Telman B. GASYMOV, Tarlan Z. GARAYEV

ON NECESSARY AND SUFFICIENT CONDITIONS FOR OBTAINING THE BASES OF BANACH SPACES

Abstract

In the paper we cite necessary and sufficient conditions for completeness, minimality and basicity of systems in a Banach space that are obtained from initial system by changing the finite number of elements.

In the paper [1] the criteria of basicity of systems obtained by excepting finite number of elemetns from over complete systems are cited. In the present paper similar criteria are cited for the system that are obtained from initial one by changing the finite number of elements.

Remind some definitions and facts stated for example in [2]. Let $\mathfrak{X}$ be a Banach space, $\mathfrak{X}^*$ be it's adjoint space. For $x \in \mathfrak{X}$ and $x^* \in \mathfrak{X}^*$ by $\langle x, x^* \rangle$ we denote the value of the functional x^* on the element x. For the system $\{x_n\}_{n\in N} \subset \mathfrak{X}$, where N is a set of natural numbers, the system for which $\langle x_n, x_k^* \rangle = \delta_{nk}$, where δ_{nk} is a Kronecker symbol, is said to be an adjoint system $\{x_n^*\}_{n\in N} \subset \mathfrak{X}^*$.

The system $\{x_n\}_{n\in N}$ is said to be minimal, if no element of this system belongs to a closed linear span of remaining elements. Minimality of the system is equivalent to the existence of the adjoint system.

The system is said to be complete in $\mathfrak{X}$ if closure of a linear span of this system concides with $\mathfrak{X}$. Completeness of the system $\{x_n\}_{n\in N}$ in $\mathfrak{X}$ is equivalent to the following condition: if for some $x^* \in \mathfrak{X}^*$ $\langle x_n, x^* \rangle = 0$ for all $n \in N$, then $x^* = 0$.

Two systems of the Banach space $\mathfrak{X}$ are said to be equivalent, if there exists a bounded and boundedly invertible operator, transferring one of these systems to another one.

Let $J_m = \{n_1, ..., n_m\}$ be some collection of different natural numbers

$$N_m = N \setminus J_m$$

Theorem 1. *Let $\{x_n\}_{n\in N} \subset \mathfrak{X}$ be a minimal system, and be $\{x_n^*\}_{n\in N} \subset \mathfrak{X}^*$ its adjoint one. If for some system $\{u_n\}_{k=1}^m \subset \mathfrak{X}$ it is fulfilled the condition $\Delta_m =$* det $A_m \neq 0$, *where*

$$A_m = \left\| \left\langle u_k, x_{n_j}^* \right\rangle \right\|_{k,j=1}^m , \tag{1}$$

the system $\{u_k\}_{k=1}^m \cup \{x_n\}_{n\in N_m}$ is minimal as well. Therewith the adjoint system $\{u_n^*\}_{n\in N}$ is determined in the following way:

$$u_{n_k}^* = \frac{1}{\Delta_m} \begin{vmatrix} 0 & x_{n_1}^* & ... & x_{n_m}^* \\ 0 & \langle u_1, x_{n_1}^* \rangle & ... & \langle u_1, x_{n_m}^* \rangle \\ \cdot & \cdot & ... & \cdot \\ 1 & \langle u_k, x_{n_1}^* \rangle & ... & \langle u_k, x_{n_m}^* \rangle \\ \cdot & \cdot & ... & \cdot \\ 0 & \langle u_m, x_{n_1}^* \rangle & ... & \langle u_m, x_{n_m}^* \rangle \end{vmatrix} \tag{2}$$

for $k=\overline{1,m}$;

$$u_n^* = \frac{1}{\Delta_m}\begin{vmatrix} x_n^* & x_{n_1}^* & ... & x_{n_m}^* \\ \langle u_1, x_n^*\rangle & \langle u_1, x_{n_1}^*\rangle & ... & \langle u_1, x_{n_m}^*\rangle \\ . & . & ... & . \\ \langle u_m, x_n^*\rangle & \langle u_m, x_{n_1}^*\rangle & ... & \langle u_m, x_{n_m}^*\rangle \end{vmatrix} \tag{3}$$

for $n \in N_m$.

