THE BASIS PROPERTY OF STURM–LIOUVILLE PROBLEMS WITH BOUNDARY CONDITIONS DEPENDING QUADRATICALLY ON THE EIGENPARAMETER

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الخلاصـة:

ندرس في هذا البحث تكوّن قاعدة الاقترانات الجذرية لمعضلات ستورم – ليوفيل تحت الشروط الحدية ذات الاعتماد التربيعي على المعاملات الطيفية. وقد حَدَّدنا بوضوح شكل النظام ثنائي التعامد، ومن ثم استخدمناه لإثبات أن نظام الاقترانات الجذرية بحذف اقترانين اعتباطَيين يُكوّن نظاماً مصَّغراً في L_2، فيما عدا بعض الحالات التي لا يكون فيها النظام كاملاً أو مُصغراً. ومن تكوّن القاعدة في L_2 أثبتنا أن جزءاً من فضاء الجذر يكون قريباً على شكل تربيعي من نظام الجيوب وجيوب التمام. كما أننا درسنا صفات هذه القواعد من خلال تعميم L_p.

ABSTRACT

We study basisness of root functions of Sturm–Liouville problems with a boundary condition depending quadratically on the spectral parameter. We determine the explicit form of the biorthogonal system. Using this we prove that the system of root functions, with arbitrary two functions removed, form a minimal system in L_2, except some cases where this system is neither complete nor minimal. For the basisness in L_2 we prove that the part of the root space is quadratically close to systems of sines and cosines. We also consider these basis properties in the context of general L_p.

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Key words: Sturm–Liouville, eigenparameter-dependent boundary conditions, basis, minimal system, completeness, quadratically close systems.

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1. INTRODUCTION

Consider the spectral problem

$$-y'' + q(x)y = \lambda y,\ 0 < x < 1, \tag{0.1}$$

$$y'(0)\sin\beta = y(0)\cos\beta,\ 0 \le \beta < \pi, \tag{0.2}$$

$$y'(1) = (a\lambda^2 + b\lambda + c)y(1),\ a \ne 0, \tag{0.3}$$

where λ is the spectral parameter, $q(x)$ is a real valued and continuous function on the interval $[0,1]$, and a, b, c are real.

In a recent paper [1] existence and asymptotics of eigenvalues of (0.1)–(0.3) were studied (see also [2, Sect. 4.1]). It was proved that the eigenvalues of (0.1)–(0.3) form an infinite sequence, accumulating only at $+\infty$, and only following cases are possible:

(*a)* All the eigenvalues are real and simple;
(*b)* All the eigenvalues are simple and all, except a conjugate pair of non-real, are real;
(*c)* All the eigenvalues are real and all, except one double, are simple;
(*d)* All the eigenvalues are real and all, except one triple, are simple.

The eigenvalues λ_n $(n \ge 0)$ will be considered to be listed according to non-decreasing real part and repeated according to algebraic multiplicity. Asymptotically the eigenvalues are as follows [3, Theorem 2.2], [1, Theorem 1]:

$$\lambda_n = \begin{cases} (n-3/2)^2\pi^2 + O(1) & \text{if } \beta \ne 0, \\ (n-1)^2\pi^2 + O(1) & \text{if } \beta = 0. \end{cases} \tag{0.4}$$

The present article concerns the basis properties in $L_p(0,1)$ $(1 < p < \infty)$ of the root function system of the boundary value problem (0.1)–(0.3). Some of the presented results concerning eigenfunctions have been announced in [4]. Basis properties of boundary value problems with a spectral parameter in boundary conditions have been studied in papers [5,6,7]. For the problems considered in these papers only the case *(a)* is possible. In [8] the problem with cases *(a)* and *(c)* was studied. See also [9–15].

2. INNER PRODUCTS AND NORMS OF EIGENFUNCTIONS

We define $y(x,\lambda)$ to be the non-zero solution of (0.1),(0.2), analytic in $\lambda \in \mathbb{C}$, and we write $\omega(\lambda) = y'(1,\lambda) - (a\lambda^2 + b\lambda + c)y(1,\lambda)$. By (0.3), λ_n is an eigenvalue if and only if $\omega(\lambda_n) = 0$, and we say that λ_n is a simple eigenvalue, if in addition $\omega'(\lambda_n) \ne 0$. The eigenvalue λ_k is multiple if $\omega'(\lambda_k) = 0$, in particular, we say that λ_k is a double eigenvalue if in addition $\omega''(\lambda_k) \ne 0$, and a triple eigenvalue if $\omega''(\lambda_k) = 0 \ne \omega'''(\lambda_k)$.

Let y_n be an eigenfunction corresponding to eigenvalue λ_n. Note that $y(x,\lambda) \to y(x,\lambda_n) = y_n$, uniformly in $x \in [0,1]$, as $\lambda \to \lambda_n$ (see [16, Section 10.72]).

We denote by $(\cdot,\cdot)$ the scalar product in $L_2(0,1)$, and by $\|\cdot\|_p$ the norm in $L_p(0,1)$.

Lemma 1.1. If y_n and y_m are eigenfunctions corresponding to eigenvalues λ_n and λ_m $(\lambda_n \ne \overline{\lambda_m})$ then

$$(y_n, y_m) = -(a\lambda_n + b + a\overline{\lambda_m})y_n(1)\overline{y_m(1)}.$$

Proof. To begin, we note that

$$\frac{d}{dx}\left(y(x,\lambda)\overline{y'(x,\mu)} - y'(x,\lambda)\overline{y(x,\mu)}\right) = (\lambda - \overline{\mu})y(x,\lambda)\overline{y(x,\mu)}.$$

By integrating this identity from 0 to 1, we obtain

$$(\lambda - \overline{\mu})(y(\cdot,\lambda), y(\cdot,\mu)) = \left(y(x,\lambda)\overline{y'(x,\mu)} - y'(x,\lambda)\overline{y(x,\mu)}\right)\Big|_0^1.$$

From (0.2), we obtain

$$y(0,\lambda)\overline{y'(0,\mu)} - y'(0,\lambda)\overline{y(0,\mu)} = 0.$$

Noting the definition of $\omega(\lambda)$ it is easy to check that

$$y(1,\lambda)\overline{y'(1,\mu)} - y'(1,\lambda)\overline{y(1,\mu)} = -(\lambda-\overline{\mu})(a\lambda+b+a\overline{\mu})y(1,\lambda)\overline{y(1,\mu)}$$
$$+y(1,\lambda)\overline{\omega(\mu)} - \overline{y(1,\mu)}\omega(\lambda).$$

From the last three equalities it follows that for $\lambda \neq \overline{\mu}$,

$$(y(\cdot,\lambda),y(\cdot,\mu)) = -(a\lambda+b+a\overline{\mu})y(1,\lambda)\overline{y(1,\mu)}$$
$$+y(1,\lambda)\frac{\overline{\omega(\mu)}}{\lambda-\overline{\mu}} - \overline{y(1,\mu)}\frac{\omega(\lambda)}{\lambda-\overline{\mu}}. \tag{1.1}$$

Since λ_n and λ_m are eigenvalues of (0.1)-(0.3) then $\omega(\lambda_n) = \omega(\lambda_m) = 0$, hence by setting $\lambda = \lambda_n$ and $\mu = \lambda_m$ in (1.1) we prove the required equality. □

Lemma 1.2. If λ_n is a real eigenvalue then

$$\|y_n\|_2^2 = (y_n, y_n) = -(2a\lambda_n + b)y_n(1)^2 - y_n(1)\omega'(\lambda_n).$$

Proof. Since $\omega(\lambda_n) = 0$ then $\omega(\lambda)/(\lambda-\lambda_n) \to \omega'(\lambda_n)$ as $\lambda \to \lambda_n$. Therefore, by setting $\mu = \lambda_n$ ($\lambda \neq \lambda_n$) and then tending $\lambda \to \lambda_n$ in (1.1) we obtain the required equality (*cf.* [16, Sect. 10.72]). □

Corollary 1.1. If λ_k is a multiple eigenvalue then

$$\|y_k\|_2^2 = (y_k, y_k) = -(2a\lambda_k + b)y_k(1)^2.$$

Corollary 1.2. If λ_r is a non-real eigenvalue then

$$\|y_r\|_2^2 = -(2a\mathrm{Re}\lambda_r + b)|y_r(1)|^2.$$

Proof. Since $\lambda_r \neq \overline{\lambda_r}$ then this equality follows at once from Lemma 1.1 if we set $\lambda_n = \lambda_m = \lambda_r$. □

For the eigenfunction y_n define

$$B_n = \|y_n\|_2^2 + (2a\mathrm{Re}\lambda_n + b)|y_n(1)|^2. \tag{1.2}$$

Corollary 1.3. $B_n \neq 0$ if and only if the corresponding eigenvalue λ_n is real and simple.

