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The basis property in L_p of the boundary value problem rationally dependent on the eigenparameter

by

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Abstract. We consider a Sturm–Liouville operator with boundary conditions rationally dependent on the eigenparameter. We study the basis property in L_p of the system of eigenfunctions corresponding to this operator. We determine the explicit form of the biorthogonal system. Using this we establish a theorem on the minimality of the part of the system of eigenfunctions. For the basisness in L_2 we prove that the system of eigenfunctions is quadratically close to trigonometric systems. For the basisness in L_p we use F. Riesz's theorem.

Consider the spectral problem

$$-y'' + q(x)y = \lambda y, \qquad 0 < x < 1, \tag{0.1}$$

$$y(0)\cos\beta = y'(0)\sin\beta, \qquad 0 \le \beta < \pi, \tag{0.2}$$

$$y'(1)/y(1) = h(\lambda), \tag{0.3}$$

where λ is the spectral parameter, q is a real-valued and continuous function on the interval $[0,1]$,

$$h(\lambda) = a\lambda + b - \sum_{k=1}^{N} \frac{b_k}{\lambda - c_k},$$

where all the coefficients are real and $a \ge 0$, $b_k > 0$, $c_1 < \cdots < c_N$, $N \ge 0$. If $h(\lambda) = \infty$ then (0.3) is interpreted as a Dirichlet condition $y(1) = 0$. If $N = 0$ then there are no c_k's and $h(\lambda)$ is affine in λ.

In a recent paper [1] existence and asymptotics of eigenvalues and oscillation of eigenfunctions of this problem were studied. It was proved that the eigenvalues of (0.1)–(0.3) are real, simple and form a sequence $\lambda_0 < \lambda_1 < \cdots$ accumulating only at ∞ and with $\lambda_0 < c_1$. Moreover, it was proved that if

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ω_n is the number of zeros in $(0,1)$ of the eigenfunction y_n, associated with the eigenvalue λ_n, then $\omega_n = n - m_n$, where m_n is the number of points $c_i \le \lambda_n$. In particular, $\omega_0 = 0$ and $\omega_n = n - N$ when $\lambda_n > c_N$.

The basis properties of eigenvectors of the self-adjoint operator on $L_2 \oplus \mathbb{C}^{N+1}$ (or on $L_2 \oplus \mathbb{C}^N$ if $a = 0$), formed by the eigenfunctions of (0.1)–(0.3) were examined in [2].

The current article concerns the basis properties in $L_p(0,1)$ $(1 < p < \infty)$ of the system of eigenfunctions of the boundary value problem (0.1)–(0.3).

Basis properties of the boundary value problem (0.1)–(0.3) in cases where h is affine or bilinear have been analyzed in [5], [6], [8].

A complete discussion of the basis properties in $L_p(0,1)$ $(1 < p < \infty)$ of the boundary value problem

$$\begin{aligned} &-y'' = \lambda y, \qquad 0 < x < 1,\\ &y(0) = 0, \qquad (a-\lambda)y'(1) = b\lambda y(1), \end{aligned}$$

where a, b are positive constants, is given in [6].

The basis properties in $L_2(0,1)$ of the boundary value problem

$$\begin{aligned} &-y'' + q(x)y = \lambda y, \qquad 0 < x < 1,\\ &b_0 y(0) = d_0 y'(0),\\ &(a_1\lambda + b_1)y(1) = (c_1\lambda + d_1)y'(1), \end{aligned}$$

where q is a real-valued continuous function on $[0,1]$ and $|b_0| + |d_0| \neq 0$, $a_1 d_1 - b_1 c_1 > 0$, were studied in more detail in [8].

1. Minimality of the system of eigenfunctions of (0.1)–(0.3). The following lemma will be needed:

LEMMA 1.1. *Let $\mu_0, \mu_1, \ldots, \mu_N, d_1, d_2, \ldots, d_N$ be pairwise different real numbers. Then*

$$\begin{vmatrix} 1 & (\mu_0 - d_1)^{-1} & \cdots & (\mu_0 - d_N)^{-1}\\ 1 & (\mu_1 - d_1)^{-1} & \cdots & (\mu_1 - d_N)^{-1}\\ \hdotsfor{4}\\ 1 & (\mu_N - d_1)^{-1} & \cdots & (\mu_N - d_N)^{-1} \end{vmatrix} = \frac{\prod\limits_{0\le i<j\le N} (\mu_i - \mu_j) \prod\limits_{1\le i<j\le N} (d_j - d_i)}{\prod\limits_{\substack{0\le i\le N\\ 1\le j\le N}} (\mu_i - d_j)}.$$

