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ON BASICITY IN $L_p(0,1)$ $(1<p<\infty)$ OF THE SYSTEM OF EIGENFUNCTIONS OF ONE BOUNDARY VALUE PROBLEM. I

Abstract

The basis properties of the spectral problem is investigated for differential operator of the second order with the spectral parameter in both boundary conditions. In this part of the paper the oscillation properties of eigenfunctions are established and the asymptotic formulae are derived for eigen values and eigenfunctions.

Introduction. Consider the spectral problem

$$-y'' + q(x)y = \lambda y, \quad 0<x<1, \tag{0.1}$$

$$(a_0\lambda + b_0)y(0) = (c_0\lambda + d_0)y'(0), \tag{0.2}$$

$$(a_1\lambda + b_1)y(1) = (c_1\lambda + d_1)y'(1), \tag{0.3}$$

where $\lambda-$ is a spectral parameter, $q(x)$ is a real-valued function from the class $C[0,1]$ and a_k, b_k, c_k, d_k $(k=0,1)$ are real constants.

The present paper is devoted to the investigation of basis properties in the spaces $L_p(0,1)$ $(1<p<\infty)$ of the system of eigenfunctions of boundary value problem (0.1)-(0.3) and consists of two parts: in the first part the oscillation properties of eigenfunctions of problem (0.1)-(0.3) are established and the asymptotic formulae for eigenvalues and eigenfunctions of this problem are derived; in the second part the basicity in $L_p(0,1)$ $(1<p<\infty)$ of system of eigen functions of boundary value problem (0.1)-(0.3) is investigated.

The boundary value problems for ordinary differential operators with the spectral parameter in boundary conditions in different statements were studied in different papers (see, for example [1-16]). In the paper [3] the list of papers is reduced in which such problems were considered in connection with the physical problems.

In [14] (see also [11, 12]) the basicity in $L_2(0,1)$ of system of eigenfunctions of the boundary value problem

$$-y'' + q(x)y = \lambda y, \quad 0<x<1,$$

$$y(0)=0, \quad (a-\lambda)y'(1) - b\lambda y(1) = 0,$$

is investigated in detail, where a and b are positive constants, $q(x)$ is continuous non-negative function on the segment $[0,1]$.

The basis properties of a system of eigenfunctions of boundary value problem (0.1)-(0.3) in $L_2(0,1)$ provided $a_0=c_0=0$, $|b_0|+|d_0|\neq 0$, $a_1d_1-b_1c_1>0$ were investigated completely in [6].

In the sequel everywhere we'll assume that $q(x)$ is a real-valued function from the class $C[0,1]$ and the following conditions

$$\sigma_0 = a_0d_0 - b_0c_0 < 0, \quad \sigma_1 = a_1d_1 - b_1c_1 > 0. \tag{0.4}$$

are fulfilled.

Note that the reality and the simplicity of eigenvalues of boundary value problem (0.1)-(0.3) under the condition (0.4) were proved in many papers (see, for example [3] and [8]).

1. Oscillation properties of eigenfunctions of boundary value problem (0.1)-(0.3).

To study the basicity properties in $L_p(0,1)$ of a system of eigenfunctions of boundary value problem (0.1)-(0.3) we need oscillation properties of solutions of this problem.

Along with boundary value problem (0.1)-(0.3) consider the following boundary value problems:

$$\begin{cases} -y'' + q(x)y = \lambda y, \quad 0 < x < 1, & (1.1') \\ y(0) = 0, \quad y(1) = 0; & (1.2') \end{cases}$$

$$\begin{cases} -y'' + q(x)y = \lambda y, \quad 0 < x < 1, & (1.1) \\ (a_0\lambda + b_0)y(0) = (c_0\lambda + d_0)y'(0), & (1.2) \\ y(1) = 0. & (1.3) \end{cases}$$

The eigenvalues of boundary value problem $(1.1')$-$(1.2')$ we denote by μ'_n $(n = 0, 1, 2, ...) : \mu'_0 < \mu'_1 < ... < \mu'_n < ....$

In case $c_0 \neq 0$ we determine the non-negative integer N_0 from the inequality

$$\mu'_{N_0-1} < -d_0/c_0 \le \mu'_{N_0} \tag{1.4}$$

(at this it is assumed that $\mu'_{-1} = -\infty$).

In [10] it is proved that the eigenvalues of boundary value problem (1.1)-(1.3) form unbounded increasing sequence $\{\mu_n\}_{n=0}^{\infty}$ $(\mu_0 < \mu_1 < \cdots < \mu_n < \cdots)$ and besides the following assertions are true:

(A) if $c_0 = 0$ then the eigenfunction $y_n(x)$ corresponding to the eigenvalue μ_n has exactly n simple zeros in the interval $(0,1)$;

(B) if $c_0 \neq 0$ then the eigenfunction $y_n(x)$ corresponding to the eigenvalue μ_n has at $n \le N_0$ exactly n, and at $n > N_0$ exactly $n-1$ simple zeros in the interval $(0,1)$, moreover from the inequality $\mu'_{N_0-1} < -d_0/c_0 < \mu'_{N_0}$ it follows the inequality at $\mu_{N_0} < -d_0/c_0 < \mu_{N_0+1}$, and from the equality $-d_0/c_0 = \mu'_{N_0}$ it follows the equality $-d_0/c_0 = \mu_{N_0+1}$.

