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Some Generalizations of Riesz-Fisher Theorem

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Abstract

In the paper are obtained the generalizations of Housdorff-Young, Riesz and Paley type theorems with respect to uniformly orthonormed system for the case of the space $L_{(p,q)}$ with the mixed norm.

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1 Introduction

One of the important problems of theory of Fourier series is estimation of Fourier coefficients by a function, and vice versa, the function by its Fourier coefficients. In this connection, in the space L_2 the Parseval inequality and Riesz-Fisher theorem are known. In principle, this means that a space of coefficients of an orthogonal basis in Hilbert space L_2 coincides with l_2. Partially these results are extended on the space L_p $(p \neq 2)$ by Hausdorff–Young statements for trigonometric systems, by F. Riesz statements for general orthogonal and Hardy Littlewood-Paley systems. According to F. Riesz theorem, to each function of the space $L_p[a, b]$ $(1 < p \leq 2)$ with respect to some orthonormed system there corresponds the sequence of its Fourier coefficients from $l_{p'}$ $\left(\frac{1}{p'} + \frac{1}{p}\right)$, and vice versa. This fact doesn't remain valid for $p > 2$.

However, for $p > 2$ the Paley theorem is associated to this result. These matters are well interpreted in [8;10;11;16;19]. Another direction in this field is study of behavior of Fourier coefficients under some smoothness conditions on a function. The Bernstein theorem on absolute convergence of Fourier series of functions of the class $Lip\alpha\,(\alpha > 1/2)$ is the basic result of this matter. Further, the Fourier series with vector coefficients, Banach spaces of Fourier type p, of type and cotype of Rademaher are of great interest in this direction. One can be acquainted with these matters in [1;4;5;15].

The Housdorff-Young, Riesz and Paley type theorems play a special role while studying many problems (for example, in grounding the Fourier method for solving partial equations, in theory of wavelet analysis and also in theory of frames and etc.) in harmonic analysis. The present paper is devoted to obtaining generalizations of these theorems for the case of b-basis. In the paper, relations between the functions of the space $L_{(p,q)}$ with appropriate sequence of coefficients of its Fourier series of the space of sequences l_p with ordinary norm are established. These results are extended to the case of the space $W_{(p,q)}$. It should be noticed that the results obtained in the papers [1-3;6;7;9;12-14;17;18] are very close to the matters considered in this paper.

2 Some notion and facts

Give some notion and facts used in the paper.

Let $L_p(a,b)$ $(1 < p < +\infty)$ be a Banach space of measurable on (a,b) functions $f(x)$ for which the norm

$$\|f\|_{L_p} = \left(\int_a^b |f(x)|^p \, dx \right)^{1/p}$$

is finite. $W_p^m\,(a,b)$ $(1 < p < +\infty)$ is a Banach of functions $u(x)$ having generalized derivatives on (a,b) m-th order, ininclusively, for which the norm

$$\|u\|_{W_p^m} = \sum_{k=0}^{m} \left\| u^{(k)} \right\|_{L_p}$$

is finite.

$L_{(p,p')}((a,b)\times(c,d))$ $\left(1 < p < +\infty,\ \frac{1}{p} + \frac{1}{p'} = 1\right)$ is a Banach space of measurable on $(a,b)\times(c,d)$ functions $f(x,y)$ for which the mixed norm

$$\|f\|_{L_{(p,p')}} = \left(\int_a^b \left(\int_c^d |f\,(x,y)|^{p'} \, dy \right)^{p/p'} dx \right)^{1/p}$$

is finite. $W^{m,n}_{(p,p')}((a,b)\times(c,d))$ $\left(1<p<+\infty, \frac{1}{p}+\frac{1}{p'}=1\right)$ is a Banach space of functions $u(x,y)$ having on (a,b) the generalized derivatives $\frac{\partial^m u}{\partial x^m}$, $\frac{\partial^n u}{\partial y^n}$ and the finite mixed norm

$$\|u\|_{W^{m,n}_{(p,p')}} = \|u\|_{L_{(p,p')}} + \left\|\frac{\partial^m u}{\partial x^m}\right\|_{L_{(p,p')}} + \left\|\frac{\partial^n u}{\partial y^n}\right\|_{L_{(p,p')}}.$$

$l_p(a,b)$ $(1<p<+\infty)$ is a Banach space of sequences $\overline{a\,(t)}=\{a_i\,(t)\}_{i\in N}$ of measurable on (a,b) functions $a_i\,(t)$, for which the norm

$$\|\bar{a}\|_{l_p(a,b)} = \left(\sum_{i=1}^{\infty}\int_a^b |a_i(t)|^p\,dt\right)^{1/p}$$

is finite.