Proof. It is known (see e.g. [12, Ch. VII, Prob. 3]) that

$$\begin{vmatrix} (\mu_0 - d_0)^{-1} & (\mu_0 - d_1)^{-1} & \cdots & (\mu_0 - d_N)^{-1}\\ (\mu_1 - d_0)^{-1} & (\mu_1 - d_1)^{-1} & \cdots & (\mu_1 - d_N)^{-1}\\ \hdotsfor{4}\\ (\mu_N - d_0)^{-1} & (\mu_N - d_1)^{-1} & \cdots & (\mu_N - d_N)^{-1} \end{vmatrix}$$

$$= \frac{\prod\limits_{0\le i<j\le N} (\mu_i - \mu_j) \prod\limits_{0\le i<j\le N} (d_j - d_i)}{\prod\limits_{\substack{0\le i\le N\\ 0\le j\le N}} (\mu_i - d_j)}.$$

Consequently,

$$\begin{vmatrix} 1 & (\mu_0 - d_1)^{-1} & \cdots & (\mu_0 - d_N)^{-1} \\ 1 & (\mu_1 - d_1)^{-1} & \cdots & (\mu_1 - d_N)^{-1} \\ \hdotsfor{4} \\ 1 & (\mu_N - d_1)^{-1} & \cdots & (\mu_N - d_N)^{-1} \end{vmatrix}$$

$$= -\lim_{d_0\to\infty} d_0 \begin{vmatrix} (\mu_0 - d_0)^{-1} & (\mu_0 - d_1)^{-1} & \cdots & (\mu_0 - d_N)^{-1} \\ (\mu_1 - d_0)^{-1} & (\mu_1 - d_1)^{-1} & \cdots & (\mu_1 - d_N)^{-1} \\ \hdotsfor{4} \\ (\mu_N - d_0)^{-1} & (\mu_N - d_1)^{-1} & \cdots & (\mu_N - d_N)^{-1} \end{vmatrix}$$

$$= -\lim_{d_0\to\infty} d_0 \frac{\prod\limits_{0\le i<j\le N} (\mu_i - \mu_j) \prod\limits_{0\le i<j\le N} (d_j - d_i)}{\prod\limits_{\substack{0\le i\le N\\ 0\le j\le N}} (\mu_i - d_j)}$$

$$= \frac{\prod\limits_{0\le i<j\le N} (\mu_i - \mu_j) \prod\limits_{1\le i<j\le N} (d_j - d_i)}{\prod\limits_{\substack{0\le i\le N\\ 1\le j\le N}} (\mu_i - d_j)}.$$

This proves the lemma.

THEOREM 1.1.

(a) *If $a \neq 0$ and if $k_0, k_1, \ldots, k_N$ are pairwise different nonnegative integers then the system $\{y_n\}$ $(n = 0, 1, \ldots; n \neq k_0, k_1, \ldots, k_N)$ is minimal in $L_p(0,1)$.*

(b) *If $a = 0$ and if $k_1, \ldots, k_N$ are pairwise different nonnegative integers then the system $\{y_n\}$ $(n = 0, 1, \ldots; n \neq k_1, \ldots, k_N)$ is minimal in $L_p(0,1)$.*

Proof. (a) It suffices to show the existence of a system $\{u_n\}$ biorthogonal to $\{y_n\}$ $(n = 0, 1, \ldots; n \neq k_0, k_1, \ldots, k_N)$ in $L_p(0,1)$.

Note that

$$\frac{d}{dx}(y_n(x)y_m'(x) - y_m(x)y_n'(x)) = (\lambda_n - \lambda_m)y_m(x)y_n(x)$$

for $0 \le x \le 1$. By integrating this identity from 0 to 1, we obtain

$$(1.1) \qquad (\lambda_n - \lambda_m)(y_n, y_m) = (y_n(x)y_m'(x) - y_m(x)y_n'(x))\big|_0^1,$$

where $(\cdot,\cdot)$ is the Hilbert space inner product on $L_2(0,1)$.

From (0.2), we obtain

$$y_n(0)y_m'(0) - y_m(0)y_n'(0) = 0 \tag{1.2}$$

for all $n, m = 0, 1, \ldots$.

