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ON OSCILLATION PROPERTIES OF THE EIGENFUNCTIONS OF A FOURTH ORDER DIFFERENTIAL OPERATOR

Abstract

The spectral problem for a fourth order ordinary differential operator is investigated. The oscillation properties of the eigenfunctions and their derivatives are established.

Let's consider the boundary-value problem

$$\left(p\left(x\right)y''\right)'' - \left(q\left(x\right)y'\right)' = \lambda\rho\left(x\right)y, \quad 0 < x < l, \tag{1}$$

$$y'\left(0\right)\cos\alpha - \left(py''\right)\left(0\right)\sin\alpha = 0, \tag{2.a}$$

$$y\left(0\right)\cos\beta + Ty\left(0\right)\sin\beta = 0, \tag{2.b}$$

$$y'\left(l\right)\cos\gamma + \left(py''\right)\left(l\right)\sin\gamma = 0, \tag{2.c}$$

$$y\left(l\right)\cos\delta - Ty\left(l\right)\sin\delta = 0, \tag{2.d}$$

where λ is a spectral parameter, the functions $p\left(x\right), q\left(x\right), \rho\left(x\right)$ are strictly positive and continuous on $[0,l]$, $p\left(x\right)$ has absolutely continuous derivative, $q\left(x\right)$ is absolutely continuous on $[0,l]$ and $\alpha,\beta,\gamma,\delta$ are real constants, such that $0 \leq \alpha,\beta,\gamma \leq \pi/2, \pi/2 < \delta < \pi$ and

$$Ty = \left(py''\right)' - qu'. \tag{3}$$

The present paper is devoted to study of oscillation properties of the eigenfunctions of oscillation properties of the eigenfunctions of boundary-value problem (1)-(2). The basic result of this paper is the oscillation theorem (theorem 4).

The oscillation properties of the eigenfunctions of boundary-value problem (1)-(2) provided $0 \leq \delta \leq \pi/2$ have been investigated in detail in [1]. In this work it is investigated only positive eigenvalues and corresponding eigenfunctions of problem (1)-(2). In this connection in the paper [1] the following two cases are excluded: (i) $\alpha = \gamma = 0$ and $\beta = \delta = \pi/2$, (ii) any three of parameters $\alpha,\beta,\gamma,\delta$ are equal to $\pi/2$. In reality, only the case $\beta = \delta = \pi/2$ is to be excluded. Let's prove this.

It is known, that the least eigenvalue of boundary-value problem (1)-(2) is a minimum of Relay's ration

$$R\left[y\right] = \left(\int\limits_0^l \left(py''^2 + qy'^2\right)dx + N\left[y\right]\right)\left(\int\limits_0^l \rho y^2 dx\right)^{-1}, \tag{4}$$

where $N\left[y\right]$ is a functional, which takes only nonnegative values (see [2, p.160) or [1, p.64]).

Let $\beta = \delta = \pi/2$. The direct testing shows, that the function $y(x) \equiv c_0 = const \neq 0$ $(x \in [0,l])$ is an eigenfunction of boundary-value problem (1)-(2), corresponding to the eigenvalue $\lambda = 0$. The simplicity of the eigenvalue $\lambda = 0$ follows from the fact, that the corresponding eigenfunction $y(x)$ must satisfy the relation $y'(x) \equiv 0$ $(x \in [0,l])$ (see (4)).

Let $\lambda = 0$ is an eigenvalue of boundary-value problem (1)-(2). From formula (4) it follows, that for the corresponding eigenfunction $y(x)$ it is true $y'(x) \equiv 0$ $(x \in [0,l])$, that is equivalent to $y(x) \equiv c_0 = const \neq 0$ $(x \in [0,l])$. Boundary conditions (2a) and (2c) are automatically satisfied at that. For fulfillment of boundary conditions (2b) and (2d) the condition $\beta = \delta = \pi/2$ is to be fulfilled.

As in [1], to study the oscillation properties of eigenfunctions and their derivatives we'll use the Prufer-type transformation

$$u(x) = r(u)\sin\psi(x)\cos\theta(x), \tag{5.a}$$

$$u'(x) = r(x)\cos\psi(x)\sin\varphi(x), \tag{5.b}$$

$$\left(pu''\right)(x) = r(x)\cos\psi(x)\cos\varphi(x), \tag{5.c}$$

$$Tu(x) = r(x)\sin\psi(x)\sin\theta(x). \tag{5.d}$$

Let's write equation (1) in equivalent form

$$U' = MU, \tag{6}$$

where

$$U = \begin{pmatrix} y \\ y' \\ py'' \\ Ty \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1/p & 0 \\ 0 & q & 0 & 1 \\ \lambda\rho & 0 & 0 & 0 \end{pmatrix}.$$

Assuming $w(x) = ctg\psi(x)$ and using transformation (5) in (6) we'll obtain the system of first order differential equations with respect to the functions r, w, θ, φ of the following form:

$$r' = [\sin 2\psi\theta\sin\varphi + \left(q + \frac{1}{p}\right)\cos^2\psi\sin 2\varphi +$$

$$+\sin 2\psi\sin\theta\cos\varphi + \frac{\lambda\rho}{2}\sin^2\psi\sin 2\theta]\frac{r}{2}, \tag{7.a}$$

$$w' = -w^2\cos\theta\sin\varphi + \frac{1}{2}\left(q + \frac{1}{p}\right)w\sin 2\varphi + \sin\theta\cos\varphi - \frac{\lambda\rho}{2}w\sin 2\theta, \tag{7.b}$$

$$\theta' = -w\sin\varphi\sin\theta + \lambda\rho\cos^2\theta, \tag{7.c}$$

$$\varphi' = \frac{1}{p}\cos^2\varphi - q\sin^2\varphi - \frac{1}{w}\sin\theta\sin\varphi. \tag{7.d}$$

Let's cite some statements from [1].