Proof. Really, acting by the functional $u_{n_k}^*$ on the element u_j from (2) we get

$$\langle u_j, u_{n_k}^*\rangle = \frac{1}{\Delta_m}\begin{vmatrix} 0 & \langle u_j, x_{n_1}^*\rangle & ... & \langle u_j, x_{n_m}^*\rangle \\ 0 & \langle u_1, x_{n_1}^*\rangle & ... & \langle u_1, x_{n_m}^*\rangle \\ . & . & ... & . \\ 1 & \langle u_k, x_{n_1}^*\rangle & ... & \langle u_k, x_{n_m}^*\rangle \\ . & . & ... & . \\ 0 & \langle u_m, x_{n_1}^*\rangle & ... & \langle u_m, x_{n_m}^*\rangle \end{vmatrix}.$$

Revealing the last determinant with respect to the elements of the first column we get

$$\langle u_j, u_{n_k}^*\rangle = \delta_{jk}, \quad j,k=\overline{1,m}.$$

Now let $k,\ n \notin J_m$, i.e $k,\ n \neq n_j,\ j=\overline{1,m}$. Then, from (3) we have

$$\langle x_k, u_n^*\rangle = \frac{1}{\Delta_m}\begin{vmatrix} \langle x_k, x_n^*\rangle & \langle x_k, x_{n_1}^*\rangle & ... & \langle x_k, x_{n_m}^*\rangle \\ \langle u_1, x_n^*\rangle & \langle u_1, x_{n_1}^*\rangle & ... & \langle u_1, x_{n_m}^*\rangle \\ . & . & ... & . \\ \langle u_m, x_n^*\rangle & \langle u_m, x_{n_1}^*\rangle & ... & \langle u_m, x_{n_m}^*\rangle \end{vmatrix} =$$

$$= \frac{1}{\Delta_m}\begin{vmatrix} \delta_{kn} & 0 & ... & 0 \\ \langle u_1, x_n^*\rangle & \langle u_1, x_{n_1}^*\rangle & ... & \langle u_1, x_{n_m}^*\rangle \\ . & . & ... & . \\ \langle u_m, x_n^*\rangle & \langle u_m, x_{n_1}^*\rangle & ... & \langle u_m, x_{n_m}^*\rangle \end{vmatrix}.$$

Revealing we last determinant with respect to the element of the first row we get $\langle x_k, u_n^*\rangle = \delta_{kn}$ for $k,\ n \notin J_m$.

It is similarly verified that $\langle u_j, u_n^*\rangle = 0$ for $n \notin J_m,\ j=\overline{1,m}$ and $\left\langle x_k, u_{n_j}^*\right\rangle = 0$ for $k \notin J_m,\ j=\overline{1,m}$. The obtained relations show that the system $\{u_k\}_{k=1}^m \cup \{x_n\}_{n\in N_m}$, is adjoint to the system $\{u_n^*\}_{n\in N}$ that is equivalent to the minimality of the last system.

Theorem 2. *If in the denotation of theorem 1 the system* $\{x_n\}_{n\in N}$ *is complete and minimal and* $\Delta_m = 0$, *the system* $\{u_k\}_{k=1}^m \cup \{x_n\}_{n\in N_m}$ *is not complete in* $\mathfrak{X}$.