Lemma 1.1. *Let $h_k(x) \in C[0,1]$ and $u_k(x)$ be a solution of the equation $u''_k + h_k(x)u_k = 0$ satisfying the conditions $u_k(0) = c_0\rho_k + d_0$, $u'_k(0) = a_0\rho_k + b_0$, $(k = 1, 2)$. Besides, let $h_1(x) < h_2(x)$ $(0 \le x \le 1)$ and one of the following conditions be fulfilled: (a') $c_0 = 0$, $\rho_1 < \rho_2$; (b') $c_0 \neq 0$, $\rho_1 < \rho_2 < -d_0/c_0$; (c') $c_0 \neq 0$, $-d_0/c_0 \le \rho_1 < \rho_2$. Then if $u_1(x)$ in interval $(0,1]$ has m zeros, then $u_2(x)$ in the same interval has no less than m zeros and k-th zero of $u_2(x)$ less than k-th zero of $u_1(x)$.*

Proof. The case (a') was considered in [8]. The cases (b') and (c') are considered absolutely analogously.

Let $y(x,\lambda)$ be a solution of equation (0.1) satisfying the initial conditions $y(0,\lambda) = c_0\lambda + d_0$, $y'(0,\lambda) = a_0\lambda + b_0$. Denote by $m_0(\lambda)$ the quantity of zeros of the function $y(x,\lambda)$ in the interval $(0,1)$.

Assume that n is an arbitrary fixed non-negative integer and $\mu_{-1}=-\infty$.

Lemma 1.2. *The following assertions are true:* (a_1) *if* $c_0=0$ *and* $\lambda\in(\mu_{n-1},\mu_n]$, *then* $m_0(\lambda)=n$; (b_1) *if* $c_0\neq0$, $\lambda\in(\mu_{n-1},\mu_n]$ *and* $n\leq N_0$, *then* $m_0(\lambda)=n$; (c_1) *if* $c_0\neq0$, $\lambda\in(\mu_{n-1},\mu_n]$ *and* $n\geq N_0+2$, *then* $m_0(\lambda)=n-1$; (d_1) *if* $c_0\neq0$ *and* $\lambda\in(\mu_{N_0},-d_0/c_0)$, *then* $m_0(\lambda)=N_0+1$; (e_1) *if* $c_0\neq0$ *and* $\lambda\in\left[-d_0/c_0,\mu_{N_0+1}\right]$, *then* $m_0(\lambda)=N_0$.

Proof. Let one of the conditions (a_1), (b_1), (c_1) be fulfilled and besides $\lambda\neq\mu_n$. Since $y(1,\lambda)\neq0$, then by virtue of lemma 1.1 the function $y(x,\lambda)$ in the interval $(0,1)$ has no less than $m_0(\mu_{n-1})+1$ and no more than $m_0(\mu_n)$ zeros. Since in the considered cases $m_0(\mu_{n-1})+1=m_0(\mu_n)$ then the truth of the assertion of the lemma follows from (A) and (B).

Let $c_0\neq0$ and $\lambda\in\left(\mu_{N_0},-d_0/c_0\right)$. Since $y(1,\lambda)\neq0$, then by virtue of lemma 1.1 the function $y(x,\lambda)$ in the interval $(0,1)$ has no less than N_0+1 zeros. Let $m_0(\lambda)\geq N_0+2$. By virtue of Sturm theorem [17, p.28] between each two zeros of the function $y(x,\lambda)$ there is at least one zero of the function $y\left(x,-d_0/c_0\right)$. Consequently, $m_0\left(-d_0/c_0\right)\geq N_0+1$. The last contradicts to the equality $m_0\left(-d_0/c_0\right)=N_0$.

The case (e_1) is considered absolutely analogously. The proof of lemma 1.2 is complete.

Let's prove that $y(x,\lambda)\neq0$ $(0\leq x\leq1)$ at $\lambda<\mu_0$. Note, that if $c_0\neq0$ then $\mu_0<-d_0/c_0$. Consequently, $y(0,\lambda)\neq0$ at $\lambda<\mu_0$. Since μ_0 is the first value of the parameter λ for which $y(1,\lambda)=0$, then it is obvious that $y(1,\lambda)\neq0$ at $\lambda<\mu_0$. Consequently if $y(x_1,\lambda)=0$, then it must be $0<x_1<1$. In this case by virtue of lemma 1.1 for some point $x_0\in(0,x_1)$ it would be true the equality $y(x_0,\mu_0)=0$, that is contradiction.

Lemma 1.3. *The relation* $\lim\limits_{\lambda\to-\infty}\frac{y'(1,\lambda)}{y(1,\lambda)}=+\infty$ *holds.*

Proof. Let λ be sufficiently large by the modulus negative number. The fundamental system of solutions of equation (1.1) we denote by $\psi_j(x,\lambda)$ $(j=1,2)$. It is evident [18, p.59] that

$$\psi_j^{(s)}(x,\lambda)=\left(\theta_j\left|\lambda\right|^{1/2}\right)^s\exp\left(\theta_j\left|\lambda\right|^{1/2}\right)^s\left(1+O\left(\left|\lambda\right|^{-1/2}\right)\right),\qquad(1.5)$$

where $j=1,2;s=0,1$ and $\theta_1=-\theta_2=1$.