Let $\{\varphi_n(t)\}$ be an orthonormed system on $[a,b]$. Then the series $\sum\limits_{i=1}^{\infty} a_i\varphi_i(t)$, where $a_i=\int\limits_a^b f(t)\varphi_i(t)dt$ is said to be a Fourier series of the function $f(t)$ by the system $\{\varphi_n(t)\}$, is written as $f(t)\sim\sum\limits_{i=1}^{\infty} a_i\varphi_i(t)$, and the numbers a_i are called the Fourier coefficients of the function $f(t)$ by the system $\{\varphi_n(t)\}$.

Give the following Riesz and Paley classic statements: (see [10.ch. 6]).

Theorem 2.1. *Let $\{\varphi_n(t)\}$ be an orthonormed system on $[a,b]$ such that almost everywhere on $[a,b]$: $|\varphi_n\,(t)|\le M$ $(n\in N)$, M is independent of n and $\frac{1}{p'}+\frac{1}{p}$. Then:*

1) if $f\in L_p[a,b]$ $\;(1<p\le 2)$, then $\|\{a_i\}\|_{l_{p'}}\le M^{\frac{2-p}{p}}\|f\|_{L_p}$, where a_i are the Fourier coefficients of the function $f(t)$ by the system $\{\varphi_n(t)\}$;

2) if $\{a_i\}\in l_p$ $\;(1<p\le 2)$, then there exists the function $f\in L_{p'}[a,b]$ for which a_i are Fourier coefficients by the system $\{\varphi_n(t)\}$, and $\|f\|_{L_{p'}}\le m_p\|\,\{a_i\}\,\|_{l_p}$, where m_p is independent of $\{a_i\}$ and f.

Theorem 2.2. *Let $\{\varphi_n(t)\}$ be an orthonormed system such that almost everywhere on $[a,b]:|\varphi_n\,(t)|\le M$ $(n\in N)$, M is independent of n. Then:*

1) if $f\in L_p[a,b]$ $\;(1<p\le 2)$, then $\sum\limits_{i=1}^{\infty}|a_i|^p\,i^{p-2}\le M_p\int\limits_a^b|f(t)|^p\,dt$, where M_p is independent of f and a_i are the Fourier coefficients of the function $f(t)$;

2) if $\{a_i\}:\sum_{i=1}^{\infty}|a_i|^q\,i^{q-2}<+\infty\,(q\ge 2)$, then there exists a function $f\in L_p\,[a,b]$ for which a_i are Fourier coefficients, and

$$\int_a^b |f\,(t)|^q\,dt \le M_q\sum_{i=1}^{\infty}|a_i|^q\,i^{q-2},$$

where M_q is independent of $\{a_i\}$.

3 $l_p(a,b)$ variants

In this section, we establish the analogues of Riesz and Paley theorems for the case of spaces $L_{(p',p)}$ and $l_p(a,b)$, and also study the analogues of the results on convergence rate of Fourier series depending on smoothness degree of the function for the functions of space $W^{m,n}_{(p,p')}$ by the system of exponents.

Let $\Pi = (a,b) \times (c,d)$. The following theorems are valid.

Theorem 3.1. *Let $\{\varphi_n(t)\}$ be an orthonormed system on $[c,d]$ such that almost everywhere on $[c,d]$:$|\varphi_n(t)| \leq (n \in N)$, M is independent of n, and $\frac{1}{p'} + \frac{1}{p}$. Then:*

1) if $f \in L_{(p',p)}(\Pi)$ $(1 < p \leq 2)$, then $\overline{a}(t) = \{a_i(t)\}_{i \in N} \in l_{p'}(a,b)$, where $a_i(t) = \int_c^d f(t,s)\varphi_i(s)\,ds$ and

$$\|\overline{a}\|_{l_{p'(a,b)}} \leq M^{\frac{2-p}{p}} \|f\|_{L_{(p',p)}}. \tag{1}$$