Proof. Let for the system $\{u_k\}_{k=1}^m$ and for the collection $J_m = \{n_1, ..., n_m\}$

$$\Delta_m = \det\ A_m = 0,$$

where the matrix A_m is determined by (1). Then there exists a non-zero vector $C = \begin{pmatrix} c_1 \\ ... \\ c_m \end{pmatrix}$ such that $A_m C = 0$. Assume $x_0^* = \sum_{k=1}^n c_k x_{n_j}^*$. Since, even if one of the numbers c_j doesn't equal zero, then $x_0^* \neq 0$ and for $n \notin J_m$

$$\langle x_n, x_0^*\rangle = \sum_{j=1}^m c_j \left\langle x_n, x_{n_j}^*\right\rangle = 0,$$

$$\langle u_k, x_0^* \rangle = \sum_{j=1}^{m} c_j \left\langle u_k, x_{n_j}^* \right\rangle = 0, \quad k = \overline{1,m}.$$

Thus, the non-zero vector x_0^* is orthogonal to all the vectors of the system $\{u_k\}_{k=1}^m \cup \{x_n\}_{n\in N_m}$, and this means that the system is not complete in $\mathfrak{X}$.

Theorem 3. *Let the system $\{x_n\}_{n\in N}$ be a basis of the space $\mathfrak{X}$, and $\{u_k\}_{k=1}^m$ be some system of vectors from $\mathfrak{X}$. Then for the basicity of the system $\{u_k\}_{k=1}^m \cup \{x_n\}_{n\in N_m}$ in the space $\mathfrak{X}$ it is necessary and sufficient to fulfill the condition $\Delta_m \neq 0$. For $\Delta_m = 0$ the system $\{u_k\}_{k=1}^m \cup \{x_n\}_{n\in N_m}$ is not complete and is not minimal.*

Proof. Let $\{x_n\}_{n\in N}$ be a basis of the space $\mathfrak{X}$ and for thesystem $\{u_k\}_{k=1}^m \subset \mathfrak{X}$ and the collection $J_m = \{n_1, ..., n_m\}$ the condition

$$\Delta_m = \det \left\| \left\langle u_k, x_{n_j}^* \right\rangle \right\|_{k,j=1}^{m} \neq 0.$$

be fulfilled. Take any $x \in \mathfrak{X}$, expand it and also each vector $u_k \in \mathfrak{X}, k = \overline{1,m}$ in a biorthogonal series

$$x = \sum_{n\in N} \langle x, x_n^* \rangle x_n, \tag{4}$$

$$u_k = \sum_{n\in N} \langle u_k, x_n^* \rangle x_n, \quad k = \overline{1,m}. \tag{5}$$

Assume

$$b_k = \sum_{n\in N_m} \langle u_k, x_n^* \rangle x_n$$

and rewrite (5) in the following form:

$$\sum_{j=1}^{m} \left\langle u_k, x_{n_j}^* \right\rangle x_{n_j} = u_k - b_k, \qquad k = \overline{1,m}. \tag{6}$$

Solving the last system of linear algebraic equations with respect to the unknown x_{n_j} we find

$$x_{n_j} = \frac{\Delta_m^j (u-b)}{\Delta_m},$$

where $\Delta_m^j (u-b)$ is a determinant obtained from the determinant Δ_m replacing the $j-$ column by the column with elements $u_k - b_k,\ k = \overline{1,m}$. Represent it in the form

$$\Delta_m^j (u-b) = \Delta_m^j (u) - \Delta_m^j (b) = \sum_{k=1}^{m} \Delta_m^{jk} u_k - \sum_{k=1}^{m} \Delta_m^{jk} \left(\sum_{n\in N_m} \langle u_k, x_n^* \rangle x_n \right),$$

where Δ_m^{jk} is an algebraic complement of the elements with the numbers (j,k) of the determinant Δ_m. Then

$$x_{n_j} = \frac{1}{\Delta_m} \sum_{k=1}^{m} \Delta_m^{jk} u_k - \sum_{n\in N_m} \left(\frac{1}{\Delta_m} \sum_{k=1}^{m} \langle u_k, x_n^* \rangle \Delta_m^{jk} \right) x_n.$$

