Doctoral School of General and Quantitative Economics

THESIS ON

László Csató

Methodological and applicational issues of paired comparison based ranking

Ph.D. dissertation

Supervisors:

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senior research fellow

Dr. József Temesi CSc
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Department of Operations Research and Actuarial Sciences

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Chapter 1

Introduction

Paired comparison based ranking problems are given by a tournament matrix representing the performance of some objects against each other. They arise in many different fields like social choice theory (Chebotarev and Shamis, 1998), sports (Landau, 1895, 1914; Zermelo, 1929) or psychology (Thurstone, 1927). The usual goal is to determine a winner (possibly a set of winners) or a complete ranking for the objects. There were some attempts to link the two areas (i.e. Bouyssou (2004)), however, they achieved a limited success. We will deal only with the latter issue, allowing for different preference intensities (including ties), incomplete and multiple comparisons among the objects.

The ranking includes three areas: representation of the practical problem as a mathematical model, its solution, and interpretation of the results. The third issue strongly depends on the actual application, therefore it is not addressed in the thesis, however, it will appear in Chapter 7.

All papers using these methods obviously discuss the first question, however, there is a lack of general review. Nevertheless, it is worth mentioning that Jiang et al. (2011) revisit the aggregation of individual evaluation into paired comparisons. Besides an extensive application, we want to overview some fields where these rankings are used.

Regarding the second issue, we will follow an axiomatic approach. A number of papers present a characterization of a specific scoring procedure, and some works discuss the problem on a domain more restricted than our (see, for example, Laslier (1997) or Altman and Tennenholtz (2008)). We mention three articles giving a similar discussion. Chebotarev and Shamis (1998) give a survey of paired comparisons based scoring procedures, and their known axiomatizations. Chebotarev and Shamis (1999) introduce the concepts of win-loss combining and win-loss unifying procedures, and argues for the use of the latter, among them the score, generalised row sum, and least squares methods. It also promises some future axiomatic construction of procedures, but this work has not been completed yet. Despite that the recent paper of González-Díaz et al. (2014) does not substitute for the lack of characterizations, it contains 14 properties and adds some new aspects for the comparison of different methods, supporting the choice of an appropriate scoring procedure.

Our discussion about the solution of paired comparison-based ranking problems follow González-Díaz et al. (2014). We define some novel axioms and thoroughly analyse their connections. Other properties can be found in Csató (2013b) and Csató (2014b). However, we focus on score, generalised row sum, and least squares. Fair bets is examined in Csató (2013d) and Csató (2014b), while maximum likelihood is disregarded due to its non-linearity.

The thesis is structured as follows. Chapter 2 presents our framework for paired comparison ranking, based on Chebotarev and Shamis (1998) and González-Díaz et al. (2014). The concepts of results and matches matrices are introduced in order to represent the structure of comparisons with graphs. We also review two different approaches of ranking, the approximation of paired comparisons by linear orders and the use of scoring procedures. Bouyssou (2004) has persuaded us that the latter direction requires more attention, thus an overview is given about the scoring procedures from an axiomatic perspective. Chapter 3 discusses some of them, with a focus on the score, generalised row sum, and least squares methods.

Chapter 4 derives a graph interpretation for generalised row sum and least squares methods on the basis of the balanced comparison multigraph, which is one of the main contributions of our thesis. We will compare it to similar methods proposed in the literature, like Brozos-Vázquez et al. (2008) or Herings et al. (2005). In Chapter 5, we define some known and novel axioms for scoring procedures, and analyse the performance of chosen methods with respect to them. The results of Chebotarev (1994) and González-Díaz et al. (2014) are supplemented, moreover, one of them is proved to be false. The connection of certain properties will also be revealed. Chapter 6 gives a characterization of the score on the set of round-robin ranking problems, by using the main theorem of Bouyssou (1992). It is presented that the axiomatization is not valid on the extended sets of balanced or unweighted ranking problems. Finally, we revisit the relationship with the score method through other properties.

In Chapter 7, an application of paired comparison-based scoring procedures for Swiss-system chess team tournaments is investigated. The competition can be represented as a ranking problem, and two examples are analysed with the proposed methods. We argue for the use of least squares method with a generalised result matrix favouring match points. Chapter 8 presents the fields where the use of these scoring procedures has been considered. Some recommendations are formulated on the basis of our knowledge and experiences.

Own results are detailed in short summaries after each topic and in Chapter 9.

Chapter 2

A model of paired comparison ranking

In order to solve problems similar to those discussed in Chapter 1, we need to formalize a model of paired comparisons and set the task.

2.1 The ranking problem

Notation 2.1. $\mathbf{0} \in \mathbb{R}^n$ is the zero vector.
$\mathbf{e} \in \mathbb{R}^n$ is the unit column vector.
$O \in \mathbb{R}^{n \times n}$ is the zero matrix.
$I \in \mathbb{R}^{n \times n}$ is the unit matrix with 1-s in the diagonal, and 0-s off the diagonal.
$J \in \mathbb{R}^{n \times n}$ is the matrix whose all elements are 1.

Definition 2.1. *Set of objects*: $N = \{X_1, X_2, \dots, X_n\}$, $n \in \mathbb{N}$ is the *set of objects.*

Definition 2.2. *Individual paired comparison matrix*: $R^{(p)} = \left(r_{ij}^{(p)}\right) \in \mathbb{R}_+^{n \times n}$ is a partially defined *individual paired comparison matrix* such that

- $r_{ij}^{(p)}$ and $r_{ji}^{(p)}$ are unknown, or $r_{ij}^{(p)} + r_{ji}^{(p)} = 1$ for all $i \neq j$;
- $r_{ii}^{(p)} = 0$ for all $i = 1, 2, \dots, n$.

$r_{ij}^{(p)}$ represents the outcome of paired comparison between objects X_i and X_j by decision maker p.

Definition 2.3. *Preference profile*: The pair $(N, \mathbf{R})$ is a *preference profile*, where N is the set of objects, and $\mathbf{R} = \left(R^{(1)}, R^{(2)}, \dots, R^{(m)}\right)$, $m \in \mathbb{N}$ is an $m \times n \times n$ array such that $R^{(p)}$ is an individual paired comparison matrix for all $p = 1, 2, \dots, m$.

Notation 2.2. The set of preference profiles with n objects is denoted by $\mathcal{R}_*^n$.

Notation 2.3. $\chi_{(N,\mathbf{R})} : \{1; 2; \ldots ; m\} \times N \times N \to \{0; 1\}$ is the indicator function of known elements in the preference profile $(N, \mathbf{R})$:

$$\chi_{(N,\mathbf{R})}(p, X_i, X_j) = \begin{cases} 1 & \text{if } X_i \neq X_j \text{ and } r_{ij}^{(p)} \text{ is known} \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.4. *Aggregated paired comparison matrix*: The *aggregated paired comparison matrix* $R = (r_{ij}) \in \mathbb{R}_+^{n \times n}$ corresponds to preference profile $(N, \mathbf{R})$, where

$$r_{ij} = \begin{cases} 0 & \text{if } \sum_{p=1}^{m} \chi_{(N,\mathbf{R})}(X_i, X_j, p) = 0 \\ \sum_{p=1,\, \chi_{(N,\mathbf{R})}(p,X_i,X_j)=1}^{m} r_{ij}^{(p)} & \text{otherwise.} \end{cases}$$

Definition 2.5. *Ranking problem*: The pair (N, R) is the *ranking problem* corresponding to preference profile $(N, \mathbf{R})$ such that R is the aggregated paired comparison matrix.

Notation 2.4. The set of ranking problems is denoted by $\mathcal{R}$. The set of ranking problems defined over set of objects N (i.e. those with n objects) is denoted by $\mathcal{R}^n$.

Our definitions follow the models of Chebotarev and Shamis (1998) and González-Díaz et al. (2014). Now another representation is given, which is not as parsimonious but useful.

Definition 2.6. *Results matrix*: The *results matrix* $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ corresponds to preference profile $(N, \mathbf{R})$, where $a_{ij} = r_{ij} - r_{ji}$ for all $X_i \neq X_j$ and $a_{ii} = 0$ for all $X_i \in N$.

Results matrix is skew-symmetric ($A^\top = -A$). It is identical to a the pairwise comparison matrix of the Analytic Hierarchy Process (AHP) (Saaty, 1980) by taking the logarithm of the latter's elements (Csató, 2012b).

Definition 2.7. *Matches matrix*: The *matches matrix* $M = (m_{ij}) \in \mathbb{R}^{n \times n}$ corresponds to preference profile $(N, \mathbf{R})$, where $m_{ij} = r_{ij} + r_{ji}$ for all $X_i, X_j \in N$.

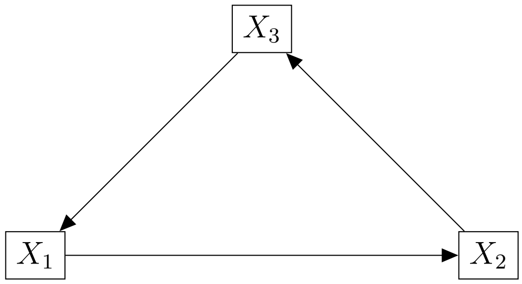
Matches matrix is symmetric ($M^\top = M$).

Remark 2.1. $a_{ij} \in [-m_{ij}, m_{ij}]$ for all $X_i, X_j \in N$, that is, $m_{ij} = 0$ implies $a_{ij} = 0$. Moreover, $m_{ij} = \sum_{p=1}^{m} \chi_{(N,\mathbf{R})}(X_i, X_j, p)$ for all $X_i, X_j \in N$, hence $M \in \mathbb{N}^{n \times n}$.

Definition 2.8. *Ranking problem*: The triple (N, A, M) is the *ranking problem* corresponding to preference profile $(N, \mathbf{R})$ such that A is the results and M is the matches matrix.

Ranking problem $(N, A, M) \in \mathcal{R}^n$ can be represented by graphs such that the nodes are the objects, k times $(X_i, X_j) \in N \times N$ undirected edge means $a_{ij} (= a_{ji}) = 0$, $m_{ij} = k$, and k times $(X_i, X_j) \in N \times N$ directed edge means $a_{ij} = k$ $(a_{ji} = -k)$, $m_{ij} = k$.

Figure 2.1: Ranking problem of Example 2.1



Example 2.1. Condorcet paradox (Condorcet, 1785) is given by the ranking problem in Figure 2.1 with the set of objects $N = \{X_1, X_2, X_3\}$.

Definition 2.9. Special ranking problems: A ranking problem $(N, A, M) \in \mathcal{R}^n$ is

- *balanced* if $\sum_{X_k \in N} m_{ik} = \sum_{X_k \in N} m_{jk}$ for all $X_i, X_j \in N$;
- *round-robin* if $m_{ij} = m_{k\ell}$ for all $X_i \neq X_j$ and $X_k \neq X_\ell$;
- *unweighted* if $m_{ij} \in \{0; 1\}$ for all $X_i, X_j \in N$;
- *tournament* if $m_{ij} = 1$ and $a_{ij} \in \{-m_{ij}; m_{ij}\}$ for all $X_i \neq X_j$.