If λ_k is a multiple (double or triple) eigenvalue ($\lambda_k = \lambda_{k+1}$) then $B_k = -y_k(1)\omega'(\lambda_k) = 0$ and B_{k+1} is not defined, so we set $B_{k+1} = -y_k(1)\omega''(\lambda_k)/2$. If λ_k is a triple eigenvalue ($\lambda_k = \lambda_{k+1} = \lambda_{k+2}$) then $B_{k+1} = 0$ and B_{k+2} is not defined, so we set $B_{k+2} = -y_k(1)\omega'''(\lambda_k)/6$.

Lemma 1.3. If λ_r and λ_s are a conjugate pair of non-real eigenvalues $\lambda_s = \overline{\lambda_r}$ then

$$(y_r, y_s) = -(2a\lambda_r + b)y_r(1)^2 - y_r(1)\omega'(\lambda_r).$$

The proof is similar to the proof of Lemma 1.2.

3. INNER PRODUCTS AND NORMS OF ASSOCIATED FUNCTIONS

In this section and in Section 3 we consider the cases **(c)** and **(d)**, where all the eigenvalues are reals.

If λ_k is a multiple eigenvalue ($\lambda_k = \lambda_{k+1}$) then for the first order associated function y_{k+1} the following relations hold [17, ch. I, §2]:

$$-y''_{k+1} + q(x)y_{k+1} = \lambda_k y_{k+1} + y_k,$$
$$y'_{k+1}(0)\sin\beta = y_{k+1}(0)\cos\beta,$$
$$y'_{k+1}(1) = (a\lambda_k^2 + b\lambda_k + c)y_{k+1}(1) + (2a\lambda_k + b)y_k(1).$$

Note that $y_{k+1} + cy_k$, where c is an arbitrary constant, is also an associated function. So the associated function y_{k+1} is not unique.

Differentiating (0.1), (0.2), and $\omega(\lambda)$ with respect to the parameter λ, we obtain

$$-y''_\lambda(x,\lambda) + q(x)y_\lambda(x,\lambda) = \lambda y_\lambda(x,\lambda) + y(x,\lambda),$$
$$y'_\lambda(0,\lambda)\sin\beta = y_\lambda(0,\lambda)\cos\beta,$$
$$\omega'(\lambda) = y'_\lambda(1,\lambda) - (a\lambda^2 + b\lambda + c)y_\lambda(1,\lambda) - (2a\lambda + b)y(1,\lambda).$$

Since $\omega(\lambda_k) = \omega'(\lambda_k) = 0$ then $y(x,\lambda) \to y_k$, $y_\lambda(x,\lambda) \to \widetilde{y}_{k+1}$, uniformly according to $x \in [0,1]$, as $\lambda \to \lambda_k$, where $\widetilde{y}_{k+1}$ is one of the first order associated functions, and it is obvious that $\widetilde{y}_{k+1} = y_{k+1} + \widetilde{c}y_k$, where $\widetilde{c} = (\widetilde{y}_{k+1}(1) - y_{k+1}(1))/y_k(1)$.

If λ_k is a triple eigenvalue ($\lambda_k = \lambda_{k+1} = \lambda_{k+2}$) then together with the first order associated function y_{k+1} there exists the second order associated function y_{k+2}, for which the following relations hold [17, ch. I, §2]:

$$-y''_{k+2} + q(x)y_{k+2} = \lambda_k y_{k+2} + y_{k+1},$$
$$y'_{k+2}(0)\sin\beta = y_{k+2}(0)\cos\beta,$$
$$y'_{k+2}(1) = (a\lambda_k^2 + b\lambda_k + c)y_{k+2}(1) + (2a\lambda_k + b)y_{k+1}(1) + ay_k(1).$$

Similar to y_{k+1} the associated function y_{k+2} is not unique, because the function $y_{k+2} + dy_k$, where d is a constant, is also an associated function of the second order. Note also that the second order associated function $y_{k+2} + cy_{k+1}$ corresponds to the first order associated function $y_{k+1} + cy_k$.

Differentiating (0.1), (0.2), and $\omega(\lambda)$ twice, we obtain

$$-y''_{\lambda\lambda}(x,\lambda) + q(x)y_{\lambda\lambda}(x,\lambda) = \lambda y_{\lambda\lambda}(x,\lambda) + 2y_\lambda(x,\lambda),$$
$$y'_{\lambda\lambda}(0,\lambda)\sin\beta = y_{\lambda\lambda}(0,\lambda)\cos\beta,$$
$$\omega''(\lambda) = y'_{\lambda\lambda}(1,\lambda) - (a\lambda^2 + b\lambda + c)y_{\lambda\lambda}(1,\lambda) - 2(2a\lambda + b)y_\lambda(1,\lambda) - 2ay(1,\lambda).$$

If λ_k is a triple eigenvalue $\omega''(\lambda_k) = 0$ then $y_{\lambda\lambda} \to 2\widetilde{y}_{k+2}$, uniformly in $x \in [0,1]$, as $\lambda \to \lambda_k$, where $\widetilde{y}_{k+2}$ is one of the second order associated functions corresponding to the first order associated function $\widetilde{y}_{k+1}$, and it is obvious that $\widetilde{y}_{k+2} = y_{k+2} + \widetilde{c}y_{k+1} + \widetilde{d}y_k$, and $\widetilde{d} = (\widetilde{y}_{k+2}(1) - y_{k+2}(1) - \widetilde{c}y_{k+1}(1))/y_k(1)$.

Lemma 2.1. If λ_k is a multiple eigenvalue and $\lambda_n \neq \lambda_k$ then

$$(y_{k+1}, y_n) = -(a\lambda_k + b + a\lambda_n)y_{k+1}(1)y_n(1) - ay_k(1)y_n(1),$$
$$(y_{k+1}, y_k) = -(2a\lambda_k + b)y_{k+1}(1)y_k(1) - ay_k(1)^2 - y_k(1)\frac{\omega''(\lambda_k)}{2}, \tag{2.1}$$
$$\|y_{k+1}\|_2^2 = (y_{k+1}, y_{k+1}) = -(2a\lambda_k + b)y_{k+1}(1)^2 - 2ay_{k+1}(1)y_k(1)$$
$$-\widehat{y}_{k+1}(1)\frac{\omega''(\lambda_k)}{2} - y_k(1)\frac{\omega'''(\lambda_k)}{6}, \tag{2.2}$$

where $\widehat{y}_{k+1} = y_{k+1} - \widetilde{c}y_k$.

