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THE OSCILLATION PROPERTIES OF THE BOUNDARY VALUE PROBLEM WITH SPECTRAL PARAMETER IN THE BOUNDARY CONDITION

Abstract

The spectral problem is investigated for the fourth order ordinary differential operator with spectral parameter in the boundary conditions. The oscillation properties of the eigenfunctions of this problem are established.

Research of boundary value problems with spectral parameter in the equation and in the boundary conditions form the important part of the spectral theory of linear differential operators. Many concrete problems of mathematical physics (see for example, [1, pp.149-152] and [2-5]) are led to such type problems.

The main purpose of this paper is to study the ocillation properties of the eigenfunctions of the boundary value problem

$$(p(x)y'')'' - (q(x)\,y')' = \lambda\rho(x)\,y,\ \ 0 < x < l, \tag{1}$$

$$y'(0)\cos\alpha - (py'')(0)\sin\alpha = 0, \tag{2.a}$$

$$y(0)\cos\beta + Ty(0)\sin\beta = 0, \tag{2.b}$$

$$y'(l)\cos\gamma + (py'')(l)\sin\gamma = 0, \tag{2.c}$$

$$(a\lambda + b)\,y(l) - (c\lambda + d)\,Ty(l) = 0, \tag{2.d}$$

where

$$Ty \equiv (py'')' - qy',$$

and λ is a spectral parameter, the functions $p(x)$, $q(x)$, $\rho(x)$ are strictly positive and continuous on $[0, l]$, $p(x)$ has an absolutely continuous derivative, $q(x)$ is absolutely continuous on $[0, l]$, $\alpha, \beta, \gamma, a, b, c, d$ are real constants, moreover $0 \le \alpha, \beta, \gamma \le \frac{\pi}{2}$ and

$$\sigma = bc - ad > 0. \tag{3}$$

As in [6-8] to study the oscillation properties of eigenfunctions and its derivatives the Prűfer type transformation is used:

$$y(x) = r(x)\sin\psi(x)\cos\theta(x), \tag{4.a}$$

$$y'(x) = r(x)\cos\psi(x)\sin\varphi(x), \tag{4.b}$$

$$(py'')(x) = r(x)\cos\psi(x)\cos\varphi(x), \tag{4.c}$$

$$Ty(x) = r(x)\sin\psi(x)\sin\theta(x). \tag{4.d}$$

We first write equation (1) as the equivalent first order system

$$U' = MU, \tag{5}$$

where

$$U = \begin{pmatrix} y \\ y' \\ py'' \\ Ty \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1/p & 0 \\ 0 & q & 0 & 1 \\ \lambda\rho & 0 & 0 & 0 \end{pmatrix}.$$

Assuming $w(x) = ctg\psi(x)$ and using transformation (4) in (5) we'll obtain the system of a first order differential equations for the functions r, w, θ, φ of the following form:

$$r' = \left[\sin 2\psi \sin(\theta+\varphi) + \left(q + \frac{1}{p}\right)\cos^2\psi \sin 2\varphi + \right.$$

$$\left. +\lambda\rho\sin^2\psi\sin 2\theta\right]\frac{r}{2}, \tag{6.a}$$

$$w' = -w^2\cos\theta\sin\varphi + \frac{1}{2}\left(q+\frac{1}{p}\right)w\sin 2\varphi +$$

$$+\sin\theta\cos\varphi - \frac{\lambda\rho}{2}w\sin 2\theta, \tag{6.b}$$

$$\theta' = -w\sin\varphi\sin\theta + \lambda\rho\cos^2\theta, \tag{6.c}$$

$$\varphi' = \frac{1}{p}\cos^2\varphi - q\sin^2\varphi - \frac{1}{w}\sin\theta\sin\varphi. \tag{6.d}$$

The following statements hold.

Lemma 1 (see lemma 3 from [8]). *For each fixed $\lambda \in \mathbf{C}$ there exists a unique (to within constant multiplier) nontrivial solution $y(x,\lambda)$ of problem (1), (2.a), (2.b), (2.c).*

Remark 1 (see remark 1 from [8]). Without loss of generality, for every fixed $x \in [0,l]$ the function $y(x,\lambda)$ can be taken as entire function of λ.