Notation 2.5. The set of balanced ranking problems is denoted by $\mathcal{R}_B$.
The set of round-robin ranking problems is denoted by $\mathcal{R}_R$.
The set of unweighted ranking problems is denoted by $\mathcal{R}_U$.
The set of tournaments is denoted by $\mathcal{R}_T$.

Remark 2.2. $\mathcal{R}_R \subset \mathcal{R}_B \subset \mathcal{R}$ and $\mathcal{R}_T \subset (\mathcal{R}_R \cap \mathcal{R}_U) \subset \mathcal{R}_B \subset \mathcal{R}$.

Definition 2.10. Special matches matrices (Brozos-Vázquez et al., 2008): Matches matrix M is called *block diagonal* and *block anti-diagonal*, respectively, if the set of objects N has a partition $N_1 \cup N_2 = N$, $|N_1| = n_1$ and $|N_2| = n_2$ such that with a possible reordering of the objects,

$$M = \begin{pmatrix} M^1_{n_1 \times n_1} & O_{n_1 \times n_2} \\ O_{n_2 \times n_1} & M^2_{n_2 \times n_2} \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} O_{n_1 \times n_1} & M^1_{n_1 \times n_2} \\ M^2_{n_2 \times n_1} & O_{n_2 \times n_2} \end{pmatrix},$$

respectively, where the subscripts denote the dimensions of (sub)matrices.

Definition 2.11. *Multiset* (multiset) (Chebotarev and Shamis, 1998): In a *multiset*, as distinct from sets, multiple occurrence of elements is allowed.

Notation 2.6. $\uplus$ denotes the union of sets such that multiple occurrence of elements is preserved.

Definition 2.12. *Opponent set and multiset*: Let $(N, A, M) \in \mathcal{R}^n$ be a ranking problem. Then

- the *pth opponent set* of object $X_i \in N$ is the set of elements compared with it in the pth round: $O_i^p = \{X_j : m_{ij} = 1\}$ for all $X_i \in N$ and $p = 1, 2, \dots, m$;
- the *opponent multiset* of object $X_i \in N$ is the multiset of other elements, which contains the replications of objects equal to the number of comparisons between them: $O_i = \{X_j : \sharp X_j = m_{ij}\}$ for all $X_i \in N$.

Objects of the opponent multiset O_i are called *opponents* of X_i.

Notation 2.7. $\{X_j\}^{m_{ij}} \subseteq O_i$ denotes that object X_j has m_{ij} replications in the opponent multiset of X_i.

Corollary 2.1. *$O_i = \uplus_{p=1}^{m} O_i^p$ or all $X_i \in N$.*

Definition 2.13. *Number of comparisons*: Let $(N, A, M) \in \mathcal{R}^n$ be a ranking problem. Then

- the *number of comparisons* of object X_i is $d_i = \sum_{X_j \in N} m_{ij} = |O_i|$ for all $X_i \in N$;
- the *maximal number of comparisons* is $\mathfrak{d} = \max\{d_i : X_i \in N\}$.

2.2 Graph representation of the matches matrix

Basic graph-theoretic concepts used can be found in Mohar (1991) and Csató (2014a, Section 2: Notations and rating methods). Here we mention only a few.

Notation 2.8. The *adjacency matrix* of multigraph $G = (N, E)$ is denoted by $T = (t_{ij}) \in \mathbb{R}^{n \times n}$.
The *Laplacian matrix* of multigraph $G = (N, E)$ is denoted by $L = (\ell_{ij}) \in \mathbb{R}^{n \times n}$.

Lemma 2.1. *The Laplacian matrix of multigraph $G = (N, E)$ is symmetric, and its eigenvalues $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n = 0$ are real, so it is positive semidefinite. The eigenvector corresponding to $\mu_n = 0$ is $\mathbf{e}$.*

Definition 2.14. *Comparison multigraph*: Undirected multigraph $G := (N, E)$ is the *comparison multigraph* of the ranking problem $(N, A, M) \in \mathcal{R}^n$ such that the number of $(X_i, X_j) \in E$ edges is m_{ij}.

Definition 2.15. *Connected ranking problem*: A ranking problem $(N, A, M) \in \mathcal{R}^n$ is *connected* if the corresponding comparison multigraph $G = (N, E)$ is connected.

Notation 2.9. The set of connected ranking problems is denoted by $\mathcal{R}_O$.

Lemma 2.2. *Let $(N, A, M) \in \mathcal{R}^n$ be a ranking problem. The matches matrix M is block diagonal if and only if the comparison multigraph G is connected.*

2.3 Some approaches for ranking

Notation 2.10. $X_i \succeq_{(N,A,M)} X_j$ means that X_i is at least as good as X_j in the ranking problem $(N, A, M) \in \mathcal{R}^n$. It defines the following binary relations:

- $X_i \succ_{(N,A,M)} X_j$ if $X_i \succeq_{(N,A,M)} X_j$ and $X_j \not\succeq_{(N,A,M)} X_i$;
- $X_i \sim_{(N,A,M)} X_j$ if $X_i \succeq_{(N,A,M)} X_j$ and $X_j \succeq_{(N,A,M)} X_i$;
- $X_i \perp_{(N,A,M)} X_j$ if $X_i \not\succeq_{(N,A,M)} X_j$ and $X_j \not\succeq_{(N,A,M)} X_i$.

Definition 2.16. *Ranking*: The reflexive and transitive (but not necessarily total) binary relation $\succeq_{(N,A,M)}$ on the object set N is a *ranking*.

Notation 2.11. The set of rankings with n objects is denoted by $\mathcal{P}^n$.

Definition 2.17. *Linear order*: The irreflexive, transitive, and total binary relation $\succ_{(N,A,M)}$ on the object set N is a *linear order*.

Notation 2.12. The set of linear orders with n objects is denoted by $\mathcal{L}^n$.

Definition 2.18. *Ranking method*: A function $\mathcal{R}^n_* \to \mathcal{P}^n$ is a *ranking method*.

Definition 2.19. *General scoring procedure*: A function $f : \mathcal{R}^n_* \to \mathbb{R}^n$ is a *general scoring procedure*.

Remark 2.3. Every general scoring procedure $f : \mathcal{R}^* \to \mathbb{R}^n$ defines a ranking method $\succeq^f : \mathcal{R} \to \mathcal{P}^n$ such that $f_i(N, A, M) \geq f_j(N, A, M) \Rightarrow X_i \succeq^f_{(N,A,M)} X_j$. This ranking is well-defined and total.

Definition 2.20. *Scoring procedure* (scoring method): A function $f : \mathcal{R}^n \to \mathbb{R}^n$ is a *scoring procedure*.

Definition 2.21. *Proportionality*: Scoring procedures $f^1, f^2 : \mathcal{R}^n \to \mathbb{R}^n$ are called *proportional* if there exists a constant $\kappa > 0$ such that $f^1(N, A, M) = \kappa f^2(N, A, M)$ for all $(N, A, M) \in \mathcal{R}^n$.

Notation 2.13. The proportionality of scoring procedures $f^1, f^2 : \mathcal{R}^n \to \mathbb{R}^n$ is denoted by $f^1 \propto f^2$.

Lemma 2.3. *Proportional scoring procedures define the same ranking.*

Definition 2.22. *Equivalence*: Scoring procedures $f^1, f^2 : \mathcal{R}^n \to \mathbb{R}^n$ are called *equivalent* if $f^1_i(N, A, M) \geq f^1_j(N, A, M) \Leftrightarrow f^2_i(N, A, M) \geq f^2_j(N, A, M)$ for all $X_i, X_j \in N$ and $(N, A, M) \in \mathcal{R}^n$.

Notation 2.14. The equivalence of scoring procedures $f^1, f^2 : \mathcal{R}^n \to \mathbb{R}^n$ is denoted by $f^1 \approx f^2$.

Lemma 2.4. *Proportional scoring procedures are equivalent, but equivalence does not imply proportionality.*

The second approach is based on the approximation of the (generalized) tournament by linear orders (Kemeny, 1959; Slater, 1961).

Definition 2.23. *Optimal linear order*: The solution $L \in \mathcal{L}^n$ of

$$\min_{L \in \mathcal{L}^n} \sum_{X_i, X_j \in N} (r_{ji} : X_i \prec X_j) = \min_{L \in \mathcal{L}^n} \sum_{X_i, X_j \in N} (0,5 a_{ji} + 0,5 m_{ji} : X_i \prec X_j)$$

is an *optimal linear order* of the ranking problem $(N, A, M) \in \mathcal{R}^n$.

It usually leads to interesting combinatorial and algorithmic problems (Ali et al., 1986; Coleman, 2005; Hudry, 2009), however, from a theoretical viewpoint these methods have two disadvantages: the possible occurrence of multiple optimal solutions and the difficulties arising in the examination of their (normative) properties (Bouyssou, 2004; Pasteur, 2010). Consequently, the thesis will follow the former approach, we will discuss scoring procedures since Chebotarev and Shamis (1999) prove that only the methods using the aggregated paired comparison matrix R can satisfy self-consistent monotonicity.

2.4 An axiomatic review of scoring procedures

See Csató (2014a, Section 1: Introduction).

2.5 Summary

In this chapter we have presented our framework for paired comparison ranking on the basis of Chebotarev and Shamis (1998) and González-Díaz et al. (2014). The concepts of results and matches matrices were introduced in order to connect the model to Saaty's pairwise comparison matrix and represent the structure of comparisons with graphs. Finally, we have reviewed two different approaches of ranking, the approximation of paired comparisons by linear orders and the use of scoring procedures. According to Bouyssou (2004), the latter method have been chosen. We have given an overview about the methods proposed in the literature, focusing on axiomatic results in Subsection 2.4.

Chapter 3

Scoring procedures

In this chapter we present some scoring procedures and discuss them.

3.1 Some scoring procedures

Definition 3.1. *Fixed-order method* (Slutzki and Volij, 2005): $\mathbf{fo} : \mathcal{R}^n \to \mathbb{R}^n$ such that $fo_i(N, A, M) = i$ for all $X_i \in N$ and $(N, A, M) \in \mathcal{R}^n$.

Definition 3.2. *Flat method* (Slutzki and Volij, 2005): $\mathbf{fl} : \mathcal{R}^n \to \mathbb{R}^n$ such that $fl_i(N, A, M) = 0$ for all $X_i \in N$ and $(N, A, M) \in \mathcal{R}^n$.

Definition 3.3. *Score method* (Borda, 1781; Copeland, 1951): $\mathbf{s} : \mathcal{R}^n \to \mathbb{R}^n$ such that $\mathbf{s}(N, A, M) = A\mathbf{e}$ for all $(N, A, M) \in \mathcal{R}^n$.

Definition 3.4. *Generalised row sum method, GRS* (Chebotarev, 1989): $\mathbf{x}(\varepsilon) : \mathcal{R}^n \to \mathbb{R}^n$ such that $(I + \varepsilon L)\mathbf{x}(\varepsilon)(N, A, M) = (1 + \varepsilon mn)\mathbf{s}$ for all $(N, A, M) \in \mathcal{R}^n$, where $\varepsilon > 0$ is a parameter.