Proof. Differentiating (1.1) with respect to λ we obtain for $\lambda \neq \mu$,

$$(y_\lambda(\cdot,\lambda), y(\cdot,\mu)) = -(a\lambda + b + a\mu)y_\lambda(1,\lambda)y(1,\mu) - ay(1,\lambda)y(1,\mu)$$
$$+y_\lambda(1,\lambda)\frac{\omega(\mu)}{\lambda-\mu} - y(1,\lambda)\frac{\omega(\mu)}{(\lambda-\mu)^2} - y(1,\mu)\frac{\omega'(\lambda)}{\lambda-\mu} + y(1,\mu)\frac{\omega(\lambda)}{(\lambda-\mu)^2}. \tag{2.3}$$

Setting $\mu = \lambda_n$ and $\lambda = \lambda_k$ in (2.3) we obtain

$$(\widetilde{y}_{k+1}, y_n) = -(a\lambda_k + b + a\lambda_n)\widetilde{y}_{k+1}(1)y_n(1) - ay_k(1)y_n(1).$$

By Lemma 1.1,

$$(y_k, y_n) = -(a\lambda_k + b + a\lambda_n)y_k(1)y_n(1).$$

Note that

$$(y_{k+1}, y_n) = (\widetilde{y}_{k+1} - \widetilde{c}y_k, y_n) = (\widetilde{y}_{k+1}, y_n) - \widetilde{c}(y_k, y_n).$$

The first equality of Lemma 2.1 follows from the last three equalities. The equality (2.1) can be proved in a similar way. In this case we set $\mu = \lambda_k$ ($\lambda \neq \lambda_k$) in (2.3) and let λ tend to λ_k.

Differentiating (2.3) with respect to μ we obtain for $\lambda \neq \mu$,

$$(y_\lambda(\cdot,\lambda), y_\mu(\cdot,\mu)) = -(a\lambda + b + a\mu)y_\lambda(1,\lambda)y_\mu(1,\mu)$$
$$-ay_\lambda(1,\lambda)y(1,\mu) - ay(1,\lambda)y_\mu(1,\mu) + y_\lambda(1,\lambda)\frac{\omega'(\mu)}{\lambda-\mu}$$
$$+y_\lambda(1,\lambda)\frac{\omega(\mu)}{(\lambda-\mu)^2} - y(1,\lambda)\frac{\omega'(\mu)}{(\lambda-\mu)^2} - y(1,\lambda)\frac{2\omega(\mu)}{(\lambda-\mu)^3} - y_\mu(1,\mu)\frac{\omega'(\lambda)}{\lambda-\mu}$$
$$-y(1,\mu)\frac{\omega'(\lambda)}{(\lambda-\mu)^2} + y_\mu(1,\mu)\frac{\omega(\lambda)}{(\lambda-\mu)^2} + y(1,\mu)\frac{2\omega(\lambda)}{(\lambda-\mu)^3}. \tag{2.4}$$

Setting $\mu = \lambda_k$ ($\lambda \neq \lambda_k$) and letting λ tend to λ_k in (2.4) we obtain

$$(\widetilde{y}_{k+1}, \widetilde{y}_{k+1}) = -(2a\lambda_k + b)\widetilde{y}_{k+1}(1)^2 - 2a\widetilde{y}_{k+1}(1)y_k(1)$$
$$-\widetilde{y}_{k+1}(1)\frac{\omega''(\lambda_k)}{2} - y_k(1)\frac{\omega'''(\lambda_k)}{6}.$$

Note that

$$(y_{k+1}, y_{k+1}) = (\widetilde{y}_{k+1}, \widetilde{y}_{k+1}) - 2\widetilde{c}(y_{k+1}, y_k) - \widetilde{c}^2(y_k, y_k).$$

Using Corollary 1.1, (2.1), and the last two equalities we obtain, after simplifications, the required equality (2.2). □

Legitimacy of differentiation and subsequent passages to limit within integrals in the proof of Lemma 2.1 is based on [18, ch. 3, §4].

Lemma 2.2. If λ_k is a triple eigenvalue and $\lambda_n \neq \lambda_k$ then

$$(y_{k+2}, y_n) = -(a\lambda_k + b + a\lambda_n)y_{k+2}(1)y_n(1) - ay_{k+1}(1)y_n(1),$$

$$(y_{k+2}, y_k) = -(2a\lambda_k + b)y_{k+2}(1)y_k(1) - ay_{k+1}(1)y_k(1) - y_k(1)\frac{\omega'''(\lambda_k)}{6}, \tag{2.5}$$

$$(y_{k+2}, y_{k+1}) = -(2a\lambda_k + b)y_{k+2}(1)y_{k+1}(1) - ay_{k+2}(1)y_k(1) - ay_{k+1}(1)^2$$
$$-\widehat{y}_{k+1}(1)\frac{\omega'''(\lambda_k)}{6} - y_k(1)\frac{\omega^{IV}(\lambda_k)}{24}, \tag{2.6}$$

$$\|y_{k+2}\|_2^2 = (y_{k+2}, y_{k+2}) = -(2a\lambda_k + b)y_{k+2}(1)^2 - 2ay_{k+2}(1)y_{k+1}(1)$$
$$-\widehat{y}_{k+2}(1)\frac{\omega'''(\lambda_k)}{6} - \widehat{y}_{k+1}(1)\frac{\omega^{IV}(\lambda_k)}{24} - y_k(1)\frac{\omega^{V}(\lambda_k)}{120}, \tag{2.7}$$

where $\widehat{y}_{k+2} = y_{k+2} - \widetilde{c}\widetilde{y}_{k+1} - \widetilde{d}y_k$.

Proof. The proof in this case is very similar to Lemma 2.1, so we only indicate the main steps. By differentiating (2.3) with respect to λ, after passages to limit, we prove the first equality and (2.5). Differentiating (2.4) with respect to λ, after passing to the limit, we obtain (2.6). Differentiating (2.4) with respect to λ and then to μ, after passages to limit, we obtain (2.7). □

4. EXISTENCE OF AUXILIARY ASSOCIATED FUNCTIONS

In this section we shall prove the existence of some associated functions y^*_{k+1}, $y^\#_{k+1}$, $y^\#_{k+2}$ with special properties, which we call *auxiliary associated functions*, or just (for short) *A functions*.

Lemma 3.1. If λ_k is a double eigenvalue then there exists an associated function $y^*_{k+1} = y_{k+1} + c_1 y_k$, where c_1 is a constant, for which

$$(y^*_{k+1}, y_{k+1}) = -(2a\lambda_k + b)y^*_{k+1}(1)y_{k+1}(1) - ay^*_{k+1}(1)y_k(1) - ay_{k+1}(1)y_k(1).$$

Proof. By summing (2.2) with (2.1) multiplied by c_1, where

$$c_1 = -\frac{y_k(1)\omega'''(\lambda_k) + 3\widehat{y}_{k+1}(1)\omega''(\lambda_k)}{3y_k(1)\omega''(\lambda_k)},$$

we obtain that

$$(y_{k+1} + c_1 y_k, y_{k+1}) = -(2a\lambda_k + b)(y_{k+1}(1) + c_1 y_k(1))y_{k+1}(1)$$
$$-a(y_{k+1}(1) + c_1 y_k(1))y_k(1) - ay_{k+1}(1)y_k(1),$$

which proves the existence of the A function y^*_{k+1}. □

Lemma 3.2. If λ_k is a triple eigenvalue then there exists an associated function $y^\#_{k+1} = y_{k+1} + c_2 y_k$, where c_2 is a constant, for which

$$(y^\#_{k+1}, y_{k+2}) = -(2a\lambda_k + b)y^\#_{k+1}(1)y_{k+2}(1) - ay^\#_{k+1}(1)y_{k+1}(1) - ay_{k+2}(1)y_k(1).$$

Proof. By summing (2.6) with (2.5) multiplied by c_2, where

$$c_2 = -\frac{y_k(1)\omega^{IV}(\lambda_k) + 4\widehat{y}_{k+1}(1)\omega'''(\lambda_k)}{4y_k(1)\omega'''(\lambda_k)},$$

we prove the existence of the A function $y^\#_{k+1}$. □

Note that for the A functions y^*_{k+1} (in the case **(c)**) and $y^\#_{k+1}$ (in the case **(d)**) first two equalities in Lemma 2.1 are also true, and

$$(y^\#_{k+1}, y_{k+1}) = -(2a\lambda_k + b)y^\#_{k+1}(1)y_{k+1}(1) - ay_k(1)(y^\#_{k+1}(1) + y_{k+1}(1)) + B_{k+2}.$$