Let $\lambda_n, \lambda_m \neq c_j$ for $j = 1, \ldots, N$. Then by (0.3),

$$\begin{aligned} y_n(1)y_m'(1) - y_m(1)y_n'(1) &= (h(\lambda_m) - h(\lambda_n))y_m(1)y_n(1) \\ &= (\lambda_m - \lambda_n)\left(a + \sum_{k=1}^{N} \frac{b_k}{(\lambda_n - c_k)(\lambda_m - c_k)}\right)y_n(1)y_m(1). \end{aligned} \tag{1.3}$$

Now suppose that $\lambda_n = c_s$ for some $s \in \{1, \ldots, N\}$. Then by (0.3), $y_n(1) = 0$. Hence

$$y_n(1)y_m'(1) - y_m(1)y_n'(1) = -y_n'(1)y_m(1) \tag{1.4}$$

for $\lambda_m \neq c_s$ $(m = 0, 1, \ldots)$.

From (1.1)–(1.4), it follows that for $m \neq n$,

$$(y_n, y_m) = \begin{cases} -\left(a + \displaystyle\sum_{k=1}^{N} \frac{b_k}{(\lambda_n - c_k)(\lambda_m - c_k)}\right)y_n(1)y_m(1) & \\ & \text{if } \lambda_n, \lambda_m \neq c_1, \ldots, c_N, \\ \dfrac{y_n'(1)y_m(1)}{\lambda_m - c_s} & \text{if } \lambda_n = c_s. \end{cases} \tag{1.5}$$

Let $\lambda_k \neq c_j$ for all $k = 0, 1, \ldots$ and $j = 1, \ldots, N$. We define elements of the system $\{u_n\}$ $(n = 0, 1, \ldots; n \neq k_0, k_1, \ldots, k_N)$ by

$$u_n(x) = \frac{A_{n,k_0,\ldots,k_N}(x)}{B_n \Delta}, \tag{1.6}$$

where

$$A_{n,k_0,\ldots,k_N}(x) = \begin{vmatrix} y_n(x) & y_n(1) & \frac{y_n(1)}{\lambda_n - c_1} & \frac{y_n(1)}{\lambda_n - c_2} & \cdots & \frac{y_n(1)}{\lambda_n - c_N} \\ y_{k_0}(x) & y_{k_0}(1) & \frac{y_{k_0}(1)}{\lambda_{k_0} - c_1} & \frac{y_{k_0}(1)}{\lambda_{k_0} - c_2} & \cdots & \frac{y_{k_0}(1)}{\lambda_{k_0} - c_N} \\ y_{k_1}(x) & y_{k_1}(1) & \frac{y_{k_1}(1)}{\lambda_{k_1} - c_1} & \frac{y_{k_1}(1)}{\lambda_{k_1} - c_2} & \cdots & \frac{y_{k_1}(1)}{\lambda_{k_1} - c_N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ y_{k_N}(x) & y_{k_N}(1) & \frac{y_{k_N}(1)}{\lambda_{k_N} - c_1} & \frac{y_{k_N}(1)}{\lambda_{k_N} - c_2} & \cdots & \frac{y_{k_N}(1)}{\lambda_{k_N} - c_N} \end{vmatrix}, \tag{1.7}$$

$$B_n = \|y_n\|^2 + \left(a + \sum_{k=1}^{N} \frac{b_k}{(\lambda_n - c_k)^2}\right)y_n^2(1), \tag{1.8}$$

$$\Delta = \frac{\displaystyle\prod_{0\le i<j\le N} (\lambda_{k_i} - \lambda_{k_j}) \cdot \prod_{1\le i<j\le N} (c_j - c_i)}{\displaystyle\prod_{\substack{0\le i\le N \\ 1\le j\le N}} (\lambda_{k_i} - c_j)} \prod_{0\le i\le N} y_{k_i}(1). \tag{1.9}$$

Let us verify that $(u_n, y_m) = \delta_{n,m}$ $(n, m = 0, 1, \ldots;\ n, m \neq k_0, k_1, \ldots, k_N)$, where $\delta_{n,m}$ is Kronecker's symbol. Indeed, from (1.6) and (1.7) we have

$$(1.10)\quad (u_n, y_m) = \frac{1}{B_n \varDelta} \begin{vmatrix} (y_n, y_m) & y_n(1) & \frac{y_n(1)}{\lambda_n - c_1} & \frac{y_n(1)}{\lambda_n - c_2} & \cdots & \frac{y_n(1)}{\lambda_n - c_N} \\ (y_{k_0}, y_m) & y_{k_0}(1) & \frac{y_{k_0}(1)}{\lambda_{k_0} - c_1} & \frac{y_{k_0}(1)}{\lambda_{k_0} - c_2} & \cdots & \frac{y_{k_0}(1)}{\lambda_{k_0} - c_N} \\ (y_{k_1}, y_m) & y_{k_1}(1) & \frac{y_{k_1}(1)}{\lambda_{k_1} - c_1} & \frac{y_{k_1}(1)}{\lambda_{k_1} - c_2} & \cdots & \frac{y_{k_1}(1)}{\lambda_{k_1} - c_N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ (y_{k_N}, y_m) & y_{k_N}(1) & \frac{y_{k_N}(1)}{\lambda_{k_N} - c_1} & \frac{y_{k_N}(1)}{\lambda_{k_N} - c_2} & \cdots & \frac{y_{k_N}(1)}{\lambda_{k_N} - c_N} \end{vmatrix}.$$