Lemma 3.1. *The score and generalised row methods are proportional if $\varepsilon \to 0$, that is, $\lim_{\varepsilon \to 0} \mathbf{x}(\varepsilon) \propto \mathbf{s}$, moreover, $\lim_{\varepsilon \to 0} \mathbf{x}(\varepsilon) = \mathbf{s}$.*

Definition 3.5. *Reasonable choice of ε* (Chebotarev, 1994, Proposition 5.1): Let $(N, A, M) \in \mathcal{R}^n$ be a ranking problem. Reasonableness for the choice of ε of generalised row sum method amounts to satisfying the constraint

$$0 < \varepsilon \leq \frac{1}{m(n-2)}.$$

Reasonable upper bound of ε is $\bar{\varepsilon} = 1/\left[m(n-2)\right]$.

Proposition 3.1. *For the generalised row sum method with a reasonable choice of ε, $-m(n-1) \leq x_i(\varepsilon) \leq m(n-1)$ for all $X_i \in N$.*

Proof. See Chebotarev (1994, Property 13). □

Proposition 3.2. *Both the score and the generalized row sum methods are well-defined for all* $(N, A, M) \in \mathcal{R}^n$.

Proof. See Chebotarev (1994, Property 1). □

For comments and other details, see Csató (2014a, Section 2: Notations and rating methods).

3.2 About the least squares method

See Csató (2014a, Section 3: The least squares method and its solution).

3.3 Existence of a well-defined solution

Regarding the least squares method, see Csató (2014a, Section 3: The least squares method and its solution).

Three main conditions are known for the existence of a unique solution of scoring procedures:

1. Irreducibility of R. It is a necessary and sufficient condition for maximum likelihood (Zermelo, 1929; Bradley and Terry, 1952), invariant, and fair bets methods (Daniels, 1969; Moon and Pullman, 1970).

2. Irreducibility of M. It is a necessary and sufficient condition for least squares (Csató, 2014a, Proposition 1). It is a weaker requirement than the former.

3. There is no need for restrictions on $\mathcal{R}$. It is the case in score, generalised row sum, the extension of maximum likelihood (Conner and Grant, 2000), or PageRank (Brin and Page, 1998).

Methods defined on irreducible problems are inapplicable to 'decomposable' preferences, where N can be ordered such that the objects of a league with a lower number are never preferred to those of a higher league. This constraint is not critical in the case of round-robin ranking problems, because the leagues have a natural order, and they may be handled separately. In fact, still Zermelo (1929) discussed this extension to a larger class (a recent version can be found in Slutzki and Volij (2005)). However, this is not true for problems with unknown or multiple comparisons: in Csató (2014a, Example 1), there exists no natural order between the leagues $\{X_1\}$ and $\{X_2\}$, but they are connected, the matches matrix is irreducible. According to Chebotarev (1994), it is an important argument for the use of generalised row sum (and therefore, least squares) method.

3.4 Summary

In Chapter 3, some scoring procedures were discussed. Two methods, fixed-order and flat are used for technical purposes only, they help in understanding the following properties. The thesis focuses on the score, generalised row sum, and least squares scoring procedures. The second is not a single method, but a parametric family of methods. Choice of ε is not a trivial question, although Chebotarev (1994) gives some idea about it. Nevertheless, according to González-Díaz et al. (2014, Example 2.1), in certain cases the reasonable upper bound is not high enough to allow for an appropriate role of opponents. Csató (2014a, Example 1) also reveals that this restriction can be contrasted with the existence of linear order.

It suggests that the least squares method is worth to analyse. We have taken this step in Subsection 3.2. After that, the issue of domain have been discussed, which offered arguments for the scoring procedures discussed as methods requiring the irreducibility of aggregated paired comparison matrix R are unable to give a ranking in some relevant cases.

Chapter 4

A graph interpretation of the least squares method

4.1 An alternative calculation

See Csató (2014a, Section 3: The least squares method and its solution).

4.2 The iterative decomposition

See Csató (2014a, Section 4: The iterative calculation of the least squares rating).

4.3 An extension of the graph interpretation

The decomposition of the least squares ranking makes possible to extend it and define a family of rating methods similarly to the generalized row sum method, where the least squares correspond to a special case of the family. The graph interpretation shows that the main idea of the least squares method is that the paths (represented by the powers of matrix C) between the objects play a correcting role in order to address the different schedules, by taking the scores of opponents into account. It is a well-known assumption in the literature to take exponentially decreasing weights with the length of paths, which is a relevant consideration since the very long paths have probably slight significance (see, for example, Katz (1953)). In our case the possible presence of loops yields inherently greater weights for shorter paths, nevertheless, it seems to be a valid generalization.

Definition 4.1. *Generalised Buchholz method*: $\mathbf{w}(\delta) : \mathcal{R}^n \to \mathbb{R}^n$ such that

$$\mathbf{w}(\delta) = \frac{1}{\mathfrak{d}} \sum_{k=0}^{\infty} \left(\frac{1}{\delta} C \right)^k \mathbf{s} = \frac{1}{\mathfrak{d}} \left[\mathbf{s} + \frac{1}{\delta} C \mathbf{s} + \left(\frac{1}{\delta} C \right)^2 \mathbf{s} + \left(\frac{1}{\delta} C \right)^3 \mathbf{s} + \dots \right]$$

for all $(N, A, M) \in \mathcal{R}^n$.

Proposition 4.1. *Generalised Buchholz method is well-defined for all* $(N, A, M) \in \mathcal{R}^n$.

Proposition 4.2. *The rating vector* $\mathbf{w}(\delta)$ *of generalised Buchholz method is the unique solution of the equation system*

$$\mathfrak{d}\left(\frac{\delta-\mathfrak{d}}{\delta}I+\frac{1}{\delta}L\right)\mathbf{w}(\delta)=\mathbf{s}.$$

Remark 4.1. The limit of generalised Buchholz method is proportional to the least squares method if $\delta \to \mathfrak{d}$, moreover, $\lim_{\delta \to \mathfrak{d}} \mathbf{w}(\delta) = \mathbf{q}$. The limit of generalised Buchholz is proportional to the score method if $\delta \to \infty$, moreover, $\lim_{\delta \to \infty} \mathbf{w}(\delta) = (1/\mathfrak{d})\mathbf{s}$.

Theorem 4.1. *Let* $(N, A, M) \in \mathcal{R}^n$ *be a ranking problem. Generalised Buchholz and generalised row sum methods are proportional, vectors* $\mathbf{w}(\delta)$ *and* $\mathbf{x}(\varepsilon)$ *can be obtained from each other by scalar multiplication with an appropriate choice of parameters:*

$$\mathbf{x}(\varepsilon)=\left(1+\frac{mn}{\delta-\mathfrak{d}}\right)\frac{\delta}{\mathfrak{d}(\delta-\mathfrak{d})}\mathbf{w}(\delta) \qquad \text{if } \varepsilon = 1/\left(\delta-\mathfrak{d}\right), \text{ and}$$

$$\mathbf{w}(\delta)=\frac{1}{1+\varepsilon mn}\frac{\mathfrak{d}}{1+\varepsilon\mathfrak{d}}\mathbf{x}(\varepsilon) \qquad \text{if } \delta = 1/\varepsilon+\mathfrak{d}, \text{ respectively.}$$

Remark 4.2. According to Definition 3.5 the reasonable choice of generalised row sum's parameter is $0 < \varepsilon \leq 1/\left[m(n-2)\right]$. For the generalised Buchholz method it corresponds to $\delta \geq m(n-2)+\mathfrak{d}$. The condition $\mathfrak{d} \leq m(n-1)$ for all $(N, A, M) \in \mathcal{R}^n$ means that in the subsequent steps of the iteration score should be corrected not by the average score of the opponents, but by at most half of it since usually $\delta \geq 2\mathfrak{d}$. If $\mathfrak{d}$ is significantly lower than the theoretical maximum $m(n-1)$, the influence of opponents should be small to remain in the reasonable interval. For instance, in Csató (2014a, Example 1), $m = 1$, $\mathfrak{d} = 3$, and $n = 7$, hence $\delta \geq 8$. The difference is significant when $\mathfrak{d} \ll m(n-2)$.

We emphasize that from the viewpoint of their origin, generalized row sum was obtained as the set of ranking methods satisfying certain conditions on how the pairwise results or the players are to be aggregated (Chebotarev, 1994), whereas the generalized Buchholz was obtained from the graph interpretation of the least squares method. The use of comparison multigraph seems to be more intuitive than the generalized row sum, and the nice graph interpretation of generalized Buchholz makes it acceptable for a wide audience.

4.4 Summary

In Chapter 4, a graph interpretation was given for generalised row sum and least squares methods on the basis of the balanced comparison multigraph. Regarding the latter, decomposition does not work if the comparison multigraph is regular bipartite. The rating vector can be obtained as the limit of an iteration process using the score of objects and the discount factor $1/\delta \leq 1/\mathfrak{d}$. Here $\mathfrak{d} = \max\{d_i : i = 1, 2, \dots, n\}$, the maximal number of comparisons is endogenously given by matches matrix M.

They are entirely new results founded on elementary linear algebra and graph theory. It has a strong connection to the suggestions of existing literature like Brozos-Vázquez et al. (2008) or Herings et al. (2005). We hope the presented graph interpretation will give incentives for practical applications since it immediately shows the calculation from the scores as well as the main idea behind the methods.

Chapter 5

Some properties of scoring procedures

Bouyssou (2004, p. 270) states that *there is a real need for a thorough study of ranking procedures*. In this chapter, similarly to González-Díaz et al. (2014), we give an axiomatic overview of scoring procedures. The focus will be on the score, generalised row sum and least squares methods. Chebotarev (1994) has analysed generalised row sum, while González-Díaz et al. (2014) have examined the score and least squares methods as well as generalised row sum with a fixed $\varepsilon = 1/\left[m(n-2)\right]$. Therefore, some results are given in these papers, but the discussion of new axioms and some complementary comments are our own contribution. Part of the results can be found in Csató (2014b), where fair bets and some derivatives of it are also analysed.

5.1 Insensitivity of the ranking

Definition 5.1. *Anonymity* (*ANO*) (Young, 1974): Let $(N, A, M) \in \mathcal{R}^n$ be a ranking problem and $\sigma : \{1; 2; \dots; m\} \to \{1; 2; \dots; m\}$ be a permutation on the set of voters. Let σA be the results matrix obtained from A by the permutation. Generalised scoring procedure $f : \mathcal{R}^n_* \to \mathbb{R}^n$ is *anonymous* if $f_i(N, A, M) \geq f_j(N, A, M) \Leftrightarrow f_i(N, \sigma A, M) \geq f_j(N, \sigma A, M)$ for all $X_i, X_j \in N$.

Lemma 5.1. *A scoring procedure $f : \mathcal{R}^n \to \mathbb{R}^n$ is anonymous.*

Lemma 5.2. *The score, generalised row sum and least squares methods satisfy ANO.*

Definition 5.2. *Neutrality* (*NEU*) (Young, 1974): Let $(N, A, M) \in \mathcal{R}^n$ be a ranking problem and $\sigma : N \to N$ be a permutation on the set of objects. Let $\sigma(N, A, M) \in$

$\mathcal{R}^n$ be the ranking problem obtained from (N, A, M) by permutation σ. Scoring procedure $f : \mathcal{R}^n \to \mathbb{R}^n$ is *neutral* if for all $X_i, X_j \in N$: $f_i(N, A, M) \geq f_j(N, A, M) \Leftrightarrow f_{\sigma i}[\sigma(N, A, M)] \geq f_{\sigma j}[\sigma(N, A, M)]$.