It holds the following

Lemma 2. *The eigenvalues of boundary value problem (1)-(2) are real.*

Proof. Let $y(x,\lambda)$ be a nontrivial solution of problem (1), (2.a), (2.b), (2.c). Then eigenvalues of problem (1)-(2) are the roots of the equation

$$(a\lambda + b)\,y(l,\lambda) - (c\lambda + d)\,Ty(l,\lambda) = 0. \tag{7}$$

If λ is a non-real eigenvalue of problem (1)-(2), then $\overline{\lambda}$ will also be eigenvalue of this problem. Really, as the coefficients $p(x), q(x), \rho(x), \alpha, \beta, \gamma, a, b, c, d,$ are real, then $y\left(x,\overline{\lambda}\right)=\overline{y\left(x,\lambda\right)}$. Consequently, if the relation (7) holds for λ, then it will be as well as true also for $\overline{\lambda}$.

By virtue of (1) we have

$$\left(Ty\left(x,\mu\right)\right)'y\left(x,\lambda\right)-\left(Ty\left(x,\lambda\right)\right)'y\left(x,\mu\right)=\left(\mu-\lambda\right)\rho\left(x\right)\left(x,\mu\right)y\left(x,\lambda\right).$$

Integrating this identity in the range from 0 to l, using formula of integration by parts and taking into account (2.a), (2.b), (2.c), we get

$$y\left(l,\lambda\right)Ty\left(l,\mu\right)-y\left(l,\mu\right)Ty\left(l,\lambda\right)=\left(\mu-\lambda\right)\int\limits_0^l\rho\left(x\right)y\left(x,\mu\right)y\left(x,\lambda\right)dx. \tag{8}$$

Assuming $\mu=\overline{\lambda}$ in (8), we'll obtain

$$y\left(l,\lambda\right)\overline{Ty\left(l,\lambda\right)}-\overline{y\left(l,\lambda\right)}Ty\left(l,\lambda\right)=\left(\overline{\lambda}-\lambda\right)\int\limits_0^l\rho\left(x\right)\left|y\left(x,\lambda\right)\right|^2dx. \tag{9}$$

We'll rewrite condition (2.d) in the form

$$Ty\left(l,\lambda\right)=\frac{a\lambda+b}{c\lambda+d}y\left(l,\lambda\right).$$

Taking into account last relation from (9) we get

$$\frac{\left(\overline{\lambda}-\lambda\right)\left(ad-bc\right)\left|y\left(l,\lambda\right)\right|^2}{\left|c\lambda+d\right|^2}=\left(\overline{\lambda}-\lambda\right)\int\limits_0^l\rho\left(x\right)\left|y\left(x,\lambda\right)\right|^2dx.$$

As $\overline{\lambda}\neq\lambda$, then hence it follows the equality

$$\frac{-\sigma\left|y\left(l,\lambda\right)\right|^2}{\left|c\lambda+d\right|^2}=\int\limits_0^1\rho\left(x\right)\left|y\left(x,\lambda\right)\right|^2dx.$$

The last equality contradicts to the conditions $\sigma>0$, $\int\limits_0^l\rho\left(x\right)\left|y\left(x,\lambda\right)\right|^2dx>0$. Hence, λ must be real. Lemma 2 is proved.

Lemma 3. *The eigenvalues of boundary value problem (1)-(2) form no more than countable set, which has no finite limit point. All eigenvalues of boundary value problem (1)-(2) are simple.*

Proof. The eigenvalues are the zeros of the entire function standing at the left-hand side of equation (7). We show (lemma 2) that this function doesn't vanish for non-real λ. Hence, it is not equal to zero identically. Therefore, its zeros form no more than countable set, which has no finite limit point.

Let's prove, that equation (7) has only simple roots. Really, if $\lambda=\lambda^*$ is a multiple root of equation (7), then it holds the equalities

$$(a\lambda^* + b)\, y\,(l,\lambda^*) - (c\lambda^* + d)\, Ty\,(l,\lambda^*) = 0, \tag{10}$$

$$ay\,(l,\lambda^*) + (a\lambda^* + b)\,\frac{\partial}{\partial\lambda} y\,(l,\lambda^*) - cTy\,(l,\lambda^*) -$$

$$- (c\lambda^* + d)\,\frac{\partial}{\partial\lambda} Ty\,(l,\lambda^*) = 0. \tag{11}$$

Dividing the both parts of (8) by $(\mu - \lambda)$ and passing to the limit as $\mu \to \lambda$ we'll obtain

$$y\,(l,\lambda)\,\frac{\partial}{\partial\lambda} Ty\,(l,\lambda) - Ty\,(l,\lambda)\,\frac{\partial}{\partial\lambda} y\,(l,\lambda) = \int\limits_0^l \rho\,(x)\, y^2\,(x,\lambda)\, dx. \tag{12}$$