Lemma 3.3. If λ_k is a triple eigenvalue then there exists an associated function $y^\#_{k+2} = y_{k+2} + c_2 y_{k+1} + d_1 y_k$, where d_1 is a constant, for which

$$(y^{\#}_{k+2}, y_{k+1}) = -(2a\lambda_k + b)y^{\#}_{k+2}(1)y_{k+1}(1) - ay^{\#}_{k+2}(1)y_k(1) - ay^{\#}_{k+1}(1)y_{k+1}(1),$$

$$(y^{\#}_{k+2}, y_{k+2}) = -(2a\lambda_k + b)y^{\#}_{k+2}(1)y_{k+2}(1) - ay^{\#}_{k+2}(1)y_{k+1}(1) - ay^{\#}_{k+1}(1)y_{k+2}(1).$$

Proof. Firstly, we find for the function $y^*_{k+2} = y_{k+2} + c_2 y_{k+1}$ that

$$(y^*_{k+2}, y_{k+1}) = -(2a\lambda_k + b)y^*_{k+2}(1)y_{k+1}(1) - ay^*_{k+2}(1)y_k(1) - ay^{\#}_{k+1}(1)y_{k+1}(1).$$

By summing this equality with (2.1) multiplied by d_1, and noting $\omega''(\lambda_k) = 0$ we see that the first equality of Lemma 3.3 is true irrespective of the value of d_1. Next we find that

$$\begin{aligned}(y^*_{k+2}, y_{k+2}) = -(2a\lambda_k + b)y^*_{k+2}(1)y_{k+2}(1) - ay^*_{k+2}(1)y_{k+1}(1) - ay^{\#}_{k+1}(1)y_{k+2}(1)\\ -\widehat{y}_{k+2}(1)\frac{\omega'''(\lambda_k)}{6} - \widehat{y}_{k+1}(1)\frac{\omega^{IV}(\lambda_k)}{24} - y_k(1)\frac{\omega^{V}(\lambda_k)}{120}\\ -c_2\left(\widehat{y}_{k+1}(1)\frac{\omega'''(\lambda_k)}{6} + y_k(1)\frac{\omega^{IV}(\lambda_k)}{24}\right).\end{aligned}$$

By summing this equality with (2.5) multiplied by d_1, where

$$d_1 = -\frac{y_k(1)\omega^{V}(\lambda_k) + 5\widehat{y}_{k+1}(1)\omega^{IV}(\lambda_k) + 20\widehat{y}_{k+2}(1)\omega'''(\lambda_k)}{20y_k(1)\omega'''(\lambda_k)} + c_2^2,$$

we obtain the second equality in Lemma 3.3. □

Note that for the A function $y^{\#}_{k+2}$ first two equalities in Lemma 2.2 are also true:

$$(y^{\#}_{k+2}, y_n) = -(a\lambda_k + b + a\lambda_n)y^{\#}_{k+2}(1)y_n(1) - ay^{\#}_{k+1}(1)y_n(1),$$

$$(y^{\#}_{k+2}, y_k) = -(2a\lambda_k + b)y^{\#}_{k+2}(1)y_k(1) - ay^{\#}_{k+1}(1)y_k(1) + B_{k+2}.$$

5. ASYMPTOTIC FORMULAS FOR EIGENFUNCTIONS

Lemma 4.1. Following asymptotic formulas are valid:

$$y_n = \begin{cases} \sqrt{2}\cos\left(n - 3/2\right)\pi x + O(1/n) & \text{if } \beta \neq 0,\\ \sqrt{2}\sin\left(n-1\right)\pi x + O(1/n) & \text{if } \beta = 0,\end{cases} \tag{4.1}$$

$$\|y_n\|_2 = 1 + O(1/n), \tag{4.2}$$

$$y_n(x) = O(1). \tag{4.3}$$

Proof. Noting (0.4) the proof of the asymptotic formula (4.1) is similar to [7, Theorem 2.1], [5, Theorem 2.1]. The formulas (4.2) and (4.3) follows directly from (4.1). □

Lemma 4.2. Following asymptotic formulas are valid:

$$y_n(1) = O(1/n^3), \tag{4.4}$$

$$B_n = 1 + O(1/n). \tag{4.5}$$

Proof. It is known that (see [9, Theorem 2], the case $q = +\infty$),

$$\max_{0\le x\le 1}|y_n'(x)| \le \text{const}\cdot\left(1 + |\sqrt{\lambda_n}|\right)\max_{0\le x\le 1}|y_n(x)|.$$

Then using (0.3), (0.4), and (4.3), we obtain

$$|(a\lambda_n^2 + b\lambda_n + c)y_n(1)| = |y_n'(1)| \le \text{const}\cdot\left(1 + |\sqrt{\lambda_n}|\right)\max_{0\le x\le 1}|y_n(x)| = O(n).$$

Using this and noting $a\lambda_n^2 + b\lambda_n + c = a\pi^4 n^4(1 + O(1/n))$ we obtain the formula (4.4). The formula (4.5) follows from (0.4), (1.2), (4.2), and (4.4). □

6. BASISNESS OF ROOT FUNCTIONS

Theorem 5.1.

(1) In the cases ***(a)*** and ***(b)*** the system $\{y_n\}$ ($n = 0, 1, \ldots;\ n \neq i, j$), where i, j are arbitrary, non-negative and different integers, form a basis in $L_p(0,1)$ $(1 < p < \infty)$.

(2) In the case ***(c)*** the system $\{y_n\}$ $(n = 0,\ 1, \ldots;\ n \neq k, k+1)$ form a basis in $L_p(0,1)$.

(3) In the case ***(c)*** the system $\{y_n\}$ $(n = 0,\ 1, \ldots;\ n \neq k+1, j)$, where $j \neq k, k+1$ is arbitrary non-negative integer, form a basis in $L_p(0,1)$.

(4) In the case ***(c)*** the system $\{y_n\}$ $(n = 0,\ 1, \ldots;\ n \neq k, j)$, where $j \neq k, k+1$ is arbitrary non-negative integer, form a basis in $L_p(0,1)$ if and only if $y^*_{k+1}(1)(\lambda_j - \lambda_k) \neq y_k(1)$.

(5) In the case ***(c)*** the system $\{y_n\}$ $(n = 0,\ 1, \ldots;\ n \neq i, j)$, where $i, j \neq k, k+1$ are arbitrary, non-negative and different integers, form a basis in $L_p(0,1)$.

(6) In the case ***(d)*** the system $\{y_n\}$ $(n = 0,\ 1, \ldots;\ n \neq k+1, k+2)$ form a basis in $L_p(0,1)$.

(7) In the case ***(d)*** the system $\{y_n\}$ $(n = 0,\ 1, \ldots;\ n \neq k, k+2)$ form a basis in $L_p(0,1)$ if and only if $y^{\#}_{k+1}(1) \neq 0$.

(8) In the case ***(d)*** the system $\{y_n\}$ $(n = 0,\ 1, \ldots;\ n \neq k, k+1)$ form a basis in $L_p(0,1)$ if and only if $y^{\#}_{k+1}(1)^2 \neq y_k(1) y^{\#}_{k+2}(1)$.

(9) In the case ***(d)*** the system $\{y_n\}$ $(n = 0,\ 1, \ldots;\ n \neq k+2, j)$, where $j \neq k, k+1, k+2$ is arbitrary non-negative integer, form a basis in $L_p(0,1)$.

(10) In the case ***(d)*** the system $\{y_n\}$ $(n = 0,\ 1, \ldots;\ n \neq k+1, j)$, where $j \neq k, k+1, k+2$ is arbitrary non-negative integer, form a basis in $L_p(0,1)$ if and only if $y^{\#}_{k+1}(1)(\lambda_j - \lambda_k) \neq y_k(1)$.

(11) In the case ***(d)*** the system $\{y_n\}$ $(n = 0,\ 1, \ldots;\ n \neq k, j)$, where $j \neq k, k+1, k+2$ is arbitrary non-negative integer, form a basis in $L_p(0,1)$ if and only if $y^{\#}_{k+2}(1)(\lambda_j - \lambda_k) \neq y^{\#}_{k+1}(1)$.