It is now immediate from (1.5) that for $m \neq n$ the first column of the determinant in (1.10) is a linear combination of the other columns; hence $(u_n, y_m) = 0$ for $n \neq m$.

Assume now that $n = m$ in (1.10). Adding to the first column the $2\text{nd}, 3\text{rd}, \ldots, (N+2)\text{th}$ columns multiplied respectively by

$$a y_n(1),\ \frac{b_1 y_n(1)}{\lambda_n - c_1},\ \ldots,\ \frac{b_N y_n(1)}{\lambda_n - c_N},$$

we obtain

$$(u_n, y_n) = \frac{1}{B_n \varDelta} \begin{vmatrix} B_n & y_n(1) & \frac{y_n(1)}{\lambda_n - c_1} & \frac{y_n(1)}{\lambda_n - c_2} & \cdots & \frac{y_n(1)}{\lambda_n - c_N} \\ 0 & y_{k_0}(1) & \frac{y_{k_0}(1)}{\lambda_{k_0} - c_1} & \frac{y_{k_0}(1)}{\lambda_{k_0} - c_2} & \cdots & \frac{y_{k_0}(1)}{\lambda_{k_0} - c_N} \\ 0 & y_{k_1}(1) & \frac{y_{k_1}(1)}{\lambda_{k_1} - c_1} & \frac{y_{k_1}(1)}{\lambda_{k_1} - c_2} & \cdots & \frac{y_{k_1}(1)}{\lambda_{k_1} - c_N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & y_{k_N}(1) & \frac{y_{k_N}(1)}{\lambda_{k_N} - c_1} & \frac{y_{k_N}(1)}{\lambda_{k_N} - c_2} & \cdots & \frac{y_{k_N}(1)}{\lambda_{k_N} - c_N} \end{vmatrix},$$

where we have used the definition (1.8) for B_n. Thus from Lemma 1.1 and the definition (1.9) for $\varDelta$ we obtain

$$(u_n, y_n) = \frac{1}{\varDelta} \begin{vmatrix} 1 & (\lambda_{k_0} - c_1)^{-1} & \cdots & (\lambda_{k_0} - c_N)^{-1} \\ 1 & (\lambda_{k_1} - c_1)^{-1} & \cdots & (\lambda_{k_1} - c_N)^{-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & (\lambda_{k_N} - c_1)^{-1} & \cdots & (\lambda_{k_N} - c_N)^{-1} \end{vmatrix} \cdot \prod_{0 \le i \le N} y_{k_i}(1) = 1.$$

Now consider the case where some of the numbers c_j $(j = 1, \ldots, N)$ are eigenvalues of (0.1)–(0.3). In this case we define

$$(1.11)\qquad u_n(x) = \frac{A'_{n,k_0,\ldots,k_N}(x)}{B'_n \varDelta'},$$

where $A'_{n,k_0,\ldots,k_N}(x)$ is a determinant of order $N+2$ which we obtain from $A_{n,k_0,\ldots,k_N}(x)$ as follows (here we also give the definitions of B'_n and $\varDelta'$):

I. If $\lambda_{k_t} \neq c_j$ ($\lambda_n \neq c_j$) for all $j = 1, \ldots, N$ then column $t+2$ (respectively, the first column) does not change.

II. If $\lambda_{k_t} = c_s$ ($\lambda_n = c_s$) for some t (respectively, n) and s then all the elements in row $t+2$ (respectively, in the first row) vanish, except the first element and $y_{k_t}(1)/(\lambda_{k_t} - c_s)$ (respectively, $y_n(1)/(\lambda_n - c_s)$); the first element does not change but $y_{k_t}(1)/(\lambda_{k_t} - c_s)$ (respectively, $y_n(1)/(\lambda_n - c_s)$) is replaced by $-y'_{k_t}(1)/b_s$ (respectively, by $-y'_n(1)/b_s$).