For the function $\varphi(x)\in C^1[0,1]$ we determine the expression $U(\varphi,\lambda)$ in the following way:

$$U(\varphi,\lambda)=(a_0\lambda+b_0)\varphi(0)-(c_0\lambda+d_0)\varphi'(0).\qquad(1.6)$$

It is easy to check that

$$y(x,\lambda)=W^{-1}\left\{U(\psi_1(x,\lambda),\lambda)\psi_2(x,\lambda)-U(\psi_2(x,\lambda),\lambda)\psi_1(x,\lambda)\right\},$$

where

$$W=\begin{vmatrix}\psi_1(0,\lambda) & \psi_2(0,\lambda)\\ \psi_1'(0,\lambda) & \psi_2'(0,\lambda)\end{vmatrix}.$$

It is obvious that

$$\frac{y'(1,\lambda)}{y(1,\lambda)}=\frac{\psi_1'(1,\lambda)-\frac{U(\psi_1(x,\lambda),\lambda)}{U(\psi_2(x,\lambda),\lambda)}\psi_2'(1,\lambda)}{\psi_1(1,\lambda)-\frac{U(\psi_1(x,\lambda),\lambda)}{U(\psi_2(x,\lambda),\lambda)}\psi_2(1,\lambda)}.\qquad(1.7)$$

The immediate calculation using formulae (1.5)-(1.7) shows that $\frac{y'(1,\lambda)}{y(1,\lambda)} = |\lambda|^{1/2} + O\left(|\lambda|^{-1/2}\right)$. Lemma 1.3 is proved (see also [13]).

In case $c_1 \neq 0$ we determine the nonnegative integer N_1 from the inequality

$$\mu_{N_1-1} < -\frac{d_1}{c_1} \leq \mu_{N_1}.$$

Let n be an arbitrary fixed non-negative integer and k_n be a quantity of eigenvalues of the boundary value problem (0.1)-(0.3) contained in the interval $(\mu_{n-1}, \mu_n]$. Below we'll show that either $k_n = 1$ (in this case we denote the corresponding eigenvalue by $\lambda_n^{(0)}$), or $k_n = 2$ (in this case we denote the corresponding eigenvalues by $\lambda_n^{(1)}$ and $\lambda_n^{(2)}$).

Theorem 1.1 *The following assertions are true:*

(a) if $c_0 = c_1 = 0$*, then* $k_n = 1$, $\lambda_n^{(0)} \in (\mu_{n-1}, \mu_n)$ *and* $m_0\left(\lambda_n^{(0)}\right) = m_0(\mu_n)$;

(b) if $c_0 = 0, c_1 \neq 0$ *and* $-d_1/c_1 \notin \left(\mu_{n-1}, \mu_n\right]$ *then* $k_n = 1, \lambda_n^{(0)} \in (\mu_{n-1}, \mu_n)$ *and* $m_0(\lambda_n^{(0)}) = = m_0(\mu_n)$;

(c) if $c_0 \neq 0$, $c_1 = 0$ *and* $-d_0/c_0 \notin (\mu_{n-1}, \mu_n]$ *then* $k_n = 1$, $\lambda_n^{(0)} \in (\mu_{n-1}, \mu_n)$ *and* $m_0(\lambda_0^{(n)}) = = m_0(\mu_n)$;

(d) if $c_0 c_1 \neq 0$, $-d_0/c_0 \notin (\mu_{n-1}, \mu_n]$ *and* $-d_1/c_1 \notin (\mu_{n-1}, \mu_n]$ *then* $k_n = 1$, $\lambda_n^{(0)} \in \in (\mu_{n-1}, \mu_n)$ *and* $m_0(\lambda_n^{(0)}) = k_0(\mu_n)$;

(e) if $c_0 \neq 0$, $c_1 = 0$ *and* $-d_0/c_0 \in (\mu_{n-1}, \mu_n]$ *then* $k_n = 1$ *and either* $\lambda_n^{(0)} \in (\mu_{n-1}, -d_0/c_0)$ *(in this case* $m_0(\lambda_n^{(0)}) = m_0(\mu_{n-1}) + 1$*) or* $\lambda_n^{(0)} \in [-d_0/c_0, \mu_n)$ *(in this case* $m_0(\lambda_n^{(0)}) = m_0(\mu_n)$*);*

(f) if $c_0 c_1 \neq 0$, $-d_0/c_0 \in (\mu_{n-1}, \mu_n]$ *and* $-d_1/c_1 \notin (\mu_{n-1}, \mu_n]$ *then* $k_n = 1$ *and either* $\lambda_n^{(0)} \in (\mu_{n-1}, -d_0/c_0)$ *(in this case* $m_0(\lambda_n^{(0)}) = m_0(\mu_{n-1}) + 1$*) or* $\lambda_n^{(0)} \in [-d_0/c_0, \mu_n)$ *(in this case* $m_0\left(\lambda_n^{(0)}\right) = m_0(\mu_n)$*);*

(g) if $c_0 = 0$, $c_1 \neq 0$ *and* $-d_1/c_1 \in \left(\mu_{n-1}, \mu_n\right)$ *then* $k_n = 2$, $\mu_{n-1} < \lambda_n^{(1)} < -d_1/c_1 < \lambda_n^{(2)} < \mu_n$ *and* $m_0(\lambda_n^{(1)}) = m_0(\lambda_n^{(2)}) = m_0(\mu_n)$;