Lemma 1. (see [1], p.59, lemma 2.1). *Let $y(x,\lambda)$ be a nontrivial solution of differential equation (1) at $\lambda > 0$. If y, y', y'' and Ty are nonnegative at $x = a$ (but not all zero), then they all are positive for $x > a$. If $y, -y', y''$* and *$-Ty$ are nonnegative at $x = a$ (but not all zero), then they all are positive for $x < a$.*

Theorem 1. (see [1], p.61, theorem 3.1). *Let $y(x,\lambda)$ be a nontrivial solution of problem (1), (2.a), (2.c) at $\lambda > 0$. Then the Jacobian $J[y] = r^3 \cos\psi$ $\sin\psi$ of the transformation (5) does not vanish in $(0,l)$.*

The following lemma holds:

Lemma 2. *At every fixed $\lambda \in \mathbf{C}$ there exits the unique (to within constant factor) nontrivial solution $y(x,\lambda)$ of problem (1), (2.a), (2.b), (2.c).*

Proof. Denote by $\varphi_k(x,\lambda)$ $\left(k = \overline{1,4}\right)$ the solutions of equation (1), normalized at $x = 0$ by Cauchy conditions

$$\varphi_k^{(s-1)}(0,\lambda) = \delta_{ks} \quad \left(s = \overline{1,3}\right), \quad T\varphi_k(0,\lambda) = \delta_{k4}, \tag{8}$$

where δ_{ks} is a Kronecker's symbol.

We'll search the function $y(x,\lambda)$ in the form

$$y(x,\lambda) = \sum_{k=1}^{4} C_k \varphi_k(x,\lambda), \tag{9}$$

where C_k $\left(k = \overline{1,4}\right)$ are some constants.

Suppose, that in boundary conditions (2.a), (2.b), (2,c) $\alpha \neq 0$, $\beta \neq 0$, $\gamma \neq 0$. From (8), (9) and from boundary conditions (2.a), (2.b) it follows, that

$$C_3 = \frac{C_2}{p(0)} ctg\alpha, \quad C_4 = -C_1 ctg\beta$$

holds. From here and from (9) we'll obtain

$$y(x,\lambda) = C_1 \{\varphi_1(x,\lambda) - \varphi_4(x,\lambda)\, ctg\beta\} + C_2 \left\{\varphi_2(x,\lambda) + \varphi_3(x,\lambda) \frac{ctg\alpha}{p(0)}\right\}. \tag{10}$$

Taking into account (8), (10) and (2.c) for definition of C_1 and C_2 we'll obtain the relation

$$C_1 \alpha^*(\lambda) + C_2 \beta^*(\lambda) = 0,$$

where

$$\alpha^*(\lambda) = \left\{\varphi_1'(l,\lambda)\, ctg\gamma + p(l)\, \varphi_1''(l.\lambda)\right\} - ctg\beta \left\{\varphi_4'(l.\lambda)\, ctg\gamma + p(l)\, \varphi_4''(l,\lambda)\right\}, \tag{11}$$

$$\beta^*(\lambda) = \left\{\varphi_2'(l,\lambda)\, ctg\gamma + p(l)\, \varphi_2''(l.\lambda)\right\} - \frac{ctg\alpha}{p(0)} \left\{\varphi_3'(l.\lambda)\, ctg\gamma + p(l)\, \varphi_3''(l,\lambda)\right\}. \tag{12}$$

To complete the proof of lemma 2 in the considered case it suffices to show, that it holds

$$|\alpha^*(\lambda)| + |\beta^*(\lambda)| > 0. \tag{13}$$

From lemma 1 and from (8) it follows, that at $\lambda > 0$ the inequalities $\varphi'_k(l,\lambda) > 0$, $\varphi''_k(l,\lambda) > 0$ $\left(k=\overline{1,4}\right)$ are true. From here and from (12) obtain the truth of (13) at $\lambda > 0$.

Let $\lambda \in \mathbf{C}/\mathbf{R}^+$. Let's prove the truth of (13). Really, otherwise the functions

$$\phi_1(x,\lambda) = \varphi_1(x,\lambda) - ctg\beta\varphi_4(x,\lambda), \ \phi_2(x,\lambda) = \varphi_2(x,\lambda) + \frac{ctg\alpha}{p(0)}\varphi_3(x,\lambda) \tag{14}$$

are the solutions of problem (1), (2.a), (2.b), (2.c). It is obvious, that any linear combination of the functions $\phi_1(x,\lambda)$ and $\phi_2(x,\lambda)$ is also the solution of this problem. The eigenvalues of boundary-value problem (1)-(2) at $\delta = 0$ are positive (see [1] or theorem 3 of the given paper). Hence, $\phi_1(l,\lambda) \neq 0$ and $\phi_2(l,\lambda) \neq 0$. Let's define the function $\upsilon(x,\lambda)$ by the following way:

$$\upsilon(x,\lambda) = \phi_1(x,\lambda)\phi_2(l,\lambda) - \phi_2(x,\lambda)\phi_1(l,\lambda).$$

It is obvious, that $\upsilon(l,\lambda) = 0$. Then the function $\upsilon(x,\lambda)$ is an eigenfunction of problem (1)-(2) at $\delta = 0$, corresponding to the eigenvalue $\lambda \in \mathbf{C}/\mathbf{R}^+$. The obtained contradiction proves the truth of (13).

The rest cases are considered similarly. Lemma 2 is proved.

Remark 1. *From proof of lemma 2 it is obvious, that solution of problem (1), (2.a), (2.b), (2.c), i.e. the function $y(x,\lambda)$ for each fixed $x \in [0,l]$ may be considered an entire function of λ. In particular, in the case $\alpha \neq 0$, $\beta \neq 0$, $\gamma \neq 0$, the function $y(x,\lambda)$ has the form*

$$y(x,\lambda) = \beta^*(\lambda)\phi_1(x,\lambda) - \alpha^*(\lambda)\phi_2(x,\lambda),$$

where $\alpha^*(\lambda), \beta^*(\lambda), \phi_1(x,\lambda), \phi_2(x,\lambda)$ are defined by relations (11), (12) and (14). As the functions $\varphi_k(x,\lambda)$ $\left(k=\overline{1,4}\right)$ and their derivatives for each fixed $x \in [0,l]$ are entire functions of λ, then $y(x,\lambda)$ for each fixed $x \in [0,l]$ is also an entire function of λ.

Lemma 3. *The eigenvalues of boundary-value problem (1)-(2) are real and form no more than countable set, having no finite limit points. All eigenvalues of boundary-value problem (1)-(2) are simple.*

Proof. The reality of eigenvalues follows from self-adjointness of boundary-value problem (1)-(2).