III. If $\lambda_n \neq c_j$ for all $j = 1, \ldots, N$ then $B'_n = B_n$.

IV. If $\lambda_n = c_s$ for some $s \in \{1, \ldots, N\}$, then $B'_n = \|y_n\|^2 + (y'_n(1))^2/b_s$.

V. Δ' is the complementary minor of the upper left element of the determinant $A'_{n,k_0,\ldots,k_N}$.

For example if $N = 2$, $a \neq 0$, $\lambda_{k_1} = c_2$, $\lambda_n, \lambda_{k_0}, \lambda_{k_2} \neq c_1, c_2$ then

$$A'_{n,k_0,k_1,k_2}(x) = \begin{vmatrix} y_n(x) & y_n(1) & \frac{y_n(1)}{\lambda_n - c_1} & \frac{y_n(1)}{\lambda_n - c_2} \\ y_{k_0}(x) & y_{k_0}(1) & \frac{y_{k_0}(1)}{\lambda_{k_0} - c_1} & \frac{y_{k_0}(1)}{\lambda_{k_0} - c_2} \\ y_{k_1}(x) & 0 & -\frac{y'_{k_1}(1)}{b_s} & 0 \\ y_{k_2}(x) & y_{k_2}(1) & \frac{y_{k_2}(1)}{\lambda_{k_2} - c_1} & \frac{y_{k_2}(1)}{\lambda_{k_2} - c_2} \end{vmatrix},$$

$$\Delta' = \begin{vmatrix} y_{k_0}(1) & \frac{y_{k_0}(1)}{\lambda_{k_2} - c_1} & \frac{y_{k_0}(1)}{\lambda_{k_0} - c_2} \\ 0 & -\frac{y'_{k_1}(1)}{b_s} & 0 \\ y_{k_2}(1) & \frac{y_{k_2}(1)}{\lambda_{k_2} - c_1} & \frac{y_{k_2}(1)}{\lambda_{k_2} - c_2} \end{vmatrix}$$

$$= \frac{\lambda_{k_0} - \lambda_{k_2}}{(\lambda_{k_0} - c_2)(\lambda_{k_2} - c_2)} \cdot y_{k_0}(1) \cdot \left(-\frac{y'_{k_1}(1)}{b_s}\right) \cdot y_{k_2}(1).$$

Let us prove that $\Delta' \neq 0$. From the construction, it follows that each row of Δ' is either of the form $(0, \ldots, 0, -y'_{k_t}(1)/b_s, 0, \ldots, 0)$ (in this case $\lambda_{k_t} = c_s$) or

$$(y_{k_t}(1), y_{k_t}(1)/(\lambda_{k_t} - c_1), \ldots, y_{k_t}(1)/(\lambda_{k_t} - c_N))$$

(in this case $\lambda_{k_t} \neq c_j$ for all $j = 1, \ldots, N$). It can easily be seen from the form of the determinant Δ' and Lemma 1.1 that $\Delta' \neq 0$. The proof now proceeds along the same lines as above.

This concludes the proof for the case $a \neq 0$.

(b) The case $N = 0$ is a classical Sturm–Liouville problem. So we can suppose $N \geq 1$. In this case we construct a biorthogonal system $\{u_n\}$ ($n = 0, 1, \ldots; n \neq k_1, \ldots, k_N$) as in part (a) with obvious modifications. In particular, we obtain the corresponding determinants $A_{n,k_1,\ldots,k_N}(x)$ and

$A'_{n,k_1,\dots,k_N}(x)$ of degree $N+1$ from $A_{n,k_0,\dots,k_N}(x)$ and $A'_{n,k_0,\dots,k_N}(x)$ by deleting the second row and second column.

The proof of Theorem 1.1 is complete.

2. Basisness in $L_p(0,1)$ of the system of eigenfunctions of the boundary value problem (0.1)–(0.3)

THEOREM 2.1.

(a) *If $a \neq 0$ and if $k_0, k_1, \dots, k_N$ are pairwise different nonnegative integers then the system $\{y_n\}$ $(n = 0, 1, \dots;\ n \neq k_0, k_1, \dots, k_N)$ is a basis of $L_p(0,1)$ $(1 < p < \infty)$; moreover if $p = 2$ then this basis is unconditional.*

(b) *If $a = 0$ and if $k_1, \dots, k_N$ are pairwise different nonnegative integers then the system $\{y_n\}$ $(n = 0, 1, \dots; n \neq k_1, \dots, k_N)$ is a basis of $L_p(0,1)$ $(1 < p < \infty)$; moreover if $p = 2$ then this basis is unconditional.*