This property is mentioned as *anonymity* in González-Díaz et al. (2014).

Remark 5.1. Let $f : \mathcal{R}^n \to \mathbb{R}^n$ be a neutral scoring procedure. If for objects $X_i, X_j \in N$ $a_{ij} = 0$, and $a_{ik} = a_{jk}$, $m_{ik} = m_{jk}$ for all $X_k \in N$, then $f_i(N, A, M) = f_j(N, A, M)$.

Lemma 5.3. *The score, generalised row sum and least squares methods satisfy* NEU.

Definition 5.3. *Centering* (CNT) (Chebotarev, 1994): Let $(N, A, M) \in \mathcal{R}^n$ be a ranking problem. Scoring procedure $f : \mathcal{R}^n \to \mathbb{R}^n$ has the property *centering* if $\sum_{X_i \in N} f_i(N, A, M) = 0$.

Lemma 5.4. *The score, generalised row sum and least squares methods satisfy* CNT.

Definition 5.4. *Linear relation with comparison results* ($LRCR$) (Chebotarev, 1994): Let $(N, A, M), (N, A', M), (N, A'', M) \in \mathcal{R}^n$ be three ranking problems with the same N set of objects and M matches matrix such that $\alpha, \beta \in \mathbb{R}$ and $\mathbf{s}(N, A'', M) = \alpha \mathbf{s}(N, A, M) + \beta \mathbf{s}(N, A', M)$. Scoring procedure $f : \mathcal{R}^n \to \mathbb{R}^n$ is in *linear relation with comparison results* if $f(N, A'', M) = \alpha f(N, A, M) + \beta f(N, A', M)$.

Proposition 5.1. *The score, generalised row sum and least squares methods satisfy* $LRCR$.

Neutrality is discussed by Csató (2014b), too.

5.2 Multiplicative transformations

Homogeneity (HOM) is analysed in Csató (2014b, Section 3: Multiplicative properties).

Definition 5.5. *Admissible transformation of the results*: Let $(N, A, M) \in \mathcal{R}^n$ be a ranking problem. An *admissible transformation of the results* provides a ranking problem $(N, kA, M) \in \mathcal{R}^n$ such that $k > 0$, $k \in \mathbb{R}$ and $ka_{ij} \in [-m_{ij}, m_{ij}]$ for all $X_i, X_j \in N$.

Definition 5.6. *Scale invariance* (SI): Let $(N, A, M), (N, kA, M) \in \mathcal{R}^n$ be two ranking problems such that (N, kA, M) is obtained from (N, A, M) through an admissible transformation of the results. Scoring procedure $f : \mathcal{R}^n \to \mathbb{R}^n$ is *scale invariant* if $f_i(N, A, M) \geq f_j(N, A, M) \Leftrightarrow f_i(N, kA, M) \geq f_j(N, kA, M)$ for all $X_i, X_j \in N$.

Corollary 5.1. *$LRCR$ implies SI.*

SI is further discussed in Csató (2014b, Section 3: Multiplicative properties).

5.3 Additive transformations

See the analysis of properties consistency (CS), flatness preservation (FP), and result consistency (RCS) in (Csató, 2014b, Section 4: Additive properties).

5.4 The connection of results matrix and ranking

Definition 5.7. *Symmetry* (SYM) (González-Díaz et al., 2014): Let $(N, A, M) \in \mathcal{R}^n$ be a ranking problem such that $A = O$. Scoring procedure $f : \mathcal{R}^n \to \mathbb{R}^n$ is *symmetric* if $f_i(N, A, M) = f_j(N, A, M)$ for all $X_i, X_j \in N$.

Lemma 5.5. *The score, generalised row sum and least squares methods satisfy SYM.*

Definition 5.8. *Inversion* (INV) (Chebotarev and Shamis, 1998): Let $(N, A, M) \in \mathcal{R}^n$ be a ranking problem. Scoring procedure $f : \mathcal{R}^n \to \mathbb{R}^n$ is *invertible* if for all $X_i, X_j \in N$, $f_i(N, A, M) \geq f_j(N, A, M) \Leftrightarrow f_i(N, -A, M) \leq f_j(N, -A, M)$.

Remark 5.2. Let $f : \mathcal{R}^n \to \mathbb{R}^n$ be a scoring procedure satisfying INV. Then $f_i(N, A, M) > f_j(N, A, M) \Leftrightarrow f_i(N, -A, M) < f_j(N, -A, M)$ for all $X_i, X_j \in N$.

Corollary 5.2. *INV implies SYM.*

Proposition 5.2. *RCS and SYM imply INV.*

Lemma 5.6. *The score method satisfies INV.*

Proposition 5.3. *$LRCR$ and SYM imply INV. Moreover, if scoring procedure f satisfies CNT, then $f_i(N, -A, M) = -f_i(N, A, M)$ for all $X_i \in N$ and $(N, A, M) \in \mathcal{R}^n$.*

Lemma 5.7. *The generalised row sum and least squares methods satisfy INV.*

A further discussion of SYM and INV can be found in Csató (2014b, Section 2: Structural invariance properties).

Definition 5.9. *Existence of a linear order of the objects*: Let $(N, A, M) \in \mathcal{R}^n$ be a ranking problem. There *exists a linear order of the objects* if

$$\min_{L \in \mathcal{L}^n} \sum_{X_i, X_j \in N} (r_{ji} : X_i \prec X_j) = \min_{L \in \mathcal{L}^n} \sum_{X_i, X_j \in N} (0.5\, a_{ji} + 0.5\, m_{ji} : X_i \prec X_j) = 0.$$

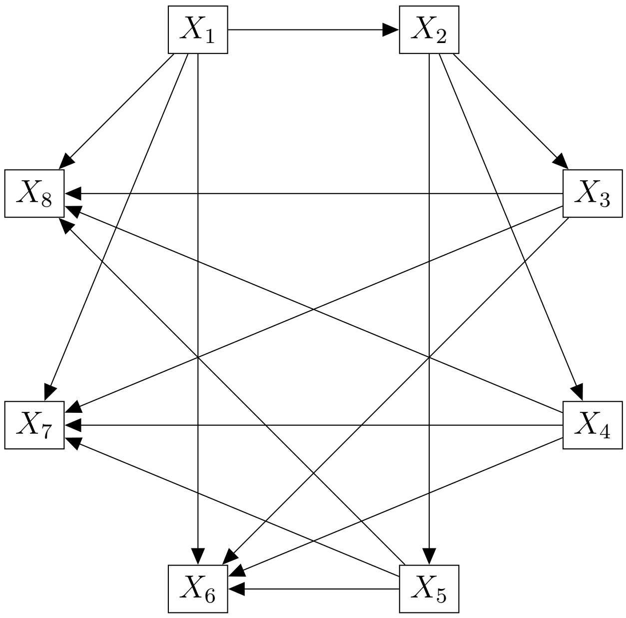
Definition 5.10. *Linear order preservation* (*LOP*): Let $(N, A, M) \in \mathcal{R}^n$ be a ranking problem with an L linear order of the objects. Scoring procedure $f : \mathcal{R}^n \to \mathbb{R}^n$ *preserves linear order* if $X_i \succ^L_{(N,A,M)} X_j \Rightarrow f_i(N, A, M) \geq f_j(N, A, M)$.

Proposition 5.4. *The score and generalised row sum methods violate LOP.*

Proof. Take Chebotarev (1994, Example 1), where LOP implies $X_3 \succeq X_5$ but $X_3 \prec^{\mathbf{s}}_{(N,A,M)} X_5$ and $X_3 \prec^{\mathbf{x}(1/5)}_{(N,A,M)} X_5$. □

Theorem 5.1. *(joint work with Lajos Rónyai) The least squares method violates LOP.*

Figure 5.1: Ranking problem of Example 5.1



Proof.

Example 5.1. Let $(N, A, M) \in \mathcal{R}^8_B \cap \mathcal{R}^8_U$ be the balanced and unweighted ranking problem in Figure 5.1. The conditions of linear orders are $X_1 \succ X_2$, $X_2 \succ X_i$, $i = 3, 4, 5$ and $X_i \succ X_j$, $i = 3, 4, 5$, $j = 6, 7, 8$.

Now the score method satisfies LOP but the least squares method gives

$$\mathbf{q}(N, A, M) = \begin{bmatrix} 11/16 & 13/16 & 3/16 & 3/16 & 3/16 & -11/16 & -11/16 & -11/16 \end{bmatrix}^\top,$$

where $X_1 \prec^{\mathbf{q}}_{(N,A,M)} X_2$, in contradiction with preservation of linear order. □

Conjecture 1. *The generalised row sum method violates LOP for all fixed ε-s.*

Conjecture 2. *Example 5.1 is minimal with respect to the number of objects.*

5.5 Irrelevant comparisons

Definition 5.11. *Independence of irrelevant matches* (IIM) (González-Díaz et al., 2014): Let $(N, A, M) \in \mathcal{R}^n$ be a ranking problem and $X_i, X_j, X_k, X_\ell \in N$ be four different objects. Let $f : \mathcal{R}^n \to \mathbb{R}^n$ be a scoring procedure such that $f_i(N, A, M) \geq f_j(N, A, M)$, and $(N, A', M') \in \mathcal{R}^n$ be a ranking problem identical to (N, A, M) except for the result $a'_{k\ell} \neq a_{k\ell}$ and match $m'_{k\ell} \in \mathbb{N}$. f is called *independent of irrelevant matches* if $f_i(N, A', M') \geq f_j(N, A', M')$.

Remark 5.3. Property IIM has a meaning if $n \geq 4$.

A somewhat weaker property is the following.

Definition 5.12. *Independence of irrelevant results* (IIR): Let $(N, A, M) \in \mathcal{R}^n$ be a ranking problem and $X_i, X_j, X_k, X_\ell \in N$ be four different objects. Let $f : \mathcal{R}^n \to \mathbb{R}^n$ be a scoring procedure such that $f_i(N, A, M) \geq f_j(N, A, M)$, and $(N, A', M) \in \mathcal{R}^n$ be a ranking problem identical to (N, A, M) except for the result $a'_{k\ell} \neq a_{k\ell}$. f is called *independent of irrelevant results* if $f_i(N, A', M) \geq f_j(N, A', M)$.

Remark 5.4. Property IIR has a meaning if $n \geq 4$.