Let $y(x,\lambda)$ be a solution of problem (1), (2.a), (2.b), (2.c). Then the eigenvalues of problem (1)-(2) are the roots of the equation

$$\Phi(\lambda) \equiv y(l,\lambda)\cos\delta - Ty(l,\lambda)\sin\delta = 0. \tag{15}$$

The entire function $\Phi(\lambda)$ doesn't vanish at nonreal λ. Consequently, it is not equal to zero identically. Therefore, its zeros form no more than countable set, having no finite limit point.

By virtue of (1) we have

$$(Ty(x,\mu))'\,y(x,\lambda) - (Ty(x,\lambda))'\,y(x,\mu) = (\mu-\lambda)\rho(x)\,y(x,\lambda)\,y(x,\mu).$$

Integrating this identity in limits from 0 to l, using the formula of integration by parts and taking into account (2.a), (2.b), (2.c) we obtain

$$y(l,\lambda)\,Ty(l,\mu)-y(l,\mu)\,Ty(l,\lambda)=(\mu-\lambda)\int_0^l\rho(x)\,y(x,\lambda)\,y(x,\mu)\,dx. \qquad (16)$$

Deriving the both parts of (16) by $(\mu-\lambda)$ and by the next limiting passage as $\mu\to\lambda$ we'll obtain

$$y(l,\lambda)\frac{\partial}{\partial\lambda}Ty(l,\lambda)-Ty(l,\lambda)\frac{\partial}{\partial\lambda}y(l,\lambda)=\int_0^l\rho(x)\,y^2(x,\lambda)\,dx\ . \qquad (17)$$

Let's prove, that equation (15) has only simple roots. Really, if $\lambda=\lambda^*$ is a multiple root of equation (15), then the equalities

$$y(l,\lambda^*)\cos\delta-Ty(l,\lambda^*)\sin\delta=0,$$

$$\cos\delta\frac{\partial}{\partial\lambda}y(l,\lambda^*)-\sin\delta\frac{\partial}{\partial\lambda}Ty(l,\lambda^*)=0$$

hold.

Using the last two equalities in (17) at $\lambda=\lambda^*$ we have $\int_0^l\rho(x)\,y^2(x,\lambda^*)\,dx=0$, that is contradiction. Lemma 3 is proved.

Lemma 4. *Let $y(x,\lambda)$ be a nontrivial solution of problem (1), (2.a), (2.b), (2.c) and one of the following conditions be fulfilled: (i) $\lambda<0$; (ii) $\lambda=0$, $\beta\in[0,\pi/2)$. Then Jacobian $J[y]=r^3\cos\psi\sin\psi$ of the transformation (5) does not vanish in $(0,l)$.*

Proof. Suppose, that the statement of lemma 4 is not true and at the some point $x_1\in(0,l)$ it holds $\sin\psi\cos\psi=0$. The following cases are possible: (a) $\sin\psi(x_1,\lambda)=0$; (b) $\cos\psi(x_1,\lambda)=0$.

Let $\lambda<0$. Let's consider case (a). Then by virtue of (5) it holds $y(x_1,\lambda)=Ty(x_1,\lambda)=0$. Suppose, that $y(x,\lambda)>0$ at the left neighbourhood $U(x_1)$ of the point x_1. Then from (1) it follows, that $(Ty(x,\lambda))'<0$ at $x\in U(x_1)$. So, $Ty(x,\lambda)>0$ at $x\in U(x_1)$. From (2.b) it follows, that $y(0,\lambda)\,Ty(0,\lambda)\le 0$. Then there exists the point $x_0\in[0,x_1)$ such that $y(x_0,\lambda)\,Ty(x_0,\lambda)=0$ and

$$y(x,\lambda)\,Ty(x,\lambda)>0 \qquad (x_0<x<x_1)\,. \qquad (18)$$

Let $Ty(x_0,\lambda)=0$. Hence, there exists the point $\xi_0\in(x_0,x_1)$ such that $(Ty(x,\lambda))'_{x=\xi_0}=0$. From here and from equation (1) we obtain $y(\xi_0,\lambda)=0$. The last equality contradicts to inequality (18).

Let $y(x_0,\lambda)=0$. Hence, there exists the point $\eta_0\in(x_0,x_1)$ such that $y'(\eta_0,\lambda)=0$. It is obvious, that $y(\eta_0,\lambda)>0$, $Ty(\eta_0,\lambda)>0$. Let's define the number $\delta_0\in\left(0,\frac{\pi}{2}\right)$ by the following way: $\delta_0=arctg\frac{y(\eta_0,\lambda)}{Ty(\eta_0,\lambda)}$. So, the function $y(x,\lambda)$ is a solution of boundary-value problem (1)-(2) at $l=\eta_0$, $\gamma=0$, $\delta=\delta_0$. As the eigenvalues of boundary-value problem (1)-(2) at $l=\eta_0$, $\gamma=0$, $\delta=\delta_0$ are positive, then we obtain the contradiction. Hence, $\sin\psi(x,\lambda)\neq 0 \quad (0<x<l)$ at $\lambda<0$.

Let $\lambda < 0$ and (b) hold. By virtue of (5) we have $y'(x_1,\lambda) = y''(x_1,\lambda) = 0$. It is obvious, that $y(x_1,\lambda) \neq 0$ and $Ty(x_1,\lambda) \neq 0$. Really, if $y(x_1,\lambda) = 0$, then $y(x,\lambda)$ is an eigenfunctions of boundary-value problem (1)-(2) at $\gamma = \pi/2$, $\delta = 0$, $l = x_1$, that contradicts to the condition $\lambda < 0$. By the similar way the case $Ty(x_1,\lambda) = 0$ is excluded.

As $Ty(x_1,\lambda) \neq 0$, then it is obvious, that the point x_1 is a point of local extremum of the function $y'(x,\lambda)$. Suppose, that $y'(x,\lambda) > 0$ at the deleted neighbourhood $V(x_1)$ of the point x_1. Then $y''(x,\lambda) < 0$ at the left neighbourhood $V^-(x_1)$ of the point x_1 and $y''(x,\lambda) > 0$ at the right neighbourhood $V^+(x_1)$ of the point x_1. From here and from condition (2.a) it follows, that there exists the point $x_0 \in [0,x_1)$ such that $y'(x_0,\lambda)\, y''(x_0,\lambda) = 0$ and

$$y'(x,\lambda) > 0, \quad y''(x,\lambda) < 0 \qquad (x \in (x_0,x_1)). \tag{19}$$

Suppose, that $y'(x_0,\lambda) = 0$. Then there exists the point $\xi_0 \in (x_0,x_1)$ such that $y''(\xi_0,\lambda) = 0$. The last relation contradicts to (19).