Proof. It was proved in [1] that

$$\lambda_n = (\pi(n+\nu))^2 + O(1),$$

where

$$\nu = \begin{cases} -1/2 - N & \text{if } a \neq 0,\ \beta \neq 0, \\ -N, & \text{if } a \neq 0,\ \beta = 0, \\ -N, & \text{if } a = 0,\ \beta \neq 0, \\ 1/2 - N, & \text{if } a = 0,\ \beta = 0. \end{cases} \tag{2.1}$$

This gives, for sufficiently large n,

$$\sqrt{\lambda_n} = \pi(n+\nu) + O(1/n). \tag{2.2}$$

Denote by $\psi_1(x,\mu)$ and $\psi_2(x,\mu)$ a fundamental system of solutions of the differential equation $u'' - q(x)u + \mu^2 u = 0$, with initial conditions

$$\psi_1(0,\mu) = 1, \quad \psi_1'(0,\mu) = i\mu, \tag{2.3}$$

$$\psi_2(0,\mu) = 1, \quad \psi_2'(0,\mu) = -i\mu. \tag{2.4}$$

It is well known (see [9] or [11, Ch. II, §4.5]) that for sufficiently large μ,

$$\psi_j(x,\mu) = \exp(\mu\omega_j x)(1 + O(1/\mu)) \quad (j = 1,2), \tag{2.5}$$

where $\omega_1 = -\omega_2 = i$.

We seek the eigenfunction y_n corresponding to the eigenvalue λ_n in the form

$$y_n(x) = P_n \begin{vmatrix} \psi_1(x,\sqrt{\lambda_n}) & \psi_2(x,\sqrt{\lambda_n}) \\ U(\psi_1(x,\sqrt{\lambda_n})) & U(\psi_2(x,\sqrt{\lambda_n})) \end{vmatrix}, \tag{2.6}$$

where

$$P_n = \begin{cases} (i\sqrt{2\lambda_n}\sin\beta)^{-1} & \text{if } \beta \neq 0, \\ (i\sqrt{2})^{-1} & \text{if } \beta = 0, \end{cases} \tag{2.7}$$

and

$$U(\psi(x)) = \psi(0)\cos\beta - \psi'(0)\sin\beta, \tag{2.8}$$

for any $\psi \in C^1[0,1]$. From (2.1)–(2.8) we easily obtain

$$y_n(x) = \begin{cases} \sqrt{2}\cos(n-1/2-N)\pi x + O(1/n) & \text{if } a \neq 0,\ \beta \neq 0, \\ \sqrt{2}\sin(n-N)\pi x + O(1/n) & \text{if } a \neq 0,\ \beta = 0, \\ \sqrt{2}\cos(n-N)\pi x + O(1/n) & \text{if } a = 0,\ \beta \neq 0, \\ \sqrt{2}\sin(n+1/2-N)\pi x + O(1/n) & \text{if } a = 0,\ \beta = 0. \end{cases} \tag{2.9}$$

From now on we shall give the details only for the case $a \neq 0$, $\beta \neq 0$. We define the elements of the system $\{\varphi_n\}$ $(n = 0, 1, \dots;\ n \neq k_0, k_1, \dots, k_N)$ as follows:

$$\varphi_n(x) = \begin{cases} \sqrt{2}\cos(j_n - 1/2)\pi x & (n = 0, 1, \dots, k^*;\ n \neq k_0, k_1, \dots, k_N), \\ \sqrt{2}\cos(n-1/2-N)\pi x & (n = k^*+1, k^*+2, \dots), \end{cases}$$

where $k^* = \max(k_0, \dots, k_N)$, and $\{j_n\}$ $(n = 0, 1, \dots, k^*; n \neq k_0, \dots, k_N)$ is an increasing (k^*-N)-term sequence of numbers from $\{1, \dots, k^*-N\}$. It is obvious that this system is identical to the system $\{\sqrt{2}\cos(n-1/2-N)\pi x\}$ $(n = N+1, N+2, \dots)$, which is a basis of $L_p(0,1)$, and in particular, an orthonormal basis of $L_2(0,1)$ (see for example [10]).

Let $\|\cdot\|_p$ denote the norm in $L_p(0,1)$.