Supposing in equality (12) $\lambda = \lambda^*$ we'll have

$$y\,(l,\lambda^*)\,\frac{\partial}{\partial\lambda} Ty\,(l,\lambda^*) - Ty\,(l,\lambda^*)\,\frac{\partial}{\partial\lambda} y\,(l,\lambda^*) = \int\limits_0^l \rho\,(x)\, y^2\,(x,\lambda^*)\, dx. \tag{13}$$

As $\sigma \neq 0$, then $(a,\lambda^* + b)^2 + (c\lambda^* + d)^2 \neq 0$. Let $c\lambda^* + d \neq 0$. From (10) and (11) it follows respectively

$$Ty\,(l,\lambda^*) = \frac{a\lambda^* + b}{c\lambda^* + d} y\,(l,\lambda^*)\,,$$

$$\frac{\partial}{\partial\lambda} Ty\,(l,\lambda^*) = \frac{a\lambda^* + b}{c\lambda^* + d}\frac{\partial}{\partial\lambda} y\,(l,\lambda^*) - \frac{\sigma}{(c\lambda^* + d)^2} y\,(l,\lambda^*)\,.$$

Using the last two equalities in (13), we get

$$-\frac{\sigma}{(c\lambda^* + d)^2} y^2\,(l,\lambda^*) = \int\limits_0^l \rho\,(x)\, y^2\,(l,\lambda^*)\, dx,$$

that is not impossible by virtue of condition (3).

The case $a\lambda^* + b \neq 0$ is considered similarly. Lemma 3 is proved.

The following statements are true.

Lemma 4 (see theorem 3.1 from [7]). *Let $y\,(x,\lambda)$ be a nontrivial solution of problem (1), (2.a), (2.b), (2.c) for $\lambda > 0$. Then the Jacobian $J\,[y] = r^3 \cos\psi \sin\psi$ of transformation (4)* does not vanish in $(0,l)$.

Lemma 5 (see lemma 5 from [8]). *Let $y\,(x,\lambda)$ be a nontrivial solution of problem (1), (2.a), (2.b), (2.c) and one of the following conditions be fulfilled: (i) $\lambda < 0$; (ii), $\lambda = 0$, $\beta \in \left[0, \frac{\pi}{2}\right)$. Then the Jacobian $J\,[y] = r^3 \cos\psi \sin\psi$ of transformation (4)* does not vanish in $(0,l)$.

Let $y\,(x,\lambda)$ be a nontrivial solution of problem (1), (2.a), (2.b), (2.c), and either $\lambda \in \mathbf{R}\backslash\{0\}$, or $\lambda = 0, \beta \in \left[0, \frac{\pi}{2}\right)$. Suppose that $\theta\,(x,\lambda)$ and $\varphi\,(x,\lambda)$ are corresponding functions from (4). Without loss of generality, we'll define the initial values of these functions in the following way

$$\theta(0,\lambda)=\beta-\frac{\pi}{2},\quad \varphi(0,\lambda)=\alpha, \tag{14}$$

where $\alpha=0$, when $\psi(0)=\frac{\pi}{2}$.

The proof of this fact is led completely by the scheme of the proof of theorem 3.1 from [6]. (see also [7, theorem 3.3]).

Lemma 6 (see theorem 4.2. from [7]). *Let $y(x,\lambda)$ be a nontrivial solution of problem (1), (2.a), (2.b), (2.c) for $\lambda>0$. Then $\theta(l,\lambda)$ is a strictly increasing continuous function of λ.*

Let's introduce the following boundary condition

$$y(l)\cos\delta-Ty(l)\sin\delta=0,\quad \delta\in\left[0,\frac{\pi}{2}\right]. \tag{2.d'}$$

Theorem 1 (see theorems 5.4 and 5.5 from [7]). *The eigenvalues of boundary value problem (1), (2.a), (2.b), (2.c), (2.d') (except the case $\beta=\delta=\frac{\pi}{2}$) form an infinitely increasing sequence $\{\lambda_n^\delta\}_1^\infty$ such that*

$$0<\lambda_1^\delta<\lambda_2^\delta<...<\lambda_n^\delta<...,$$

$$\theta\left(l,\lambda_n^\delta\right)=(2n-1)\frac{\pi}{2}-\delta,\quad n\in\mathbf{N}. \tag{15}$$

Besides, the eigenfunction $v_n^\delta(x)$, corresponding to the eigenvalue λ_n^δ, has exactly $n-1$ simple zeros in the interval $(0,l)$, and the function $Tv_n^\delta(x)$ has exactly n zeros on the segment $[0,l]$.