Let $y''(x_0,\lambda) = 0$. Then there exists the point $\xi_0 \in (x_0,x_1)$ such that $(p(x)\, y''(x,\lambda))'_{x=\xi_0} = 0$. From (19) it follows, that

$$Ty(x_0,\lambda) = \big(p(x)\, y''(x,\lambda)\big)'_{x=\xi_0} - q(\xi_0)\, y'(\xi_0,\lambda) < 0.$$

Besides, $Ty(x_1,\lambda) = (p(x)\, y''(x,\lambda))'_{x=x_1} - q(x_1)\, y'(x_1,\lambda) = p(x_1)\, y'''(x_1,\lambda) > 0$. Hence, there exists the point $\eta_0 \in (\xi_0,x_1)$ such that $Ty(\eta_0,\lambda) = 0$.

We'll define the number $\gamma_0 \in \left(0, \frac{\pi}{2}\right)$ by the following equality:

$$\gamma_0 = -arctg \frac{p(\eta_0)\, y''(\eta_0,\lambda)}{y'(\eta_0,\lambda)}.$$

It is easy to check, that $y(x,\lambda)$ is an eigenfunction of boundary-value problem (1)-(2) at $\gamma = \gamma_0$, $\delta = \pi/2$, $l = \eta_0$, that contradicts to the condition $\lambda < 0$.

Let now $\lambda = 0$, $\beta \in [0,\pi/2)$. Let's consider case (a). From (1) it follows, that $Ty(x,0) \equiv const$ $(0 \le x \le l)$. Hence, by virtue (5.d) we have: $Ty(x,0) \equiv 0$ $(0 \le x \le l)$. Multiplying this equality by the function $y(x,\lambda)$ and integrating the obtained identity from 0 to l, we obtain

$$p(l)\, y''(l,0)\, y'(l,0) - p(0)\, y''(0,0)\, y'(0,0) -$$
$$- \int_0^l \big(p(x)\, y''^2(x,0) + q y'^2(x,0)\big)\, dx = 0. \tag{20}$$

By virtue of conditions (2.a) and (2.c) we have

$$p(l)\, y''(l,0)\, y'(l,0) \le 0, \quad p(0)\, y''(0,0)\, y'(0,0) \ge 0. \tag{21}$$

From here and from (20) we obtain, that $y(x,0) \equiv const$. As $\sin\psi(x_1,0) = 0$, then $y(x,0) \equiv 0$ $(0 \le x \le l)$, that is contradiction.

Let $\lambda = 0$, $\beta \in [0, \pi/2)$ and $\cos\psi(x_1, 0) = 0$, where x_1 is some point from $(0, l)$. By virtue of (5) we have

$$y'(x_1, 0) = y''(x_1, 0) = 0. \tag{22}$$

Let's prove, that in the considered case $Ty(0,0) \neq 0$. Really, if $Ty(0,0) = 0$, then from (2.b) it follows, that $y(0,0) = 0$. Besides, from (1) obtain, that $Ty(x,0) \equiv const = 0$ $(0 \le x \le l)$. Using (20), (21) and taking into account the equality $y(0,0) = 0$, we conclude, that $y(x,0) \equiv 0$ $(0 \le x \le l)$. The last is contradiction.

As $Ty(x,0) = Ty(0,0) \neq 0$ $(0 \le x \le l)$, then from (3) it follows, that $y'''(x_1, 0) \neq 0$. So, x_1 is a double zero of the function $y'(x, \lambda)$. Without losing generality, it is possible to consider, that $y'''(x_1, 0) > 0$. Hence, $Ty(x_1, 0) = p(x_1)\, y'''(x_1, 0) > 0$ and besides, at the some right neighbourhood of the point x_1 it holds

$$y'(x,0) > 0, \quad y''(x,0) > 0. \tag{23}$$

Let's assume, that (x_1, l_0) is an interval of maximum length, where inequality (23) is true. It is obvious, that $y'(l_0, 0) \ge 0$, $y''(l_0, 0) \ge 0$.

Let $y'(l_0, 0) = 0$. Then from (22) it follows, that for some point $\xi \in (x_1, l_0)$ it holds $y''(\xi, 0) = 0$. The last contradicts to (23).

Let $y''(l_0, 0) = 0$. As $p(x_1)\, y''(x_1, 0) = p(l_0)\, y''(l_0, 0) = 0$, then again there exists the point $\xi \in (x_1, l_0)$ such that $(p(x)\, y''(x,0))'_{x=\xi} = 0$. Hence $Ty(\xi, 0) = (p(x)\, y''(x,0))'_{x=\xi} - q(\xi)\, y'(\xi, 0) < 0$. On the other hand it holds $Ty(x,0) \equiv const = Ty(x_1, 0)$ $(0 \le x \le l)$, that is contradiction.

So, we've shown, that $l_0 = l$ and $y'(l, 0) > 0$, $y''(l, 0) > 0$. The last contradicts to condition (2.c). The proof of lemma 4 is completed.

Let $y(x, \lambda)$ be a nontrivial solution of problem (1), (2.a), (2.b), (2.c) and either $\lambda \in \mathbf{R}/\{0\}$, or $\lambda = 0$ and $\beta \in [0, \pi/2)$. Suppose, that $\theta(x, \lambda)$ and $\varphi(x, \lambda)$ are corresponding functions from (5). Without losing generality, we can define the initial value of these functions by the following way:

$$\theta(0, \lambda) = \beta - \frac{\pi}{2}, \tag{24}$$

$$\varphi(0, \lambda) = \alpha. \tag{25}$$

The proof of this fact is completely made by scheme of the proof of theorem 3.1 from [3] (see theorem 3.3 from [1]).

The following two statements are proved in [1].