(h) if $c_0 = 0$, $c_1 \neq 0$ *and* $-d_1/c_1 = \mu_n$ *then* $k_n = 2$, $\mu_{n-1} < \lambda_n^{(1)} < \lambda_n^{(2)} = -d_1/c_1$ *and* $m_0(\lambda_n^{(1)}) = m_0(\lambda_n^{(2)}) = m_0(\mu_n)$;

(i) if $c_0 c_1 \neq 0$,$-d_0/c_0 \notin (\mu_{n-1}, \mu_n]$ *and* $-d_1/c_1 \in (\mu_{n-1}, \mu_n]$ *then* $k_n = 2$, $\mu_{n-1} < \lambda_n^{(1)} < -d_1/c_1 < \lambda_n^{(2)} < \mu_n$ *and* $m_0(\lambda_n^{(1)}) = m_0(\lambda_n^{(2)}) = m_0(\mu_n)$;

(j) if $c_0 c_1 \neq 0$, $-d_0/c_0 \notin (\mu_{n-1}, \mu_n]$ *and* $-d_1/c_1 = \mu_n$ *then* $k_n = 2$, $\mu_{n-1} < \lambda_n^{(1)} < \lambda_n^{(2)} = = -d_1/c_1$ *and* $m_0(\lambda_n^{(1)}) = m_0(\lambda_n^{(2)}) = m_0(\mu_n)$;

(k) if $c_0 c_1 \neq 0$ *and* $\mu_{n-1} < -d_0/c_0 < -d_1/c_1 < \mu_n$ *then* $k_n = 2$, $\lambda_n^{(2)} \in (-d_1/c_1, \mu_n)$, $m_0(\lambda_n^{(2)}) = m_0(\mu_n)$ *and either* $\lambda_n^{(1)} \in (\mu_{n-1}, -d_0/c_0)$*; (in this case* $m_0(\lambda_n^{(1)}) = m_0(\mu_{n-1}) + 1$) *or* $\lambda_n^{(1)} \in [-d_0/c_0, -d_1/c_1)$ *(in this case* $m_0(\lambda_n^{(1)}) = m_0(\mu_n)$*);*

(l) if $c_0 c_1 \neq 0$ *and* $\mu_{n-1} < -d_1/c_1 < -d_0/c_0 \leq \mu_n$ *then* $k_n = 2$, $\lambda_n^{(1)} \in (\mu_{n-1}, -d_1/c_1)$, $m_0\left(\lambda_n^{(1)}\right) = m_0(\mu_{n-1})$ *and either* $\lambda_n^{(2)} \in (-d_1/c_1, -d_0/c_0)$ *(in this case* $m_0(\lambda_n^{(2)}) = m(\mu_{n-1}) + 1$) *or* $\lambda_n^{(2)} \in [-d_0/c_0, \mu_n)$ *(in this case* $m_0(\lambda_n^{(2)}) = m_0(\mu_n)$*);*

(m) *if* $c_0c_1 \neq 0$ *and* $\mu_{n-1} < -d_0/c_0 = -d_1/c_1 < \mu_n$, *then* $k_n = 2$ *and* $\lambda_n^{(1)} \in (\mu_{n-1}, -d_0/c_0)$, $\lambda_n^{(2)} \in (-d_0/c_0, \mu_n)$, $m_0(\lambda_n^{(1)}) = m_0(\mu_{n-1}) + 1$, $m_0(\lambda_n^{(2)}) = m_0(\mu_n)$;

(n) *if* $c_0c_1 \neq 0$ *and* $\mu_{n-1} < -d_0/c_0 < -d_1/c_1 = \mu_n$, *then* $k_n = 2$, $\lambda_n^{(2)} = \mu_n$, $m_0(\lambda_n^{(2)}) = m_0(\mu_n)$ *and either* $\lambda_n^{(1)} \in (\mu_{n-1}, -d_0/c_0)$ *(in this case* $m_0(\lambda_n^{(1)}) = m_0(\mu_{n-1}) + 1$*) or* $\lambda_n^{(1)} \in [-d_0/c_0, \mu_n)$ *(in this case* $m_0(\lambda_n^{(1)}) = m_0(\mu_n)$*);*

(0) *if* $c_0c_1 \neq 0$ *and* $\mu_{n-1} < -d_0/c_0 = -d_1/c_1 = \mu_n$, *then* $k_n = 2$, $\lambda_n^{(1)} \in (\mu_{n-1}, \mu_n)$, $\lambda_n^{(2)} = -d_0/c_0$, $m_0(\lambda_n^{(1)}) = m_0(\mu_{n-1}) + 1$, $m_0(\lambda_n^{(2)}) = m_0(\mu_n)$.

Proof. At first we'll prove that the function $\frac{y'(1,\lambda)}{y(1,\lambda)}$ in the interval (μ_{n-1}, μ_n) strongly decreases. For this it is sufficient to prove that at $\lambda \in (\mu_{n-1}, \mu_n)$ the inequality $\frac{\partial}{\partial\lambda}\left(\frac{y'(1,\lambda)}{y(1,\lambda)}\right) < 0$ is true.