Firstly we prove that the system $\{y_n\}$ $(n = 0, 1, \dots;\ n \neq k_0, \dots, k_N)$ is an unconditional basis of $L_2(0,1)$. For this we compare the system

$$\{y_n\}\ (n = 0, 1, \dots;\ n \neq k_0, k_1, \dots, k_N) \tag{2.10}$$

with $\{\varphi_n\}$ $(n = 0, 1, \dots;\ n \neq k_0, k_1, \dots, k_N)$. From (2.9) it follows that for sufficiently large n,

$$\|y_n - \varphi_n\|_2 \leq \mathrm{const}/n.$$

Therefore the series

$$\sum_{n=0;\, n\neq k_0,\dots,k_N}^{\infty} \|y_n - \varphi_n\|_2^2$$

is convergent. Hence in this case the system (2.10) is quadratically close to $\{\varphi_n\}$ $(n = 0, 1, \dots; n \neq k_0, k_1, \dots, k_N)$, which is an orthonormal basis of $L_2(0,1)$ as mentioned above. Since the system (2.10) is minimal in $L_2(0,1)$, our claim is established for $p = 2$ (see [4, Sect. 9.9.8 of the Russian translation]).

For the remaining part of the theorem the following asymptotic formula will be needed:

$$u_n(x) = y_n(x) + O(1/n), \tag{2.11}$$

for sufficiently large n.

It follows from (2.9) that

$$\|y_n\|_2 = 1 + O(1/n), \tag{2.12}$$

$$y_n(1) = O(1/n). \tag{2.13}$$

Let $\lambda_n \neq c_j$ for all $n = 0, 1, \ldots$ and $j = 1, \ldots, N$. For this case the system $\{u_n\}$ $(n = 0, 1, \ldots;\, n \neq k_0, k_1, \ldots, k_N)$ is defined by (1.6)–(1.9). Then by (1.8), (2.12) and (2.13),

$$B_n = 1 + O(1/n). \tag{2.14}$$

Expanding the determinant (1.7) along the first row and taking into account that all elements in other rows are either bounded functions or fixed real numbers, we deduce from (1.6)–(1.9), (2.13) and (2.14) that the formula (2.11) is true.

The case in which some of the numbers c_j $(j = 1, \ldots, N)$ are eigenvalues of the boundary value problem (0.1)–(0.3) can be treated in a similar way. In this case for the proof of (2.11) we use the corresponding representations for the functions $\{u_n\}$ $(n = 0, 1, \ldots;\, n \neq k_0, k_1, \ldots, k_N)$ for sufficiently large n (see **I**–**III**, **V** from the previous section).

The asymptotic formulas

$$y_n(x) = \varphi_n(x) + O(1/n), \tag{2.15}$$

$$u_n(x) = \varphi_n(x) + O(1/n), \tag{2.16}$$

are also valid for sufficiently large n. This follows immediately from (2.9) and (2.11).

We are now ready to prove our claim for $p \neq 2$. Let $1 < p < 2$ be fixed. It was seen above that the system (2.10) is a basis of $L_2(0, 1)$. Thus this system is complete in $L_p(0, 1)$. Hence, for basisness in $L_p(0, 1)$ of the system (2.10) it is sufficient to show the existence of a constant $M > 0$ such that

$$\Big\| \sum_{n=1;\, n\neq k_0,\ldots,k_N}^{T} (f, u_n) y_n \Big\|_p \leq M \cdot \|f\|_p \quad (T = 1, 2, \ldots) \tag{2.17}$$

for all $f \in L_p(0, 1)$ (see [7, Ch. I, §4]).

By (2.15) and (2.16),

$$\begin{aligned} \Big\| \sum_{n=1;\, n\neq k_0,\ldots,k_N}^{T} (f, u_n) y_n \Big\|_p &\leq \Big\| \sum_{n=1;\, n\neq k_0,\ldots,k_N}^{T} (f, \varphi_n) \varphi_n \Big\|_p \\ &+ \Big\| \sum_{n=1;\, n\neq k_0,\ldots,k_N}^{T} (f, u_n) O(1/n) \Big\|_p + \Big\| \sum_{n=1;\, n\neq k_0,\ldots,k_N}^{T} (f, O(1/n)) \varphi_n \Big\|_p. \end{aligned} \tag{2.18}$$

We shall now prove that all the summands on the right hand side of (2.18) are bounded from above by $\text{const} \cdot \|f\|_p$.