Substitute the obtained expression for x_{n_j} in (4) :

$$x = \sum_{j=1}^{m} \left\langle x, x_{n_j}^* \right\rangle x_{n_j} + \sum_{n\in N_m} \langle x, x_n^* \rangle x_n = \sum_{j=1}^{m} \left\langle x, x_{n_j}^* \right\rangle \frac{1}{\Delta_m} \sum_{k=1}^{m} \Delta_m^{jk} u_k -$$

$$-\sum_{j=1}^{m}\left\langle x,x_{n_j}^*\right\rangle \sum_{n\in N_m}\left(\frac{1}{\Delta_m}\sum_{k=1}^{m}\left\langle u_k,x_n^*\right\rangle \Delta_m^{jk}\right)x_n+\sum_{n\in N_m}\left\langle x,x_n^*\right\rangle x_n=$$

$$=\sum_{k=1}^{m}\left\langle x,\frac{1}{\Delta_m}\sum_{j=1}^{m}\Delta_m^{jk}x_{n_j}^*\right\rangle u_k+\sum_{n\in N_m}\left\langle x,x_n^*-\frac{1}{\Delta_m}\sum_{j=1}^{m}\sum_{k=1}^{m}\left\langle u_k,x_n^*\right\rangle \Delta_m^{jk}\right\rangle x_n=$$

$$=\sum_{k=1}^{m}\left\langle x,u_{n_k}^*\right\rangle u_n+\sum_{n\in N_m}\left\langle x,u_n^*\right\rangle x_n;$$

here $u_{n_k}^*$ and u_n^* are determined by formulae (2) and (3), respectively.

Thus, we showed that we can expand any element $x\in\mathfrak{X}$ in biorthogonal series by the system $\{u_k\}_{k=1}^m\cup\{x_n\}_{n\in N_m}$, that means the basicity of this system. Sufficiency of the theorem is proved.

Necessity of the theorem follows from theorem 2, since for $\Delta_m=0$ the system is not complete and all the more is not a basis. Now, let's show that for $\Delta_m=0$ the system $\{u_k\}_{k=1}^m\cup\{x_n\}_{n\in N_m}$ is not minimal. Let $C=\begin{pmatrix} c_1\\ ...\\ c_m\end{pmatrix}$ be a non-zero vector for which $A_mC=0$. Assume

$$u_0=\sum_{k=1}^{m}c_ku_k$$

and expand the vector u_0 in the basis of $\{x_k\}_{k\in N}$:

$$u_0=\sum_{n\in N}\left\langle u_0,x_n^*\right\rangle x_n=\sum_{j=1}^{m}\left\langle u_0,x_{n_j}^*\right\rangle x_{n_j}^*+\sum_{n\in N_m}\left\langle u_0,x_n^*\right\rangle x_n=$$

$$=\sum_{j=1}^{m}\left(\sum_{k=1}^{m}c_k\left\langle u_k,x_{n_j}^*\right\rangle\right)x_{n_j}+\sum_{n\in N_m}\left\langle u_0,x_n^*\right\rangle x_n=\sum_{n\in N_m}\left\langle u_0,x_n^*\right\rangle x_n \qquad (7)$$

Let $c_{k_0}\neq 0$. Then from (7) we get

$$u_{k_0}=-\sum_{\substack{k=1\\k\neq k_0}}^{m}\frac{c_k}{c_{k_0}}u_k+\sum_{n\in N_m}\frac{\left\langle u_0,x_n^*\right\rangle}{c_{k_0}}x_n,$$

i.e. the element u_{k_0} belongs to the closure of a linear span of other elements of the system $\{u_k\}_{k=1}^m\cup\{x_n\}_{n\in N_m}$, that means non-minimality of the last system.

Corollary 1. *If in the conditions of theorem 3 $\Delta_m\neq 0$ then the system $\{u_k\}_{k=1}^m\cup\{x_n\}_{n\in N_m}$ is equivalent to the system $\{x_n\}_{n\in N}$.*

Proof. Let $\{x_n\}_{n\in N}$ be a basis of the space $\mathfrak{X}$. Take an arbitrary $x\in\mathfrak{X}$ and expand it in series

$$x=\sum_{n\in N}\left\langle x,x_n^*\right\rangle x_n$$

We determine the operator T in the following way:

$$Tx=\sum_{k=1}^{m}\left\langle x,x_{n_k}^*\right\rangle u_k+\sum_{n\in N_m}\left\langle x,x_n^*\right\rangle x_n \qquad (8)$$

The operator T is determined on the whole of the space $\mathfrak{X}$ and is bounded.

