

so that

$$\Delta_1^2 - 4\Delta\Delta_2 = 0, \tag{4.5}$$

which is the *explicit secular equation for non-principal surface waves in deformed incompressible materials.*

Upon inspection of (2.7) and (2.10), we find that $K_{13}^{(1)} = \kappa$, $K_{33}^{(1)} = X - \mu$, $K_{22}^{(1)} = X - \nu$, $K_{11}^{(1)} = X - \eta$. Computing $\boldsymbol{N}^{-1}$, we find that up to a common disposable factor, $K_{13}^{(-1)}$, $K_{33}^{(-1)}$, $K_{22}^{(-1)}$, $K_{11}^{(-1)}$ are proportional to polynomials of degree 1, 2, 3, 2 in X, respectively. Similarly, computing $\boldsymbol{N}^3$, we find that $K_{13}^{(3)}$, $K_{33}^{(3)}$, $K_{22}^{(3)}$, $K_{11}^{(3)}$ are polynomials of degree 1, 2, 1, 2, respectively. We conclude from the definitions (4.4) of Δ, Δ_1, Δ_2 that the secular equation (4.5) is a polynomial of degree 12 in $X = \rho v^2$, just as for monoclinic crystals in linear anisotropic elasticity [19]. It is too long to reproduce here but it was obtained in a formal manner with Maple and with Mathematica.

The numerical resolution of the polynomial (4.5) yields a priori 12 roots for X. From these, we discard at once the complex roots, the negative real roots, and the roots corresponding to supersonic surface waves (faster than bulk waves). Out of the remaining roots, at most one will yield attenuation coefficients q_1, q_2, q_3 (with a positive imaginary part) from the propagation condition (2.13), such that the exact secular equation (2.21) is satisfied.

To conclude this Section, we check that the rationalized secular equation (4.5) is consistent with the known secular equation for principal surface waves. When $\theta = 0$, it is easy to see that $K_{13}^{(-1)} = K_{13}^{(1)} = K_{13}^{(3)} = 0$, therefore $\Delta = \Delta_2 = 0$. Thus, (4.5) reduces to $\Delta_1 = 0$. By substituting $X = \gamma_{12} - \gamma_{21}\zeta^2$, we find that Δ_1 factorizes into the product of a quadratic in ζ^2 and two cubics in ζ, one of which

is indeed Dowaikh and Ogden's [5] equation (3.10).

4.2 Example: Varga materials

The standard Varga strain energy function [29, 30] is defined as

$$W = \mathcal{C}(\lambda_1 + \lambda_2 + \lambda_3 - 3), \tag{4.6}$$

where the material parameter $\mathcal{C}$ is constant. This strain energy function has been introduced to describe natural rubber vulcanizates. It leads to the following expressions for the quantities γ_{ij} and β_{ij}, defined in (2.8),

$$\gamma_{ij} = \mathcal{C}\lambda_i^2/(\lambda_i + \lambda_j), \quad \beta_{ij} = \mathcal{C}\lambda_i\lambda_j/(\lambda_i + \lambda_j). \tag{4.7}$$
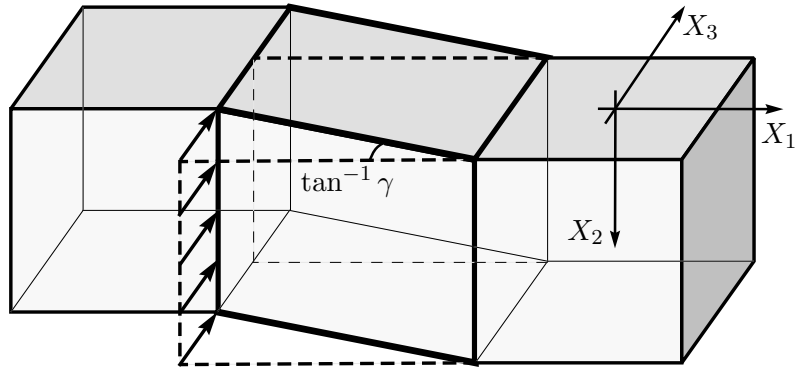

Figure 4: An elastic material under simple shear; the dashed lines represent the body at rest.

Simple shear plays an important role in the experimental determination of a strain energy function [22]. Consider now a half-space made of Varga material, subject to an amount of shear γ along X_3, see Figure 4. With a view to the

possible non-destructive evaluation of sheared rubber, we are interested in the propagation of a surface wave in any direction in the plane of shear $\mathcal{O}X_1X_3$. We assume that this plane is free of normal load, thus $\sigma_2 = 0$.