Corollary 5.3. *IIM implies IIR.*

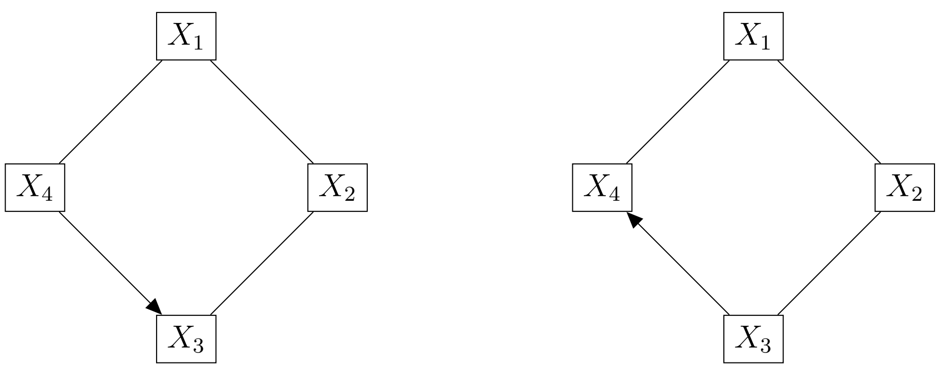
Proposition 5.5. *The score method satisfies IIM.*

Lemma 5.8. *The score method satisfies IIR.*

Proposition 5.6. *The generalised row sum and least squares methods violate IIR.*

Figure 5.2: Ranking problems of Example 5.2

(a) Ranking problem (N, A, M) (b) Ranking problem (N, A', M)



Proof.

Example 5.2. Let $(N, A, M), (N, A', M) \in \mathcal{R}^4_B \cap \mathcal{R}^4_U$ be the balanced and unweighted ranking problems in Figure 5.2.

Here

$$x_1(\varepsilon) = x_2(\varepsilon)' = (1+\varepsilon mn)\frac{\varepsilon}{(1+2\varepsilon)(1+4\varepsilon)} = \frac{\varepsilon}{1+2\varepsilon} \text{ and}$$

$$x_1(\varepsilon)' = x_2(\varepsilon) = (1+\varepsilon mn)\frac{-\varepsilon}{(1+2\varepsilon)(1+4\varepsilon)} = \frac{-\varepsilon}{1+2\varepsilon},$$

that is $X_1 \succ^{\mathbf{x}(\varepsilon)}_{(N,A,M)} X_2$, but $X_1 \prec^{\mathbf{x}(\varepsilon)}_{(N,A',M)} X_2$.

Regarding the least squares method, on the basis of Proposition Csató (2014a, Corollary 1):

$$q_1 = \frac{\lim_{\varepsilon\to\infty} x_1(\varepsilon)}{mn} = q_2' = \frac{\lim_{\varepsilon\to\infty} x_2(\varepsilon)'}{mn} = \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{8} \text{ and}$$

$$q_1' = \frac{\lim_{\varepsilon\to\infty} x_1(\varepsilon)'}{mn} = q_2 = \frac{\lim_{\varepsilon\to\infty} x_2(\varepsilon)}{mn} = -\frac{1}{2}\cdot\frac{1}{4} = -\frac{1}{8}.$$

Hence $X_1 \succ^{\mathbf{q}}_{(N,A,M)} X_2$, but $X_1 \prec^{\mathbf{q}}_{(N,A',M)} X_2$. □

Lemma 5.9. *The generalised row sum and least squares methods violate IIM.*

Theorem 5.2. *NEU, SYM and CS imply IIM.*

Proof. For the round-robin case, see Nitzan and Rubinstein (1981, Lemma 3).

Assume to the contrary: let $(N,A,M) \in \mathcal{R}^n$ be a ranking problem, $X_i, X_j, X_k, X_\ell \in N$ be four different objects such that $f_i(N,A,M) \geq f_j(N,A,M)$, $(N,A',M) \in \mathcal{R}^n$ be identical to (N,A,M) except for the result $r'_{k\ell} \neq r_{k\ell}$ but $f_i(N,A',M) < f_j(N,A',M)$.

Proposition 5.2 implies that a symmetric and consistent scoring procedure satisfies INV, hence $f_i(N,-A,M) \leq f_j(N,-A,M)$. Denote $\sigma: N \to N$ the permutation $\sigma(X_i) = X_j$, $\sigma(X_j) = X_i$, and $\sigma(X_k) = X_k$ for all $X_k \in N \setminus \{X_i, X_j\}$. By neutrality, $f_i[\sigma(N,A,M)] \leq f_j[\sigma(N,A,M)]$. $f_i[\sigma(N,-A',M)] < f_j[\sigma(N,-A',M)]$ due to INV and Remark 5.2. With the definition $A'' = \sigma(A) - \sigma(A') - A + A' = O$,

$$(N,A'',M'') = \sigma(N,A,M) + \sigma(N,-A',M) + (N,-A,M) + (N,A',M).$$

Symmetry implies $f_i(N,A'',M) = f_j(N,A'',M)$, whereas consistency results in $f_i(N,A'',M) < f_j(N,A'',M)$, a contradiction. □

Proposition 5.7. *NEU, SYM and RCS imply IIR.*

Proof. It is almost the same as the proof of Theorem 5.2. According to Proposition 5.2 a symmetric and result consistent scoring procedure also satisfies INV. However, $f_i(N,A'',M'') < f_j(N,A'',M'')$ holds only if $M = M'$. □

González-Díaz et al. (2014, p. 165) refers to IIM as a drawback of the score method. Since NEU and SYM are difficult to debate, CS is an axiom one would rather not have in the general case. It highlights the significance of Section 5.3 as weakening of consistency seems to be desirable in axiomatizations valid on the whole set of $\mathcal{R}$.

Definition 5.13. *Admissible transformation of draws*: Let $(N, A, M) \in \mathcal{R}^n$ be a ranking problem and $X_i, X_j \in N$ two objects. An *admissible transformation of draws* between X_i and X_j provides a ranking problem $(N, A, M') \in \mathcal{R}^n$ such that $m'_{ij} \geq |a_{ij}|$, and $m'_{k\ell} = m_{k\ell}$ for all $(X_k, X_\ell) \neq (X_i, X_j)$.

Definition 5.14. *Independence of draws* (ID): Let $(N, A, M), (N, A, M') \in \mathcal{R}^n$ be two ranking problems such that (N, A, M') is obtained from (N, A, M) through an admissible transformation of draws between X_i and X_j. Scoring procedure $f : \mathcal{R}^n \to \mathbb{R}^n$ is *independent of draws* if $f_i(N, A, M) > f_j(N, A, M) \Leftrightarrow f_i(N, A, M') > f_j(N, A, M')$.

Remark 5.5. Let $f : \mathcal{R}^n \to \mathbb{R}^n$ be a scoring procedure satisfying ID. Then for all $X_i, X_j \in N$, $f_i(N, A, M) = f_j(N, A, M) \Leftrightarrow f_i(N, A, M') = f_j(N, A, M')$.

Corollary 5.4. *$SIIM$ implies ID.*

Lemma 5.10. *The score method satisfies ID.*

Proposition 5.8. *The generalised row sum, and least squares methods satisfy ID.*

Proposition 5.9. *Let $(N, A, M), (N, A, M') \in \mathcal{R}^n$ be two ranking problems such that (N, A, M') is obtained from (N, A, M) through an admissible transformation of draws between X_i and X_j. Let $\Delta f_k = f_k(N, A, M') - f_j(N, A, M)$ for all $X_k \in N$. For the score, generalised row sum, and least squares methods:*

- *$f_i(N, A, M) = f_j(N, A, M)$ implies $f_k(N, A, M') = f_k(N, A, M)$ for all $X_k \in N$;*
- *$f_i(N, A, M) > f_j(N, A, M)$, $m_{ij} < m'_{ij}$ imply $f_i(N, A, M') > f_j(N, A, M')$ and $\Delta f_i \leq \Delta f_k \leq \Delta f_j$ for all $X_k \in N$;*
- *$f_i(N, A, M) > f_j(N, A, M)$, $m_{ij} > m'_{ij}$ imply $f_i(N, A, M') > f_j(N, A, M')$ and $\Delta f_i \geq \Delta f_k \geq \Delta f_j$ for all $X_k \in N$.*

Remark 5.6. Condition (2) of Chebotarev (1994, Property 14)'s axiom of *dynamic monotonicity* partly corresponds to ID as it implies Proposition 5.9 for generalised row sum with a reasonable parameter $0 \leq \varepsilon \leq 1/\left[m(n-2)\right]$.

Proposition 5.10. *An admissible transformation of the draws can influence the ranking of the objects in the case of the least squares method.*

Conjecture 3. *An admissible transformation of the draws can influence the ranking of the objects in the case of the generalised row sum method with all fixed ε-s.*

5.6 Summary

In this chapter we have reviewed a number of properties for scoring procedures, and have given an analysis similar to González-Díaz et al. (2014). However, we have restricted our focus to the score, generalise row sum, and least squares methods. We have proved that the generalised row sum does not satisfy HOM, which makes its use questionable. Property SI is a new one with some significance for practical applications. The additivity of methods were widely examined through axioms CS, FP, and RCS (the latter is our contribution), differentiating among the three procedures.

We have discussed the connection of known properties SYM and INV. Furthermore, LOP was defined in order to relate scoring procedures to the other approach, the approximation of paired comparison results with linear orders (Kemeny, 1959; Slater, 1961). In this respect, the findings are mostly negative. Finally, the role of irrelevant comparisons were analysed, and dynamic axiom of Chebotarev (1994, Property 14) was also extended. Since IIM is an unfavourable property in the presence of unknown and multiple comparisons (González-Díaz et al., 2014, p. 165), the use of score method may be debated. Theorem 5.2 reveals that with approving NEU and SYM (two natural properties), CS also becomes adverse in the general case.

Our main results are as follow:

1. On the basis of homogeneity and result consistency, generalised row sum method should be used by a variable parameter ε depending on the number of matches (Csató, 2014b, Proposition 4.1 and 4.2; Lemma 5.8, Proposition 5.5);[1]

2. Introduction of scale invariance (Definition 5.6), a property with great theoretical and practical significance (Csató, 2014b, Section 4: Multiplicative properties);

3. Generalised row sum break consistency (Csató, 2014b, Proposition 5.1);[2]

4. Definition of result consistency (Csató, 2014b, Definition 5.4) and its examination (Csató, 2014b, Proposition 5.4; Lemma 5.8, Proposition 5.5);

5. Connection of SYM and INV (Propositions 5.2 and 5.3);

6. Introduction of linear order preservation (Definition 5.10) and its examination (Proposition 5.4, Theorem 5.1, joint work with Lajos Rónyai);

[1] Although homogeneity was introduced by González-Díaz et al. (2014, p. 145), they do not mention this issue since the use of the reasonable upper bound $\bar{\varepsilon} = 1/\left[m(n-2)\right]$.

[2] Note that González-Díaz et al. (2014, Example 4.2) have shown the violation of a somewhat weaker property called *order preservation* in the case of $\varepsilon = 1/\left[m(n-2)\right]$.

7. Introduction of axiom IIR (Definition 5.12), connection of CS (RCS) and IIM (IIR) on the set $\mathcal{R}$ (Theorem 5.2, Proposition 5.7);

8. Introduction of independence of draws (Definition 5.14) and its examination (Proposition 5.8 and 5.9).

Publication of main results is in progress (Csató, 2014b).

Chapter 6

Connection with the score method

In the following, we give a characterization of the score method on a subset of ranking problems, and discuss its relation to other scoring procedures.