Let λ^* be a fixed point from the interval (μ_{n-1}, μ_n). By virtue of (1.1) we have

$$\frac{d}{dx}\left\{y'(x,\lambda^*)y(x,\mu) - y'(x,\mu)y(x,\lambda^*)\right\} = (\mu - \lambda^*)y(x,\mu)y(x,\lambda^*),$$

where μ is an arbitrary real number. Integrating this identity in the range from 0 to 1 we obtain

$$y'(1,\lambda^*)y(1,\mu) - y(1,\lambda^*)y'(1,\mu) = (\mu - \lambda^*)\left\{\int_0^1 y(x,\lambda^*)y(x,\mu)dx - \sigma_0\right\}.$$

By dividing the both parts of the latter into $(\mu - \lambda^*)y(1,\lambda^*)y(1,\mu)$ and by the subsequent limit passage $\mu \to \lambda^*$ we'll obtain

$$\frac{\partial}{\partial\lambda}\left(\frac{y'(1,\lambda)}{y(1,\lambda)}\right)_{\lambda=\lambda^*} = \frac{\sigma_0}{y^2(1,\lambda^*)} - \frac{1}{y^2(1,\lambda^*)}\int_0^1 y^2(x,\lambda^*)dx < 0.$$

Hence from lemma 1.3 and from the equalities $y(1,\mu_k) = 0$ $(k = 0,1,...)$ it follows that in the interval (μ_{n-1}, μ_n) the function $\frac{y'(1,\lambda)}{y(1,\lambda)}$ must strongly decrease from $(+\infty)$ to $(-\infty)$.

Let $P(\lambda) = (a_1\lambda + b_1)/(c_1\lambda + d_1)$. We have $P'(\lambda) = \sigma_1(c_1\lambda + d_1)^{-2}$. Since by the conjecture of (0.4) $\sigma_1 > 0$, then under the condition $c_1 = 0$ the function $P(\lambda)$ strongly increases in the interval $(-\infty; +\infty)$, and under the condition $c_1 \neq 0$ the function $P(\lambda)$ strongly increases at each of the intervals $(-\infty; -d_1/c_1)$ and $(-d_1/c_1; +\infty)$, moreover $\lim\limits_{\lambda\to -d_1/c_1 - 0} P(\lambda) = +\infty$, $\lim\limits_{\lambda\to -d_1/c_1 + 0} P(\lambda) = -\infty$..

Let $c_1 = 0$ or $c_1 \neq 0$, $-d_1/c_1 \notin (\mu_{n-1}, \mu_n]$. From aforesaid it follows that in the interval (μ_{n-1}, μ_n) it will be found the unique value $\lambda = \lambda_n^{(0)}$ for which

$$\frac{y'(1,\lambda)}{y(1,\lambda)} = P(\lambda), \tag{1.8}$$

i.e., condition (0.3) is fulfilled. Consequently, $\lambda_n^{(0)}$ is an eigenvalue of the boundary value problem (0.1)-(0.3) and $y\left(x, \lambda_n^{(0)}\right)$ is a corresponding eigenfunction. Hence and from lemma 1.2 we obtain the validity of the assertions $(a) - (f)$.

Let $c_1 \neq 0$ and $-d_1/c_1 \in (\mu_{n-1}, \mu_n]$. By an analogous way at each of intervals $(\mu_{n-1}, -d_1/c_1)$ and $(-d_1/c_1, \mu_n)$ the unique value λ ($\lambda_n^{(1)}$ and $\lambda_n^{(2)}$, respectively) will be found for which (1.8) is fulfilled, i.e., condition (0.3) is fulfilled. Since in the considered case $(-d_1/c_1)$ is not the eigenvalue of boundary value problem (0.1)-(0.3), then from here and from lemma 1.2 it follows the validity of the assertions $(g), (i), (k) - (m)$.

The case $c_1 \neq 0$, $-d_1/c_1 = \mu_n$ (see $(h), (j), (n), (0)$) is considered absolutely analogously. Here it is used the fact that in this case μ_n is also an eigenvalue of boundary value problem (0.1)-(0.3). The proof of theorem 1.1 is complete.

The existence of unbounded increasing sequences of the eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ of boundary value problem (0.1)-(0.3) and oscillation properties of the eigenfunctions $y_n(x)(n = 0, 1, ...)$ follows from theorem 1.1.

Corollary 1.1. *The following assertions are true:*

(a_2) *if* $c_0 = c_1 = 0$ *then the eigenfunction* $y_n(x)$ *has exactly* n *simple zeros in the interval* $(0, 1)$*;*

(b_2) *if* $c_0 = 0$ *and* $c_1 \neq 0$ *then the eigenfunction* $y_n(x)$ *at* $n \leq N_1$ *has exactly* n*, and at* $n > N_1$ *exactly* $n - 1$ *simple zeros in the interval* $(0, 1)$*, moreover either* $\lambda_{N_1} < -d_1/c_1 < \lambda_{N_1+1}$ *(provided* $\mu_{N_1} \neq -d_1/c_1$*) or* $\lambda_{N_1} < -d_1/c_1 = \lambda_{N_1+1}$ *(provided* $\mu_{N_1} = -d_1/c_1$*);*

(c_2) *if* $c_0 \neq 0$ *and* $c_1 = 0$ *then the eigenfunction* $y_n(x)$ *at* $n \leq N_0$ *has exactly* n *simple zeros, and at* $n > N_0 + 1$ *exactly* $n - 1$ *simple zeros in the interval* $(0, 1)$*; the eigenfunction* $y_{N_0+1}(x)$ *has either exactly* $N_0 + 1$ *(in this case* $\mu_{N_0} < \lambda_{N_0+1} < -d_0/c_0$*) or exactly* N_0 *(in this case* $-d_{0/C_0} \leq \lambda_{N_0+1} < \mu_{N_0+1}$*) simple zeros in the interval (0,1).*

Noteci that items (a_2) and (b_2) of corollary 1.1 were proved as well as in the papers [8] and [13].