The principal stretches associated with the amount of shear γ may be determined using the following equations [7],

$$\lambda_1 - \lambda_1^{-1} = \gamma\,, \qquad \lambda_1 > 1\,, \qquad \lambda_2 = 1\,, \qquad \lambda_3 = \lambda_1^{-1}\lambda_2^{-1}\,. \qquad (4.8)$$

We substitute the parameters (4.7) into the secular equation (4.5), and then solve that equation numerically for the amounts of shear: $\gamma = 0.5, 1.0, 1.5$, corresponding to the angles of shear: $\tan^{-1}\gamma = 26.56°, 45°, 56.31°$, respectively. We obtain the velocity of the surface wave propagating in a direction making an angle $\theta + \psi$ with $\mathcal{O}X_1$, where $\psi = (1/2)\tan^{-1}(2/\gamma) = 37.98°, 31.72°, 26.56°$, respectively. Note that here $\mathcal{O}X_1$ is not a principal axis, for the two principal axes $\mathcal{O}x_1$ and $\mathcal{O}x_3$ in the plane of shear are at an angle ψ with $\mathcal{O}X_1$ and with $\mathcal{O}X_3$, respectively.

For the selection of the relevant root of the secular equation out of the 12 possible ones, we followed the checking procedure described at the end of the previous Section. For example, at $\gamma = 1$ and $\theta + \psi = 0$, the rationalized secular equation (4.5) has only 2 real roots, giving $\sqrt{\rho v^2/\mathcal{C}} = 0.848$ or $\sqrt{\rho v^2/\mathcal{C}} = 0.884$. Both roots yield three attenuation factors with a positive imaginary part from the propagation condition (2.13). However, the exact secular equation (2.21) is satisfied only with the first root. Hence, with a 32 digit precision, Maple finds that $|n\omega_{\mathrm{I}}/[\omega_{\mathrm{III}}(m - \omega_{\mathrm{II}})] - 1| < 10^{-22}$ when $\sqrt{\rho v^2/\mathcal{C}} = 0.848$, indicating that (2.21) is satisfied; on the other hand, $|n\omega_{\mathrm{I}}/[\omega_{\mathrm{III}}(m - \omega_{\mathrm{II}})] - 1| > 1.96$ when $\sqrt{\rho v^2/\mathcal{C}} = 0.884$, indicating that (2.21) is not satisfied then.

Figure 5 displays the dependence of $\sqrt{\rho v^2/\mathcal{C}}$ on the angle $\theta + \psi$ over the range
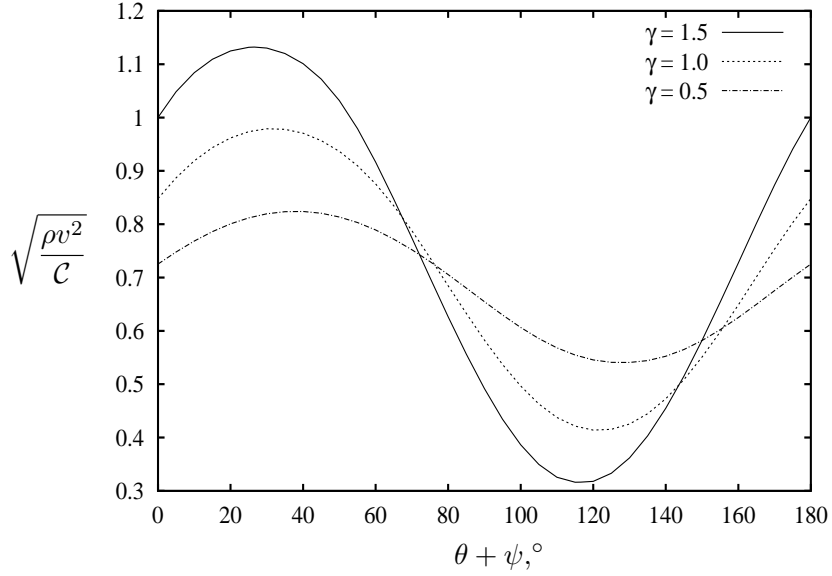
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Figure 5: Scaled surface wave speed as a function of $\theta + \psi$ for a Varga material subjected to simple shear deformations.

$[0° - 180°]$. The solid, dot, and dash-dot curves correspond to an amount of shear of 0.5, 1.0, 1.5, respectively. The figure confirms what is to be expected intuitively: as the half-space is more and more sheared, the strain-induced anisotropy increases, and its influence on the surface wave speed is more and more marked. It also shows that the surface wave travels at its fastest (slowest) speed along the direction of greatest (least) stretch, thus allowing for an acoustic determination of the directions of the principal stretches in sheared rubber.

5   Acknowledgements*

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