Remark 2 (see [8]). In the case $\beta=\delta=\frac{\pi}{2}$ the first eigenvalue of boundary value problem (1), (2.a), (2.b), (2.c), (2.d') is equal to zero and the corresponding eigenfunction is a constant. In this case the statement of theorem 1 is valid for $n\geq 2$.

It is obvious, that the eigenvalues $\mu_n=\lambda_n^0$ and $\nu_n=\lambda_n^{\pi/2}$ $(n\in\mathbf{N})$ are zeros of the entere functions $y(l,\lambda)$ and $Ty(l,\lambda)$ respectively.

Let's consider the function $\dfrac{Ty(l,\lambda)}{y(l,\lambda)}$ for $\lambda\in K\equiv\bigcup\limits_{n=1}^{\infty}\left(\mu_{n-1},\mu_n\right)$, where $\mu_0=-\infty$.

It holds the following

Lemma 7 (see lemma 6 from [8]). *The function $\dfrac{Ty(l,\lambda)}{y(l,\lambda)}$ in each of the interval $\left(\mu_{n-1},\mu_n\right)$ $(n=1,2,...)$ is strictly increasing function of λ.*

Lemma 8 (see lemma 7 from [8]). *Let $y(x,\lambda)$ be a nontrivial solution of problem (1), (2.a), (2.b), (2.c). Then it holds the relation*

$$\lim_{\lambda\to-\infty}\frac{Ty(l,\lambda)}{y(l,\lambda)}=-\infty.$$

Remark 3 (see [8]). It is easy to observe that if $\lambda<0$ or $\lambda=0$, $\beta\in\left[0,\frac{\pi}{2}\right)$, then $\dfrac{Ty(l,\lambda)}{y(l,\lambda)}<0$; if $\lambda=0$ and $\beta=\frac{\pi}{2}$, then $Ty(l,\lambda)=0$. Besides, the relation $\nu_1<\mu_1<\nu_2<\mu_2<...$ is valid.

Lemma 9 (see lemma 8 from [8]). *Let $y(x,\lambda)$ be a nontrivial solution of problem (1), (2.a), (2.b), (2.c). If $\lambda \le 0$, then $y(x,\lambda) \ne 0$ for $x \in (0,l)$.*

Let $y(x,\lambda)$ be a nontrivial solution of problem (1), (2.a), (2.b), (2.c). Denote by $m(\lambda)$ the quantity of zeros of function $y(x,\lambda)$ in the interval $(0,l)$.

Lemma 10. *If $\lambda \in (\mu_{n-1},\mu_n]$, $n \in \mathbf{N}$, then $m(\lambda) = n-1$.*

Proof. It is easy to observe, that the statement of the lemma for $\lambda \in (\mu_0, 0]$ follows from lemma 9.

Let $\lambda > 0$ and $\theta(x,\lambda)$ be a corresponding angular function determined by transformation (4). From (14) it follows that $\theta(0,\lambda) = \beta - \frac{\pi}{2} \in \left[-\frac{\pi}{2}, 0\right]$. From (15) we have $\theta(l,\mu_n) = (2n-1)\frac{\pi}{2}$ $(n \in \mathbf{N})$. It is known (see theorems 5.1 and 5.2 from [7]) that if $\lambda > 0$, then the function $\theta(x,\lambda)$ takes the values of the form $\frac{k\pi}{2}$ $(k \ge -1)$ only strictly increasing. Taking into account the statement of lemma 6 we get that $\theta(x,\lambda) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for $\lambda \in (0,\mu_1]$, $x \in (0,l)$; $\theta(x,\lambda) \in \left(-\frac{\pi}{2}, (2n-1)\frac{\pi}{2}\right)$ for $\lambda \in (\mu_{n-1},\mu_n]$, $x \in (0,l)$, $n \ge 2$. Hence, we obtain the truth of statement of lemma 10 for $\lambda > 0$. Lemma 10 is proved.

In the case $c \ne 0$ we'll define the natural number N_0 from the inequality

$$\mu_{N_{0-1}} < -\frac{d}{c} \le \mu_{N_0}.$$

The following theorem are the main results of the paper.