Really,

$$\|Tx\| = \left\| \sum_{k=1}^{m} \left\langle x, x_{n_k}^* \right\rangle (u_k - x_{n_k}) + \sum_{n\in N} \left\langle x, x_n^* \right\rangle x_n \right\| \leq$$

$$\leq \sum_{k=1}^{n} \left| \left\langle x, x_{n_k}^* \right\rangle \right| \|u_k - x_{n_k}\| + \|x\| \leq (M+1) \|x\|,$$

where

$$m = \max_{1\leq k\leq m} \|x_{n_k}\| \|u_k - x_{n_k}\|.$$

Besides, the operator T has the inverse for which

$$x = \sum_{k=1}^{n} \left\langle x, u_{n_k}^* \right\rangle u_k + \sum_{n\in N_m} \left\langle x, u_n^* \right\rangle x_n$$

is given by the formula

$$T^{-1}x = \sum_{k=1}^{m} \left\langle x, u_{n_k}^* \right\rangle x_{n_k} + \sum_{n\in N_m} \left\langle x, u_n^* \right\rangle x_n.$$

Obviously, T^{-1} is also a bounded operator. Since by definition (8) $Tx_{n_k} = u_k,\ k = \overline{1,m}$ and $Tx_n = x_n,\ n \in N_m$, hence we get the equivalence of the systems $\{u_k\}_{k=1}^{m} \cup \{x_n\}_{n\in N_m}$ and $\{x_n\}_{n\in N}$.

Corollary 2. *If in the conditions of the theorem 3 $\mathfrak{X}$ is a Hilbert space, $\{x_n\}_{n\in N}$ is a Riesz basis in $\mathfrak{X}$, and $\Delta_m \neq 0$ the system $\{u_k\}_{k=1}^{m} \cup \{x_n\}_{n\in N_m}$ is also a Riesz basis in $\mathfrak{X}$.*

Now, let's apply the obtained results to the spaces of the form $\mathfrak{X} = \mathfrak{X}_0 \oplus C^m$, where $\mathfrak{X}_0$ is some Banach space, and C^m is m copy of a complex plane C. Let $\{e_k\}_{k=1}^{m}$ be a natural basis in C^m, i.e $e_1 = (1,0,...,0)$, $e_2 = (0,1,....,0)$, ..., $e_m = (0,0,...,1)$ and assume $\widehat{e}_k = (0, e_k) \in \mathfrak{X}$.

Theorem 4. *Let $\widehat{x}_n = (x_n, \alpha_{n1}, ..., \alpha_{nm})$, $n \in N$, be a basis of the space $\mathfrak{X}$, $\widehat{x}_n^* = (x_n^*, \beta_{n1}, ..., \beta_{nm})$, $n \in N$, be an adjoint system. For the basicity of the system $\{\widehat{e}_k\}_{k=1}^{m} \cup \{\widehat{x}_n\}_{n\in N_m}$ in the space $\mathfrak{X}$ it is necessary and sufficient to fulfill the condition*

$$\Delta_m = \det \left\| \beta_{n_{kj}} \right\|_{k,j=1}^{m} \neq 0$$

If $\Delta_m = 0$ the system $\{\widehat{e}_k\}_{k=1}^{m} \cup \{\widehat{x}_n\}_{n\in N_m}$ is not complete and minimal.

The proof of the theorem directly follows from theorem 3, if as the system $\{u_k\}_{k=1}^{m}$ we take the system $\{e_k\}_{k=1}^{m}$, since in this case $\left\langle \widehat{e}_k, \widehat{x}_{n_j}^* \right\rangle = \beta_{n_j k}$.