(12) In the case ***(d)*** the system $\{y_n\}$ $(n = 0,\ 1, \ldots;\ n \neq i, j)$, where $i, j \neq k, k+1, k+2$ are arbitrary, non-negative and different integers, form a basis in $L_p(0,1)$.

Moreover, if $p = 2$ then in all the considered cases the basis is unconditional.

Proof. ***(1)*** Firstly we consider the case ***(a)*** and prove that in this case the system

$$\{y_n\}\ \ (n = 0,\ 1, \ldots;\ n \neq i, j), \tag{5.1}$$

is minimal in $L_p(0,1)$. For this it suffices to show the existence of a system

$$\{u_n\}\ \ (n = 0,\ 1, \ldots;\ n \neq i, j), \tag{5.2}$$

biorthogonal to (5.1). We define the elements of (5.2) by

$$u_n(x) = \frac{1}{B_n \Delta_{ij}} \begin{vmatrix} y_n(x) & y_n(1) & \lambda_n y_n(1) \\ y_i(x) & y_i(1) & \lambda_i y_i(1) \\ y_j(x) & y_j(1) & \lambda_j y_j(1) \end{vmatrix}, \tag{5.3}$$

where $\Delta_{ij} = (\lambda_j - \lambda_i) y_i(1) y_j(1)$ is the complementary minor of the upper left element of the above determinant. Let us verify that $(u_n, y_m) = \delta_{nm}$ $(n, m \neq i, j)$, where δ_{nm} denotes as usually, Kronecker's symbol: $\delta_{nm} = 0$ if $n \neq m$ and $\delta_{nn} = 1$. We have

$$(u_n, y_m) = \frac{1}{B_n \Delta_{ij}} \begin{vmatrix} (y_n, y_m) & y_n(1) & \lambda_n y_n(1) \\ (y_i, y_m) & y_i(1) & \lambda_i y_i(1) \\ (y_j, y_m) & y_j(1) & \lambda_j y_j(1) \end{vmatrix}. \tag{5.4}$$

It now follows immediate by from Lemma 1.1 that for $m \neq n$ the first column of the determinant in (5.4) is a linear combination of the other columns; hence $(u_n, y_m) = 0$.

Assume now that $m = n$ in (5.4). Adding to the first column of the determinant in (5.4) the 2nd and 3rd columns multiplied respectively by $(a\lambda_n + b) y_n(1)$ and $a y_n(1)$ we obtain

$$(u_n, y_n) = \frac{1}{B_n \Delta_{ij}} \begin{vmatrix} B_n & y_n(1) & \lambda_n y_n(1) \\ 0 & y_i(1) & \lambda_i y_i(1) \\ 0 & y_j(1) & \lambda_j y_j(1) \end{vmatrix} = 1.$$

We shall now prove unconditional basisness of the system (5.1) in $L_2(0,1)$. For this we compare this system with the system

$$\{\varphi_n\}(n = 2, 3, \ldots), \tag{5.5}$$

where

$$\varphi_n(x) = \begin{cases} \sqrt{2} \cos{(n - 3/2)\,\pi x} & \text{if } \beta \neq 0, \\ \sqrt{2} \sin{(n-1)\,\pi x} & \text{if } \beta = 0. \end{cases}$$

The system (5.5) is a basis of $L_p(0,1)$, and in particular, an orthonormal basis of $L_2(0,1)$ (see *e.g.* [19]). By (4.1),

$$\|y_n - \varphi_n\|_2 \le \text{const}/n. \tag{5.6}$$

for sufficiently large n. Without loss of generality we may suppose that $i < j$. From (5.6) it follows that the series

$$\sum_{n=2}^{i+1} \|y_{n-2} - \varphi_n\|_2^2 + \sum_{n=i+2}^{j} \|y_{n-1} - \varphi_n\|_2^2 + \sum_{n=j+1}^{\infty} \|y_n - \varphi_n\|_2^2,$$

is convergent. Hence the system (5.1) is quadratically close to the system (5.5). Since the system (5.1) is minimal in $L_2(0,1)$ then our claim is established for $p = 2$ (see [20], Sect. 9.9.8 of the Russian translation). The proof of the remaining case $p \neq 2$ will be essentially same with [7, Theorem 2.1], so we only indicate the following asymptotic formula

$$u_n(x) = y_n(x) + O(1/n),$$

for sufficiently large n. Indeed, expanding the determinant in (5.3) along the first row and taking into account that all the elements in 2nd and 3rd rows are either bounded functions or fixed real numbers, we deduce from (0.4), Lemma 4.2 that this asymptotic formula is valid.

Now we consider the case ***(b)***. Let λ_r and λ_s are a conjugate pair of non-real eigenvalues $\lambda_s = \overline{\lambda_r}$. The biorthogonal system of

$$\{y_n\} \ \ (n = 0,\ 1, \ldots;\ n \neq r, s),$$

is defined by

$$u_n(x) = \frac{1}{B_n \Delta_{rs}} \begin{vmatrix} y_n(x) & y_n(1) & \lambda_n y_n(1) \\ y_r(x) & y_r(1) & \lambda_r y_r(1) \\ y_s(x) & y_s(1) & \lambda_s y_s(1) \end{vmatrix},$$

for $n \neq r, s$. The equality $(u_n, y_m) = \delta_{nm}$ for $n, m \neq r, s$ can be verified using Lemma 1.1. Basisness can be proved as in case ***(a)***.

The biorthogonal system of

$$\{y_n\} \ \ (n = 0,\ 1, \ldots;\ n \neq r, j),$$

where $j \neq r, s$ is an arbitrary non-negative integer, is defined by

$$u_n(x) = \frac{1}{B_n \Delta_{rj}} \begin{vmatrix} y_n(x) & y_n(1) & \lambda_n y_n(1) \\ y_r(x) & y_r(1) & \lambda_r y_r(1) \\ y_j(x) & y_j(1) & \lambda_j y_j(1) \end{vmatrix},$$

for $n \neq r, s, j$, and

$$u_s(x) = -\frac{1}{y_r(1)\omega'(\lambda_r)\Delta_{sj}} \begin{vmatrix} y_r(x) & y_r(1) & \lambda_r y_r(1) \\ y_s(x) & y_s(1) & \lambda_s y_s(1) \\ y_j(x) & y_j(1) & \lambda_j y_j(1) \end{vmatrix}.$$

The biorthogonal system of

$$\{y_n\} \ \ (n = 0,\ 1, \ldots;\ n \neq i, j),$$

where $i, j \neq r, s$ are arbitrary non-negative different integers, is defined by (5.3) for $n \neq r, s, i, j$, and

$$u_r(x) = -\frac{1}{y_s(1)\omega'(\lambda_s)\Delta_{ij}} \begin{vmatrix} y_s(x) & y_s(1) & \lambda_s y_s(1) \\ y_i(x) & y_i(1) & \lambda_i y_i(1) \\ y_j(x) & y_j(1) & \lambda_j y_j(1) \end{vmatrix},$$

$$u_s(x) = -\frac{1}{y_r(1)\omega'(\lambda_r)\Delta_{ij}} \begin{vmatrix} y_r(x) & y_r(1) & \lambda_r y_r(1) \\ y_i(x) & y_i(1) & \lambda_i y_i(1) \\ y_j(x) & y_j(1) & \lambda_j y_j(1) \end{vmatrix}.$$

The equality $(u_n, y_m) = \delta_{nm}$ for $n, m \neq i, j$ can be verified using Lemma 1.1, Lemma 1.3, and Corollary 1.2.

Thus we have studied all the cases of ***(b)***: when both of the removed eigenfunctions correspond to non-real eigenvalues, when only one, and when none.

(2) The biorthogonal system is defined by

$$u_n(x) = \frac{1}{B_n y_k(1)^2} \begin{vmatrix} y_n(x) & y_n(1) & \lambda_n y_n(1) \\ y_k(x) & y_k(1) & \lambda_k y_k(1) \\ y_{k+1}(x) & y_{k+1}(1) & \lambda_k y_{k+1}(1) + y_k(1) \end{vmatrix}, \tag{5.7}$$

for $n \neq k, k+1$. The equality $(u_n, y_m) = \delta_{nm}$ for $n, m \neq k, k+1$ can be verified using Lemma 1.1, Lemma 2.1.