Theorem 2. (see theorem 4.2 from [1]). *Let $y(x, \lambda)$ be a nontrivial solution of problem (1), (2.a), (2.b), (2.c) at $\lambda > 0$. Then $\theta(l, \lambda)$ is a strictly increasing continuous function of λ.*

Theorem 3. (see theorems 5.4 and 5.5 from [1]). *The eigenvalues of boundary-value problem (1)-(2) at $\delta \in [0, \pi/2]$ (except the case $\beta = \delta = \pi/2$) form infinitely increasing sequence $\{\mu_k(\delta)\}_1^\infty$ such that*

$$0 < \mu_1(\delta) < \mu_2(\delta) < ... < \mu_n(\delta) < ...,$$

$$\theta\left(1,\mu_n(\delta)\right)=(2n-1)\frac{\pi}{2}-\delta. \tag{26}$$

Besides, the eigenfunction $\vartheta_n^{\delta}(x)$, *corresponding to the eigenvalue* $\mu_n(\delta)$, *has exactly* $(n-1)$ *simple zeros in the interval* $(0,l)$, *and the function* $T\vartheta_n^{\delta}(x)$ *has exactly* n *zeros on the segment* $[0,l]$.

Remark 2. *In the case* $\beta=\delta=\pi/2$ *the first eigenvalue of boundary problem (1)-(2) is equal to zero and the corresponding eigenfunction is constant. In this case the statement of theorem 3 is true at* $n\geq 2$.

Obviously, the eigenvalues $\mu_n=\mu_n(0)$ and $\nu_n=\mu_n\left(\frac{\pi}{2}\right)$ $(n\in\mathbf{N})$ are zeros of the entire functions $y(l,\lambda)$ and $Ty(l,\lambda)$, respectively. Besides we note that by theorem 2 and equality (23) the relation $\nu_1<\mu_1<\nu_2<\mu_2<...$ is valid.

Let's consider the function $\dfrac{Ty(l,\lambda)}{y(l,\lambda)}$ at $\lambda\in K\equiv\bigcup\limits_{k=0}^{\infty}\left(\mu_k,\mu_{k+1}\right)$, where $\mu_0=-\infty$. From (16) at $\lambda,\mu\in K$ we have

$$\frac{Ty(l,\mu)}{y(l,\mu)}-\frac{Ty(l,\lambda)}{y(l,\lambda)}=(\mu-\lambda)\frac{\int\limits_0^l\rho(x)\,y(x,\mu)\,y(x,\lambda)\,dx}{y(l,\mu)\,y(l,\lambda)}. \tag{27}$$

Deriving both parts of (27) by $(\mu-\lambda)$ and by the next limiting passage as $\mu\to\lambda$ we'll obtain

$$\frac{\partial}{\partial\lambda}\left(\frac{Ty(l,\lambda)}{y(l,\lambda)}\right)=\frac{\int\limits_0^l\rho(x)\,y^2(x,\lambda)\,dx}{y^2(l,\lambda)}>0. \tag{28}$$

So, we proved the following statement.

Lemma 5. *The function* $\dfrac{Ty(l,\lambda)}{y(l,\lambda)}$ *in each of the interval* $\left(\mu_k,\mu_{k+1}\right)$ $(k=0,1,2,...)$ *is a strictly increasing function of* λ.

Lemma 6. *Let* $y(x,\lambda)$ *be a nontrivial solution of problem (1), (2.a), (2.b), (2.c). Then it holds the relation*

$$\lim_{\lambda\to-\infty}\frac{Ty(l,\lambda)}{y(l,\lambda)}=-\infty. \tag{29}$$

Proof. Without losing generality, it may be considered that $\int\limits_0^l\rho(x)\,y^2(x,\lambda)\,dx=1$. As it is proved in [4, p.353-354] it holds the inequality

$$y^2(l,\lambda)\leq c_0\sqrt{\int\limits_0^1 q(x)\,y'^2(x,\lambda)\,dx}+c_1, \tag{30}$$

where c_0 and c_1 are positive constants, dependent only on the functions $q(x)$ and $\rho(x)$.

Multiplying both parts of (1) by the function $y(x,\lambda)$ and integrating this identity by x in the limits from 0 to l, we'll obtain

$$y(l,\lambda)\,Ty(l,\lambda)-y(0,\lambda)\,Ty(0,\lambda)-p(l)\,y'(l,\lambda)\,y''(l,\lambda)+$$

$$+p(0)\,y'(0,\lambda)\,y''(0,\lambda)+\int_0^l q(x)\,y'^2(x,\lambda)\,dx+\int_0^l \rho(x)\,y''^2(x,\lambda)\,dx=\lambda. \qquad (31)$$

By virtue of boundary conditions (2.a), (2.b), (2.c) the inequalities

$$p(l)\,y'(l,\lambda)\,y''(l,\lambda)\le 0,\quad y(0,\lambda)\,Ty(0,\lambda)\le 0,\quad p(0)\,y'(0,\lambda)\,y''(0,\lambda)\ge 0$$

are true. From here and from (31) it follows, that

$$\lim_{\lambda\to-\infty} y(l,\lambda)\,Ty(l,\lambda)=-\infty. \qquad (32)$$

From lemma 5 it implies, that as $\lambda\to-\infty$, the ratio $\dfrac{Ty(l,\lambda)}{y(l,\lambda)}$ has finite or infinite limit. Suppose, that

$$\lim_{\lambda\to-\infty}\frac{Ty(l,\lambda)}{y(l,\lambda)}=-a_0, \qquad (33)$$

where $0<a_0<+\infty$. Taking into account (32) and (33) we'll obtain, that $\lim\limits_{\lambda\to-\infty} y^2(l,\lambda)=+\infty$. From here and from (30) we have

$$\lim_{\lambda\to-\infty}\int_0^l q(x)\,y'^2(x,\lambda)\,dx=+\infty. \qquad (34)$$

By virtue of (33) at the sufficiently large by module negative values of λ the inequality $\left|\dfrac{Ty(l,\lambda)}{y(l,\lambda)}\right|\le a_0$ is true. From here and from (31), (30) at those values of λ we'll obtain

$$\lambda\ge\int_0^l q(x)\,y'^2(x,\lambda)\,dx-|y(l,\lambda)\,Ty(l,\lambda)|\ge\int_0^l q(x)\,y'^2(x,\lambda)\,dx-a_0y^2(l,\lambda)\ge$$

$$\ge\int_0^l q(x)\,y'^2(x,\lambda)\,dx-a_0c_0\sqrt{\int_0^l q(x)\,y'^2(x,\lambda)\,dx}-a_0c_1\ge$$

$$\ge\sqrt{\int_0^l q(x)\,y'^2(x,\lambda)\,dx}\left(\sqrt{\int_0^l q(x)\,y'^2(x,\lambda)\,dx}-a_0c_0\right)-a_0c_1,$$

that by virtue of (34) is contradiction. Lemma 6 is proved.