2) if $a(t) = \{a_i(t)\} \in l_p(a,b)$ $(1 < p \leq 2)$, then there exists a function $f \in L_{(p,p')}(\Pi)$ for which $a_i(t) = \int_c^d f(t,s)\varphi_i(s)\,ds$ and

$$\|f\|_{L_{(p,p')}} \leq M^{\frac{2-p}{p}} \|a\|_{l_{p(a,b)}}. \tag{2}$$

Proof. Prove statement 1). Since the function $f \in L_{(p',p)}(\Pi)$, then it follows from the Foubini theorem that for almost all $t \in [a,b]$ the function $f(t,\cdot) \in L_p([c,d])$. Therefore, by the Riesz theorem for $\{a_i(t)\}$, where $a_i(t) = \int\limits_c^d f(t,s)\varphi_i(s)ds$, almost everywhere on $[a,b]$ it holds $\sum\limits_{i=1}^{\infty} |a_i(t)|^{p'} < +\infty$ $\left(\frac{1}{p'} + \frac{1}{p}\right)$ and

$$\left(\sum_{i=1}^{\infty} |a_i(t)|^{p'}\right)^{1/p'} \leq M^{\frac{2-p}{p}} \left(\int_c^d |f(t,s)|^p\,ds\right)^{1/p}. \tag{3}$$

Raising the both hand sides of (3) in p'-th degree and integrating with respect to t on the segment $[a,b]$, we get $\sum\limits_{i=1}^{\infty} \int\limits_a^b |a_i(t)|^{p'}\,dt < +\infty$, moreover

$$\sum_{i=1}^{\infty} \int_a^b |a_i(t)|^{p'}\,dt \leq M^{\frac{2-p}{p} \cdot p'} \int_a^b \left(\int_c^d |f(t,s)|^p\,ds\right)^{p'/p} dt. \tag{4}$$

Obviously, (1) follows from (4).

Now prove validity of statement 2). Let $\{a_i(t)\}$ be such that $\sum\limits_{i=1}^{\infty}\int\limits_a^b |a_i(t)|^p\,dt < +\infty$ $(1 < p \le 2)$. Then, according to the corollary of B. Levi theorem (see [12]) for almost all $t \in [a,b]$: $\sum\limits_{i=1}^{\infty} |a_i(t)|^p < +\infty$. Applying theorem 1.1, we get that almost everywhere on $[a,b]$ there exists a function $f(t,\cdot) \in L_p[c,d]$ for which $a_i(t) = \int\limits_c^d f(t,s)\varphi_i(s)ds$ and

$$\left(\int_c^d |f(t,s)|^{p'}\,ds\right)^{1/p'} \le M^{\frac{2-p}{p}}\left(\sum_{i=1}^{\infty}|a_i(t)|^p\right)^{1/p}. \tag{5}$$

Further, raise the both hand sides of (5) in p-th degree and integrate with respect to t on the segment $[a,b]$. We have

$$\int\limits_a^b\left(\int\limits_c^d |f(t,s)|^{p'}\,ds\right)^{p'/p} dt \le M^{p-2}\sum_{i=1}^{\infty}\int\limits_a^b |a_i(t)|^p\,dt. \tag{6}$$

(2) follows from (6). The theorem is proved.

Theorem 3.2. *Let $\{\varphi_n(t)\}$ be an orthonormed system on $[c,d]$ such that almost everywhere on $[c,d]$: $|\varphi_n(t)| \le M$ $(n \in N)$, M is independent of n. Then:*

1) if $f \in L_p(\Pi)$ $(1 < p \le 2)$, then

$$\sum_{i=1}^{\infty} i^{p-2}\int\limits_a^b |a_i(t)|^p\,dt \le M_p\int\limits_a^b\int\limits_c^d |f(t,s)|^p\,dsdt, \tag{7}$$

where M_p is independent of f and $a_i(t) = \int\limits_c^d f(t,s)\varphi_i(s)ds$;

2) if $\{a_i(t)\}$ is such that $\sum\limits_{i=1}^{\infty} i^{q-2}\int\limits_a^b |a_i(t)|^q\,dt < +\infty$ $(q \ge 2)$, then there exists a function $f \in L_q(\Pi)$ for which $a_i(t) = \int\limits_c^d f(t,s)\varphi_i(s)ds$ and

$$\int\limits_a^b\int\limits_c^d |f(t,s)|^q\,dsdt \le M_q\sum_{i=1}^{\infty} i^{q-2}\int\limits_a^b |a_i(t)|^q\,dt, \tag{8}$$

where M_q is independent of $\{a_i(t)\}$.

At first prove validity of 1). According to Foubini theorem, it follows from $f \in L_p(\Pi)$ that for almost all $t \in [a,b]$ the function $f(t,\cdot) \in L_p[c,d]$. Then, it follows from theorem 2.2 that for $\{a_i(t)\}$, $a_i(t) = \int\limits_c^d f(t,s)\varphi_i(s)ds$, almost everywhere on $[a,b]$ it holds

$$\sum_{i=1}^{\infty} |a_i(t)|^p \, i^{p-2} \le M_p \int\limits_c^d |f(t,s)|^p \, ds. \tag{9}$$

Having integrated the both hand sides of (9) in the segment $[a,b]$, we get validity of (7).