6.1 A characterization in the round-robin case

Definition 6.1. *Strong monotonicity* (SM) (Bouyssou, 1992): Let $(N, A, M) \in \mathcal{R}^n$ be a ranking problem. Let $f : \mathcal{R}^n \to \mathbb{R}^n$ be a scoring procedure such that $f_i(N, A, M) \geq f_j(N, A, M)$, and $(N, A', M) \in \mathcal{R}^n$ be a ranking problem identical to (N, A, M) except for the result $a'_{ik} > a_{ik}$. f is *strongly monotonic* if $f_i(N, A', M) \geq f_j(N, A', M)$.

Definition 6.2. *Close strong monotonicity* (CSM) (Rubinstein, 1980): Let $(N, A, M) \in \mathcal{R}^n$ be a ranking problem. Let $f : \mathcal{R}^n \to \mathbb{R}^n$ be a scoring procedure such that $f_i(N, A, M) \geq f_j(N, A, M)$, and $(N, A', M) \in \mathcal{R}^n$ be a ranking problem identical to (N, A, M) except for the result $a'_{ik} > a_{ik}$ between objects $X_i \in N$ and $X_k \in N \setminus \{X_i, X_j\}$. f is *closely strong monotonic* if $f_i(N, A', M) \geq f_j(N, A', M)$.

Corollary 6.1. *SM implies CSM.*

Lemma 6.1. *The score, and generalised row sum methods satisfy SM.*

Lemma 6.2. *The score, and generalised row sum methods satisfy CSM.*

Proposition 6.1. *The least squares method violates CSM.*

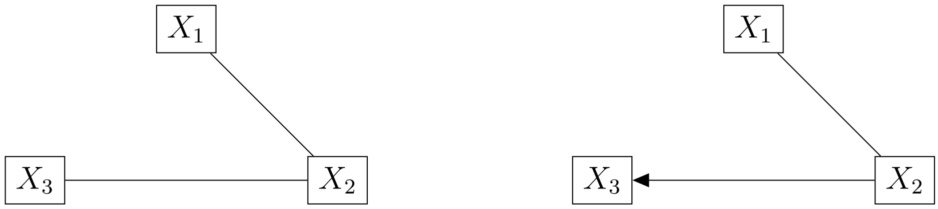
Proof.

Example 6.1. Let $(N, A, M), (N, A', M) \in \mathcal{R}^3_U$ be the ranking problems in Figure 6.1.

The least squares method results in $\mathbf{q}(N, A, M) = [0, 0, 0]^\top$ and $\mathbf{q}(N, A', M) = [1/3, 1/3, -2/3]^\top$, that is, $X_1 \sim^{\mathbf{q}}_{(N,A,M)} X_2$ and $X_1 \sim^{\mathbf{q}}_{(N,A',M)} X_2$, however, $1 = a'_{23} > a_{23} = 0$. □

Figure 6.1: Ranking problems of Example 6.1

(a) Ranking problem (N, A, M) (b) Ranking problem (N, A', M)



Lemma 6.3. *The least squares method violates SM.*

Remark 6.1. The axiom, called *monotonicity* by Chebotarev (1994, Property 11) contains much more information about the generalised row sum method than SM.

Definition 6.3. *Admissible transformation on an elementary circuit* (Bouyssou, 1992): Let $(N, A, M) \in \mathcal{R}^n$ be a ranking problem and $(X_i = X_{k_0}, X_{k_1}, \dots, X_{k_t} = X_i)$ an elementary circuit in the corresponding G comparison multigraph. An *admissible transformation on an elementary circuit* provides a ranking problem $(N, A', M) \in \mathcal{R}^n$ such that $a'_{ij} = a_{ij}$ for all $(X_i, X_j) \neq (X_{k_\ell} X_{k_{\ell+1}})$ and $a'_{k_\ell k_{\ell+1}} = a_{k_\ell k_{\ell+1}} + g$ for all $\ell = 0, 1, \dots, t-1$, where $g \in \mathbb{R}$ and $a_{k_\ell k_{\ell+1}} + g$, $a_{k_{\ell+1} k_\ell} - g \in \left[-m_{k_\ell k_{\ell+1}}, m_{k_\ell k_{\ell+1}} \right]$ for all $\ell = 0, 1, \dots, t-1$.

Definition 6.4. *Independence of circuits* (IC) (Bouyssou, 1992): Let (N, A, M), $(N, A', M) \in \mathcal{R}^n$ be two ranking problems such that (N, A', M) is obtained from (N, A, M) through an admissible transformation on an elementary circuit. Scoring procedure $f : \mathcal{R}^n \to \mathbb{R}^n$ is *independent of circuits* if $f_i(N, A, M) \geq f_j(N, A, M) \Leftrightarrow f_i(N, A', M) \geq f_j(N, A', M)$ for all $X_i, X_j \in N$.

Remark 6.2. IC means that the ranking is not sensitive to the direction of cyclic triads. For example, if $m_{ij} = m_{ik} = m_{jk} = 1$ and $a_{ij} = a_{jk} = a_{ki} = 1$, that is, X_i defeated X_j, X_j defeated X_k, and X_k defeated X_i, then $g = -2$ is an admissible transformation on the elementary circuit (X_i, X_j, X_k, X_i), resulting in a ranking problem with $a'_{ij} = a'_{jk} = a'_{ki} = -1$, where X_j won over X_i, X_k over X_j, and X_i over X_k. If $g = -1$, then all results along the elementary circuit (X_i, X_j, X_k, X_i) is a draw.

Lemma 6.4. *The score, generalised row sum, and least squares methods satisfy IC.*

Proposition 6.2. *A neutral (NEU), strongly monotonic (SM) and independent of the circuits (IC) scoring procedure $f : \mathcal{R}^n_R \to \mathbb{R}^n$ is equivalent to the score method.*

Proof. According to Lemmata 5.3 (NEU), 6.1 (SM), and 6.4 (IC), the score method satisfies the three axioms.

Bouyssou (1992) examines directed graphs, where the flow $R_{ij} \in [0, 1]$ between nodes X_i and X_j reflects the paired comparison result, and unknown comparisons are not allowed. In this model, all scoring procedures satisfying NEU, SM, and IC is equivalent to the score method.

$a_{ij} := R_{ij} - R_{ji}$ is a one-to-one correspondence with our model. However, it is not the whole set $\mathcal{R}_R$ since $R_{ij} - R_{ji} \in [-1, 1]$. It can be checked that the proof of Bouyssou (1992) remains valid if $R_{ij} \in [0, m]$. □

Lemma 6.5. *On the set of round-robin ranking problems $\mathcal{R}_R$, the three properties of Proposition 6.2 are independent.*

Proposition 6.3. *The characterization of Proposition 6.2 is valid neither on the set of balanced ranking problems $\mathcal{R}_B$ nor the set of unweighted ranking problems $\mathcal{R}_U$.*

Proof. According to Lemmata 5.3 (NEU), 6.1 (SM), and 6.4 (IC), generalised row sum satisfies the three axioms. Because of Example 5.2, it is not equivalent to the score method on the set of balanced and unweighted ranking problems as $s_1(N, A, M) = s_2(N, A, M)$, but $x_1(\varepsilon)(N, A, M) > x_2(\varepsilon)(N, A, M)$, and $(N, A, M) \in \mathcal{R}_B \cap \mathcal{R}_U$. □

6.2 Scoring procedures as an extension of score

Definition 6.5. *Score consistency* (SCC) (González-Díaz et al., 2014): Scoring procedure $f : \mathcal{R}^n \to \mathbb{R}^n$ is *score consistent* if it is equivalent to the score method for all round-robin ranking problems $(N, A, M) \in \mathcal{R}^n_R$.

Remark 6.3. Regarding the generalised row sum method, Chebotarev (1994, Property 3) introduces a more general axiom called *agreement*: if $(N, A, M) \in \mathcal{R}^n_R$ is a round-robin ranking problem, then $\mathbf{x}(N, A, M) = \mathbf{s}(N, A, M)$. Moreover, according to *local agreement* (Chebotarev, 1994, Property 4), if $r_{ij}^{(p)}$ is known for all $X_j \in N$ and $p = 1, 2, \dots, m$, then $x_i(N, A, M) = s_i(N, A, M)$.

Definition 6.6. *Homogeneous treatment of victories* (HTV) (González-Díaz et al., 2014): Let $(N, A, M) \in \mathcal{R}^n$ be a ranking problem such that $m_{ik} = m_{jk}$ for all $X_k \in N \setminus \{X_i, X_j\}$. Scoring procedure $f : \mathcal{R}^n \to \mathbb{R}^n$ *treats victories homogeneously* if $f_i(N, A, M) \geq f_j(N, A, M) \Leftrightarrow s_i(N, A, M) \geq s_j(N, A, M)$.

Corollary 6.2. *HTV implies SCC.*

Remark 6.4. Regarding the generalised row sum method, Chebotarev (1994, Property 10) introduces a more general axiom called *domination*: if for objects $X_i, X_j \in N$, $m_{ik} = m_{jk}$ for all $X_k \in N \setminus \{X_i, X_j\}$, then with the notation $d_0 = d_i = d_j$:

$$x_i(N, A, M) - x_j(N, A, M) = \frac{1 + mn}{1 + d_0 + m_{ij}} \left[s_i(N, A, M) - s_j(N, A, M) \right].$$

Notation 6.1. Let $g : N \leftrightarrow N$ be a one-to-one correspondence. In this case $\mathfrak{g} : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ such that $X_{\mathfrak{g}(k)} = g(X_k)$.

Definition 6.7. *Homogeneous treatment of opponents* (*HTO*): Let $(N, A, M) \in \mathcal{R}^n$ be a ranking problem, and $f : \mathcal{R}^n \to \mathbb{R}^n$ be a scoring procedure such that there is a one-to-one correspondence $g : O_i \setminus \{X_j\}^{m_{ij}} \leftrightarrow O_j \setminus \{X_i\}^{m_{ij}}$ between the opponents of $X_i, X_j \in N$, where $f_k(N, A, M) = f_{\mathfrak{g}(k)}(N, A, M)$ for all $(X_k, g(X_k)) \in (O_i \setminus \{X_j\}^{m_{ij}}) \times (O_j \setminus \{X_i\}^{m_{ij}})$. f *treats opponents homogeneously* if $f_i(N, A, M) \geq f_j(N, A, M) \Leftrightarrow s_i(N, A, M) \geq s_j(N, A, M)$.

Corollary 6.3. *HTO implies HTV (therefore SCC).*

Proposition 6.4. *The score, generalised row sum and least squares methods satisfy HTO.*

Remark 6.5. For generalised row sum and least squares methods, *HTO* may be written in a more general form if equivalence is disregarded.
Let $(N, A, M) \in \mathcal{R}^n$ be a ranking problem, and $f : \mathcal{R}^n \to \mathbb{R}^n$ be a scoring procedure such that there is a one-to-one correspondence $g : O_i \setminus \{X_j\}^{m_{ij}} \leftrightarrow O_j \setminus \{X_i\}^{m_{ij}}$ between the opponents of $X_i, X_j \in N$, where $f_k(N, A, M) \geq f_{\mathfrak{g}(k)}(N, A, M)$ for all $(X_k, g(X_k)) \in (O_i \setminus \{X_j\}^{m_{ij}}) \times (O_j \setminus \{X_i\}^{m_{ij}})$. Then $s_i(N, A, M) \geq s_j(N, A, M) \Rightarrow f_i(N, A, M) \geq f_j(N, A, M)$, moreover, $f_i(N, A, M) > f_j(N, A, M)$ if any inequality is strict ($>$).