In the case $c_0 c_1 \neq 0$ the oscillation properties of eigenfunctions of boundary value problem (0.1)-(0.3) have more difficult character.

Consider the following spectral problem:

$$-y'' = \lambda y, \quad 0 < x < 1,$$

$$y(0) = (2/\pi)^3 \lambda y'(0), \quad y(1) = -(2/\pi)^3 \lambda y'(1).$$

It is easy to be sure that for this problem $a_0 = d_0 = a_1 = d_1 = 0$, $b_0 = b_1 = 1$, $c_0 = -c_1 = (2/\pi)^3$, $\sigma_0 = -(2/\pi)^3 < 0$, $\sigma_1 = (2/\pi)^3 > 0$. The immediate calculations show that

$$\lambda_0 = -\mu_0^2, \quad y_0(x) = \exp(\mu_0(x-1)) + \exp(-\mu_0 x); \quad \lambda_1 = -\mu_1^2,$$

$$y_1(x) = \exp(\mu_1(x-1)) - \exp(-\mu_1 x); \; \lambda_2 = \left(\frac{\pi}{2}\right)^2, \quad y_2(x) = \sin\frac{\pi x}{2} + \cos\frac{\pi x}{2},$$

where μ_0 and μ_1 are corresponding unique positive roots of the equations

$$\exp(-\mu_0) = \frac{(2\mu_0/\pi)^3 - 1}{(2\mu_0/\pi)^3 + 1}, \quad \exp(-\mu_1) = \frac{1 - (2\mu_1/\pi)^3}{1 + (2\mu_1/\pi)^3}.$$

It is obvious that $\lambda_0 < -(\pi/2)^2 < \lambda_1$. Note that the functions $y_0(x)$ and $y_2(x)$ in the interval $(0, 1)$ have no zeros, and the function $y_1(x)$ in the interval $(0, 1)$ has a unique zero $x_0 = \frac{1}{2}$.

Corollary 1.2. *Let $c_0c_1 \neq 0_1$ and $N_0+2 \leq N_1$. Then the eigenfunction $y_n(x)$ has at $n \leq N_0$ exactly n, at $N_0+2 \leq n \leq N_1$ exactly $n-1$, and at $n \geq N_1+1$ exactly $n-1$ zeros in the interval $(0,1)$; the eigenfunction $y_{N_0+1}(x)$ has either exactly N_0+1 (in this case $\mu_{N_0} < \lambda_{N_0+1} < -d_0/c_0$) or exactly N_0 (in this case $-d_0/c_0 \leq \lambda_{N_0+1} < \mu_{N_0}$) zeros in the interval $(0,1)$. Besides, if $-d_1/c_1 \neq \mu_{N_1}$ then $\lambda_{N_1} < -d_1/c_1 < \lambda_{N_1+1}$ and if $-d_1/c_1 = \mu_{N_1}$ then $\lambda_{N_1} < -d_1/c_1 = \lambda_{N_1+1}$.*

Corollary 1.3. *Let $c_0c_1 \neq 0$, $N_0+1 = N_1$ and $-d_0/c_0 \leq -d_1/c_1$. Then the eigenfunction $y_n(x)$ has at $n \leq N_0$ exactly n, and at $n \geq N_1+1$ exactly $n-2$ zeros in the interval $(0,1)$; the eigenfunction $y_{N_0+1}(x)$ has either exactly N_0+1 (in this case $\mu_{N_0} < \lambda_{N_0+1} < -d_0/c_0$) or exactly N_0 (in this case $-d_0/c_0 \leq \lambda_{N_0+1} < -d_1/c_1$) zeros in the interval $(0,1)$. Besides if $-d_1/c_1 \neq \mu_{N_1}$ then $-d_1/c_1 < \lambda_{N_1+1} < \mu_{N_1}$ and if $-d_1/c_1 = \mu_{N_1}$ then $\lambda_{N_1+1} = -d_1/c_1$.*

Corollary 1.4. *Let $c_0c_1 \neq 0$, $N_0+1 = N_1$ and $-d_1/c_1 < -d_0/c_0$. Then the eigenfunction $y_n(x)$ has at $n \leq N_0+1$ exactly n (at this $\lambda_{N_0+1} < -d_1/c_1$), and at $n \geq N_1+2$ exactly $n-1$ zeros in the interval $(0,1)$; the eigenfunction $y_{N_0+2}(x)$ has either exactly N_0+1 (in this case $-d_1/c_1 \leq \lambda_{N_0+2} < -d_0/c_0$) or exactly N_0 (in this case $-d_0/C_0 \leq \lambda_{N_0+2} < \mu_{N_0+1}$) zeros in the interval $(0,1)$.*

Corollary 1.5. *Let $c_0c_1 \neq 0$ and $N_1 \leq N_0+1$. Then the eigenfunction $y_n(x)$ has at $n \leq N_1$ exactly n (at this $\lambda_{N_1} < -d_1/c_1$) at $N_1+1 \leq n \leq N_0+1$ exactly $n-1$ (at this either $-d_1/c_1 < \lambda_{N_1+1} < \mu_{N_1}$ or $\lambda_{N_1+1} = -d_1/c_1 = \mu_{N_1}$), and at $n \geq N_0+3$ exactly $n-2$ zeros in the interval $(0,1)$; the eigenfunction $y_{N_0+2}(x)$ has exactly either N_0+1 (in this case $\mu_{N_0} < \lambda_{N_0+2} < -d_0/c_0$) or exactly N_0 (in this case $-d_0/c_0 \leq \lambda_{N_0+2} < \mu_{N_0+1}$) zeros in the interval $(0,1)$.*

2. Asymptotic formulae for eigenvalues and eigenfunctions of boundary value problem (0.1)-(0.3).

Let $y_n(x)$ be an eigenfunction of boundary value problem (0.1)-(0.3) corresponding to the eigenvalue λ_n.