Remark 3. *It is easy to note, that if* $\lambda<0$ *or* $\lambda=0$ *and* $\beta\in[0,\frac{\pi}{2})$, *then* $\dfrac{Ty(l,\lambda)}{y(l,\lambda)}<0$; *besides, if* $\lambda=0$ *and* $\beta=\dfrac{\pi}{2}$, *then* $Ty(l,\lambda)=0$.

Lemma 7. *Let* $y(x,\lambda)$ *be a nontrivial solution of problem (1), (2.a), (2.b), (2.c). If* $\lambda\le 0$, *then* $y(x,\lambda)\ne 0$ *at* $0<x<l$; *if* $\lambda<0$ *or* $\lambda=0$, $\beta\in[0,\frac{\pi}{2})$, *then* $Ty(x,\lambda)\ne 0$ *at* $0<x<l$.

Proof. Let $\theta(x,\lambda)$ be corresponding function from (4), where either $\lambda<0$, or $\lambda=0$ and $\beta\in[0,\frac{\pi}{2})$. From (24) it follows, that $\theta(0,\lambda)=\beta-\dfrac{\pi}{2}\in\left[-\dfrac{\pi}{2},0\right]$.

Let $\lambda = 0$ and $\beta \in [0, \frac{\pi}{2})$. By virtue of (1) we have $Ty(x,0) \equiv const$ $(0 \le x \le l)$. As on the base of remark 3 it is true the $y(l,0)\,Ty(l,0) < 0$, then it is obvious, that $Ty(x,0) \equiv c_0 \ne 0$ $(0 \le x \le l)$. So, $\theta(x,0) \ne k\pi$ $(k \in Z)$ at $0 \le x \le l$.

Let's note, that by virtue of equality (5.a) and (5.d) the following equality is true:

$$sgn\,(y(l,0)\,Ty(l,0)) = sgn\,(\sin\theta(l,0)\cos\theta(l,0)).$$

Hence

$$\theta(l,0) \in \left(-\frac{\pi}{2}, 0\right). \tag{35}$$

Let $\lambda < 0$. Let's prove, that $\theta(l,\lambda) \in \left(-\frac{\pi}{2}, 0\right)$. First of all suppose, that $\beta \in [0, \frac{\pi}{2})$. From (7.c) it follows, that the function $\theta(x,\lambda)$ takes the value of the form $k\pi$ $(k \in \mathbf{Z})$ strictly decreasing and therefore

$$\theta(x,\lambda) < 0 \quad (0 < x < l).$$

Let $\theta(l,\lambda) \in (-(m_0+1)\pi, -m_0\pi)$, where m_0 is some fixed nonnegative integer. As $y(l,\lambda)\,Ty(l,\lambda) < 0$, then it is obvious, that it holds

$$\theta(l,\lambda) \in \left(-m_0\pi - \frac{\pi}{2}, -m_0\pi\right). \tag{36}$$

If $m_0 = 0$, then $\theta(l,\lambda) \in \left(-\frac{\pi}{2}, 0\right)$. Suppose, that $m_0 \ge 1$. As $\theta(l,\lambda)$ is a continuous function of $\lambda \in (-\infty, +\infty)$, then by virtue of (35) and (36) we can state the existence of the point $\lambda_0 \in (\lambda, 0)$ such that $\theta(l,\lambda_0) \in \left(-\pi, -\frac{\pi}{2}\right)$. Hence and from (5.a), (5.d) we have $y(l,\lambda_0)\,Ty(l,\lambda_0) > 0$, that contradicts to remark 3. Consequently, in the considered case

$$\theta(l,\lambda) \in \left(-\frac{\pi}{2}, 0\right). \tag{37}$$

It is obvious, that $\theta(l,\lambda)$ is a continuous function on $\beta \in \left[0, \frac{\pi}{2}\right]$. Since $\theta(l,\lambda) \in \left(-\frac{\pi}{2}, 0\right)$ at $\lambda < 0$ and $\beta \in [0, \frac{\pi}{2})$, then $\theta(l,\lambda)\,|_{\beta=\pi/2} = \lim\limits_{\beta \to \frac{\pi}{2}-0} \theta(l,\lambda) \in \left[-\frac{\pi}{2}, 0\right]$. Then on the base of inequality $y(l,\lambda)\,Ty(l,\lambda) < 0$ we'll obtain, that $\theta(l,\lambda) \in \left(-\frac{\pi}{2}, 0\right)$ at $\beta = \frac{\pi}{2}$.

Suppose, that the statement of lemma, relating to the function $y(x,\lambda)$ is not true and let $x_1 \in (0,l)$ be nearest point to zero, at which $y(x_1,\lambda) = 0$.

Let's consider 5 cases.

Case 1. Let $\lambda < 0$ and $\beta \in \left(0, \frac{\pi}{2}\right)$. On the base of Lemma 4 from (5.a) it follows, that $\theta(x_1,\lambda) = -\frac{\pi}{2}$. Under the condition $y'(x_1,\lambda) = 0$ the function $y(x,\lambda)$ is a solution of boundary-value problem (1)-(2), where $l = x_1$ and $\gamma = \delta = 0$, that contradicts to the condition $\lambda < 0$. Hence, $y'(x_1,\lambda) \ne 0$. From here and from (5.b) we'll obtain, that $\varphi(x_1,\lambda) \ne 0$. On the base of (7.c), lemma 4 and definition of the function $w(x,\lambda)$ it holds the relation $\theta'(x_1,\lambda) = -w(x_1,\lambda)\sin\varphi(x_1,\lambda) \ne 0$. Hence, $\theta'(x_1,\lambda) < 0$. As $\theta(l,\lambda) \in \left(-\frac{\pi}{2}, 0\right)$, then there exists the point $x_2 \in (x_1, 1)$ such that $\theta(x_2,\lambda) = -\frac{\pi}{2}$ (it is considered, that x_2 is a point, having this property and closest to x_1). So, $y(x_1,\lambda) = y(x_2,\lambda) = 0$. Then at the some point $\xi \in (x_1, x_2)$ we have $y'(\xi,\lambda) = 0$. Let's note, that at $x \in (x_1, x_2)$ it is true the $\theta(x,\lambda) \in \left(-\pi, -\frac{\pi}{2}\right)$. From here and from relations (5.a), (5.d) we'll obtain

$$y(x,\lambda)\,Ty(x,\lambda) = r^2(x,\lambda)\sin^2\psi(x,\lambda)\cos\theta(x,\lambda)\sin\theta(x,\lambda) > 0, \tag{38}$$

where $0 < x_1 < x < x_2 < l$.