It is a possible extension of self-consistency (Chebotarev and Shamis, 1997; Csató, 2013b).

Lemma 6.6. *The score, generalised row sum and least squares methods satisfy SCC and HTV.*

Lemma 6.7. *The generalised row sum method satisfies HOM and RCS on the set of round-robin ranking problems $\mathcal{R}_R$.*

Lemma 6.8. *The generalised row sum, and least squares methods satisfy CS, IIR and IIM on the set of round-robin ranking problems $\mathcal{R}_R$.*

Lemma 6.9. *The least squares method satisfies SM and CSM on the set of round-robin ranking problems $\mathcal{R}_R$.*

6.3 Summary

In this chapter we have dealt with the score method and its relation to other scoring procedures. Based on Bouyssou (1992, Theorem 1), a characterization was given on the set of round-robin ranking problems. It requires three independent, more or less natural axioms, NEU, SM, and IC. However, we have proved that the axiomatization is not valid on the extended sets of balanced or unweighted ranking problems.

The characterization suggests that the score method is appropriate for ranking in the round-robin case. With respect to this fact, a chain of axioms was introduced to ensure the equivalence with the score on this set. SCC and HTV were still defined by González-Díaz et al. (2014) (and, implicitly, by Chebotarev (1994)). We were able to extend them, comparing not the opponents, but their valuations. It was demonstrated that all the three scoring procedures discussed in the thesis satisfy these properties.

Other characterizations on the general domain of $\mathcal{R}$ are also known. On the basis of Slutzki and Volij (2005), González-Díaz et al. (2014) show that fair bets is the only scoring procedure satisfying ANO, HOM, SYM, FP, and nonnegative response to losses (NRL). The first four of them are satisfied by score and least squares, so NRL plays a key role in the characterization. However, it is a property with a subjective background, defined for giving the right eigenvector of the appropriate matrix. A similar axiom can also yield the left eigenvector Slutzki and Volij (2005), but both procedures violate INV, a more natural property.

Characterizations of the score method in social choice theory Young (1974); Hansson and Sahlquist (1976); Nitzan and Rubinstein (1981) are difficult (but probably not impossible) to build in our model. However, all of them contains CS, which is strongly connected to IIM: according to Theorem 5.2, they are equivalent in the case of neutral and symmetric scoring procedures. Since González-Díaz et al. (2014, p. 165) claims that it is an unfavourable axiom in the general model, the significance of these extensions seems to be moderate from a practical point of view.

Chapter 7

Ranking in Swiss-system chess team tournaments

Csató (2014c) uses paired comparison-based scoring procedures in order to determine the result of Swiss-system chess team tournaments. We present the main challenges, the features of individual and team competitions as well as the failures of official lexicographical orders. Our model is discussed with respect to the properties of the score, generalised row sum and least squares methods. The proposed method is illustrated with a detailed analysis of the two recent chess team European championships. Final rankings are compared through their distances on the basis of Can (2014) and visualized with multidimensional scaling (MDS). Rankings are evaluated by prediction accuracy, retrodictive performance, and stability.

It is actually a more thorough analysis of the problem discussed in Csató (2013a). Publication of main findings is in progress.

7.1 Modelling of the problem

See Csató (2014c, Summary: Modelling of the problem).

7.2 An application: chess team European championships

See Csató (2014c, Summary: An application: chess team European championships).

7.3 Summary

See Csató (2014c, Summary: Conclusion).

Chapter 8

Further applications of paired comparisons

In this part we review some problems where the use of paired comparisons proved to be fruitful.

8.1 Promising fields of use

Paired comparisons were adapted on the following research areas:

- Statistics, international price comparisons. In purchasing power calculations, the non-transitivity of the Fisher index (Fisher, 1922) implies that the derivation of a common scale is not a trivial task. The problem is usually addressed by the EKS (Éltető-Köves-Szulc) method (Éltető and Köves, 1964; Szulc, 1964). Weighting can be based on external information (Rao and Timmer, 2003).

- Scientometrics, where citations represent paired comparisons (Pinski and Narin, 1976; Palacios-Huerta and Volij, 2004; Kóczy and Nichifor, 2013; Kóczy and Strobel, 2010).

- Ranking of web pages via hyperlinks (Brin and Page, 1998).

- Psychology as individuals are often not able to judge on the same scale (Thurstone, 1927; Gulliksen, 1956; Kaiser and Serlin, 1978).

- Rankings from user ratings (Bozóki et al., 2014; London and Csendes, 2013; Jiang et al., 2011).

- Voting theory (Chebotarev and Shamis, 1998).

- University rankings based on the preferences of applicants (Avery et al., 2013; Telcs et al., 2013a; Csató, 2013c; Telcs et al., 2013b).

- Sport (Zermelo, 1929; Radicchi, 2011; Temesi et al., 2012; Csató, 2013a).

8.2 Paired comparisons from individual ratings

Definition 8.1. *Voter-alternative matrix*: Let q_{pi} be the evaluation of $X_i \in N$ by decision maker p. Then $Q = (q_{pi}) \in \mathbb{R}^{m \times n}$, $p = 1, 2, \dots, m$, $i = 1, 2, \dots, n$ is a *voter-alternative matrix.*

Four ideas about the construction of paired comparisons from raw datasets on the basis of Jiang et al. (2011) are given. They are arithmetic mean of valuation differences, geometric mean of valuation ratios, binary comparison, and logarithmic odds ratio.

8.3 A framework for the solution of paired comparison problems

A central issue of applications is modelling as a ranking problem $(N, A, M) \in \mathcal{R}^n$. Usually, the choice of matches matrix M is obvious like in Swiss system tournaments, or scientometrics. However, multiple comparisons may arise due to various causes. In certain cases, matrix M is worth to derive together with the results matrix A since overwhelming comparison outcomes may lead to weird rankings (Temesi et al., 2012).

In order to determine matrix A, one should consider different features of the problem. Monotonicity is a natural condition, namely, a larger victory corresponds to a greater value. If it does not provide an exact definition, alternative representations are worth to compare. For example, Csató (2013a) shows that its rankings for the Chess Olympiad 2010 are not sensitive to scaling. Sometimes the construction can be founded axiomatically, like in the application presented in Chapter 7.

Another important question is the differentiation between known and unknown comparisons, especially if the objects are individuals behaving strategically, or paired comparisons are derived from a voter-alternative matrix. In this case, taking the motivations and informations of decision makers into account becomes indispensable. It is not a simple task, for a debate, see Telcs et al. (2013a), Csató (2013c), and Telcs et al. (2013b).

Therefore we propose to take the following steps in the investigation of paired comparisons problems:

1. Check the preconditions of mathematical models. For example, if the objects can influence the outcomes, were they prompted to achieve a better result, or can they manipulate them.

2. Define the number of comparisons through matches matrix M.

3. Transform paired comparison outcomes into the results matrix A.

4. Choose a suitable scoring procedure focusing on the axiomatic approach (if it is possible).

5. Examine the sensitivity of rankings with respect to research hypothesis.

6. Analyse the results, compare them to known rankings and solutions.

This process is not necessarily sequential, the calculated rankings can show the problems of initial assumptions or point to some properties of the method chosen. The weighting of comparisons is not independent from the derivation of results (Temesi et al., 2012). In some cases, some steps may be left out, too.

8.4 Summary

In this chapter we have dealt with applications of paired comparisons models. Some relevant areas have been presented, as well as the transformation of a voter-alternative matrix into our model has been investigated. Finally, we have formulated some suggestions for users on the basis of our knowledge and experiences.

Chapter 9

Conclusions

In the introduction we have mentioned two aims addressed by this thesis: to give an overview about how to represent practical problems in a mathematical model as well as about the solution concepts of the latter. The first issue was discussed in Chapters 7 and 8, and the second question was investigated by Chapters 2-6.

Chapter 2 presented two representations of the ranking problem. The first is analogous to the models of Chebotarev and Shamis (1998) and González-Díaz et al. (2014), it is favourable for analysing examples and used by some scoring procedures (invariant, fair bets, maximum likelihood). The second is not as parsimonious but it helps in the understanding and makes possible the graph representation.

We also reviewed two approaches of ranking, the approximation of paired comparisons by linear orders and the application of scoring procedures. On the basis of Bouyssou (2004) we argued for use of the latter. Some of them were examined in Chapter 3. Our focus was on generalised row sum and least squares, for which a new graph interpretation was given in Chapter 4.

In the following two chapters González-Díaz et al. (2014) were followed by a theoretical investigation of the score, generalised row sum, and least squares scoring procedures.

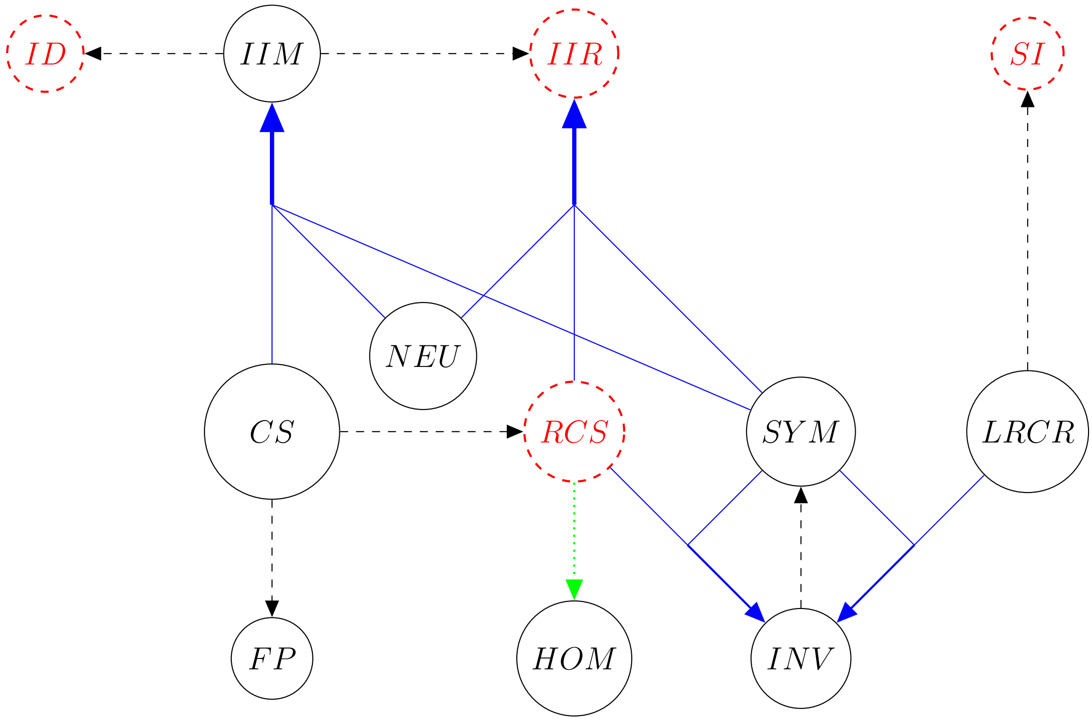
Figure 9.1 gives a comprehensive picture about the axioms discussed in Chapter 5. Homogeneity (HOM) can only be partially derived from result consistency (RCS), exploration of their connection remains the topic of further studies. However, common roots are revealed by the generalised row sum method, which satisfies both conditions only for certain ε-s.[3]

In Chapter 6 we examined a characterization of the score method (Bouyssou, 1992). It is shown that this result is not valid in our general model as both generalised row sum and least squares satisfy the necessary axioms. An analysis of the connection with the score method was carried out on the basis of some known and novel axioms.