Denote by Δ the greatest element of the set of roots of the functions $a_j\lambda + b_j$, $c_j\lambda + d_j$, $(j=0,1)$. It is obvious that the function $P_j(\lambda) = \frac{c_j\lambda+d_j}{a_j\lambda+b_j}$ has a constant sign in the interval $(\Delta, +\infty)$. Let $p_j = \lim\limits_{\lambda\to+\infty} sgn\ P_j(\lambda)$ and $\widetilde{N}_0$ be such a natural number that at $n \geq \widetilde{N}_0$ the inequality $\lambda_n > \Delta_0$ is true where $\Delta_0 = \max\{\Delta, 2C+1\}$, $C = \max\limits_{0\leq x\leq 1} |q(x)|$.

Everywhere in future we'll assume that $n \geq \widetilde{N}_0$.

Introduce the corner variable $\theta_n(x) = Arctg\frac{y_n(x)}{y_n'(x)}$ or exactly

$$\theta_n(x) = \arg\left\{y_n'(x) + iy_n(x)\right\}. \tag{2.1}$$

Allowing for (0.2) we'll determine the initial value $\theta_n(0)$ by the equality

$$\theta_n(0) = arctg\frac{c_0\lambda_n + d_0}{a_0\lambda_n + b_0} + \frac{1-p_0}{2}\pi. \tag{2.2}$$

For the other x the function $\theta_n(x)$ is given by formula (2.1) to within arbitrary addend divisible by 2π since the functions $y_n(x)$ and $y_n'(x)$ can't vanish simultaneously. This expression divisible by 2π is to be fixed such that the function $\theta_n(x)$

satisfy condition (2.2) and be continuous by x. By this the function $\theta_n(x)$ is determined by a unique way [19, p.244].

Lemma 2.1 [9]. *The function $\theta_n(x)$ satisfies the differential equation*

$$\theta_n'(x) = \cos^2\theta_n(x) + (\lambda_n - q(x))\sin^2\theta_n(x) \tag{2.3}$$

and increases on the segment $[0,1]$.

From (2.1) it is clear that the zeros of the function $y_n(x)$ coincide with the points in which $\theta_n(x)$ is divisible by π. Considering the function $y_n(x)$ when x increases from 0 to 1 we see that it has zero at the point $x \in (0,1)$ iff at this point $\theta_n(x)$ increasing passes the value divisible by π.

Since $0 < \theta_n(0) < \pi$ we see that by increasing x from 0 to 1 the function $\theta_n(x)$ takes the finite number of values $\pi, 2\pi, ...,$ consecutively. As the function $\theta_n(x)$ can't decreasing converge to angle divisible by π then it achieves the angles divisible by π in increase order.

Denote by $x_{n,k}$ $(k = \overline{1, m_k})$ the zeros of the eigenfunction $y_n(x)$ in the interval $(0,1)$.

From the oscillation properties of eigenfunctions of boundary value problem (0.1)-(0.3) (see corollaries 1.1-1.5) it follows that at $n \geq \widetilde{N}_0$

$$m_n = n - sgn\,|c_0| - sgn\,|c_1| \tag{2.4}$$

holds.

Besides it is easy to see that

$$\theta_n(1) = arctg\frac{c_1\lambda_n + d_1}{a_1\lambda_n + b_1} + \frac{1-p_1}{2}\pi + \pi m_n. \tag{2.5}$$

Lemma 2.2 (see [9] and [20]). *For eigenvalues λ_n of boundary value problem (0.1)-(0.3) at $n \geq \widetilde{N}_0$ the estimation*

$$C_1 n^2 \leq \lambda_n \leq C_2 n^2, \tag{2.6}$$

is true, where C_1 and C_2 are some positive constants.

Lemma 2.3. *The asymptotic formulae*

$$\lambda_n = (\pi(n-\sigma))^2 + O(1), \tag{2.7}$$

$$2^{-1/2}y_n(x) = sgn\,|c_0| \cdot \cos(n-\sigma)\pi x + (1 - sgn\,|c_0|) \cdot \sin(n-\sigma)\pi x + O(n^{-1}), \tag{2.8}$$

are valid, where $\sigma = 1 + \frac{1}{2}(sgn\,|c_0| + sgn\,|c_1|)$.

Proof. We'll prove formula (2.7) only in the case $c_0 c_1 \neq 0$. The other cases are considered analogously.