Since $\{\varphi_n\}$ $(n=0,1,\dots;\, n\neq k_0,k_1,\dots,k_N)$ is a basis of $L_p(0,1)$, we have

$$\left\|\sum_{n=1;\, n\neq k_0,\dots,k_N}^{T} (f,\varphi_n)\varphi_n\right\|_p \leq \text{const}\cdot\|f\|_p, \tag{2.19}$$

for all $f\in L_p(0,1)$ (see [7, Ch. I, §4]). Applying Hölder's and Minkowski's inequalities, and (2.16), we obtain

$$\begin{aligned}\left\|\sum_{n=1;\, n\neq k_0,\dots,k_N}^{T} (f,u_n)O(1/n)\right\|_p &\leq \text{const}\cdot\sum_{n=1;\, n\neq k_0,\dots,k_N}^{T} |(f,u_n)|\,\frac{1}{n} \\ &\leq \text{const}\cdot\Big(\sum_{n=1;\, n\neq k_0,\dots,k_N}^{T} |(f,u_n)|^q\Big)^{1/q}\cdot\Big(\sum_{n=1;\, n\neq k_0,\dots,k_N}^{T} \frac{1}{n^p}\Big)^{1/p} \\ &\leq \text{const}\cdot\Big[\Big(\sum_{n=1;\, n\neq k_0,\dots,k_N}^{T} |(f,\varphi_n)|^q\Big)^{1/q} \\ &\quad+\Big(\sum_{n=1;\, n\neq k_0,\dots,k_N}^{T} |(f,O(1/n))|^q\Big)^{1/q}\Big],\end{aligned} \tag{2.20}$$

where $1/p+1/q=1$.

Note that $\{\varphi_n\}$ $(n=0,1,\dots;\, n\neq k_0,k_1,\dots,k_N)$ is an orthonormal uniformly bounded function system. Thus by F. Riesz's theorem (see [13, Ch. XII, Theorem 2.8]),

$$\Big(\sum_{n=1;\, n\neq k_0,\dots,k_N}^{T} |(f,\varphi_n)|^q\Big)^{1/q} \leq \text{const}\cdot\|f\|_p. \tag{2.21}$$

Using the well known fact (see e.g. [3, Sect. 2.2.4]) that $\|f\|_p$ is a non-decreasing function of p, we have

$$\begin{aligned}&\Big(\sum_{n=1;\, n\neq k_0,\dots,k_N}^{T} |(f,O(1/n))|^q\Big)^{1/q} \\ &\qquad\leq \text{const}\cdot\|f\|_1\cdot\Big(\sum_{n=1}^{T}\frac{1}{n^q}\Big)^{1/q} \leq \text{const}\cdot\|f\|_p.\end{aligned} \tag{2.22}$$

Similarly, for the third summand of (2.18), using Parseval's equality we have

$$\left\|\sum_{n=1;\, n\neq k_0,\dots,k_N}^{T} (f,O(1/n))\varphi_n\right\|_p \leq \left\|\sum_{n=1;\, n\neq k_0,\dots,k_N}^{T} (f,O(1/n))\varphi_n\right\|_2 \tag{2.23}$$

$$= \Big(\sum_{n=1;\, n\neq k_0,\ldots,k_N}^{T} |(f, O(1/n))|^2 \Big)^{1/2}$$

$$\le \text{const} \cdot \|f\|_1 \cdot \left(\sum_{n=1}^{T} \frac{1}{n^2} \right)^{1/2} \le \text{const} \cdot \|f\|_p.$$

Finally, (2.17) follows from (2.18)–(2.23). Hence the system (2.10) is a basis of $L_p(0,1)$ $(1 < p < 2)$.

Let $2 < p < \infty$. It is obvious that the system $\{u_n\}$ $(n = 0, 1, \ldots; n \neq k_0, k_1, \ldots, k_N)$ is a basis of $L_p(0,1)$. Therefore this system is complete in $L_q(0,1)$, where $1/p + 1/q = 1$. Note that $1 < q < 2$.

Using the same kind of argument, one can prove that $\{u_n\}$ $(n = 0, 1, \ldots; n \neq k_0, k_1, \ldots, k_N)$ is a basis of $L_q(0,1)$. It follows that (2.10) is a basis of $L_p(0,1)$ $(2 < p < \infty)$.

The proofs for the cases $a \neq 0$, $\beta = 0$; $a = 0$, $\beta \neq 0$; $a = 0$, $\beta = 0$ are similar if we note the fact that each of the systems

$$\begin{aligned} &\{\sqrt{2}\sin(n-N)\pi x\} && (n = N+1, N+2, \ldots), \\ &\{\sqrt{2}\cos(n-N)\pi x\} && (n = N, N+1, \ldots), \\ &\{\sqrt{2}\sin(n+1/2-N)\pi x\} && (n = N, N+1, \ldots), \end{aligned}$$

is a basis of $L_p(0,1)$ $(1 < p < \infty)$, and in particular, an orthonormal basis of $L_2(0,1)$ (see e.g. [10]).

The proof of Theorem 2.1 is complete.

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