Theorem 5. *In the conditions of theorem 4 for the basicity of the system $\{x_n\}_{n\in N_m}$ in the space $\mathfrak{X}_0$ it is sufficient and necessary to fulfill the condition*

$$\Delta_m = \det \left\| \beta_{n_{kj}} \right\|_{k,j=1}^{m} \neq 0$$

Therewith the adjoint system has the form

$$u_n^* = \frac{1}{\Delta_m} \begin{vmatrix} x_n^* & x_{n_1}^* & ... & x_{n_m}^* \\ \beta_{n1} & \beta_{n_1 1} & ... & \beta_{n_m 1} \\ \cdot & \cdot & ... & \cdot \\ \beta_{nm} & \beta_{n_1 m} & ... & \beta_{n_m m} \end{vmatrix}, \quad n \in N_m \tag{9}$$

For $\Delta_m = 0$ *the system* $\{x_n\}_{n\in N_m}$ *is not complete and minimal.*

Proof. By theorem 4 for $\Delta_m \neq 0$ the sysyem $\{\widehat{e}_k\}_{k=1}^m \cup \{\widehat{x}_n\}_{n\in N_m}$ is a basis in $\mathfrak{X} = \mathfrak{X}_0 \oplus C^m$. Take an arbitrary $x \in \mathfrak{X}_0$ and put $\widehat{x} = (x, 0, ..., 0)$ and expand $\widehat{x}$ by basis $\{\widehat{e}_k\}_{k=1}^m \cup \{\widehat{x}_n\}_{n\in N_m}$:

$$\widehat{x} = \sum_{k=1}^{m} \left\langle \widehat{x}, u_{n_k}^* \right\rangle \widehat{e}_k + \sum_{n\in N_m} \left\langle \widehat{x}, \widehat{u}_n^* \right\rangle \widehat{x}_n$$

Passing to the first coordinates we get

$$x = \sum_{n\in N_m} \left\langle x, u_n^* \right\rangle x_n \qquad (10)$$

where u_n^* is the first coordinate of the vector $\widehat{u}_n^*$. By formulae (2) and (3) $\widehat{u}_n^*$ is of the form:

$$\widehat{u}_n^* = \frac{1}{\Delta_m} \begin{vmatrix} \widehat{x}_n^* & \widehat{x}_{n_1}^* & ... & \widehat{x}_{n_m}^* \\ \langle \widehat{e}_1, x_n^* \rangle & \left\langle \widehat{e}_1, x_{n_1}^* \right\rangle & ... & \left\langle \widehat{e}_1, x_{n_m}^* \right\rangle \\ \cdot & \cdot & ... & \cdot \\ \langle \widehat{e}_m, x_n^* \rangle & \left\langle \widehat{e}_m, x_{n_1}^* \right\rangle & ... & \left\langle \widehat{e}_m, x_{n_m}^* \right\rangle \end{vmatrix}, \quad n \in N_m \qquad (11)$$

On the other hand,

$$\begin{aligned} \langle \widehat{e}_k, x_n^* \rangle &= \beta_{nk}, \quad k = \overline{1, m}; \\ \left\langle \widehat{e}_k, x_{n_j}^* \right\rangle &= \beta_{n_j k}, \quad j, k = \overline{1, m}. \end{aligned}$$

Taking into account the last equalities in (11) and passing to the first components we get (9) . Biorthogonality conditions are verified immediately.

Now, all the statements of the theorem follow from expansion (10) and appropriate statements of theorem 4.

Remark. Statement of theorem 4 by another method is shown in the paper [1] as well.

We note also the papers [3, 4] concerning the theme of the paper.

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Telman B. Gasymov, Tarlan Z. Garayev
Institute of Mathematics and Mechanics of NAS of Azerbaijan.
9, F. Agayev str., AZ1141, Baku, Azerbaijan.
Tel.: (99412) 439-47-20 (off.).

Received November 22, 2006 ; Revised February 15, 2007.