Let's define the angle δ_1 by the following way: $\delta_1 = arctg\frac{Ty(\xi,\lambda)}{y(\xi,\lambda)}$. By virtue of (38) it holds $\delta_1 \in \left(0, \frac{\pi}{2}\right)$.

It is easy to note, that the function $y(x,\lambda)$ is nontrivial solution of boundary-value problem (1)-(2), where $l = \xi$ and $\gamma = 0$, $\delta = \delta_1$. The last contradictions to the condition $\lambda < 0$.

Case 2. Let $\lambda = 0$ and $\beta \in \left(0, \frac{\pi}{2}\right)$. Then $Ty(x,\lambda) \equiv c_0 \neq 0$ $(0 \le x \le l)$, $\theta(0,\lambda) \in \left(-\frac{\pi}{2}, 0\right)$, $\theta(l,\lambda) \in \left(-\frac{\pi}{2}, 0\right)$. Hence, $\theta(x,\lambda) \in (-\pi, 0)$. Then the proof is made similarly to the proof of case 1.

Case 3. Let $\lambda < 0$ and $\beta = 0$. Then $\theta(0,\lambda) = -\frac{\pi}{2}$. By virtue of (37) and by virtue of the fact that $\theta(x,\lambda)$ takes the value of the form $k\pi$ $(k \in \mathbf{Z})$ strictly decreasing, then it holds either

$$-\frac{\pi}{2} < \theta(x,\lambda) < 0 \qquad (0 < x < x_1), \tag{39}$$

or inequality

$$-\pi < \theta(x,\lambda) < -\frac{\pi}{2} \qquad (0 < x < x_1). \tag{40}$$

At fulfillment of inequality (39) the proof of the statement $y(x,\lambda) \neq 0$ $(0 < x < l)$ is made similarly to the proof of case 1.

Let (40) hold. As $y(0,\lambda) = y(x_1,\lambda) = 0$, then at the some point $\xi \in (0, x_1)$ it holds $y'(\xi,\lambda) = 0$. Besides, relation (38) will be satisfied at $x \in (0, x_1)$. Then the proof of the statement $y(x,\lambda) \neq 0$ $(0 < x < l)$ is made similarly to the proof of case 1.

Case 4. Let $\lambda = 0$, $\beta = 0$. Then relations $\theta(0,0) = -\frac{\pi}{2}$, $\theta(l,0) \in -\left(\frac{\pi}{2}, 0\right)$, $Ty(x,0) \equiv c_0 \neq 0$ $(0 \le x \le l)$, $\theta(x,0) \in (-\pi, 0)$ $(0 < x < l)$ are true. Then again the proof is made similarly to the proof of case 1.

Case 5. And now let $\lambda = 0$ and $\beta = \frac{\pi}{2}$. From (2.b) it follows, that $Ty(0,0) = 0$. By virtue of (1) we have $Ty(x,0) \equiv 0$ $(0 \le x \le l)$ We have met the similar situation by proving lemma 4 (see (20) and (21))and there it was established, that $y(x,0) \equiv const$ $(0 \le x \le l)$. As $y(x_1,0) = 0$, then we have $y(x,0) \equiv 0$ $(0 \le x \le l)$. We obtain the contradiction.

In cases 1-4 practically it is proved, that if $\lambda < 0$ or $\lambda = 0$, $\beta \in [0, \frac{\pi}{2})$, then $\theta(x,\lambda) \in \left(-\frac{\pi}{2}, 0\right)$ at $0 < x < l$. Hence, by virtue of (5.d) we have $Ty(x,\lambda) \neq 0$ at $x \in (0, l)$. The proof of lemma 7 completed.

Now let's prove the basic result of the present paper.

Theorem 4. *The eigenvalues of boundary-value problem (1)-(2) at* $\delta \in \left(\frac{\pi}{2}, \pi\right)$ *form the infinitely increasing sequence* $\{\lambda_n(\delta)\}_{n=1}^{\infty}$ *such that*

$$\lambda_1(\delta) < \lambda_2(\delta) < ... < \lambda_n(\delta) < ...,$$

at that $\lambda_n(\delta) > 0$ *at* $n \geq 2$. *Besides*

a) the eigenfunction $y_n^\delta(x)$, *corresponding to the eigenvalue* $\lambda_n(\delta)$ *has exactly* $(n-1)$ *simple zeros in the interval* $(0, l)$;

b) if $\beta \in [0, \frac{\pi}{2})$, *then the function* $Ty_n^\delta(x)$ *has exactly* $(n-1)$ *simple zeros in the interval* $(0, l)$;

c) if $\beta = \frac{\pi}{2}$, *then the function* $Ty_1^\delta(x)$ *has no zeros in the interval* $(0, l)$, *and the function* $Ty_n^\delta(x)$ $(n \geq 2)$ *has exactly* $(n-2)$ *simple zeros in the interval* $(0, l)$;

d) if $\beta \in [0, \frac{\pi}{2})$, *then there exists* $\delta_0 \in (\pi/2, \pi)$ *such that* $\lambda_1(\delta) > 0$ *at* $\delta \in \left(\frac{\pi}{2}, \delta_0\right)$, $\lambda_1(\delta) = 0$ *at* $\delta = \delta_0$ *and* $\lambda_1(\delta) < 0$ *at* $\delta \in (\delta_0, \pi)$;

e) if $\beta = \frac{\pi}{2}$*,then* $\lambda_1(\delta) < 0$.