Theorem 2. *The eigenvalues of boundary value problem (1)-(2) form the infinitely increasing sequence $\{\lambda_n\}_1^\infty$, such that $\lambda_1 < \lambda_2 < ... < \lambda_n < ...$. Besides,*

(a) if $c = 0$, then the eigenfunction $y_n(x)$, corresponding to the eigenvalue λ_n, has exactly $n-1$ simple zeros in the interval $(0,l)$;

(b) if $c \ne 0$, then the eigenfunction $y_n(x)$, corresponding to the eigenvalue λ_n, has exactly $n-1$ simple zeros in $(0,l)$ for $n \le N_0$, and exactly $n-2$ simple zeros in $(0,l)$ for $n > N_0$.

Proof. Let $y(x,\lambda)$ be a nontrivial solution of problem (1), (2.a), (2.b), (2.c). By virtue of lemma 7 $\Phi(\lambda) = \frac{Ty(l,\lambda)}{y(l,\lambda)}$ is continuous strictly increasing function in the interval (μ_{n-1},μ_n) $(n \in \mathbf{N})$. From lemma 8 and equality $y(l,\mu_n) = 0$ $(n \in \mathbf{N})$ it follows that

$$\lim_{\lambda \to \mu_{n-1}+0} \Phi(\lambda) = -\infty, \quad \lim_{\lambda \to \mu_n - 0} \Phi(\lambda) = +\infty,$$

and besides this function takes each value from $(-\infty,+\infty)$ only at a unique point of the interval (μ_{n-1},μ_n).

Let $H(\lambda) = \frac{a\lambda + b}{c\lambda + d}$. We have $H'(\lambda) = \frac{-\sigma}{(c\lambda+d)^2}$. As by supposition (3) $\sigma > 0$, then the function $H(\lambda)$ strictly decreases in the interval $(-\infty,+\infty)$ for $c = 0$, and the function $H(\lambda)$ $c \ne 0$ strictly decreases in each of the intervals $\left(-\infty, -\frac{d}{c}\right)$ and $\left(-\frac{d}{c}, +\infty\right)$ for $c \ne 0$; moreover

$$\lim_{\lambda \to -\frac{d}{c}-0} h(\lambda) = -\infty, \quad \lim_{\lambda \to -\frac{d}{c}+0} H(\lambda) = +\infty. \tag{16}$$

Let $c = 0$ or $c \neq 0, -d/c \notin \left(\mu_{n-1}, \mu_n\right]$. From the above-stated it follows, that in the interval $\left(\mu_{n-1}, \mu_n\right)$ there will be found a unique value $\lambda = \lambda_n^*$, for which

$$\Phi(\lambda) = H(\lambda), \tag{17}$$

i.e. condition (2.d) is fulfilled. Consequently, λ_n^* is the eigenvalue of boundary value problem (1)-(2) and $y(x, \lambda_n^*)$ is a corresponding eigen function. From lemma 10 it follows that $m(\lambda_n^*) = n - 1$. It is easy to observe, that if $c = 0$ or $c \neq 0,\ n < N_0$ then λ_n^* is the $n-$th eigenvalue of boundary value problem (1)-(2). Hence we obtain the truth of statement (a) and the truth of a part of statement (b) concerning to the case $n < N_0$.

Let $c \neq 0$ and $-d/c \in \left(\mu_{N_0-1}, \mu_{N_0}\right)$. In a similar way, in each of the intervals $\left(\mu_{N_0-1}, -d/c\right)$ and $\left(-d/c, \mu_{N_0}\right)$ there can be found a unique value (λ_{N_0} and λ_{N_0+1}, respectively) which (17) is satisfied i.e. condition (2.d) is fulfilled. From lemma 10 it follows that $m(\lambda_{N_0}) = m(\lambda_{N_0+1}) = N_0 - 1$.

The case $c \neq 0, -d/c = \mu_{N_0}$ is considered in a completely similar way. At that it is used the fact, that μ_{N_0} in this case is also eigenvalue of boundary value problem (1)-(2). In the given case $\lambda_{N_0} \in \left(\mu_{N_0-1}, \mu_{N_0}\right), \lambda_{N_0+1} = \mu_{N_0}$. By virtue of the lemma 10 we have $m(\lambda_{N_0}) = m(\lambda_{N_0+1}) = N_0 - 1$.

It is easy to see that for $c \neq 0$ and $n > N_0$ a unique solution λ_n^* of equation (17) in the interval $\left(\mu_{n-1}, \mu_n\right]$ is the $(n+1)$ -th eigenvalue of boundary value problem (1)-(2), i.e. $\lambda_n^* = \lambda_{n+1}$. As $m(\lambda_n^*) = n - 1$, then $m(\lambda_{n+1}) = n - 1 = (n+1) - 2$.

From the above-stated discussions we obtain the truth of estimation (b).

Theorem 2 is proved.

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