(3) The biorthogonal system is defined by

$$u_n(x) = \frac{1}{B_n \Delta_{kj}} \begin{vmatrix} y_n(x) & y_n(1) & \lambda_n y_n(1) \\ y_k(x) & y_k(1) & \lambda_k y_k(1) \\ y_j(x) & y_j(1) & \lambda_j y_j(1) \end{vmatrix}, \tag{5.8}$$

for $n \neq k, k+1, j$, and

$$u_k(x) = \frac{1}{B_{k+1} \Delta_{kj}} \begin{vmatrix} y_{k+1}(x) & y_{k+1}(1) & \lambda_k y_{k+1}(1) + y_k(1) \\ y_k(x) & y_k(1) & \lambda_k y_k(1) \\ y_j(x) & y_j(1) & \lambda_j y_j(1) \end{vmatrix},$$

The equality $(u_n, y_m) = \delta_{nm}$ for $n, m \neq k+1, j$ can be verified using Lemma 1.1, Lemma 2.1.

(4) The biorthogonal system is defined by

$$u_n(x) = \frac{1}{B_n \Delta^*_{kj}} \begin{vmatrix} y_n(x) & y_n(1) & \lambda_n y_n(1) \\ y^*_{k+1}(x) & y^*_{k+1}(1) & \lambda_k y^*_{k+1}(1) + y_k(1) \\ y_j(x) & y_j(1) & \lambda_j y_j(1) \end{vmatrix},$$

for $n \neq k, k+1, j$, and

$$u_{k+1}(x) = \frac{1}{B_{k+1} \Delta^*_{kj}} \begin{vmatrix} y_k(x) & y_k(1) & \lambda_k y_k(1) \\ y^*_{k+1}(x) & y^*_{k+1}(1) & \lambda_k y^*_{k+1}(1) + y_k(1) \\ y_j(x) & y_j(1) & \lambda_j y_j(1) \end{vmatrix},$$

where $\Delta^*_{kj} = (\lambda_j - \lambda_k) y^*_{k+1}(1) y_j(1) - y_k(1) y_j(1)$. The equality $(u_n, y_m) = \delta_{nm}$ for $n, m \neq k, j$ can be verified using Lemma 1.1, Lemma 2.1, and Lemma 3.1. If $\Delta^*_{kj} = 0$ then the function defined by the last determinant or, which is the same, the function (*cf.* [8, Theorem 3])

$$\begin{vmatrix} y_k(x) & y_k(1) & \lambda_k y_k(1) \\ y_{k+1}(x) & y_{k+1}(1) & \lambda_k y_{k+1}(1) + y_k(1) \\ y_j(x) & y_j(1) & \lambda_j y_j(1) \end{vmatrix},$$

is orthogonal to all the elements of the system $\{y_n\}$ $(n \neq k, j)$ and therefore this system is not complete in $L_2(0,1)$. The condition $\Delta^*_{kj} = 0$ also implies that this system is not minimal in $L_2(0,1)$. Indeed, on the contrary, using minimality in $L_2(0,1)$, we can prove, as in case **(a)**, basisness of this system in $L_2(0,1)$, which contradicts incompleteness of this system in $L_2(0,1)$.

(5) The biorthogonal system is defined by (5.3) for $n \neq k, k+1, i, j$, and

$$u_k(x) = \frac{1}{B_{k+1} \Delta_{ij}} \begin{vmatrix} y^*_{k+1}(x) & y^*_{k+1}(1) & \lambda_k y^*_{k+1}(1) + y_k(1) \\ y_i(x) & y_i(1) & \lambda_i y_i(1) \\ y_j(x) & y_j(1) & \lambda_j y_j(1) \end{vmatrix},$$

$$u_{k+1}(x) = \frac{1}{B_{k+1} \Delta_{ij}} \begin{vmatrix} y_k(x) & y_k(1) & \lambda_k y_k(1) \\ y_i(x) & y_i(1) & \lambda_i y_i(1) \\ y_j(x) & y_j(1) & \lambda_j y_j(1) \end{vmatrix}.$$

(6) The biorthogonal system is defined by (5.7) for $n \neq k, k+1, k+2$, and

$$u_k(x) = \frac{1}{B_{k+2} y_k(1)^2} \begin{vmatrix} y_{k+2}(x) & y_{k+2}(1) & \lambda_k y_{k+2}(1) + y_{k+1}(1) \\ y_k(x) & y_k(1) & \lambda_k y_k(1) \\ y_{k+1}(x) & y_{k+1}(1) & \lambda_k y_{k+1}(1) + y_k(1) \end{vmatrix}.$$

The equality $(u_n, y_m) = \delta_{nm}$ for $n, m \neq k+1, k+2$ can be verified using Lemma 1.1, Corollary 1.1, Lemma 2.1, and Lemma 2.2.

(7) The biorthogonal system is defined by

$$u_n(x) = \frac{1}{B_n \Delta^{\#}_{k,k+2}} \begin{vmatrix} y_n(x) & y_n(1) & \lambda_n y_n(1) \\ y_k(x) & y_k(1) & \lambda_k y_k(1) \\ y^{\#}_{k+2}(x) & y^{\#}_{k+2}(1) & \lambda_k y^{\#}_{k+2}(1) + y^{\#}_{k+1}(1) \end{vmatrix},$$

for $n \neq k, k+1, k+2$, and

$$u_{k+1}(x) = \frac{1}{B_{k+2} \Delta^{\#}_{k,k+2}} \begin{vmatrix} y_{k+1}(x) & y_{k+1}(1) & \lambda_k y_{k+1}(1) + y_k(1) \\ y_k(x) & y_k(1) & \lambda_k y_k(1) \\ y^{\#}_{k+2}(x) & y^{\#}_{k+2}(1) & \lambda_k y^{\#}_{k+2}(1) + y^{\#}_{k+1}(1) \end{vmatrix},$$

where $\Delta^{\#}_{k,k+2} = y^{\#}_{k+1}(1) y_k(1)$. The equality $(u_n, y_m) = \delta_{nm}$ for $n, m \neq k, k+2$ can be verified using the mentioned results of Sections 1 and 2 and Lemma 3.3. If $\Delta^{\#}_{k,k+2} = 0$ then the function defined by the last determinant is orthogonal to all the elements of the system $\{y_n\}$ $(n \neq k, k+2)$ and therefore this system is neither complete nor minimal.

(8) The biorthogonal system is defined by

$$u_n(x) = \frac{1}{B_n \Delta^{\#}_{k+1,k+2}} \begin{vmatrix} y_n(x) & y_n(1) & \lambda_n y_n(1) \\ y^{\#}_{k+1}(x) & y^{\#}_{k+1}(1) & \lambda_k y^{\#}_{k+1}(1) + y_k(1) \\ y^{\#}_{k+2}(x) & y^{\#}_{k+2}(1) & \lambda_k y^{\#}_{k+2}(1) + y^{\#}_{k+1}(1) \end{vmatrix},$$

for $n \neq k, k+1, k+2$, and

$$u_{k+2}(x) = \frac{1}{B_{k+2} \Delta^{\#}_{k+1,k+2}} \begin{vmatrix} y_k(x) & y_k(1) & \lambda_k y_k(1) \\ y^{\#}_{k+1}(x) & y^{\#}_{k+1}(1) & \lambda_k y^{\#}_{k+1}(1) + y_k(1) \\ y^{\#}_{k+2}(x) & y^{\#}_{k+2}(1) & \lambda_k y^{\#}_{k+2}(1) + y^{\#}_{k+1}(1) \end{vmatrix},$$

where $\Delta^{\#}_{k+1,k+2} = y^{\#}_{k+1}(1)^2 - y_k(1) y^{\#}_{k+2}(1)$. The equality $(u_n, y_m) = \delta_{nm}$ for $n, m \neq k, k+1$ can be verified using the Lemma 3.2 and Lemma 3.3. If $\Delta^{\#}_{k+1,k+2} = 0$ then the function defined by the last determinant is orthogonal to all the elements of the system $\{y_n\}$ $(n \neq k, k+1)$ and therefore this system is neither complete nor minimal.