Note that by virtue of (2.4) in the considered case $m_n = n-2$. It is obvious that

$$\theta_n(x_{n,k}) = \pi k \quad (1 \leq k \leq n-2). \tag{2.9}$$

From (2.3) subject to lemma 2.2 we'll obtain

$$\frac{\theta_n'(x)}{\cos^2\theta_n(x) + \lambda_n \sin^2\theta_n(x)} = 1 + O(n^{-2}).$$

Integrate the both parts of the last equality from 0 to $x_{n,1}$:

$$\int_0^{x_{n,1}} \frac{\theta_n'(x)}{\cos^2\theta_n(x)+\lambda_n\sin^2\theta_n(x)} = x_{n,1}\left(1+O(n^{-2}\right).$$

Making substitution $\theta_n(x)=\varphi$ in integral and allowing for (2.9) we'll obtain:

$$\int_{\theta_n(0)}^{\pi} \frac{d\varphi}{\cos^2\varphi+\lambda_n\sin^2\varphi} = x_{n,1}\left(1+O(n^{-2})\right).$$

Acting by the analogous way we have

$$\int_{\pi(n-2)}^{\theta_n(1)} \frac{d\varphi}{\cos^2\varphi+\lambda_n\sin^2\varphi} = \left(1-x_{n,n-2}\right)\left(1+O(n^{-2})\right),$$

$$\int_{\pi k}^{\pi(k+1)} \frac{d\varphi}{\cos^2\varphi+\lambda_n\sin^2\varphi} = \left(x_{n,k+1}-x_{n,k}\right)\left(1+O(n^{-2})\right) \quad (1\le k\le n-3).$$

Allowing for (2.2), (2.5) and lemma 2.2 by the immediate calculation it is easy to be sure that

$$\int_{\theta_n(0)}^{\pi} \frac{d\varphi}{\cos^2\varphi+\lambda_n\sin^2\varphi} = \frac{\pi}{2\sqrt{\lambda_n}}+O(n^{-2}),$$

$$\int_{\pi(n-2)}^{\theta_n(1)} \frac{d\varphi}{\cos^2\varphi+\lambda_n\sin^2\varphi} = \frac{\pi}{2\sqrt{\lambda_n}}+O(n^{-2}),$$

$$\int_{\pi k}^{\pi(k+1)} \frac{d\varphi}{\cos^2\varphi+\lambda_n\sin^2\varphi} = \frac{\pi}{\sqrt{\lambda_n}} \quad (1\le k\le n-3).$$

From the last six equalities it follows that

$$x_{n,1} = \frac{\pi}{2\sqrt{\lambda_n}}+O\left(n^{-3}\right), \tag{2.10}$$

$$1-x_{n,n-2} = \frac{\pi}{2\sqrt{\lambda_n}}+O\left(n^{-3}\right), \tag{2.11}$$

$$x_{n,k+1}-x_{n,k} = \frac{\pi}{2\sqrt{\lambda_n}}+O\left(n^{-3}\right) \quad (1\le k\le n-3). \tag{2.12}$$

By virtue of (2.12) we have

$$x_{n,n-2}-x_{n,1} = \frac{\pi(n-3)}{\sqrt{\lambda_n}}+O(n^{-2}).$$

By summing the last equality with equalities (2.10) and (2.11) we'll obtain $1 = \frac{\pi(n-2)}{\sqrt{\lambda_n}} + O\left(n^{-2}\right)$ that is equivalent to $\sqrt{\lambda_n} = \pi(n-2) + O\left(n^{-1}\right)$. The derivation of formulae (2.2) in the case $c_0c_1 \neq 0$ is complete.

Prove formula (2.8). We denote by $\varphi_1(x,\mu)$ and $\varphi_2(x,\mu)$ the fundamental system of solutions of the equation $u'' - q(x)u + \mu^2 u = 0$ determined by the initial conditions

$$\begin{gathered}\varphi_1(0,\mu) = 1, \quad \varphi_1'(0,\mu) = i\mu, \\ \varphi_2(0,\mu) = 1, \quad \varphi_2'(0,\mu) = -i\mu.\end{gathered} \tag{2.13}$$

It is known that (see [21] or [18, p.59]) at sufficiently large values of μ it holds

$$\varphi_j(x,\mu) = \exp(\mu w_j x)\left(1 + O\left(\mu^{-1}\right)\right), \tag{2.14}$$

where $w_1 = -w_2 = i$.

The eigenfunction $y_n(x)$ we'll seek in the following form

$$y_n(x) = S_n \begin{vmatrix} \varphi_1(x,\mu_n), & \varphi_2(x,\mu_n) \\ U\left(\varphi_1(x,\mu_n),\mu_n^2\right) & U\left(\varphi_2(x,\mu_n),\mu_n^2\right) \end{vmatrix}, \tag{2.15}$$

where for arbitrary functions $\varphi(x) \in C^1[0,1]$ the expression $U(\varphi(x),\lambda)$ is determined by equality (1.6), $\mu_n = \sqrt{\lambda_n}$ and

$$S_n = \begin{cases} \frac{1}{\sqrt{2}ic_0\mu_n^3}, & c_0 \neq 0; \\ \frac{1}{\sqrt{2}ia_0\mu_n^2}, & c_0 = 0. \end{cases} \tag{2.16}$$

Immediate calculation with using (2.13)-(2.16) and formulae $\sqrt{\lambda_n} = \pi(n-\sigma) + O\left(n^{-1}\right)$ shows that

$$y_n(x) = \begin{cases} \sqrt{2}\cos\pi(n-\sigma)x + O(n^{-1}), & c_0 \neq 0; \\ \sqrt{2}\sin\pi(n-\sigma)x + O(n^{-1}), & c_0 = 0. \end{cases}$$

The proof of theorem (2.8) is complete.

Note that formula (2.7) was proved in the papers [20] and [9].

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