[3] See the green, dotted line in 9.1. HOM is certainly not equivalent to RCS as fair bets satisfies only the first.

Figure 9.1: Connections among the axioms of Chapter 5

Arrows sign implications. In certain cases an axiom can be derived from a set of properties like $NEU + SYM + CS \Rightarrow IIM$. Red, dashed nodes mean novel axioms; blue solid lines represent our contributions; black, dashed lines indicate trivial or known results; the green, dotted line is a special relationship (see the text).



Their consideration seems to be important for the potential use of our procedures.

Axiomatic results are summarized in Table 9.1. We have collected some known properties (in certain cases with minor modifications and extensions) as well as six new ones: scale invariance (SI), result consistency (RCS), preservation of linear order (LOP), independence of irrelevant results (IIR), independence of draws (ID), and homogeneous treatment of opponents (HTO). Score satisfies all axioms beyond LOP but we have seen that IIM is not favourable in the presence of missing and multiple comparisons. According to a central theorem it essentially excludes consistency, adding relevance to the differentiation of results and matches matrices.

Our findings recommend the use of generalised row sum with a variable parameter, somewhat proportional to the number of matches like the reasonable upper bound. It is not surprising given the statistical background of the method (Chebotarev, 1994). Then generalised row sum and least squares are distinguished by monotonicity.[4] Our scoring procedures do not satisfy the axiom LOP which is probably true for a much larger set of these measures.

Three main directions occur for future research. The first is to extend the scope of the analysis to other scoring procedures. For example, Slikker et al. (2012) suggest

[4] Some of their other differences are highlighted by González-Díaz et al. (2014); Csató (2013b).

Table 9.1: Axiomatic comparison of scoring procedures

Axiom	Fixed-order	Flat	Score†	Generalised row sum‡	Least squares
(ANO)	✔	✔	(✔)	(✔)	(✔)
(NEU)	✗	✔	(✔)	(✔)	(✔)
(CNT)	✗	✔	(✔)	(✔)	(✔)
$(LRCR)$	✗	✔	✔	(✔)	✔
(HOM)	✔	✔	(✔)	(✔)✗*	(✔)
SI	✔	✔	✔	✔	✔
(CS)	✔	✗	(✔)	(✗)	(✗)
(FP)	✔	✔	(✔)	(✔)	(✔)
RCS	✔	✗	✔	✔✗*	✔
(SYM)	✗	✔	(✔)	(✔)	(✔)
(INV)	✗	✔	(✔)	(✔)	(✔)
LOP	✗	✗	✗	✗	✗
(IIM)	✔	✔	(✔)	(✗)	(✗)
IIR	✔	✔	✔	✗	✗
$ID^{\diamond}$	✔	✔	✔	✔	✔
(SM)	✗	✗	(✔)	(✔)	(✗)
(CSM)	✗	✗	(✔)	(✔)	(✗)
(IC)	✔	✔	(✔)	✔	✔
(SCC)	✗	✗	(✔)	(✔)	(✔)
(HTV)	✗	✗	(✔)	(✔)	(✔)
HTO	✗	✗	✔	✔	✔

Known axioms and results are in parentheses, others are our contributions

† González-Díaz et al. (2014) defines score method differently; their results are in parentheses

‡ González-Díaz et al. (2014) examines only the case of $\bar{\varepsilon} = [1/m(n-2)]$; their results are in parentheses

* Depends on the choice of ε

◊ An extension of condition (2) of Chebotarev (1994, Property 14)

a general framework for ranking the nodes of directed graphs, resulting in fair bets as a limit. Positional power (Herings et al., 2005) is also worth to examine since its links to least squares from a graph-theoretic point of view (Csató, 2014a).

The second course is the introduction of new axioms with the final aim to get new characterizations, an intended end goal of our analysis. Some additional properties are discussed by Csató (2013b). However, it remains doubtful whether exact results may be expected in this general framework.

The third issue is the meaning of a unique ranking in the presence of inconsistency, an issue considered by Jiang et al. (2011). These methods behave relatively well from a mathematical point of view, therefore an error estimation seems to be possible, which can provide the probability that an object is ranked above the other (Horváth et al., 2013).

The first topic gets a somewhat restricted attention since we have still examined similar problems (Csató, 2012a,b, 2013a,c, 2014c; Temesi et al., 2012). In Chapter 8 a number of areas were reviewed, as well as some suggestions were given on the basis of our experiences. Chapter 7 applied paired comparison-based scoring procedures for ranking in Swiss-system chess team tournaments. Two examples were analysed with the proposed methods. Our results give strong arguments for the use of least squares method with a generalised result matrix favouring match points.

To conclude, we think that essential contributions were added to both issues investigated. We do not want to appreciate their relative importance, it depends on the reader. However, our later research will probably follow the analysis of relevant problems. In this way we can avoid the difficulties arising from the lack of characterizations.

Appendix: The case of regular bipartite comparison multigraph

Csató (2014a, Theorem 1) contains the condition that the comparison multigraph is not regular bipartite. This issue was not addressed in Section 4.2. Then the matches matrix M is block anti-diagonal, number of objects is even, and the two subsets of N have the same number of objects. Now Neumann series is not convergent, but the critical eigenvalue $\lambda = -1$ of B is on the boundary of the unit circle, therefore it is bounded and oscillates.

Conjecture 4. *Let $(N, A, M) \in \mathcal{R}^n$ be a ranking problem. Csató (2014a, Theorem 1) is valid if and only if the comparison multigraph is not regular bipartite or* $\mathbf{s} = \mathbf{0}$.

Conjecture 5. *If the iteration process*

$$\mathbf{q}^{(0)} = (1/\mathfrak{d})\mathbf{s}$$

$$\mathbf{q}^{(k)} = \mathbf{q}^{(k-1)} + \frac{1}{\mathfrak{d}} \left(\frac{1}{\mathfrak{d}} C \right)^k \mathbf{s}, \quad k = 1, 2, \dots,$$

does not converge for a regular bipartite comparison multigraph G, then the rating vectors obtained after an even and an odd number of steps converge, respectively. Furthermore, the arithmetic mean of the two limit gives the least squares ranking $\mathbf{q}$.

Formally, for all $\tau > 0$, there exists $t \in \mathbb{N}$ such that

$$\left\| \mathbf{q}^{(2z+2)} - \mathbf{q}^{(2z)} \right\|_2 < \tau,$$

$$\left\| \mathbf{q}^{(2z+3)} - \mathbf{q}^{(2z+1)} \right\|_2 < \tau$$

and

$$\left\| \mathbf{q} - \left(\mathbf{q}^{(2z+1)} + \mathbf{q}^{(2z)} \right) /2 \right\|_2 < \tau,$$

for all $z \geq t$.

There are also some open questions regarding the decomposition of the least squares rating vector $\mathbf{q}$.

Problem 1. Are there exactly three possible outcomes of the iteration if G is a regular bipartite comparison multigraph:

I. The iterated rating vectors $\mathbf{q}^{(z)}$ are unchanged;

II. The same iterated rating vector occurs with period 2, that is, $\mathbf{q}^{(z)} = \mathbf{q}^{(z+2)}$ for all $z \in \mathbb{N}$;

III. The iterated rating vectors $\mathbf{q}^{(z)}$ converge separately for even and odd t values?

If the answer is affirmative, does it hold that the least squares rating vector can be obtained as the arithmetic mean of rating vectors for even and odd t values in case II and as the limit of rating vectors for even and odd t values in case III?

What are the specific features of ranking problems (N, A, M) in cases I, II and III, respectively?

For a given matches matrix M associated with a regular bipartite comparison multigraph G, is it possible to produce all the three cases above by an appropriate choice of the results? As the convergence of $\mathbf{q}^{(z)}$ is provided when $\mathbf{s} = 0$, for case I the answer is affirmative.

From the viewpoint of potential applications, the oscillation of the iterated ratings does not seem to be a serious problem since regular bipartite graphs determine a very special comparison structure. In particular, they imply the lack of cycles of odd length, specifically, the existence of triplets (X_i, X_j, X_k) with all possible comparisons $(X_i, X_j), (X_i, X_k), (X_j, X_k)$ are excluded.

Regarding the necessary conditions in (Csató, 2014a, Theorem 1), the connectedness of G can be interpreted as for the uniqueness of the least squares rating, it is a natural requirement for the objects to be comparable. Lack of convergence is due to bipartiteness and regularity. For these graphs the calculation of optimal weights is cyclic: in order to determine the aggregated rating of the second group of objects, it should be known for the first group, and vice versa.

It is less restrictive than the requirement for recursive performance by Brozos-Vázquez et al. (2008), since now only regular bipartite comparison multigraphs are excluded, not all bipartite graphs. A possible reason can be the presence of loops in bipartite but not regular decomposition multigraphs; however, critical cases arise strictly from mathematical argumentation. Convergence for some regular bipartite graphs is guaranteed by the results matrix A, not by Csató (2014a, Lemma 3).

List of publications

Publications in English

Scientific articles

1) Csató, L. [2013a]: Ranking by pairwise comparisons for Swiss-system tournaments. *Central European Journal of Operations Research*, 21(4):783–803. DOI: `http://dx.doi.org/10.1007/s10100-012-0261-8`.

2) Csató, L. [2014a]: A graph interpretation of the least squares ranking method. *Social Choice and Welfare*. Forthcoming. DOI: `http://dx.doi.org/10.1007/s00355-014-0820-0`.

Book chapters

3) Csató, L. [2012a]: A pairwise comparison approach to ranking in chess team championships. In Fülöp, P. (ed.): *Tavaszi Szél 2012 Konferenciakötet*. Doktoranduszok Országos Szövetsége, Budapest, pp. 514–519.

Conference papers, working papers

4) Csató, L. [2012b]: A paired comparisons ranking and Swiss-system chess team tournaments. 6th Annual Conference of Hungarian Society of Economists. URL: `http://media.coauthors.net/konferencia/conferences/7/LLSM_Buch_ranking__.pdf`.

5) Csató, L. [2014b]: Additive and multiplicative properties of scoring methods for preference aggregation. Corvinus Economics Working Papers 3/2014, Corvinus University of Budapest, Budapest. URL: `http://unipub.lib.uni-corvinus.hu/1562/`.

Other

6) Bozóki, S., Csató, L., Rónyai, L. and Tapolcai, J. [2014]: Robust peer review decision process. Manuscript.

Publications in Hungarian

Scientific articles

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