(9) The biorthogonal system is defined by (5.8) for $n \neq k, k+1, k+2, j$, and

$$u_k(x) = \frac{1}{B_{k+2} \Delta_{kj}} \begin{vmatrix} y^{\#}_{k+2}(x) & y^{\#}_{k+2}(1) & \lambda_k y^{\#}_{k+2}(1) + y^{\#}_{k+1}(1) \\ y_k(x) & y_k(1) & \lambda_k y_k(1) \\ y_j(x) & y_j(1) & \lambda_j y_j(1) \end{vmatrix},$$

$$u_{k+1}(x) = \frac{1}{B_{k+2} \Delta_{kj}} \begin{vmatrix} y_{k+1}(x) & y_{k+1}(1) & \lambda_k y_{k+1}(1) + y_k(1) \\ y_k(x) & y_k(1) & \lambda_k y_k(1) \\ y_j(x) & y_j(1) & \lambda_j y_j(1) \end{vmatrix}.$$

(10) The biorthogonal system is defined by

$$u_n(x) = \frac{1}{B_n \Delta^{\#}_{k+1,j}} \begin{vmatrix} y_n(x) & y_n(1) & \lambda_n y_n(1) \\ y^{\#}_{k+1}(x) & y^{\#}_{k+1}(1) & \lambda_k y^{\#}_{k+1}(1) + y_k(1) \\ y_j(x) & y_j(1) & \lambda_j y_j(1) \end{vmatrix},$$

for $n \neq k, k+1, k+2, j$, and

$$u_k(x) = \frac{1}{B_{k+2} \Delta^{\#}_{k+1,j}} \begin{vmatrix} y^{\#}_{k+2}(x) & y^{\#}_{k+2}(1) & \lambda_k y^{\#}_{k+2}(1) + y^{\#}_{k+1}(1) \\ y^{\#}_{k+1}(x) & y^{\#}_{k+1}(1) & \lambda_k y^{\#}_{k+1}(1) + y_k(1) \\ y_j(x) & y_j(1) & \lambda_j y_j(1) \end{vmatrix},$$

$$u_{k+2}(x)=\frac{1}{B_{k+2}\Delta^{\#}_{k+1,j}}\begin{vmatrix} y_k(x) & y_k(1) & \lambda_k y_k(1) \\ y^{\#}_{k+1}(x) & y^{\#}_{k+1}(1) & \lambda_k y^{\#}_{k+1}(1)+y_k(1) \\ y_j(x) & y_j(1) & \lambda_j y_j(1) \end{vmatrix},$$

where $\Delta^{\#}_{k+1,j}=(\lambda_j-\lambda_k)y^{\#}_{k+1}(1)y_j(1)-y_k(1)y_j(1)$. If $\Delta^{\#}_{k+1,j}=0$ then the function defined by the last determinant is orthogonal to all the elements of the system $\{y_n\}$ $(n\neq k+1,j)$ and therefore this system is neither complete nor minimal.

(11) The biorthogonal system is defined by

$$u_n(x)=\frac{1}{B_n\Delta^{\#}_{k+2,j}}\begin{vmatrix} y_n(x) & y_n(1) & \lambda_n y_n(1) \\ y^{\#}_{k+2}(x) & y^{\#}_{k+2}(1) & \lambda_k y^{\#}_{k+2}(1)+y^{\#}_{k+1}(1) \\ y_j(x) & y_j(1) & \lambda_j y_j(1) \end{vmatrix},$$

for $n\neq k,k+1,k+2,j$, and

$$u_{k+1}(x)=\frac{1}{B_{k+2}\Delta^{\#}_{k+2,j}}\begin{vmatrix} y^{\#}_{k+1}(x) & y^{\#}_{k+1}(1) & \lambda_k y^{\#}_{k+1}(1)+y_k(1) \\ y^{\#}_{k+2}(x) & y^{\#}_{k+2}(1) & \lambda_k y^{\#}_{k+2}(1)+y^{\#}_{k+1}(1) \\ y_j(x) & y_j(1) & \lambda_j y_j(1) \end{vmatrix},$$

$$u_{k+2}(x)=\frac{1}{B_{k+2}\Delta^{\#}_{k+2,j}}\begin{vmatrix} y_k(x) & y_k(1) & \lambda_k y_k(1) \\ y^{\#}_{k+2}(x) & y^{\#}_{k+2}(1) & \lambda_k y^{\#}_{k+2}(1)+y^{\#}_{k+1}(1) \\ y_j(x) & y_j(1) & \lambda_j y_j(1) \end{vmatrix},$$

where $\Delta^{\#}_{k+2,j}=(\lambda_j-\lambda_k)y^{\#}_{k+2}(1)y_j(1)-y^{\#}_{k+1}(1)y_j(1)$. If $\Delta^{\#}_{k+2,j}=0$ then the function defined by the last determinant is orthogonal to all the elements of the system $\{y_n\}$ $(n\neq k,j)$ and therefore this system is neither complete nor minimal.

(12) The biorthogonal system is defined by (5.3) for $n\neq k,k+1,k+2,i,j$, and

$$u_k(x)=\frac{1}{B_{k+2}\Delta_{ij}}\begin{vmatrix} y^{\#}_{k+2}(x) & y^{\#}_{k+2}(1) & \lambda_k y^{\#}_{k+2}(1)+y^{\#}_{k+1}(1) \\ y_i(x) & y_i(1) & \lambda_i y_i(1) \\ y_j(x) & y_j(1) & \lambda_j y_j(1) \end{vmatrix},$$

$$u_{k+1}(x)=\frac{1}{B_{k+2}\Delta_{ij}}\begin{vmatrix} y^{\#}_{k+1}(x) & y^{\#}_{k+1}(1) & \lambda_k y^{\#}_{k+1}(1)+y_k(1) \\ y_i(x) & y_i(1) & \lambda_i y_i(1) \\ y_j(x) & y_j(1) & \lambda_j y_j(1) \end{vmatrix},$$

$$u_{k+2}(x)=\frac{1}{B_{k+2}\Delta_{ij}}\begin{vmatrix} y_k(x) & y_k(1) & \lambda_k y_k(1) \\ y_i(x) & y_i(1) & \lambda_i y_i(1) \\ y_j(x) & y_j(1) & \lambda_j y_j(1) \end{vmatrix}.$$

7. EXAMPLES

We shall demonstrate our theory in two examples inspired by [2,8].

Example 7.1. Consider the spectral problem

$$-y''=\lambda y,\ 0<x<1,$$

$$y'(0)=0,\ y'(1)=\left(\frac{\lambda^2}{\pi^2}-\lambda\right)y(1).$$

For this problem

$$y(x,\lambda)=\cos\sqrt{\lambda}x,\quad y'(x,\lambda)=-\sqrt{\lambda}\sin\sqrt{\lambda}x,$$

and

$$\omega(\lambda)=-\sqrt{\lambda}\sin\sqrt{\lambda}-\left(\frac{\lambda^2}{\pi^2}-\lambda\right)\cos\sqrt{\lambda}.$$

It is easy to check that

$$\omega(0)=\omega'(0)=0,\quad \omega''(0)=-\frac{2}{3}-\frac{2}{\pi^2},\quad \omega'''(0)=\frac{1}{5}+\frac{3}{\pi^2},\quad \omega(\pi^2)=0\neq\omega'(\pi^2).$$

Thus $\lambda_0=\lambda_1=0$ is a double eigenvalue, $\lambda_2=\pi^2$ is a simple eigenvalue and all other simple eigenvalues $\lambda_3<\lambda_4<\dots$ are the solutions of the equation

$$\tan\sqrt{\lambda}=\sqrt{\lambda}\left(1-\frac{\lambda}{\pi^2}\right). \tag{7.1}$$