Proof. Let $y(x, \lambda)$ be a nontrivial solution of problem (1), (2.a), (2.c). The function $F(\lambda) = \dfrac{Ty(l, \lambda)}{y(l, \lambda)}$ by virtue of lemma 5 is a strictly increasing continuous function in the interval $(-\infty, \mu_1)$. From lemma 6 and from the equality $y(1, \mu_1) = 0$ it follows, that $\lim\limits_{\lambda \to -\infty} F(\lambda) = -\infty$, $\lim\limits_{\lambda \to -\mu_1 - 0} F(\lambda) = +\infty$ and besides, this function takes each value from $(-\infty, +\infty)$ only at unique point of the interval $(-\infty, \mu_1)$. Hence, there will be found a unique value $\lambda_1(\delta) \in (-\infty, \mu_1)$, for which $\dfrac{Ty(l, \lambda_1(\delta))}{y(l, \lambda_1(\delta))} = ctg\ \delta$, i.e. condition (2.d) is fulfilled. It is obvious, that $\lambda_1(\delta)$ is the first eigenvalue of problem (1)-(2). At $\beta \in [0, \frac{\pi}{2})$ it is easy to remark (see remark 3), that if $ctg\delta > \dfrac{Ty(l, 0)}{y(l, 0)}$, then $\lambda_1(\delta) > 0$; if $ctg\delta = \dfrac{Ty(l, 0)}{y(l, 0)}$, then $\lambda_1(\delta) = 0$; if $ctg\delta < \dfrac{Ty(l, 0)}{y(l, 0)}$ then $\lambda_1(\delta) < 0$. Let's note that the number δ_0 appearing in the formulation of theorem 4, is defined by equality $\delta_0 = arcctg\dfrac{Ty(l, 0)}{y(l, 0)}$.

Statement e) follows from the fact, that if $\beta = \frac{\pi}{2}$ and $\lambda = 0$, then $Ty(l, \lambda) = 0$ (see again remark 3).

Let $\beta \in [0, \frac{\pi}{2})$. The function $F(\lambda)$ at $\lambda \in [0, \mu_1)$ continuously increase from the negative value $\dfrac{Ty(l, 0)}{y(l, 0)}$ to $(+\infty)$. Then the equation $F(\lambda) = 0$ has unique solution $\nu_1 \in (0, \mu_1)$, which is the eigenvalue of problem (1)-(2) at $\delta = \frac{\pi}{2}$.

Let $\dfrac{Ty(l, 0)}{y(l, 0)} < ctg\delta$. Then it is true the inequality

$$0 < \lambda_1(\delta) < \nu_1 < \mu_1. \tag{41}$$

On the base of theorem 2 from (41) it follows, that $\theta(l, \lambda_1(\delta)) < \theta(l, \nu_1)$. Besides, by virtue of (26) we have $\theta(l, \nu_1) = 0$. Consecuently, $\theta(l, \lambda_1(\delta)) < 0$. It is obvious, that $\theta(l,\lambda_1(\delta)) > -\dfrac{\pi}{2}$. Really, otherwise for some $\lambda^* \in [\lambda_1(\delta),\mu_1)$ the equality $\theta(l,\lambda^*) = -\dfrac{\pi}{2}$ would be true and λ^* would be an eigenvalue of boundary-value problem (1)-(2) at $\delta = 0$, that is contradiction. So,

$$-\frac{\pi}{2} < \theta(l,\lambda_1(\delta)) < 0. \qquad (42)$$

It is known (see theorem 5.1 and 5.2 from [1]), that if $\lambda > 0$, that the function $\theta(x,\lambda)$ takes value of the form $\dfrac{k\pi}{2}$ $(k \in Z)$ only strictly increasing. Hence, from (42) it follows, that $-\frac{\pi}{2} < \theta(x,\lambda_1(\delta)) < 0$ at $0 < x < l$. The last is equivalent to that the functions $y_1^\delta(x) = y(x,\lambda_1(\delta))$ and $Ty_1^\delta(x)$ have no zeros in the interval $(0,l)$.

As was proved above, if $ctg\delta = \dfrac{Ty(l,0)}{u(l,0)}$, then $\lambda_1(\delta) = 0$; if $ctg\delta < \dfrac{Ty(l,0)}{u(l,0)}$, then $\lambda_1(\delta) < 0$.Then on the bases of lemma 7 the functions $y_1^\delta(x)$ and $Ty_1^\delta(x)$ have no zeros in the interval $(0,l)$.

In case $\beta = \dfrac{\pi}{2}$ we have $\lambda_1(\delta) < 0$. Consequently again by lemma 7 the functions $y_1^\delta(x)$ and $Ty_1^\delta(x)$ have no zeros in the interval $(0,l)$.

The function $F(\lambda)$ is strictly increasing continuous function in the interval (μ_k,μ_{k+1}), where k is a fixed natural number. As above, it is easy to be convinced, that there exists the unique value $\lambda_{k+1}(\delta) \in (\mu_k,\mu_{k+1})$, for which $0 > \dfrac{Ty(l,\lambda_{k+1}(\delta))}{y(l,\lambda_{k+1}(\delta))} = ctg\delta$. It is obvious, that $\lambda_{k+1}(\delta)$ is the $(k+1)$ the eigenvalue of problem (1)-(2).

In the interval (μ_k,μ_{k+1}) the equation $F(\lambda) = 0$ has a unique solution $\nu_{k+1} = \mu_{k+1}\left(\dfrac{\pi}{2}\right)$, where

$$\mu_k < \lambda_{k+1}(\delta) < \nu_{k+1} < \mu_{k+1}. \qquad (43)$$

On the base of theorem 2 from (43) it follows the inequality

$$\theta(l,\mu_k) < \theta(l,\lambda_{k+1}(\delta)) < \theta(l,\nu_{k+1}). \qquad (44)$$

Hence, by virtue of (26) from (44) we'll obtain

$$(2k-1)\frac{\pi}{2} < \theta(l,\lambda_{k+1}(\delta)) < 2k\frac{\pi}{2}. \qquad (45)$$

As above, using theorems 5.1., 5.2 from [1] and equalities (24), (25), it is easy conclude, that at $x \in (0,l)$ it holds

$$-\frac{\pi}{2} < \theta(x,\lambda_{k+1}(\delta)) < 2k\frac{\pi}{2}$$

and the function $\theta(x,\lambda_{k+1})$ in turn takes the values of the form $\dfrac{m\pi}{2}$ $(m = 1,2,...,2k)$ at increasing of the argument $x \in (0,l)$. It is obvious, that the eigenfunction $y_{k+1}^\delta(x)$ corresponding to the eigenvalue $\lambda_{k+1}(\delta)$, in the interval $(0,l)$ has k simple zeros; at the $\beta \in [0,\frac{\pi}{2})$ function $Ty_{k+1}^\delta(x)$ has k simple zeros in the interval $(0,l)$; at $\beta = \dfrac{\pi}{2}$ the function $Ty_{k+1}^\delta(x)$ has $(k-1)$ simple zeros in the interval $(0,l)$. Theorem 4 is proved.

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