Show validity of 2). Let $\{a_i(t)\}$: $\sum\limits_{i=1}^{\infty} i^{q-2} \int\limits_a^b |a_i(t)|^q \, dt < +\infty$ $(q \ge 2)$. It is clear that almost everywhere on $[a,b]$:$\sum\limits_{i=1}^{\infty} |a_i(t)|^q \, i^{q-2} < +\infty$. Therefore, by theorem 2.2. there exists a function $f \in L_q[c,d]$ for which $a_i(t) = \int\limits_c^d f(t,s)\varphi_i(s)ds$ and

$$\int\limits_a^b |f\,(t,s)|^q \, ds \le M_q \sum_{i=1}^{\infty} |a_i\,(t)|^q \, i^{q-2}. \tag{10}$$

Integrating the both hand sides of (10) in the segment $[a,b]$, we get validity of (8). The theorem is proved.

Give the analogue of the known result on the relation between the smoothness degree of the function and convergence rate of its Fourier series by the system $\{e^{int}\}_{n=-\infty}^{+\infty}$.

Theorem 3.3. *Let* $f \in W^{0,m+1}_{(p',p)}((-\pi,\pi) \times (-\pi,\pi))$ $(1 < p \le 2)$ *and* $\frac{\partial^k f(t,-\pi)}{\partial s^k} = \frac{\partial^k f(t,\pi)}{\partial s^k}$ $(k = 0,..,m)$ *almost everywhere on* $(-\pi,\pi)$. *Then, the series* $\sum\limits_{n=-\infty}^{+\infty} |n|^m \int\limits_{-\pi}^{\pi} |c_n(t)| \, dt$ *converges, where*

$$c_n(t) = \int\limits_{-\pi}^{\pi} f(t,s)e^{-ins}ds, \quad n = 0, \pm 1, \pm 2, ..., \tag{11}$$

moreover,

$$\sum_{n=-\infty}^{+\infty} |n|^m \int_{-\pi}^{\pi} |c_n\,(t)| \, dt \le (2\pi)^{1/p} \left(\sum_{\substack{n=-\infty \\ n \ne 0}}^{+\infty} \frac{1}{|n|^p} \right)^{1/p} \left\| \frac{\partial^{m+1} f}{\partial s^{m+1}} \right\|_{L_{(p',p)}}. \tag{12}$$

Proof. Integrate expression (11) by parts. We have

$$c_n(t)=\int_{-\pi}^{\pi}f(t,s)\,e^{-ins}ds=-\frac{1}{in}f(t,s)\,e^{-ins}\,|_{s=-\pi}^{s=\pi}+\frac{1}{in}\int_{-\pi}^{\pi}\frac{\partial f(t,s)}{\partial s}e^{-ins}ds=$$

$$=\frac{1}{in}\int_{-\pi}^{\pi}\frac{\partial f(t,s)}{\partial s}e^{-ins}ds=-\frac{1}{(in)^2}\frac{\partial f}{\partial s}e^{-ins}\,|_{s=-\pi}^{s=\pi}+$$

$$+\frac{1}{(in)^2}\int_{-\pi}^{\pi}\frac{\partial^2 f(t,s)}{\partial s^2}e^{-ins}ds=...=\frac{1}{(in)^{m+1}}\int_{-\pi}^{\pi}\frac{\partial^{m+1}f(t,s)}{\partial s^{m+1}}e^{-ins}ds=\frac{c_n^{m+1}(t)}{(in)^{m+1}}.$$

So, almost everywhere $c_n(t)=\frac{c_n^{m+1}(t)}{(in)^{m+1}}$, where

$$c_n^{m+1}(t)=\int_{-\pi}^{\pi}\frac{\partial^{m+1}f(t,s)}{\partial s^{m+1}}e^{-ins}ds\,(n=0,\pm1,\pm2,....)\,.$$

Since by the condition $\frac{\partial f^{m+1}(t,s)}{\partial s^{m+1}}\in L_{(p',p)}((-\pi,\pi)\times(-\pi,\pi))$, then by theorem 3.1 we get $\{c_n^{m+1}(t)\}_{n\in N}\in l_{p'}(a,b)$ and

$$\left\|\left\{c_n^{m+1}(t)\right\}\right\|_{l_{p'}(-\pi,\pi)}\le\left\|\frac{\partial^{m+1}f}{\partial s^{m+1}}\right\|_{L_{(p',p)}}. \tag{13}$$