Note that $y_0=1$, $y_2=\cos\pi x$, $y_i=\cos\sqrt{\lambda_i}x$ $(i\geq 3)$, and

$$\widetilde{y}_1(x)=\lim_{\lambda\to 0}y_\lambda(x,\lambda)=\lim_{\lambda\to 0}\left(-\frac{x\sin\sqrt{\lambda}x}{2\sqrt{\lambda}}\right)=-\frac{x^2}{2}.$$

We define the first order associated function by $y_1=-\frac{x^2}{2}+c$, that is $c=-\widetilde{c}$ (c is a constant). Then $\widehat{y}_1=y_1-\widetilde{c}y_0=-\frac{x^2}{2}+2c$. For the determination of the A function y_1^* we find

$$c_1=-\frac{y_0(1)\omega'''(0)+3\widehat{y}_1(1)\omega''(0)}{3y_0(1)\omega''(0)}=-2c+\frac{3}{5}\cdot\frac{\pi^2+5}{\pi^2+3},$$

and therefore

$$y_1^*=y_1+c_1y_0=-\frac{x^2}{2}+\frac{3}{5}\cdot\frac{\pi^2+5}{\pi^2+3}-c.$$

It is possible to find the A function y_1^* directly from the equality in Lemma 3.1. Indeed, if we seek the auxiliary associated function y_1^* in the form $y_1^*=-\frac{x^2}{2}+c'$, then by Lemma 3.1,

$$\int_0^1\left(-\frac{1}{2}x^2+c'\right)\left(-\frac{1}{2}x^2+c\right)\mathrm{d}x=\left(-\frac{1}{2}+c'\right)\left(-\frac{1}{2}+c\right)$$
$$-\frac{1}{\pi^2}\left(-\frac{1}{2}+c'\right)-\frac{1}{\pi^2}\left(-\frac{1}{2}+c\right).$$

From this equality it follows that

$$c'=\frac{3}{5}\cdot\frac{\pi^2+5}{\pi^2+3}-c,$$

which agrees with $c'=c+c_1$.

We shall now apply part 4 of Theorem 5.1 to our problem. The system

$$\left\{-\frac{x^2}{2}+c,\quad \cos\sqrt{\lambda_i}x\quad (i=3,4,5,\dots)\right\}, \tag{7.2}$$

that is the system of root functions without removed functions $y_0=1$ and $y_2=\cos\pi x$, is a basis in $L_p(0,1)$ if and only if

$$y_1^*(1)(\lambda_2-\lambda_0)\neq y_0(1),$$

or more explicitly, if

$$c\neq\frac{\pi^2+15}{10(\pi^2+3)}-\frac{1}{\pi^2}.$$

If

$$c=\frac{\pi^2+15}{10(\pi^2+3)}-\frac{1}{\pi^2}.$$

then the function

$$\begin{vmatrix} y_0(x) & y_0(1) & \lambda_0y_0(1)\\ y_1^*(x) & y_1^*(1) & \lambda_0y_1^*(1)+y_0(1)\\ y_2(x) & y_2(1) & \lambda_2y_2(1)\end{vmatrix}=\pi^2\left(-\frac{x^2}{2}+\frac{1}{2}\right)+\cos\pi x+1,$$

is orthogonal to all the elements of the system (6.2). Indeed,

$$\int_0^1 \left(\pi^2\left(-\frac{x^2}{2}+\frac{1}{2}\right)+\cos\pi x+1\right)\left(-\frac{x^2}{2}+\frac{\pi^2+15}{10(\pi^2+3)}-\frac{1}{\pi^2}\right)\mathrm{d}x=0,$$

and for $i=3,4,\ldots$ by (6.1),

$$\int_0^1 \left(\pi^2\left(-\frac{x^2}{2}+\frac{1}{2}\right)+\cos\pi x+1\right)\cos\sqrt{\lambda_i}x\ \mathrm{d}x=$$

$$-\frac{\pi^2\left(\pi^2\sin\sqrt{\lambda_i}+\lambda_i^{3/2}\cos\sqrt{\lambda_i}-\pi^2\sqrt{\lambda_i}\cos\sqrt{\lambda_i}\right)}{\lambda_i^{3/2}\left(\lambda_i-\pi^2\right)}=0.$$

Therefore the system (7.2) is not complete.

Example 7.2. Consider the problem

$$-y''=\lambda y,\ 0<x<1,$$

$$y'(0)=0,\ y'(1)=\left(-\frac{\lambda^2}{3}-\lambda\right)y(1).$$

For this problem

$$y(x,\lambda)=\cos\sqrt{\lambda}x,\quad y'(x,\lambda)=-\sqrt{\lambda}\sin\sqrt{\lambda}x,$$

$$\omega(\lambda)=-\sqrt{\lambda}\sin\sqrt{\lambda}+\left(\frac{\lambda^2}{3}+\lambda\right)\cos\sqrt{\lambda},$$

$$\omega(0)=\omega'(0)=\omega''(0)=0,\ \omega'''(0)=-\frac{4}{5},\ \omega^{IV}(0)=\frac{32}{105},\ \omega^{V}(0)=-\frac{10}{189}.$$

Thus $\lambda_0=\lambda_1=\lambda_2=0$ is a triple eigenvalue, and all other simple eigenvalues $\lambda_3<\lambda_4<\ldots$ are the solutions of the equation $\tan\sqrt{\lambda}=\sqrt{\lambda}\left(1+\frac{\lambda}{3}\right)$.

As for the previous problem

$$y_0=1,\ y_i=\cos\sqrt{\lambda_i}x\ \ (i\geq 3),\ \widetilde{y}_1(x)=-\frac{x^2}{2},\ \widetilde{y}_2(x)=\lim_{\lambda\to 0}\frac{y_{\lambda\lambda}(x,\lambda)}{2}=\frac{x^4}{24},$$

so we define the first and second order associated functions by $y_1=-\frac{x^2}{2}+c$, $y_2=\frac{x^4}{24}+c\left(-\frac{x^2}{2}+c\right)+d$, that is $c=-\widetilde{c}$, $d=-\widetilde{d}$ (c,d are constants). Further calculations shows that

$$c_2=\frac{25}{42}-2c,\ d_1=\frac{1385}{5292}-\frac{25}{21}c+c^2-2d,$$

and therefore the A functions are

$$y_1^{\#}=-\frac{x^2}{2}+\frac{25}{42}-c,\ y_2^{\#}=\frac{x^4}{24}-\left(\frac{25}{42}-c\right)\frac{x^2}{2}+\frac{1385}{5292}-\frac{25}{42}c-d.$$

By part 7 of Theorem 5.1, the system

$$\left\{-\frac{x^2}{2}+c,\ \ \cos\sqrt{\lambda_i}x\ \ (i=3,4,5,\ldots)\right\},\tag{7.3}$$

is a basis in $L_p(0,1)$ if and only if $y_1^{\#}(1)\neq 0$ or more explicitly if $c\neq\frac{2}{21}$. If $c=\frac{2}{21}$ then the function $\frac{x^4}{24}-\frac{x^2}{4}+\frac{5}{24}$ is orthogonal to all the elements of the system (7.3).

By part 8 of Theorem 5.1 the system

$$\left\{\frac{x^4}{24}+c\left(-\frac{x^2}{2}+c\right)+d,\ \ \cos\sqrt{\lambda_i}x\ \ (i=3,4,5,\ldots)\right\},\tag{7.4}$$

is a basis in $L_p(0,1)$ if and only if $y_1^{\#}(1)^2\neq y_0(1)y_2^{\#}(1)$ or more explicitly if $d\neq-\frac{5}{1512}+\frac{2}{21}c-c^2$. If $d=-\frac{5}{1512}+\frac{2}{21}c-c^2$ then the same function $\frac{x^4}{24}-\frac{x^2}{4}+\frac{5}{24}$ is orthogonal to all the elements of the system (6.4).

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