Further, using the Holder inequality, we get

$$\sum_{n=-\infty}^{+\infty}|n|^m\,|c_n(t)|=\sum_{\substack{n=-\infty\\n\neq0}}^{+\infty}\frac{|c_n^{m+1}(t)|}{|n|}\le\le\left(\sum_{\substack{n=-\infty\\n\neq0}}^{+\infty}\frac{1}{|n|^p}\right)^{1/p}\left(\sum_{n=-\infty}^{+\infty}|c_n^{m+1}(t)|^{p'}\right)^{1/p'}. \tag{14}$$

Hence, integrating the both hand sides of (14) with respect to t from $-\pi$ to π, we get

$$\sum_{n=-\infty}^{+\infty}|n|^m\int_{-\pi}^{\pi}|c_n(t)|\,dt\le\left(\sum_{\substack{n=-\infty\\n\neq0}}^{+\infty}\frac{1}{|n|^p}\right)^{1/p}\times$$

$$\times\int_{-\pi}^{\pi}\left(\sum_{n=-\infty}^{+\infty}\left|c_n^{m+1}(t)\right|^{p'}\right)^{1/p'}dt. \tag{15}$$

Applying in (15) the Holder inequality, we have

$$\int_{-\pi}^{\pi}\left(\sum_{n=-\infty}^{+\infty}\left|c_n^{m+1}(t)\right|^{p'}\right)^{1/p'}dt\le(2\pi)^{1/p}\left(\sum_{n=-\infty}^{+\infty}\int_{-\pi}^{\pi}\left|c_n^{m+1}(t)\right|^{p'}dt\right)^{1/p'}. \quad (16)$$

Thus, it follows from (15), (16) and (13) that

$$\sum_{n=-\infty}^{+\infty}|n|^m\int_{-\pi}^{\pi}|c_n(t)|\,dt\le(2\pi)^{1/p}\left(\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}\frac{1}{|n|^p}\right)^{1/p}\times$$

$$\times\left(\sum_{n=-\infty}^{+\infty}\int_{pi}^{pi}\left|c_n^{m+1}(t)\right|^{p'}dt\right)^{1/p'}\le$$

$$\le(2\pi)^{1/p}\left(\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}\frac{1}{|n|^p}\right)^{1/p}\left\|\frac{\partial^{m+1}f}{\partial s^{m+1}}\right\|_{L_{(p',p)}}.$$

The theorem is proved.

Theorem 3.4. *Let $f\in W_{(p',p)}^{0,m}\left((-\pi,\pi)\times(-\pi,\pi)\right)$, and*

$$\frac{\partial^k f(t,-\pi)}{\partial s^k}=\frac{\partial^k f(t,\pi)}{\partial s^k}\,(k=0,...,m-1)$$

almost everywhere on $(-\pi,\pi)$. Then the series $\sum\limits_{n=-\infty}^{+\infty}|n|^{mp'}\int\limits_{-\pi}^{\pi}|c_n(t)|^{p'}dt$ converges, where

$$c_n(t)=\int\limits_{-\pi}^{\pi}f(t,s)e^{-ins}ds,\quad n=0,\pm1,\pm2,...., \quad (17)$$

moreover,

$$\left(\sum_{n=-\infty}^{+\infty}|n|^{mp'}\int_{-\pi}^{\pi}|c_n(t)|^{p'}dt\right)^{1/p}\le\left\|\frac{\partial^m f}{\partial s^m}\right\|_{L_{(p',p)}}. \quad (18)$$

Proof. Having integrated by parts in (17), as in the proof of theorem 3.3 for $c_n(t)$ we get

$$c_n(t) = \frac{c_n^m(t)}{(in)^m}, c_n^m(t) = \int\limits_{-\pi}^{\pi} \frac{\partial^m f(t,s)}{\partial s^m} e^{-ins} ds \ (n = 0, \pm 1, \pm 2, ...).$$

Consequently,

$$c_n^m(t) = (in)^m c_n(t). \tag{19}$$

Since $\frac{\partial f^m(t,s)}{\partial s^m} \in L_{(p',p)}((-\pi,\pi)\times(-\pi,\pi))$, then it follows from theorem 3.1 that $\{c_n^m(t)\}_{n\in N} \in l_{p'}(a,b)$ and as a result of (1) we get

$$\left(\sum_{n=-\infty}^{+\infty} \int_{-\pi}^{\pi} \left| c_n^m(t) \right|^{p'} dt \right)^{1/p'} \leq \left\| \frac{\partial^m f}{\partial s^m} \right\|_{L_{(p',p)}}. \tag{20}$$

Substituting (19) into (20), we get inequality (18). The theorem is proved.

Remark. The obtained results may be used in grounding the Fourier method for the solution of partial equations.

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